Filomat 38:17 (2024), 6059–6086
https://doi.org/10.2298/FIL2417059B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

# Asymptotic formula for the sum of eigenvalues of fourth order differential operator on Banach space

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**Abstract.**
In this paper, we derive a trace formula for self adjoint fourth order differential operator with bounded operator coefficient in the separable Banach space. Compared with the regularized trace problems studied in Hilbert space, the fact that the asymptotic formulas obtained are on Banach space is an important result of the theoretical setting which we considered.

## 1. Introduction and History

Spectral theory of differential operators is an extremely rich field which has been studied by many mathematicians. This area of study has significant applications in both physics and mathematics. One particular aspect of this field that has garnered attention is the study of the regularized trace of differential operators, which was started in the middle of the 20th century with the work of Gelfand and Levitan [16]. The results obtained in the study employed by many mathematicians. After the pioneering work of Gelfand and Levitan, Gasymov [19], Dikiy [18], Levitan [1], Levitan and Sargsyan [2] and Halberg and Kramer [3] found the regularized trace formulas for different type of differential operators. The list of these works on the subject is given by Sadovnichi and Podolsky [32] and Fulton and Pruess [30].
On the other hand trace formulas of abstract self adjoint operators are studied by researchers Gohberg and Krein [15], Gul [8], Bayramoglu and Aslanova [20], Baksi and Sezer [24], Baksi, Karayel and Sezer [23]. Among the publications about regularized trace for Sturm-Liouville type are Chalilova [27], Maksudov, Bayramoglu and Adıgüzelov[10], Adıgüzelov, Avcı and Gul [6], Adıgüzelov [7], Zaki and Hagag [9] and Aslanova [22]. The regularized trace theory allows for the approximate calculation of eigenvalues of such differential operators, which is useful for solving inverse problems and studying integrable systems. Applications of this theory have been studied in various research works by authors such as Guan and Yang [28], Gesztesy, Holden, et al [12]. Isozaki and Korotyaev [14], Kerimov [21], Ismail, Majid and Ibrahim [11] and Allogmany, Ismail, et al [26].
Considering the previous studies on the subject, the regularized traces of higher order differential operators take an important place for understanding the behaviour of physical systems and deriving mathematical

2020 *Mathematics Subject Classification*. 47A10; 34L05; 46B10
*Keywords*. Separable Banach space; trace class operator, spectrum, dense embedding.
Received: 04 September 2023; Revised: 20 January 2024; Accepted: 29 February 2024
Communicated by Dijana Mosić
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results. In particular, the fourth order self adjoint differential operators have significant applications in quantum mechanics, quantum field theory and mathematical physics.

One of the important applications of fourth-order self-adjoint differential operators is in the study of quantum systems with higher-order interactions. Such systems becomes naturally in many physical models, including the study of atoms, molecules, solid-state materials and vibration theory [13], [33].

In our paper, we obtain the second regularized trace formula of the fourth order differential operator and we extend this formula for the first time on an infinite dimensional separable Banach space. Applying the theoretical setting to boundary value problem in [29], we have the extension of the first regularized trace formula in Hilbert space to Banach space.

The main purpose of this article is to show, using the continuous dense embedding theory, and the results about uniqueness of adjoint operator, that the analysis and operator theory for regularized trace in Hilbert space is provided in Banach space.

This paper is structured as follows. In the next section, after we recall some theorems as well as definitions and notations used in this article, we introduce our problem. In section three, we start with the formulation of the regularized trace and give the main results. In the last section, we present the proofs of theorems and give some examples of operators acting on Banach space, which we calculate the regularized trace of them.

## 2. Preliminaries

We begin this section with introducing our problem. Let $X$ be a separable Banach space with dual space $X^*$ and $H$ be a separable Hilbert space with dual space $H^*$ such that $X$ is continuous densely embedded in $H$ , [17]. Consider the operators $L_0$ and $L$ in the Hilbert space $H_1 = L_2(0,\pi; H)$ defined by the differential expressions:

$$\ell_0(y) := y^{IV}(x)$$

$$\ell(y) := y^{IV}(x) + Q(x)y(x) \tag{2.1}$$

with the same boundary conditions

$$y(0) = y''(0) = y'(\pi) = y'''(\pi) = 0, \tag{2.2}$$

respectively. Suppose that the operator function $Q(x)$ in (2.1) satisfies the conditions for every $x \in [0,\pi]$:

Q1. $Q^{(i)}(x) : H \to H \quad (i = 0,1,2,3,4)$ are self adjoint kernel operators. $Q(x)$ has continuous derivative of order 4 with respect to norm $S_1[H]$. $Q^{(i)}(x)$ is the ith derivative with respect to $x$. The functions $\|Q^{(i)}(x)\|_{S_1[H]}$ $(i = 0,1,2,3,4)$ are bounded and measurable.

Q2. $\|Q\|_{H_1} < \frac{5}{2}$ and there is an orthonormal basis $\{\varphi_n\}_{n=1}^{\infty}$ in the space $H$ such that $\sum\limits_{n=1}^{\infty} \|Q(x)\varphi_n\|_{H_1} < \infty$.

Here, $S_1[H]$ is Banach space consisting of all kernel operators from $H$ to $H$, [15]. Further, the sum of eigenvalues of a kernel operator $T$ is denoted by $trT = traceT$.

Every point of the spectrum of the operator $L_0$ is an eigenvalue with infinite multiplicity:

$$\sigma(L_0) = \Big\{(\frac{1}{2})^4, (\frac{3}{2})^4, \cdots, (m+\frac{1}{2})^4, \cdots\Big\},$$

which is also point spectrum $\sigma_p(L_0)$ of the operator $L_0$.

The orthonormal eigenfunctions corresponding to eigenvalue $(m+\frac{1}{2})^4$ have the form:

$$\psi_{mn}(x) = \sqrt{\frac{2}{\pi}} sin(m+\frac{1}{2})x.\varphi_n \qquad (n = 1,2,...). \tag{2.3}$$

We take the space $X_1 = Y_2(0,\pi,X)$ to consist of all measurable functions defined on interval $[0,\pi]$ with values in Banach space $X$ so that the square of the function's norm is integrable with respect to the norm of $X$ and focus on the differential operators $L_0$ and $L$ in $X_1$ with the boundary conditions (2.2).

Assume that $Q(x)$ holds the following conditions:

Q1. $Q(x): X \to X$ is a self-adjoint, kernel operator for every $x \in [0, \pi]$.
Q2. The functions $\|Q^{(i)}(x)\|_{S_1[X]}$ $(i = 0, 1, 2, 3, 4)$ are bounded and measurable for every $x \in [0, \pi]$.
Here, $S_1[X]$ is Banach space consisting of all kernel operators (trace class operators) from $X$ to $X$.

Now, we mention about a new structure which helps us to lift our theory from Hilbert space to Banach space. Accordingly, we employ the following definitions and theorems related to self adjoint operators on Banach Space.

**Theorem 2.1.** *Suppose that X is a separable Banach space, continuously and densely embedded in a Hilbert space H, and A is a bounded linear operator on X and symmetric with respect to the inner product of H (i.e.,* $(Ax, y)_H = (x, Ay)_H$ *for all* $x, y \in X$*). Then,*

1. *A is bounded with respect to the norm in H and* $\|A^*A\|_H = \|A\|_H^2 \le c\|A\|_X^2$ *, where c is a positive constant,*
2. *The spectrum of A in H is the subset of the spectrum of A in X,*
3. $\sigma_p(A)$ *in H is equal to* $\sigma_p(A)$ *in X,*

*[25].*

Let $\mathbf{J}: H \to H^*$ be the (conjugate) isometric isomorphism between Hilbert space and its dual, and $\mathbf{J}_X = \mathbf{J}|_X$ be the restriction of $\mathbf{J}$ on $X$. Since $X$ is densely and continuously embedded in $H$, $\mathbf{J}_X$ is a (conjugate) bijective mapping of $X$ onto $\mathbf{J}_X(X) \subset H^*$ as a continuous dense embedding. It is defined a new norm on $\mathbf{J}_X(X)$, to obtain a continuous dense embedding from $\mathbf{J}_X(X)$ to $H^*$ same as between the spaces $X$ and $H$ .

**Definition 2.2.** *For* $u \in X$ *let* $u_h^* = \mathbf{J}_X(u)$ *and* $u_z^* = \frac{\|u\|_X^2}{\|u\|_H^2} u_h^*$ *and define* $X_z^* = \{u_z^* : u \in X\}$*, with norm* $\left\|u_z^*\right\|_{X_z^*} = \|u\|_X$*.* $X_z^*$ *is called Zachary representation for X in* $H^*$*.*

Here, $X_z^*$ is a separable Banach space such that $X_z^* \subset H^*$ continuous densely embedded and a (conjugate) isometric isomorphic copy of $X$, [4], [5].

**Theorem 2.3.** *Let the functional* $[.,.]$ *on* $X \times X$ *be* $[v, u] = u_z^*(v)$*. Then the functional defines a semi-inner product structure on X , [4], [5].*

Let $B[X]$, $B[H]$ denote the spaces of bounded linear operators on $X$, $H$, respectively. $C[X]$ denote the space of closed densely defined linear operators on $X$. If $A \in C[X]$ and $A'$ is its dual mapping on $X^*$, then there is a unique operator $A^* = \mathbf{J}_X^{-1}A'\mathbf{J}_X \in C[X]$. If $A \in C[X]$ is in $B[X]$, then $\|A^*A\|_X \le M\|A\|_X^2$ , for some constant $M$ and $A$ has a bounded extension to $B[H]$ , [31].

Let $U, V$ be subspaces of $X$. Then $U$ is orthogonal to $V$ if and only if, for each $v \in V, u_z^*(v) = 0, \forall u \in U$ and for each $u \in U, u_z^*(u) = 0, \forall v \in V$.

**Theorem 2.4.** *(Polar Representation) If* $A \in C[X]$*, then there exists a partial isometry U and a self-adjoint operator T with* $D(T) = D(A)$ *and* $A = UT$*. Furthermore,* $T = [A^*A]^{1/2}$ *in a well-defined sense, [31].*

**Theorem 2.5.** *For every* $\phi \in H$*, there exists an element* $\varphi_\phi^* \in X^*$ *and a constant* $c_\phi > 0$ *depending on* $\phi$ *such that* $\left(f, \phi\right)_H = c_\phi^{-1}\left\langle f, \varphi_\phi^*\right\rangle_{X^*}$ *for all* $f \in X$*, [31].*

Let $S_\infty[X]$ be the set of all compact operators on X and $A = U[A^*A]^{1/2} \in S_\infty[X]$ and let $\overline{A}$ be its extension to $H$. By [25], $\sigma_p(\overline{A}) = \sigma_p(A)$, so that $\overline{A}$ is also compact. Thus, without loss of generality there exists a orthonormal family $\left\{\phi_n | n \in \mathbb{N}\right\} \subset X$ such that $\overline{A} = \sum\limits_{n=1}^{\infty} s_n(\overline{A})(., \phi_n)_H \overline{U}\phi_n$.
From Theorem 2.5. and the fact that $s_n(\overline{A}) = s_n(A)$ [25], we state $A$ in the form

$A = \sum_{n=1}^{\infty} s_n(\overline{A})c_n^{-1} \left\langle \cdot, \varphi^*_{\phi_n} \right\rangle_{X^*} U\phi_n.$

If $\overline{A} \in S_p[H]$, the Schatten class of order $p$ in $B[H]$, its norm is defined by

$$\left\|\overline{A}\right\|_p^H = \left\{tr[(\overline{A^*})(\overline{A})]^{p/2}\right\}^{1/p} = \left\{\sum_{n=1}^{\infty} ((\overline{A^*})(\overline{A})\phi_n, \phi_n)_H^{p/2}\right\}^{1/p} = \left\{\sum_{n=1}^{\infty} |s_n(\overline{A})|^p\right\}^{1/p}.$$

**Definition 2.6.** *$S_p[X]$, the Schatten class of order $p$ in $B[X]$, is defined by*

$$S_p[X] = \left\{A \in S_\infty[X] : \|A\|_p^X = \left\{\sum_{n=1}^{\infty} |s_n(A)|^p\right\}^{1/p} < \infty\right\}, 1 \le p < \infty.$$

If $A \in S_p[X]$, from $s_n(A) = s_n(\overline{A})$ then $\overline{A} \in S_p[H]$ and $\|A\|_p^X = \left\|\overline{A}\right\|_p^H$, [31].

If $A \in S_\infty[X]$ then, by the polar representation theorem, $A^*A$ is a non negative self-adjoint operator and $|A| = [A^*A]^{1/2} \in S_\infty[X]$, where $A^*$ is the adjoint of $A$. Let $s_1(A) \ge s_2(A) \ge \cdots \ge s_k(A)$ $(1 \le k \le \infty)$ be the non zero eigenvalues of $|A|$ with each eigenvalue is repeated as many times as its multiplicity (s-numbers). When $k < \infty$, we assume that $s_j(A) = 0$ for $j = k+1, k+2, ....$

## 3. Main Results

In this section, we derive the second regularized trace formula for the operator $L$. We begin this operation by establishing relations between the resolvents and eigenvalues of $L_0$ and $L$. We denote the resolvent operators of $L_0$ and $L$ by $R_\lambda^0$ and $R_\lambda$.

Since the operator function $Q(x) : X \to X$ in (2.1) holds conditions $Q1, Q2$, we can show the following expressions:

a. $QR_\lambda^0 \in S_1[H_1]$ for every $\lambda \notin \sigma(L_0)$.

b. The spectrum of the operator $L$ is a subset of the union of intervals $F_m = \left[(m+\frac{1}{2})^4 - \|Q\|, (m+\frac{1}{2})^4 + \|Q\|\right]$ $(m = 0, 1, 2, ...)$, which are pairwise disjoint, i.e. $\sigma(L) \subset \bigcup_{m=0}^{\infty} F_m$.

c. Every point different from $(m+\frac{1}{2})^4$ on $F_m$ is a discrete eigenvalue with finite multiplicity in the spectrum of $L$.

d. The series $\sum_{n=1}^{\infty}\left[\lambda_{mn} - (m+\frac{1}{2})^4\right]$ $(m = 0, 1, 2, ...)$ are absolutely convergent, where $\{\lambda_{mn}\}_{n=1}^{\infty}$ is the set of eigenvalues of the operator $L$ in the interval $F_m$.

Let $\rho(L)$ be resolvent set of the operator $L$ i.e. $\rho(L) = \mathbf{C}\backslash\sigma(L)$. Since $QR_\lambda^0 \in S_1[H_1]$ for every $\lambda \in \rho(L)$, from the equation

$$R_\lambda = R_\lambda^0 - R_\lambda Q R_\lambda^0, \tag{3.1}$$

we obtain $R_\lambda - R_\lambda^0 \in S_1[H_1]$ .

On the other hand, if we consider the series $\sum_{n=1}^{\infty}\left[\lambda_{mn} - (m+\frac{1}{2})^4\right]$ $(m = 0, 1, 2, ...)$ are absolutely convergent then we get

$$tr(R_\lambda - R_\lambda^0) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{\lambda_{mn} - \lambda} - \frac{1}{(m+\frac{1}{2})^4 - \lambda}\right],$$

for every $\lambda \in \rho(L)$ [27]. Multiplying both sides of this equality with $\frac{\lambda^2}{2\pi i}$ and integrating over the circle $|\lambda| = b_p = (p+1)^4$ $(p = 1, 2, ...)$ , then we have

$$\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2 tr(R_\lambda - R_\lambda^0)d\lambda = \frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2\sum_{m=0}^{p}\sum_{n=1}^{\infty}\Big[\frac{1}{\lambda_{mn}-\lambda}-\frac{1}{(m+\frac{1}{2})^4-\lambda}\Big]d\lambda + \frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2\sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\Big[\frac{1}{\lambda_{mn}-\lambda}-\frac{1}{(m+\frac{1}{2})^4-\lambda}\Big]d\lambda.$$

If we compute the value of integrals above by using the cases $|\lambda_{mn}| < b_p$ for $m = 0, 1, 2, ..., p$ and $|\lambda_{mn}| > b_p$ for $m = p+1, p+2, ...$ , $n = 1, 2, 3, ...$, then we obtain

$$\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2 tr(R_\lambda - R_\lambda^0)d\lambda = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\Big[(m+\frac{1}{2})^8 - \lambda_{mn}^2\Big]. \tag{3.2}$$

Using (3.1) and (3.2), we find

$$\sum_{m=0}^{p}\sum_{n=1}^{\infty}\Big[\lambda_{mn}^2-(m+\frac{1}{2})^8\Big] = \frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2 tr[R_\lambda^0(QR_\lambda^0)]d\lambda - \frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2 tr[R_\lambda^0(QR_\lambda^0)^2]d\lambda + \frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2 tr[R_\lambda^0(QR_\lambda^0)^3]d\lambda - \frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2 tr[R_\lambda^0(QR_\lambda^0)^4]d\lambda + \frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2 tr[R_\lambda(QR_\lambda^0)^5]d\lambda$$

$$\sum_{m=0}^{p}\sum_{n=1}^{\infty}\Big[\lambda_{mn}^2-(m+\frac{1}{2})^8\Big] = \sum_{j=1}^{4}\frac{(-1)^{j+1}}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2 tr[R_\lambda^0(QR_\lambda^0)^j]d\lambda + \frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2 tr[R_\lambda(QR_\lambda^0)^5]d\lambda. \tag{3.3}$$

If we set

$$M_{pj} = \frac{(-1)^{j+1}}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2 tr[R_\lambda^0(QR_\lambda^0)^j]d\lambda, \quad (p = 1, 2, ..., \quad j = 1, 2, 3, 4)$$

$$M_p = \frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\lambda^2 tr[R_\lambda(QR_\lambda^0)^5]d\lambda, \quad (p = 1, 2, ...), \tag{3.4}$$

then the equality (3.3) becomes

$$\sum_{m=0}^{p}\sum_{n=1}^{\infty}\Big[\lambda_{mn}^2-(m+\frac{1}{2})^8\Big] = M_{p1} + M_{p2} + M_{p3} + M_{p4} + M_p. \tag{3.5}$$

Since the operator function $Q(x)$ satisfies the condition Q2, we have

$$M_{pj} = \frac{(-1)^j}{\pi i j} \int\limits_{|\lambda|=b_p} \lambda tr[(QR^0_\lambda)^j] d\lambda. \tag{3.6}$$

Before we give the main result, we need to give the following theorems :

**Theorem 3.1.** *If the operator function $Q(x)$ holds conditions Q1 and Q2, then*

$$M_{p1} = \frac{2}{\pi}\sum_{m=0}^{p}(m+\frac{1}{2})^4 \int_0^\pi trQ(x)dx - \frac{2}{\pi}\sum_{m=0}^{p}(m+\frac{1}{2})^4 \int_0^\pi trQ(x)\cos(2m+1)xdx.$$

**Theorem 3.2.** *If the operator function $Q(x)$ holds conditions Q1 and Q2, then*

$$M_{p2} = \frac{p+1}{2\pi}\int_0^\pi trQ^2(x)dx + \frac{p+1}{2\pi^2} tr\Big(\int_0^\pi Q(x)dx\Big)^2 - \frac{1}{\pi}\sum_{m=0}^{p}\int_0^\pi trQ^2(x)\cos(2m+1)xdx + O(p^{-1}),$$

*where, $O(p^{-1})$ is a function depending on p and i, satisfying the inequality $|O(p^{-1})| < cp^{-1}$ (c=constant).*

Now we are ready to give the main result of our paper:

**Theorem 3.3.** *If the operator function $Q(x)$ holds conditions Q1 and Q2, then we have the second regularized trace formula of L on Banach space $X_1$*

$$\sum_{m=0}^{\infty}\Bigg[\sum_{n=1}^{\infty}\Big(\lambda^2_{mn} - (m+\frac{1}{2})^8\Big) - \frac{2}{\pi}(m+\frac{1}{2})^4\int_0^\pi trQ(x)dx - \frac{1}{8\pi}(2m+1)^2\Big(trQ'(\pi)+trQ'(0)\Big)$$

$$+\frac{1}{8\pi}\Big(trQ'''(\pi)+trQ'''(0)\Big) - C\Bigg] = -\frac{1}{32\pi}\Big(trQ^{IV}(0) + 8trQ^2(0) - trQ^{IV}(\pi) - 8trQ^2(\pi)\Big),$$

*where, $C = \frac{1}{2\pi}\int_0^\pi trQ^2(x)dx + \frac{1}{2\pi^2}tr\Big(\int_0^\pi trQ(x)dx\Big)^2$ is a constant.*

## 4. Proofs

**Proof of Theorem 3.1.** By (2.3), (3.6) and the condition Q1, we have

$$\begin{aligned}
M_{p1} &= -\frac{1}{\pi i}\int\limits_{|\lambda|=b_p}\lambda tr(QR^0_\lambda)d\lambda = -\frac{1}{\pi i}\int\limits_{|\lambda|=b_p}\lambda\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(QR^0_\lambda\Psi_{mn},\Psi_{mn})_{X_1}d\lambda \\
&= 2\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(Q\Psi_{mn},\Psi_{mn})_{X_1}\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\frac{\lambda}{\lambda-(m+\frac{1}{2})^4}d\lambda = 2\sum_{m=0}^{p}\sum_{n=1}^{\infty}(m+\frac{1}{2})^4(Q\Psi_{mn},\Psi_{mn})_{X_1} \\
&= 2\sum_{m=0}^{p}\sum_{n=1}^{\infty}(m+\frac{1}{2})^4\int_0^\pi\Bigg(Q(x)\sqrt{\frac{2}{\pi}}\sin(m+\frac{1}{2})x\varphi_n, \sqrt{\frac{2}{\pi}}\sin(m+\frac{1}{2})x\varphi_n\Bigg)_z dx \\
&= 2\sum_{m=0}^{p}\sum_{n=1}^{\infty}(m+\frac{1}{2})^4\frac{2}{\pi}\int_0^\pi\sin^2(m+\frac{1}{2})x\,(Q(x)\varphi_n,\varphi_n)_z\,dx
\end{aligned}$$

$$= 2\sum_{m=0}^{p}(m+\frac{1}{2})^4\frac{1}{\pi}\int_0^{\pi}\left(\sum_{n=1}^{\infty}\left(Q(x)\varphi_n,\varphi_n\right)_z(1-\cos(2m+1)x)\right)dx$$

$$= \frac{2}{\pi}\sum_{m=0}^{p}(m+\frac{1}{2})^4\left(\int_0^{\pi}\sum_{n=1}^{\infty}\left(Q(x)\varphi_n,\varphi_n\right)_z dx-\int_0^{\pi}\sum_{n=1}^{\infty}\left(Q(x)\varphi_n,\varphi_n\right)_z\cos(2m+1)xdx\right)$$

$$= \frac{2}{\pi}\sum_{m=0}^{p}(m+\frac{1}{2})^4\left(\int_0^{\pi}trQ(x)dx-\frac{2}{\pi}\sum_{m=0}^{p}(m+\frac{1}{2})^4\int_0^{\pi}trQ(x)\cos(2m+1)xdx\right),$$

where, the Zachary functional on $X$ is $(.,.)_z$ . □

**Proof of Theorem 3.2.** We now evaluate $M_{p2}$ by using (3.6)

$$M_{p2}=\frac{-1}{2\pi i}\int_{|\lambda|=b_p}\lambda tr(QR_\lambda^0)^2d\lambda=\frac{-1}{2\pi i}\int_{|\lambda|=b_p}\lambda\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}((QR_\lambda^0)^2\Psi_{mn},\Psi_{mn})_{X_1}d\lambda. \tag{4.1}$$

Moreover, we know that $(QR_\lambda^0)\Psi_{mn}=Q\left(\left(m+\frac{1}{2}\right)^4-\lambda\right)^{-1}\Psi_{mn}$ and

$$(QR_\lambda^0)^2\Psi_{mn}=QR_\lambda^0\left(QR_\lambda^0\Psi_{mn}\right)=QR_\lambda^0\frac{Q\Psi_{mn}}{\left(m+\frac{1}{2}\right)^4-\lambda}=\frac{1}{\left(m+\frac{1}{2}\right)^4-\lambda}Q\left(R_\lambda^0Q\Psi_{mn}\right)$$

$$=\frac{1}{\left(m+\frac{1}{2}\right)^4-\lambda}Q\left[\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\frac{(Q\Psi_{mn},\Psi_{rq})_{X_1}\Psi_{rq}}{(r+\frac{1}{2})^4-\lambda}\right]$$

$$=\frac{1}{\left(m+\frac{1}{2}\right)^4-\lambda}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\frac{(Q\Psi_{mn},\Psi_{rq})_{X_1}Q\Psi_{rq}}{(r+\frac{1}{2})^4-\lambda}. \tag{4.2}$$

If we substitute (4.2) in (4.1), then we obtain

$$M_{p2} = \frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{\left(m+\frac{1}{2}\right)^4-\lambda}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\frac{(Q\Psi_{mn},\Psi_{rq})_{X_1}Q\Psi_{rq}}{(r+\frac{1}{2})^4-\lambda},\Psi_{mn}\right)_{H_1}d\lambda$$

$$= \frac{1}{2\pi i}\int_{|\lambda|=b_p}\lambda\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\frac{(Q\Psi_{mn},\Psi_{rq})_{X_1}(Q\Psi_{rq},\Psi_{mn})_{X_1}}{[(m+\frac{1}{2})^4-\lambda][(r+\frac{1}{2})^4-\lambda]}d\lambda$$

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2\frac{1}{2\pi i}\int_{|\lambda|=b_p}\frac{\lambda}{\left(\lambda-\left(m+\frac{1}{2}\right)^4\right)\left(\lambda-\left(r+\frac{1}{2}\right)^4\right)}d\lambda. \tag{4.3}$$

By separating the series according to $m$ and $r$ into four series and applying the Cauchy Integral Formula, we obtain $M_{p2}$ in the following form

$$\begin{aligned} M_{p2} &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=0}^{p}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\frac{\lambda}{\left(\lambda-\left(m+\frac{1}{2}\right)^4\right)\left(\lambda-\left(r+\frac{1}{2}\right)^4\right)}d\lambda \\ &+ \sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{p}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\frac{\lambda}{\left(\lambda-\left(m+\frac{1}{2}\right)^4\right)\left(\lambda-\left(r+\frac{1}{2}\right)^4\right)}d\lambda \\ &+ \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\frac{\lambda}{\left(\lambda-\left(m+\frac{1}{2}\right)^4\right)\left(\lambda-\left(r+\frac{1}{2}\right)^4\right)}d\lambda \\ &+ \sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\frac{\lambda}{\left(\lambda-\left(m+\frac{1}{2}\right)^4\right)\left(\lambda-\left(r+\frac{1}{2}\right)^4\right)}d\lambda. \end{aligned} \tag{4.4}$$

If $m, r \leq p$, then $\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\frac{\lambda}{\left(\lambda-\left(m+\frac{1}{2}\right)^4\right)\left(\lambda-\left(r+\frac{1}{2}\right)^4\right)}d\lambda = 1$ and if $m, r > p$, then $\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\frac{\lambda}{\left(\lambda-\left(m+\frac{1}{2}\right)^4\right)\left(\lambda-\left(r+\frac{1}{2}\right)^4\right)}d\lambda = 0$.
If we use the results of the integrals into (4.4), we find

$$\begin{aligned} M_{p2} &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=0}^{p}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2 \\ &+ \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\frac{\lambda}{\left(\lambda-\left(m+\frac{1}{2}\right)^4\right)\left(\lambda-\left(r+\frac{1}{2}\right)^4\right)}d\lambda \\ &+ \sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{p}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2\frac{1}{2\pi i}\int\limits_{|\lambda|=b_p}\frac{\lambda}{\left(\lambda-\left(m+\frac{1}{2}\right)^4\right)\left(\lambda-\left(r+\frac{1}{2}\right)^4\right)}d\lambda \\ &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=0}^{p}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2 + \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2\frac{-\left(m+\frac{1}{2}\right)^4}{\left(r+\frac{1}{2}\right)^4-\left(m+\frac{1}{2}\right)^4} \\ &+ \sum_{m=p+1}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{p}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2\frac{-\left(r+\frac{1}{2}\right)^4}{\left(m+\frac{1}{2}\right)^4-\left(r+\frac{1}{2}\right)^4} \\ &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=0}^{p}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2 - 2\sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2\frac{\left(m+\frac{1}{2}\right)^4}{\left(r+\frac{1}{2}\right)^4-\left(m+\frac{1}{2}\right)^4} \\ &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2 - \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2\left(1+\frac{2\left(m+\frac{1}{2}\right)^4}{\left(r+\frac{1}{2}\right)^4-\left(m+\frac{1}{2}\right)^4}\right) \\ &= \sum_{m=0}^{p}\sum_{n=1}^{\infty}\|Q\Psi_{mn}\|_{X_1}^2 - \sum_{m=0}^{p}\sum_{n=1}^{\infty}\sum_{r=p+1}^{\infty}\sum_{q=1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2\frac{\left(r+\frac{1}{2}\right)^4+\left(m+\frac{1}{2}\right)^4}{\left(r+\frac{1}{2}\right)^4-\left(m+\frac{1}{2}\right)^4}. \end{aligned}$$

Let $\alpha_p(n,q) = \sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\left|\left(Q\Psi_{mn},\Psi_{rq}\right)_{X_1}\right|^2 \frac{\left(r+\frac{1}{2}\right)^4+\left(m+\frac{1}{2}\right)^4}{\left(r+\frac{1}{2}\right)^4-\left(m+\frac{1}{2}\right)^4}$. Hence we get

$$M_{p2} = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\|Q\Psi_{mn}\|^2_{X_1} - \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_p(n,q), \tag{4.5}$$

here, we rewrite $\alpha_p(n,q)$ as a sum of three series denoted by $\alpha_{p1},\alpha_{p2},\alpha_{p3}$ , respectively:

$$\begin{aligned}\alpha_p(n,q) &= \frac{1}{\pi^2}\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{\left(r+\frac{1}{2}\right)^4+\left(m+\frac{1}{2}\right)^4}{\left(r+\frac{1}{2}\right)^4-\left(m+\frac{1}{2}\right)^4}\left|\int_0^{\pi}\cos(m-r)x(Q(x)\varphi_n,\varphi_q)_z dx\right|^2 \\ &- \frac{2}{\pi^2}\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{\left(r+\frac{1}{2}\right)^4+\left(m+\frac{1}{2}\right)^4}{\left(r+\frac{1}{2}\right)^4-\left(m+\frac{1}{2}\right)^4}Re\left[\int_0^{\pi}\cos(m-r)x(Q(x)\varphi_n,\varphi_q)_z dx\int_0^{\pi}\cos(m+r+1)x\overline{(Q(x)\varphi_n,\varphi_q)_z}dx\right] \\ &+ \frac{1}{\pi^2}\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\frac{\left(r+\frac{1}{2}\right)^4+\left(m+\frac{1}{2}\right)^4}{\left(r+\frac{1}{2}\right)^4-\left(m+\frac{1}{2}\right)^4}\left|\int_0^{\pi}\cos(m+r+1)(Q(x)\varphi_n,\varphi_q)_z dx\right|^2 \\ &= \alpha_{p1}+\alpha_{p2}+\alpha_{p3}\ .\end{aligned} \tag{4.6}$$

If we substitute (4.6), in (4.5), we find

$$M_{p2} = \sum_{m=0}^{p}\sum_{n=1}^{\infty}\|Q\Psi_{mn}\|^2_{X_1} - \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}(\alpha_{p1}+\alpha_{p2}+\alpha_{p3}). \tag{4.7}$$

Set $E_{pi} = \{(r,m) : r,m \in N, r-m = i, m \le p, r > p\}$, where $p$ and $i$ are positive integer numbers.
First we start with calculating $\alpha_{p1}$

$$\alpha_{p1} = \frac{1}{\pi^2}\sum_{i=1}^{\infty}\left|\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_z\cos ixdx\right|\sum_{m,r\in E}\frac{(2r+1)^4+(2m+1)^4}{(2r+1)^4-(2m+1)^4}\ . \tag{4.8}$$

We focus on the second sum in (4.8). If $i \le p+1 \quad (m \ge 1)$, then

$$\sum_{m,r\in E}\frac{(2r+1)^4+(2m+1)^4}{(2r+1)^4-(2m+1)^4} = p+1+i^2O(\frac{1}{p}), \tag{4.9}$$

here, $O(\frac{1}{p})$ depends on $p$, also satisfies the inequality $0 < O(\frac{1}{p}) < \frac{c}{p} \quad (i \ge 1, p \ge 2) \ (c = constant)$.
If $i \ge p+1$, then

$$\sum_{m,r\in E}\frac{(2r+1)^4+(2m+1)^4}{(2r+1)^4-(2m+1)^4} = O(p), \tag{4.10}$$

here, $O(p)$ satisfies the condition $0 < O(p) < 4p \quad (p \ge 2)$.
We use the results (4.9) and (4.10) in the equality (4.8)

$$\begin{aligned}\alpha_{p1} &= \frac{1}{\pi^2}\sum_{i=1}^{p+1}\left(\sum_{m,r\in E}\frac{(2r+1)^4+(2m+1)^4}{(2r+1)^4-(2m+1)^4}\right)\left|\int_0^\pi (Q(x)\varphi_n,\varphi_q)_z\cos ixdx\right|^2\\ &+ \frac{1}{\pi^2}\sum_{i=p+2}^{\infty}\left(\sum_{m,r\in E}\frac{(2r+1)^4+(2m+1)^4}{(2r+1)^4-(2m+1)^4}\right)\left|\int_0^\pi (Q(x)\varphi_n,\varphi_q)_z\cos ixdx\right|^2\\ &= \frac{1}{\pi^2}(p+1)\sum_{i=1}^{\infty}\left|\int_0^\pi (Q(x)\varphi_n,\varphi_q)_z\cos ixdx\right|^2\\ &+ \frac{1}{\pi^2}\sum_{i=1}^{p+1} i^2 O(p^{-1})\left|\int_0^\pi (Q(x)\varphi_n,\varphi_q)_z\cos ixdx\right|^2\\ &+ \frac{1}{\pi^2}\sum_{i=p+2}^{\infty} O(p)\left|\int_0^\pi (Q(x)\varphi_n,\varphi_q)_z\cos ixdx\right|^2.\end{aligned}$$

Now we obtain an asymptotic formula for the sum $\sum\limits_{n=1}^{\infty}\sum\limits_{q=1}^{\infty}\alpha_{p1}$ in (4.7)

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{p1} &= \pi^{-2}(p+1)\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{i=1}^{\infty}\left|\int_0^\pi (Q(x)\varphi_n,\varphi_q)_z\cos ixdx\right|^2\\ &+ \pi^{-2}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{i=1}^{p+1} i^2O(p^{-1})\left|\int_0^\pi (Q(x)\varphi_n,\varphi_q)_z\cos ixdx\right|^2\\ &+ \pi^{-2}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{i=p+2}^{\infty} O(p)\left|\int_0^\pi (Q(x)\varphi_n,\varphi_q)_z\cos ixdx\right|^2\\ &= \frac{p+1}{\pi}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\left\{\left|\frac{1}{2}\int_0^\pi (Q(x)\varphi_n,\varphi_q)_z dx\right|^2 - \frac{1}{2\pi}\left|\int_0^\pi (Q(x)\varphi_n,\varphi_q)_z dx\right|^2\right\} + O(p^{-1})\\ &= \frac{p+1}{2\pi}\int_0^\pi trQ^2(x)dx - \frac{p+1}{2\pi^2}\sum_{n=1}^{\infty}\left\|\left(\int_0^\pi Q(x)dx\right)\varphi_n\right\|^2 + O(p^{-1})\\ &= \frac{p+1}{2\pi}\int_0^\pi trQ^2(x)dx - \frac{p+1}{2\pi^2}tr\left(\int_0^\pi Q(x)dx\right)^2 + O(p^{-1}).\end{aligned} \tag{4.11}$$

Since $Q(x)$ holds the conditions $Q1$ and $Q2$, we have

$$\left|\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\alpha_{pj}\right| \le \frac{c}{p} \qquad (j=2,3)\ (c = constant). \tag{4.12}$$

Substitude (4.11) and (4.12) in (4.7), we have

$$M_{p2} = \sum_{m=0}^{p}\sum_{n=1}^{\infty} \|Q\Psi_{mn}\|_{X_1}^2 - \frac{p+1}{2\pi}\int_0^\pi trQ^2(x)dx + \frac{p+1}{2\pi^2} tr\left(\int_0^\pi Q(x)dx\right)^2 + O(p^{-1}). \quad (4.13)$$

We calculate the sum in (4.13)

$$\begin{aligned}\sum_{m=0}^{p}\sum_{n=1}^{\infty} \|Q\Psi_{mn}\|_{X_1}^2 &= \frac{2}{\pi}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\int_0^\pi \sin^2(m+\frac{1}{2})x(Q^2(x)\varphi_n,\varphi_n)_z dx \\ &= \frac{p+1}{\pi}\int_0^\pi \sum_{n=1}^{\infty}(Q^2(x)\varphi_n,\varphi_n)_z - \frac{1}{\pi}\sum_{m=0}^{p}\int_0^\pi\sum_{n=1}^{\infty}(Q^2(x)\varphi_n,\varphi_n)_z\cos(2m+1)xdx \\ &= \frac{p+1}{\pi}\int_0^\pi trQ^2(x)dx - \frac{p+1}{\pi}\int_0^\pi trQ^2(x)\cos(2m+1)xdx. \quad (4.14)\end{aligned}$$

Substitude (4.14) in (4.13)

$$M_{p2} = \frac{p+1}{\pi}\int_0^\pi trQ^2(x)dx + \frac{p+1}{2\pi^2}tr\Big(\int_0^\pi Q(x)\Big)^2 dx - \frac{1}{\pi}\sum_{m=0}^{p}\int_0^\pi trQ^2(x)\cos(2m+1)xdx + O(p^{-1}),$$

here, $O(p^{-1})$ depends on $p$, also satisfies the inequality $0 < O(p^{-1}) < c.p^{-1}$ ( c=constant). □

**Proof of Theorem 3.3.**

First, we calculate the limit of $M_{p3}$:

$$M_{p3} = -\frac{1}{3\pi i}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\sum_{s=0}^{\infty}\sum_{k=1}^{\infty}\Big[\int_{|\lambda|=b_p}\frac{\lambda}{((m+\frac{1}{4})^4-\lambda)((r+\frac{1}{4})^4-\lambda)((s+\frac{1}{4})^4-\lambda)}d\lambda\Big]$$
$$.(Q\psi_{mn},\psi_{rq})_{X_1}(Q\psi_{rq},\psi_{sk})_{X_1}(Q\psi_{sk},\psi_{mn})_{X_1},$$

which is absolutely convergent. If we define

$$F(m,r,s) := \sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{k=1}^{\infty}(Q\psi_{mn},\psi_{rq})_{X_1}(Q\psi_{rq},\psi_{sk})_{X_1}(Q\psi_{sk},\psi_{mn})_{X_1},$$
$$G(m,r,s) := \int_{|\lambda|=b_p}\frac{\lambda}{(\lambda-(m+\frac{1}{4})^4)(\lambda-(r+\frac{1}{4})^4)(\lambda-(s+\frac{1}{4})^4)}d\lambda,$$

then we express $M_{p3}$ as:

$$M_{p3} = \frac{1}{3\pi i}\sum_{m=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}G(m,r,s)F(m,r,s). \quad (4.15)$$

By using the inner product in $X_1$, we have the following formula for $F(m,r,s)$:

$$F(m,r,s)=\frac{1}{\pi^3}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{k=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_z[\cos(m-r)x-\cos(m+r+1)x\,]dx$$
$$\times\int_0^{\pi}(Q(x)\varphi_q,\varphi_k)_z[\cos(r-s)x-\cos(r+s+1)x\,]dx\;\int_0^{\pi}(Q(x)\varphi_k,\varphi_n)_z[\cos(s-m)x-\cos(s+m+1)x\,]dx \tag{4.16}$$

and also

$$F(m,r,s)=F(m,s,r)=F(r,s,m)=F(r,m,s)=F(s,m,r)=F(s,r,m).$$

By applying the Cauchy integral formula in (4.15), we obtain

$$M_{p3}=4\sum_{m=0}^{p}\sum_{r=0}^{p}\sum_{s=p+1}^{\infty}\frac{(m+\frac{1}{2})^4}{[(m+\frac{1}{2})^4-(r+\frac{1}{2})^4][(m+\frac{1}{2})^4-(s+\frac{1}{2})^4]}F(m,r,s)$$
$$-2\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4]^2}F(m,m,s)$$
$$+2\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\sum_{s=p+1}^{\infty}\frac{(m+\frac{1}{2})^4}{[(m+\frac{1}{2})^4-(r+\frac{1}{2})^4][(m+\frac{1}{2})^4-(s+\frac{1}{2})^4]}F(m,r,s).$$

Define the following equalities by

$$F_1(m,r,s):=\frac{1}{\pi^3}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{k=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_z\cos(m-r)x\,dx$$
$$\times\int_0^{\pi}(Q(x)\varphi_q,\varphi_k)_z\cos(r-s)x\,dx\;\int_0^{\pi}(Q(x)\varphi_k,\varphi_n)_z\cos(s-m)x\,dx, \tag{4.17}$$

$$F_2(m,r,s):=F(m,r,s)-F_1(m,r,s), \tag{4.18}$$

$$A_i:=\sum_{m=0}^{p}\sum_{r=p+1}^{\infty}\sum_{s=p+1}^{\infty}\frac{(m+\frac{1}{2})^4}{[(m+\frac{1}{2})^4-(r+\frac{1}{2})^4][(m+\frac{1}{2})^4-(s+\frac{1}{2})^4]}F_i(m,r,s),\qquad (i=1,2), \tag{4.19}$$

$$B_i:=\sum_{m=0}^{p}\sum_{r=0}^{p}\sum_{s=p+1}^{\infty}\frac{(m+\frac{1}{2})^4}{[(m+\frac{1}{2})^4-(r+\frac{1}{2})^4][(m+\frac{1}{2})^4-(s+\frac{1}{2})^4]}F_i(m,r,s),\qquad (i=1,2), \tag{4.20}$$

$$C_i:=\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4]^2}F_i(m,m,s),\qquad (i=1,2). \tag{4.21}$$

We rewrite $M_{p3}$ by using $A_i, B_i$ and $C_i$

$$M_{p3}=2A_1+4B_1-2C_1+2A_2+4B_2-2C_2. \tag{4.22}$$

The relations

$$F_1(m,r,s)=F_1(r,s,m)=F_1(r,m,s)=F_1(s,m,r)=F_1(s,r,m),$$
$$F_1(m,s,s)=F_1(m,m,s)\quad(m\geq 1)$$

are satisfied. If we consider the last two relations in $A_1$, we have

$$\begin{aligned} A_1 =& 2\sum_{m=0}^{p}\sum_{\substack{r=p+1\\ r>s}}^{\infty}\sum_{s=p+1}^{\infty}\frac{(m+\frac{1}{2})^4}{[(m+\frac{1}{2})^4-(r+\frac{1}{2})^4][(m+\frac{1}{2})^4-(s+\frac{1}{2})^4]}F_1(m,r,s)\\ &+\sum_{m=0}^{p}\sum_{\substack{s=p+1\\ r=s}}^{\infty}\frac{(m+\frac{1}{2})^4}{((m+\frac{1}{2})^4-(s+\frac{1}{2})^4)^2}F_1(m,m,s). \end{aligned} \tag{4.23}$$

Similarly, by (4.20), for $i=1$, we have

$$\begin{aligned} B_1 =& \sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m}}^{p}\sum_{s=p+1}^{\infty}\frac{(m+\frac{1}{2})^4}{((m+\frac{1}{2})^4-(r+\frac{1}{2})^4)((m+\frac{1}{2})^4-(s+\frac{1}{2})^4)}F_1(m,r,s)\\ &+\sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m}}^{p}\sum_{s=p+1}^{\infty}\frac{(r+\frac{1}{2})^4}{((r+\frac{1}{2})^4-(m+\frac{1}{2})^4)((r+\frac{1}{2})^4-(s+\frac{1}{2})^4)}F_1(m,r,s)\\ &=\sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m}}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(m+\frac{1}{2})^4-(r+\frac{1}{2})^4)}\left[\frac{(m+\frac{1}{2})^4}{(m+\frac{1}{2})^4-(s+\frac{1}{2})^4}-\frac{(r+\frac{1}{2})^4}{(r+\frac{1}{2})^4-(s+\frac{1}{2})^4}\right]F_1(m,r,s)\\ &=-\sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)}F_1(m,r,s). \end{aligned}$$

If we substitude $A_1$ and $A_2$ in (4.22), we have $M_{p3}$ as:

$$M_{p3}=4T_1-4T_2-2T_3+2A_2+4B_2-2C_2, \tag{4.24}$$

where

$$T_1=\sum_{m=0}^{p}\sum_{\substack{r=p+1\\ r>s}}^{\infty}\sum_{s=p+1}^{\infty}\frac{(m+\frac{1}{2})^4}{((m+\frac{1}{2})^4-(r+\frac{1}{2})^4)((m+\frac{1}{2})^4-(s+\frac{1}{2})^4)}F_1(m,r,s), \tag{4.25}$$

$$T_2=\sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)}F_1(m,r,s), \tag{4.26}$$

$$T_3=\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)}F_1(m,m,s). \tag{4.27}$$

For any integers $p$, $i$ and $j$ such that $i>j$ and $p\geq j$, let

$$E_1=\{(m,r,s): m,r,s\in\mathbb{N}\,; r-m=i\,; s-m=j\,; m\leq p\,; r,s>p\}.$$

If we consider

$$\sum_{m=0}^{p}\sum_{\substack{\mathrm{r=p+1}\\ r>s}}^{\infty}\sum_{\mathrm{s=p+1}}^{\infty}\frac{(m+\frac{1}{2})^4}{((m+\frac{1}{2})^4-(r+\frac{1}{2})^4)((m+\frac{1}{2})^4-(s+\frac{1}{2})^4)}|F_1(m,r,s)|<\infty$$

and

$$\gamma_{ij} = \frac{1}{\pi^3}\sum_{n=1}^{\infty}\sum_{q=1}^{\infty}\sum_{k=1}^{\infty}\int_0^{\pi}(Q(x)\varphi_n,\varphi_q)_z\cos ix\,dx\int_0^{\pi}(Q(x)\varphi_q,\varphi_k)_z\cos(i-j)x\,dx \times\int_0^{\pi}(Q(x)\varphi_k,\varphi_n)_z\cos jx\,dx, \tag{4.28}$$

we express $T_1$ as

$$\begin{aligned} T_1 &= \sum_{i=2}^{\infty}\sum_{j=1}^{p}\Big[\Big(\sum_{m,r,s\in E_1}\frac{(m+\frac{1}{2})^4}{((m+\frac{1}{2})^4-(r+\frac{1}{2})^4)((m+\frac{1}{2})^4-(s+\frac{1}{2})^4)}\Big)\gamma_{ij}\Big] \\ &+\sum_{m=0}^{p}\sum_{\substack{r=p+1\\ r>s,\,s-m>p}}^{\infty}\sum_{s=p+1}^{\infty}\frac{(m+\frac{1}{2})^4}{((m+\frac{1}{2})^4-(r+\frac{1}{2})^4)((m+\frac{1}{2})^4-(s+\frac{1}{2})^4)}F_1(m,r,s) \\ &= T_{11}+T_{12}. \end{aligned} \tag{4.29}$$

By using the set $E_1$, since

$$\begin{aligned} &\sum_{m,r,s\in E_1}\frac{(m+\frac{1}{2})^4}{((m+\frac{1}{2})^4-(r+\frac{1}{2})^4)((m+\frac{1}{2})^4-(s+\frac{1}{2})^4)} \\ &=\sum_{m,r,s\in E_1}\frac{(m+\frac{1}{2})^2}{((m+\frac{1}{2})^2-(r+\frac{1}{2})^2)((m+\frac{1}{2})^2-(s+\frac{1}{2})^2)((m+\frac{1}{2})^2+(r+\frac{1}{2})^2)((m+\frac{1}{2})^2+(s+\frac{1}{2})^2)} \\ &<\sum_{m,r,s\in E_1}\frac{1}{((m+\frac{1}{2})-(r+\frac{1}{2}))((m+\frac{1}{2})+(r+\frac{1}{2}))((m+\frac{1}{2})-(s+\frac{1}{2}))((m+\frac{1}{2})+(s+\frac{1}{2})} \\ &=\sum_{m,r,s\in E_1}\frac{1}{(m-r)(m+r+1)(m-s)(m+s+1)}=\frac{1}{ij}\sum_{m,r,s\in E_1}\frac{1}{(m+r+1)(m+s+1)} \\ &=\frac{1}{ij}\sum_{m=p+1-j}^{p}\frac{1}{(2m+i+1)(2m+j+1)}<\frac{1}{ij}\sum_{m=p+1-j}^{p}\frac{1}{(2m)(2m+j)} \\ &<\frac{1}{ij}\sum_{m=p+1-j}^{p}\frac{1}{2(p+1-j)(p+2+p-j)}<\frac{1}{ij}\sum_{m=p+1-j}^{p}\frac{1}{2(p+2)}<\frac{1}{ij}\sum_{m=p+1-j}^{p}\frac{1}{p}=\frac{1}{ip}\ , \end{aligned}$$

we obtain

$$\sum_{m,r,s\in E_1}\frac{(m+\frac{1}{2})^4}{((m+\frac{1}{2})^4-(r+\frac{1}{2})^4)((m+\frac{1}{2})^4-(s+\frac{1}{2})^4)}=\frac{1}{ip}O(1),$$

where $O(1)$ is a function, which depends on $p$,$i$ and $j$ and $|O(1)|<1$. Hence, we find

$$T_{11}=\sum_{\substack{i=2\\ i>j}}^{\infty}\sum_{j=1}^{p}\Big(\frac{1}{i\pi}O(1)\Big)\gamma_{ij}. \tag{4.30}$$

For any integers $p,i,j$ such that $i<j\le p$, let

$$E_2=\{(m,r,s):m,r,s\in\mathbb{N}\,;r-m=i\,;s-m=j\,;m,r\le p\,;s>p\}.$$

Consider

$$\sum_{m=0}^{p}\sum_{\mathrm{r=0}}^{\infty}\sum_{\mathrm{s=p+1}}^{\infty}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)}|F_1(m,r,s)|<\infty$$

and rewrite the equality (4.26) by using (4.28)

$$\begin{aligned} T_2 =& \sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m,\ s-m\leq p}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)}F_1(m,r,s)\\ &+\sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m,\ s-m>p}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)}F_1(m,r,s)\\ &=\sum_{j=2}^{p}\sum_{i=1}^{p-1}\Big[\Big(\sum_{m,r,s\in E_2}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)}\Big)\gamma_{ij}\Big]\\ &+\sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m,s-m>p}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)}F_1(m,r,s)\\ &=T_{21}+T_{22}. \end{aligned} \tag{4.31}$$

By using the set $E_2$, we focus on the following sum in $T_{21}$:

$$\begin{aligned} &\sum_{m,r,s\in E_2}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)}\\ &=\sum_{m,r,s\in E_2}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^2-(m+\frac{1}{2})^2)((s+\frac{1}{2})^2-(r+\frac{1}{2})^2)((s+\frac{1}{2})^2+(m+\frac{1}{2})^2)((s+\frac{1}{2})^2+(r+\frac{1}{2})^2)}\\ &<\sum_{m,r,s\in E_2}\frac{1}{((s+\frac{1}{2})-(m+\frac{1}{2}))((s+\frac{1}{2})+(m+\frac{1}{2}))((s+\frac{1}{2})-(r+\frac{1}{2}))((s+\frac{1}{2})+(r+\frac{1}{2}))}\\ &=\sum_{m,r,s\in E_2}\frac{1}{(s-m)(s+m+1)(s-r)(s+r+1)}\\ &=\sum_{m,r,s\in E_2}\frac{1}{j(s+m+1)(j-i)(s+r+1)}\\ &=\frac{1}{j(j-i)}\sum_{m,r,s\in E_2}\frac{1}{(s+m)(s+r)}\\ &<\frac{1}{j(j-i)}\sum_{m=p+1-j}^{p-i}\frac{1}{(2m+j)(2m+i+j)}\\ &=\frac{1}{ij(j-i)}\sum_{m=p+1-j}^{p-i}\Big[\frac{1}{2m+j}-\frac{1}{2m+i+j}\Big]\\ &=\frac{1}{ij}\Big[\frac{1}{j-i}\sum_{m=p+1-j}^{p-i}\frac{1}{2m+j}-\frac{1}{j-i}\sum_{m=p+1-j}^{p-i}\frac{1}{2m+i+j}\Big]. \end{aligned} \tag{4.32}$$

Here,

$$\frac{1}{j-i}\sum_{m=p+1-j}^{p-i}\frac{1}{2m+j} < \frac{1}{j-i}\cdot\frac{j-i}{2(p+1-j)+j} = \frac{1}{2p+2-j} < \frac{1}{2p-j} = \frac{1}{p+p-j} < \frac{1}{p} \tag{4.33}$$

and

$$\frac{1}{j-i}\sum_{m=p+1-j}^{p-i}\frac{1}{2m+i+j} < \frac{1}{j-i}\cdot\frac{j-i}{2(p-i)+i+j} = \frac{1}{2p-i+j} < \frac{1}{2p+j-i} < \frac{1}{3p}. \tag{4.34}$$

If we substitude (4.33) and (4.34) in (4.32), we have

$$\sum_{m,r,s\in E_2}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)} < \frac{1}{ij}(\frac{1}{p}-\frac{1}{3p}) = \frac{2}{ijp}.$$

So, we get

$$\sum_{m,r,s\in E_2}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)} = \frac{1}{ijp}O(1). \tag{4.35}$$

Hence, we have

$$T_{21} = \sum_{\substack{j=2 \\ j>i}}^{p}\sum_{i=1}^{p-1}\Big(\frac{1}{ijp}O(1)\Big)\gamma_{ij}. \tag{4.36}$$

Since $\gamma_{ij} = \gamma_{ji}$, by (4.24), (4.27), (4.29), (4.30), (4.31) and (4.36), we obtain:

$$M_{p3} = 4\sum_{\substack{i=2 \\ i>j}}^{\infty}\sum_{j=1}^{p}\Big(\frac{O(1)}{ip}\Big)\gamma_{ij} - 4\sum_{\substack{i=2 \\ i>j}}^{p}\sum_{j=1}^{p-1}\Big(\frac{O(1)}{ijp}\Big)\gamma_{ij} + 4T_{12} - 4T_{22} - 2T_3 + 2A_2 + 4B_2 - 2C_2.$$

Moreover, since $Q(x)$ has continuous derivative with respect to the norm $S_1[X]$ on the interval $[0,\pi]$, $|\gamma_{ij}| \le \frac{const}{i^2 j^2}$. So, we have

$$\Big|\sum_{i=2}^{\infty}\sum_{j=1}^{p}\Big(\frac{O(1)}{ip}\Big)\gamma_{ij}\Big| \le \frac{1}{p}\sum_{i=2}^{\infty}\sum_{j=1}^{p}\Big(\frac{1}{i}\Big)|\gamma_{ij}| < const\frac{1}{p}\Big(\sum_{i=2}^{\infty}\frac{1}{i^3}\Big)\Big(\sum_{j=1}^{p}\frac{1}{j^2}\Big) = O(\frac{1}{p}),$$
$$\Big|\sum_{i=2}^{p}\sum_{j=1}^{p-1}\Big(\frac{1}{ij}O(\frac{1}{p})\Big)\gamma_{ij}\Big| \le \frac{2}{p}\sum_{i=2}^{p}\sum_{j=1}^{p-1}\Big(\frac{1}{ij}\Big)|\gamma_{ij}| < const\frac{1}{p}\Big(\sum_{i=2}^{p}\frac{1}{i^3_{<\infty}}\Big)\Big(\sum_{j=1}^{p-1}\frac{1}{j^3_{<\infty}}\Big) = O(\frac{1}{p}),$$

and we write $M_{p3}$ as

$$M_{p3} = 4T_{12} - 4T_{22} - 2T_3 + 2A_2 + 4B_2 - 2C_2 + O(p^{-1}). \tag{4.37}$$

Now, we prove that the limits of every term in (4.37) are zero as $p \to \infty$.
By (4.16),(4.17) and (4.18), and the formula of integration by parts, for $m \neq r \neq s$, we have

$$
\begin{aligned}
|F_2(m,r,s)| \le\ & const.\Big[\frac{1}{(m-r)^2(r-s)^2(s+m+1)^2}+\frac{1}{(m-r)^2(r+s+1)^2(s-m)^2}\\
&+\frac{1}{(m-r)^2(r+s+1)^2(s+m+1)^2}+\frac{1}{(m+r+1)^2(r-s)^2(s-m)^2}\\
&+\frac{1}{(m+r+1)^2(r-s)^2(s+m+1)^2}+\frac{1}{(m+r+1)^2(r+s+1)^2(s-m)^2}\\
&+\frac{1}{(m+r+1)^2(r+s+1)^2(s+m+1)^2}\Big]\\
&\le const.\Big[\frac{1}{(m-r)^2(r-s)^2(s+m)^2}+\frac{1}{(m-r)^2(r+s)^2(s-m)^2}+\frac{1}{(m+r)^2(r-s)^2(s-m)^2}\Big] \qquad (4.38)\\
&\le const.\Big[\frac{1}{(m-r)^2(s+m)^2}+\frac{1}{(m-r)^2(r+s)^2(s-m)^2}+\frac{1}{(m+r)^2(s-m)^2}\Big] \qquad (4.39)\\
&\le const.\Big[\frac{1}{(s-m)^4}+\frac{1}{(s-m)^2(s+m)^2}\Big]. \qquad (4.40)
\end{aligned}
$$

Moreover by (4.17), (4.28) and $|\gamma_{ij}| \le \frac{const}{i^2j^2}$, we get

$$|F_1(m,r,s)| \le\ const.\frac{1}{(m-r)^2(m-s)^2}, \quad (for\ m \ne r, m \ne s) \tag{4.41}$$

and

$$|F_1(m,r,s)| \le\ cons.\frac{1}{(s-m)^2(s-r)^2}, \quad (for\ m \ne s, r \ne s). \tag{4.42}$$

From (4.29) and (4.41), we obtain

$$
\begin{aligned}
|T_{12}| \le & \sum_{m=0}^{p}\sum_{\substack{r=p+1\\ r>s,\, s-m>p}}^{\infty}\sum_{s=p+1}^{\infty}\frac{(m+\frac12)^4.|F_1(m,r,s)|}{[(m+\frac12)^4-(r+\frac12)^4][(m+\frac12)^4-(s+\frac12)^4]}\\
&\le const\sum_{m=0}^{p}\sum_{\substack{r=p+1\\ r>s,\, s-m>p}}^{\infty}\sum_{s=p+1}^{\infty}\frac{(m+\frac12)^4}{[(r+\frac12)^4-(m+\frac12)^4][(s+\frac12)^4-(m+\frac12)^4]}.\frac{1}{(m-r)^2.(m-s)^2}\\
&\le const\sum_{m=0}^{p}\sum_{\substack{r=p+1\\ r>s,\, s-m>p}}^{\infty}\sum_{s=p+1}^{\infty}\frac{1}{[(r+\frac12)^4-(m+\frac12)^4][(s+\frac12)^4-(m+\frac12)^4]}\\
&< const\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{[(s+\frac12)^4-(m+\frac12)^4]}.\sum_{r=p+1}^{\infty}\frac{1}{[(r+\frac12)^4-(m+\frac12)^4]}\\
&< const\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{[(s+\frac12)^4-(m+\frac12)^4]}.\sum_{r=p+1}^{\infty}\frac{1}{[(r+\frac12)^4-(p+\frac12)^4]}\\
&= const\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{[(s+\frac12)^4-(m+\frac12)^4]}.\sum_{r=p+1}^{\infty}\frac{1}{[(r+\frac12)^2-(p+\frac12)^2][(r+\frac12)^2+(p+\frac12)^2]}
\end{aligned}
$$

$$\leq \frac{const}{p^2}\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4]}.\sum_{r=p+1}^{\infty}\frac{1}{[(r+\frac{1}{2})-(p+\frac{1}{2})][(r+\frac{1}{2})+(p+\frac{1}{2})]}$$

$$=\frac{const}{p^2}\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4]}.\sum_{r=p+1}^{\infty}\frac{1}{(r-p)(r+p+1)}$$

$$<\frac{const}{p^2}\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4]}.\sum_{r=p+1}^{\infty}\frac{1}{(r-p)(r-p)}<\frac{const}{p^2}\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4]}$$

$$<\frac{const}{p^2}(p+1).\sum_{s=p+1}^{\infty}\frac{1}{(s+\frac{1}{2})^4-(p+\frac{1}{2})^4}<\frac{const}{p^2}(p+1)\frac{1}{p^2}=\frac{const}{p^3}. \tag{4.43}$$

Then we get

$$\lim_{p\to\infty} T_{12}=0. \tag{4.44}$$

Now, we calculate $\lim_{p\to\infty} T_{22}$. By (4.31) and (4.42), we have

$$|T_{22}|\leq\sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m,\, s-m>p}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4.|F_1(m,r,s)|}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4][(s+\frac{1}{2})^4-(r+\frac{1}{2})^4]}$$

$$\leq\sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m,\, s-m>p}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4][(s+\frac{1}{2})^4-(r+\frac{1}{2})^4]}.\frac{1}{(m-s)^2.(r-s)^2}$$

$$=\sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m,\, s-m>p}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^2-(m+\frac{1}{2})^2)((s+\frac{1}{2})^2-(r+\frac{1}{2})^2)((s+\frac{1}{2})^2+(m+\frac{1}{2})^2)}$$

$$.\frac{1}{((s+\frac{1}{2})^2+(r+\frac{1}{2})^2)(m-s)^2.(s-r)^2}$$

$$\leq\sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m,\, s-m>p}}^{p}\sum_{s=p+1}^{\infty}\frac{1}{((s+\frac{1}{2})^2-(m+\frac{1}{2})^2)((s+\frac{1}{2})^2-(r+\frac{1}{2})^2)(m-s)^2.(s-r)^2}$$

$$=\sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m,\, s-m>p}}^{p}\sum_{s=p+1}^{\infty}\frac{1}{((s+m+1)((s+r+1)(s-m)^3.(s-r)^3}$$

$$<\sum_{m=0}^{p}\sum_{\substack{r=0\\ r>m,\, s-m>p}}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s+m)((s+r)(s-m)^3.(s-r)^3}<\frac{1}{p^2}\sum_{m=0}^{p}\sum_{r=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)^3.(s-r)^3}$$

$$<\frac{1}{p^5}\sum_{m=0}^{p}\sum_{r=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-r)^3}=\frac{p+1}{p^5}\sum_{r=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-r)^3}<\frac{p+1}{p^5}\sum_{r=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-r)^3}$$

$$=\frac{(p+1)^2}{p^5}.\sum_{s=p+1}^{\infty}\frac{1}{(s-p)^3}<\frac{(p+1)^2}{p^5}.\sum_{s=p+1}^{\infty}\frac{1}{s^3}<\frac{const}{p^3}. \tag{4.45}$$

Then we obtain

$$\lim_{p\to\infty} T_{22}=0. \tag{4.46}$$

By using (4.19) and (4.39), we have an inequality of absolute value of $A_2$:

$$
\begin{aligned}
|A_2| \leq & \sum_{m=0}^{p} \sum_{\substack{r=p+1 \\ r>s}}^{\infty} \sum_{s=p+1}^{\infty} \frac{(m+\frac{1}{2})^4}{[(m+\frac{1}{2})^4-(r+\frac{1}{2})^4][(m+\frac{1}{2})^4-(s+\frac{1}{2})^4]} |F_2(m,r,s)|, \\
\leq & \sum_{m=0}^{p} \sum_{\substack{r=p+1 \\ r>s}}^{\infty} \sum_{s=p+1}^{\infty} \frac{(m+\frac{1}{2})^4}{[(m+\frac{1}{2})^4-(r+\frac{1}{2})^4][(m+\frac{1}{2})^4-(s+\frac{1}{2})^4]} \Big[ \frac{1}{(m-r)^2(r+s+1)^2(s-m)^2} \\
& + \frac{1}{(m-r)^2(r+s+1)^2(s+m+1)^2} + \frac{1}{(m+r+1)^2(r-s)^2(s-m)^2} \Big] \\
\leq & \sum_{m=0}^{p} \sum_{\substack{r=p+1 \\ r>s}}^{\infty} \sum_{s=p+1}^{\infty} \frac{(m+\frac{1}{2})^4}{[(m+\frac{1}{2})^4-(r+\frac{1}{2})^4][(m+\frac{1}{2})^4-(s+\frac{1}{2})^4]} \Big[ \frac{1}{(m-r)^2(r+s)^2(s-m)^2} \\
& + \frac{1}{(m-r)^2(r+s)^2(s+m)^2} + \frac{1}{(m+r)^2(r-s)^2(s-m)^2} \Big] \\
\leq & \sum_{m=0}^{p} \sum_{\substack{r=p+1 \\ r>s}}^{\infty} \sum_{s=p+1}^{\infty} \frac{(m+\frac{1}{2})^4}{[(m+\frac{1}{2})^4-(r+\frac{1}{2})^4][(m+\frac{1}{2})^4-(s+\frac{1}{2})^4]} \Big[ \frac{1}{(m-r)^2(r+s)^2(s-m)^2} \\
& + \frac{1}{(r-m)^2(s+m)^2} + \frac{1}{(m+r)^2(s-m)^2} \Big] \\
= & \sum_{m=0}^{p} \sum_{\substack{r=p+1 \\ r>s}}^{\infty} \sum_{s=p+1}^{\infty} \frac{(m+\frac{1}{2})^4}{[(m+\frac{1}{2})^4-(r+\frac{1}{2})^4][(m+\frac{1}{2})^4-(s+\frac{1}{2})^4](r-m)^2(s+r)^2(s-m)^2} \\
& + 2 \sum_{m=0}^{p} \sum_{\substack{r=p+1 \\ r>s}}^{\infty} \sum_{s=p+1}^{\infty} \frac{(m+\frac{1}{2})^4}{[(m+\frac{1}{2})^4-(r+\frac{1}{2})^4][(m+\frac{1}{2})^4-(s+\frac{1}{2})^4](m+r)^2(s-m)^2} \\
< & \frac{p+1}{4p^4} \sum_{r=p+1}^{\infty} \sum_{s=p+1}^{\infty} \frac{1}{(s-p)^3(r-p)^3} + \frac{2}{p^2} \sum_{m=0}^{p} \sum_{r=p+1}^{\infty} \sum_{s=p+1}^{\infty} \frac{1}{[(r+\frac{1}{2})^2-(p+\frac{1}{2})^2](r-s)\, 2\, p\, (s-p)^2} \\
< & \, const \Big[ \frac{1}{p^3} \sum_{r=p+1}^{\infty} \frac{1}{r^3} \sum_{s=p+1}^{\infty} \frac{1}{s^3} + \frac{1}{p^2} \sum_{r=p+1}^{\infty} \frac{1}{[(r+\frac{1}{2})^2-(p+\frac{1}{2})^2]} \sum_{s=p+1}^{\infty} \frac{1}{s^2} \Big] < \frac{const}{p^2}.
\end{aligned}
\tag{4.47}
$$

Then we find

$$
\lim_{p\to\infty} A_2 = 0. \tag{4.48}
$$

Using (4.21) and (4.40), we take absolute value of $C_2$ and write as

$$
\begin{aligned}
|C_2| &\le \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4]^2}|F_2(m,m,s)| \\
&< const.\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4]^2}\Big[\frac{1}{(s-m)^4}+\frac{1}{(s-m)^2(s+m)^2}\Big] \\
&= const.\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4]^2(s-m)^2(s+m)^2} \\
&\quad + const.\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4]^2(s-m)^4} = const.(C_{21}+C_{22}),
\end{aligned}
\tag{4.49}
$$

where

$$
C_{21} = \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4]^2(s-m)^2(s+m)^2},
$$

$$
C_{22} = \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{[(s+\frac{1}{2})^4-(m+\frac{1}{2})^4]^2(s-m)^4}.
$$

Now we obtain inequalities for $C_{21}$ and $C_{22}$:

$$
\begin{aligned}
C_{21} &= \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{[(s+\frac{1}{2})^2-(m+\frac{1}{2})^2]^2[(s+\frac{1}{2})^2+(m+\frac{1}{2})^2]^2(s-m)^2(s+m)^2} \\
&< \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{[(s+\frac{1}{2})^2-(m+\frac{1}{2})^2]^2(s^2+m^2)^2(s-m)^2(s+m)^2} \\
&< \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{(s^2+m^2)^2(s-m)^4(s+m)^4} \\
&< \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)^4(s+m)^4} < \sum_{m=0}^{p}\frac{1}{p^4}\sum_{s=p+1}^{\infty}\frac{1}{s^4} < const\frac{p+1}{p^4} < const\frac{1}{p^3},
\end{aligned}
\tag{4.50}
$$

and

$$
\begin{aligned}
C_{22} &= \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{[(s+\frac{1}{2})^2-(m+\frac{1}{2})^2]^2[(s+\frac{1}{2})^2+(m+\frac{1}{2})^2]^2(s-m)^4} \\
&< \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{(s-m)^2(s+m+1)^2[(s+\frac{1}{2})^2+(m+\frac{1}{2})^2]^2(s-m)^4} \\
&\le \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)^6(s+m)^2} \\
&< \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)^4(s+m)^4} < \frac{1}{p^2}\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)^6} = \frac{p+1}{p^2}\sum_{s=p+1}^{\infty}\frac{1}{s^6}. < const\frac{1}{p}.
\end{aligned}
\tag{4.51}
$$

From (4.49), (4.50) and (4.51), we find

$$\lim_{p\to\infty} C_2 = 0. \tag{4.52}$$

By (4.27) and (4.42), we obtain

$$\begin{aligned}
|T_3| &= \Big|\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s+\frac{1}{2})^4-(m+\frac{1}{2})^4}F_1(m,m,s)\Big| \le \sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s+\frac{1}{2})^4-(m+\frac{1}{2})^4}|F_1(m,m,s)|\\
&\le const\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)(s-m)^4} \le const\sum_{m=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(p+1-m)^4.((s+\frac{1}{2})^4-(p+\frac{1}{2})^4)}\\
&= const\sum_{m=0}^{p}\frac{1}{(p+1-m)^4}.\sum_{s=p+1}^{\infty}\frac{1}{(s+\frac{1}{2})^4-(p+\frac{1}{2})^4} < \frac{const}{p^2}\sum_{m=0}^{p}\frac{1}{m^4} < \frac{const}{p^2}.
\end{aligned} \tag{4.53}$$

So, we get

$$\lim_{p\to\infty} T_3 = 0. \tag{4.54}$$

By using (4.20) and (4.38), we find

$$\begin{aligned}
|B_2| &\le \sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4\,|F_2(m,m,s)|}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)}\\
&\le const\sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)}\\
&\quad .\Big[\frac{1}{(m-r)^2(s-m)^2(r+s)^2}+\frac{1}{(m+r)^2(s-m)^2(r-s)^2}+\frac{1}{(r-s)^2(m-r)^2(s+m)^2}\Big]\\
&= const\Big(\sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)(m-r)^2(s-m)^2(r+s)^2}\\
&+\sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)(m+r)^2(s-m)^2(r-s)^2}\\
&+\sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)(r-s)^2(m-r)^2(s+m)^2}\Big)\\
&= const(B_{21}+B_{22}+B_{23}).
\end{aligned} \tag{4.55}$$

Let us start with $B_{21}$

$$\begin{aligned}
B_{21} &= \sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^2+(m+\frac{1}{2})^2)((s+\frac{1}{2})^2+(r+\frac{1}{2})^2)(s-m)(s+m+1)}\\
&\le \sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)^3(s+m)(s-r)(s+r)^3(m-r)^2}\\
&< \sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)^3.s.s^3.(s-r)(m-r)^2} < \sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)^3(s-r).s^4}\\
&< \sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)^2(s-m)(s-r).s^4} < \sum_{m=0}^{p}\sum_{\substack{r=0\\ m<r}}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(s-m)^2(s-r)^2.s^4}\\
&< \sum_{m=0}^{p}\sum_{r=0}^{p}\sum_{s=p+1}^{\infty}\frac{1}{(p+1-m)^2(p+1-r)^2.s^4}\\
&= (\sum_{m=0}^{p}\frac{1}{(p+1-m)^2}).(\sum_{r=0}^{p}\frac{1}{(p+1-r)^2})(\sum_{s=p+1}^{\infty}\frac{1}{s^4})\\
&= (\sum_{m=0}^{p}\frac{1}{(p+1-m)^2})^2(\sum_{s=p+1}^{\infty}\frac{1}{s^4}) \le \frac{const}{p^4}.
\end{aligned} \tag{4.56}$$

We limit the formula of $B_{22}$, for simplicity, plug $2p$ into $p$:

$$\begin{aligned}
B_{22} &= \sum_{m=0}^{2p}\sum_{\substack{r=0\\ m<r}}^{2p}\sum_{s=2p+1}^{\infty}\Big[\frac{(s+\frac{1}{2})^4}{(s+m)(s+m+1)((s+\frac{1}{2})^2+(r+\frac{1}{2})^2)(s+r)(s+r+1)((s+\frac{1}{2})^2+(r+\frac{1}{2})^2)}\\
&\qquad .\frac{1}{(s-m)(s-r)(m+r)^2(s-m)^2(r-s)^2}\Big]\\
&\le \sum_{m=0}^{2p}\sum_{\substack{r=0\\ m<r}}^{2p}\sum_{s=2p+1}^{\infty}\frac{1}{(s-m)^3(s+m)(s-r)^3(s+r)(m+r)^2}\\
&< \frac{1}{8p^2}\sum_{m=0}^{2p}\sum_{\substack{r=0\\ m<r}}^{2p}\sum_{s=2p+1}^{\infty}\frac{1}{(s-m)^3(s-r)^3(m+r)^2}\\
&< \frac{1}{8p^2}\Big(\sum_{m=0}^{p}\sum_{r=0}^{2p}\sum_{s=2p+1}^{\infty}\frac{1}{(s-m)^3(s-r)^3(m+r)^2}+\sum_{m=p+1}^{2p}\sum_{r=0}^{2p}\sum_{s=2p+1}^{\infty}\frac{1}{(s-m)^3(s-r)^3(m+r)^2}\Big)\\
&= \frac{1}{8p^2}\Big(B_{22}^1+B_{22}^2\Big).
\end{aligned} \tag{4.57}$$

$$
\begin{aligned}
B_{22}^{1} &= \sum_{m=0}^{p}\sum_{r=0}^{2p}\sum_{s=2p+1}^{\infty}\frac{1}{(s-m)^3(s-r)^3(m+r)^2} \leq \sum_{m=0}^{p}\sum_{r=0}^{2p}\sum_{s=2p+1}^{\infty}\frac{1}{(s-m)^3(s-r)^3} \\
&\leq \sum_{s=2p+1}^{\infty}\sum_{m=0}^{p}\frac{1}{(s-p)^3}\sum_{r=0}^{2p}\frac{1}{(s-2p)^3} \\
&< \frac{(2p+1)^2}{p^3}\sum_{s=2p+1}^{\infty}\frac{1}{(s-2p)^3} < \frac{const}{p},
\end{aligned} \tag{4.58}
$$

and

$$
\begin{aligned}
B_{22}^{2} &= \sum_{m=p+1}^{2p}\sum_{r=0}^{2p}\sum_{s=2p+1}^{\infty}\frac{1}{(s-m)^3(s-r)^3(m+r)^2} \leq \sum_{m=p+1}^{2p}\sum_{r=0}^{2p}\sum_{s=2p+1}^{\infty}\frac{1}{(s-r)^6\,4m^2} \\
&= \sum_{r=0}^{2p}\sum_{s=2p+1}^{\infty}\frac{1}{(s-r)^6}\sum_{m=p+1}^{2p}\frac{1}{4p^2} \\
&= \frac{1}{4p}\sum_{r=0}^{2p}\sum_{s=2p+1}^{\infty}\frac{1}{(s-r)^6} \\
&= \frac{1}{4p}\Big[\sum_{r=0}^{2p}\Big(\frac{1}{(2p+1-r)^6}+\sum_{s=2p+2}^{\infty}\frac{1}{(s-r)^6}\Big)\Big] \\
&\leq \frac{1}{4p}\Big[\sum_{r=0}^{2p}\Big(\frac{1}{(2p+1-r)^6}+\sum_{r=0}^{2p}\int_{2p+2}^{\infty}\frac{1}{(x-r)^6}\,dx\Big)\Big] \\
&\leq \frac{1}{4p}\Big[\frac{1}{(2p+1)^5}+\frac{1}{5}\sum_{r=0}^{2p}\frac{1}{(2p+2-r)^5}\Big] \\
&\leq \frac{1}{4p}\Big[\frac{1}{(2p+1)^5}+\frac{1}{10(p+1)}\sum_{r=1}^{2p+1}\frac{1}{(r-(2p+3))^4}\Big] \\
&< \frac{1}{4p}\Big[\frac{1}{(2p+1)^5}+\frac{1}{10(p+1)}\sum_{r=1}^{2p+1}\frac{1}{r^4}\Big] = \frac{const}{p^6}+\frac{const}{p^2} < \frac{const}{p^2}.
\end{aligned} \tag{4.59}
$$

By substituding (4.58) and (4.59) in (4.57), we obtain

$$
B_{22} \leq \frac{1}{8p^2}\Big(\frac{const}{p}+\frac{const}{p^2}\Big) < \frac{1}{8p^2}\Big(\frac{const}{p}\Big) < \frac{const}{p^3}. \tag{4.60}
$$

Consider $B_{23}$

$$
\begin{aligned}
B_{23} &= \sum_{m=0}^{p} \sum_{\substack{r=0 \\ m<r}}^{p} \sum_{s=p+1}^{\infty} \frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^4-(m+\frac{1}{2})^4)((s+\frac{1}{2})^4-(r+\frac{1}{2})^4)(r-s)^2(m-r)^2(s+m)^2} \\
&= \sum_{m=0}^{p} \sum_{\substack{r=0 \\ m<r}}^{p} \sum_{s=p+1}^{\infty} \Big(\frac{(s+\frac{1}{2})^4}{((s+\frac{1}{2})^2+(m+\frac{1}{2})^2))((s+\frac{1}{2})^2+(r+\frac{1}{2})^2)(r-s)^2(m-r)^2(s+m)^2} \\
&\qquad \cdot \frac{1}{(s-m)(s+m+1)(s-r)(s+r+1)}\Big) \\
&\leq \sum_{m=0}^{p} \sum_{\substack{r=0 \\ m<r}}^{p} \sum_{s=p+1}^{\infty} \frac{1}{(s-m)(s+m)^3(s-r)^3(m-r)^2(s+r)} \\
&\leq \sum_{m=0}^{p} \sum_{\substack{r=0 \\ m<r}}^{p} \sum_{s=p+1}^{\infty} \frac{1}{(p+1-m)(p+1-r)^3\, s^4} \\
&\leq \sum_{m=0}^{p} \frac{1}{(p+1-m)} \sum_{\substack{r=0 \\ m<r}}^{p} \frac{1}{(p+1-r)^3} \sum_{s=p+1}^{\infty} \frac{1}{s^4} \\
&\leq \sum_{m=0}^{p} \frac{1}{(p+1-p)} \sum_{\substack{r=0 \\ m<r}}^{p} \frac{1}{(p+1-r)^2(p+1)} \sum_{s=p+1}^{\infty} \frac{1}{s^4} \\
&\leq \frac{p+1}{p} \sum_{r=0}^{p} \frac{1}{r^2} \int_{p+1}^{\infty} \frac{1}{x^4}\, dx < \frac{const}{p^3}. \qquad (4.61)
\end{aligned}
$$

By using (4.56) and (4.60) in (4.61), we obtain, we have

$$\lim_{p\to\infty} B_2 = 0. \qquad (4.62)$$

From (4.44), (4.46),(4.48),(4.52),(4.54) and (4.62) , we find

$$\lim_{p\to\infty} M_{p3} = 0. \qquad (4.63)$$

We also see that

$$\lim_{p\to\infty} M_{pj} = 0, \quad for \;\; j > 3. \qquad (4.64)$$

We can prove that

$$\left\|QR_\lambda^0\right\|_{S_1[X_1]} < \frac{c}{p^2},\ (c = constant), \quad \left\|R_\lambda^0\right\|_{S_1[X_1]} < \frac{1}{p^3}, \quad \|R_\lambda\|_{X_1} < \frac{1}{p^3}, \quad \left\|QR_\lambda^0\right\|_{X_1} < 1. \qquad (4.65)$$

For $|\lambda| = (p+1)^4$ , we limit the formula (3.4) for $M_p$ by using (4.65)

$$
\begin{aligned}
\left|M_p\right| &= \frac{1}{2\pi}\left|\int_{|\lambda|=b_p} \lambda^2 tr R_\lambda (QR_\lambda^0)^5 d\lambda\right| \leq \frac{b_p^2}{2\pi} \int_{|\lambda|=b_p} \left\|R_\lambda (QR_\lambda^0)^5\right\|_{S_1[X_1]} |d\lambda| \\
&\leq \frac{b_p^2}{2\pi} \int_{|\lambda|=b_p} \|R_\lambda\|_{X_1} \left\|(QR_\lambda^0)^4\right\|_{X_1} \left\|QR_\lambda^0\right\|_{S_1[X_1]} |d\lambda| \leq \frac{c}{p^9}.
\end{aligned}
$$

Therefore, we obtain

$$\lim_{p\to\infty} M_p = 0. \tag{4.66}$$

By Theorem 3.1. and Theorem 3.2. we restate the formula (3.4)

$$\begin{aligned}\sum_{m=0}^{p}\sum_{n=1}^{\infty}\left(\lambda_{mn}^2-(m+\frac{1}{2})^8\right) &= \frac{2}{\pi}\sum_{m=0}^{p}(m+\frac{1}{2})^4\int_0^{\pi} trQ(x)dx\\ &+ \frac{1}{8\pi}\left[trQ^{'}(\pi)+trQ^{'}(0)\right]\sum_{m=0}^{p}(2m+1)^2-\frac{1}{8\pi}\left[trQ^{'''}(\pi)+trQ^{'''}(0)\right]\sum_{m=0}^{p}1\\ &- \frac{1}{8\pi}\sum_{m=0}^{p}\int_0^{\pi} trQ^{IV}(x)\cos(2m+1)xdx+\frac{p+1}{2\pi}\int_0^{\pi} trQ^2(x)dx+\frac{p+1}{2\pi^2}tr\left(\int_0^{\pi} Q(x)dx\right)^2\\ &- \frac{1}{\pi}\sum_{m=0}^{p}\int_0^{\pi} trQ^2(x)\cos(2m+1)xdx+O(p^{-1})+M_{p3}+M_{p4}+M_p.\end{aligned}$$

$$\begin{aligned}&\sum_{m=0}^{p}\sum_{n=1}^{\infty}\left(\lambda_{mn}^2-(m+\frac{1}{2})^8\right) - \frac{2}{\pi}(m+\frac{1}{2})^4\int_0^{\pi} trQ(x)dx-\frac{1}{8\pi}(2m+1)^2\left[trQ^{'}(\pi)+trQ^{'}(0)\right]\\ &+ \frac{1}{8\pi}[trQ^{'''}(\pi)+trQ^{'''}(0)]-\left[\frac{1}{2\pi}\int_0^{\pi} trQ^2(x)dx+\frac{1}{2\pi^2}tr\left(\int_0^{\pi} Q(x)dx\right)^2\right]\\ &= \frac{1}{8\pi}\sum_{m=0}^{p}\int_0^{\pi}\left(trQ^{IV}(x)+8trQ^2(x)\right)\cos(2m+1)xdx+O(p^{-1})+M_{p3}+M_{p4}+M_p.\end{aligned} \tag{4.67}$$

Taking the limit of (4.67) as $p\to\infty$, and using (4.66) and (4.64), we obtain

$$\begin{aligned}&\sum_{m=0}^{\infty}\Bigg[\sum_{n=1}^{\infty}\left(\lambda_{mn}^2-(m+\frac{1}{2})^8\right)-\frac{2}{\pi}(m+\frac{1}{2})^4\int_0^{\pi} trQ(x)dx-\frac{1}{8\pi}(2m+1)^2\left(trQ'(\pi)+trQ'(0)\right)\\ &+\frac{1}{8\pi}\left(trQ'''(\pi)+trQ'''(0)\right)-C\Bigg]=-\frac{1}{32\pi}\left(trQ^{IV}(0)+8trQ^2(0)-trQ^{IV}(\pi)-8trQ^2(\pi)\right),\end{aligned}$$

where, $C=\frac{1}{2\pi}\int_0^{\pi} trQ^2(x)dx+\frac{1}{2\pi^2}tr\left(\int_0^{\pi} Q(x)dx\right)^2$ is a constant.
This completes the proof. □

**Example 1:** Take $X_1 = Y_2(0,\pi,X)$, where X is a separable Banach space, which is continuous dense embedding in a Hilbert space $H$. Consider the operator function $Q(x)$ for every $x\in[0,\pi]$ defined by $Q(x)=xA$, where $A$ is an operator for every $t\in X$ from $X$ to $X$ defined by $At=\sum_{n=1}^{\infty}\frac{1}{n^4}(t,e_n)_z e_n$ such that $\{e_n\}_{n=1}^{\infty}$ is orthonormal basis in $H$. Here $(.,.)_z$ is the Zachary functional on $X$.

For every $t, s \in [0,\pi]$ we have

$$\begin{aligned}(At,s)_z &= \sum_{n=1}^{\infty}\frac{1}{n^4}(t,e_n)_z e_n, s)_z = \sum_{n=1}^{\infty}\frac{1}{n^4}(t,e_n)_z(e_n,s)_z \\ &= \sum_{n=1}^{\infty}(t,\frac{1}{n^4}(s,e_n)_z e_n)_z = (t,\sum_{n=1}^{\infty}\frac{1}{n^4}(s,e_n)_z e_n)_z = (t,As)_z.\end{aligned}$$

Since $A$ has eigenvalues $\{\frac{1}{n^4}\}_{n=1}^{\infty}$, $A$ is a kernel operator. Therefore $Q(x)$ is a kernel operator on $X$.
Moreover $Q(x)$ has continuous derivatives of order 4 with respect to norm $S_1[X]$. For $i = 0,1,2,3,4$, $Q^{(i)}(x)$ are self adjoint kernel operators. The functions $\|Q^{(i)}(x)\|_{S_1[X]}$ $(i = 0,1,2,3,4)$ are bounded and measurable for every $x \in [0,\pi]$, where $S_1[X]$ is Banach space consisting of all kernel operators from $X$ to $X$. For every $y = y(x) \in X_1$

$$\left\|Qy\right\|_{X_1}^2 = \int_0^{\pi}\left\|Q(x)y(x)\right\|_X^2 dx \le \int_0^{\pi}\|Q(x)\|_X^2\left\|y(x)\right\|_X^2 dx \le c^2\int_0^{\pi}\left\|y(x)\right\|_X^2 dx = c^2\left\|y(x)\right\|_{X_1}^2$$

$$\left\|Qy\right\|_{X_1} \le c\left\|y\right\|_{X_1},$$

which follows that the operator Q is a bounded operator on $X_1$.
**Example 2:** We take the Banach space $X = C[0,\pi]$ defined by the set of all continuous functions on $[0,\pi]$. Let $X$ be a continuous dense embedding in $H = L_2[0,\pi]$ ; and $X_1 = Y_2(0,\pi,X) = C([0,\pi]x[0,\pi])$ as a continuous dense embedding in $H_1 = L_2(0,\pi,L_2[0,\pi]) = L_2([0,\pi]x[0,\pi])$
We define the operator function $Q(x)$ on $X$ for every $x \in [0,\pi]$:

$$Q(x)\varphi(t) = sinx\sum_{m=1}^{\infty}m^{-4}sinmt\int_0^{\pi}\varphi(s)sinmsds.$$

It is clear that $Q(x)$ has derivatives of all order with respect to the norm in the space $S_1[X]$.
We have the derivatives in the forms
$Q^{(2i+1)}(x) = (-1)^i cosx\sum_{m=1}^{\infty}m^{-4}sinmt\int_0^{\pi}\varphi(s)sinmsds,$ for i=0,1;
$Q^{(2i)}(x) = (-1)^i sinx\sum_{m=1}^{\infty}m^{-4}sinmt\int_0^{\pi}\varphi(s)sinmsds,$ for i=0,1.
The eigenvalues of the operators $Q^{(2i+1)}(x)$ and $Q^{(2i)}(x)$ $(i = 0,1)$ are

$$\left\{\frac{(-1)^i\pi cosx}{2n^4}sinnt\right\}_{n=1}^{\infty} \quad\text{and}\quad \left\{\frac{(-1)^i\pi sinx}{2n^4}sinnt\right\}_{n=1}^{\infty},$$

respectively. Therefore $Q^{(i)}(x)$ $(i = 0,1,2,3,4)$ are self adjoint operators. On the other hand, we have

$$\left\|Q^{(2i+1)}(x)\right\|_{S_1[X]} = \frac{\pi sinx}{2}\sum_{n=1}^{\infty}n^{-4}, \quad (i = 0,1);$$

$$\left\|Q^{(2i)}(x)\right\|_{S_1[X]} = \frac{\pi sinx}{2}\sum_{n=1}^{\infty}n^{-4}, \quad (i = 0,1).$$

So, we see that $Q(x)$ holds the condition Q2.

Now, we define the self adjoint linear operators $L_0$ and $L$ on $X_1$,

$$L_0 u = \frac{\partial^4 u(x,t)}{\partial x^4},$$

$$Lu = L_0 u + Qu = \frac{\partial^4 u(x,t)}{\partial x^4} + Q(x)u(x,t),$$

with same boundary conditions

$$u(0,t) = u_{xx}(0,t) = u_x(\pi,t) = u_{xxx}(\pi,t) = 0$$

$$u(x,0) = u_{tt}(x,0) = u(x,\pi) = u_{tt}(x,\pi) = 0,$$

respectively. The eigenvalues of the operator $L_0$ have the form $(m+\frac{1}{2})^4 \quad (m = 0,1,...)$ and the corresponding orthonormal eigenfunctions are $\sqrt{\frac{2}{\pi}} sin(m+\frac{1}{2})x.\varphi_n \qquad (n = 1,2,...)$.
$u(x,t)$ has continuous partial derivatives of order 4 on $[0,\pi]x[0,\pi]$. Since $L_0$ is self adjoint operator and $Q$ is bounded and self adjoint operator, $L$ is also self adjoint operator.

$$trQ(x) = \frac{\pi sinx}{2}\sum_{n=1}^{\infty} n^{-4} = \frac{\pi^3 sinx}{180}$$

,

$$trQ'(x) = \frac{\pi^3 cosx}{180} \qquad trQ'''(x) = -\frac{\pi^3 cosx}{180} \qquad trQ^{IV}(x) = \frac{\pi^3 sinx}{180}$$

$$trQ(0) = 0, \quad trQ(\pi) = 0, \quad \int_0^{\pi} trQ(x)dx = \frac{\pi^3}{90}$$

Therefore the regularized trace formula of the opertor $L$ is

$$\sum_{m=0}^{\infty}\left[\sum_{n=1}^{\infty}\left(\lambda_{mn}^2 - (m+\frac{1}{2})^8\right) - \frac{2}{\pi}(m+\frac{1}{2})^4\frac{\pi^3}{90} - C\right] = 0$$

where, $C = \frac{1}{4} + \frac{1}{2\pi^2} tr\left(\int_0^{\pi} Q(x)dx\right)^2$ is a constant. □

**Funding:** This research was supported by Scientific Research Project Coordination Unit of Yildiz Technical University (Project ID:5418, Project Code:FBA-2023-5418).
**Conflicts of Interest:** The authors declare no conflict of interest.

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