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Periodic solutions for parabolic fractional p-Laplacian problems via topological degree

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Abstract. In this work, we consider the nonlinear parabolic initial boundary value problem involving the fractional p-Laplacian operator. We employ topological degree methods to establish the existence of a time periodic nontrivial weak solutions in the appropriate space $\mathcal{X} := L^p(0,T;W^{s,p}(\Omega))$. Our approach to proving the main result is based on transforming this nonlinear parabolic problem into an operator equation of the shape $\mathcal{K}u + \mathcal{B}u = h$, where $\mathcal{B}$ is of type (S_+) relative in the domain of densely defined linear maximal monotone operator $\mathcal{K}$.

1. Introduction and motivation

Our purpose in this study is to examine the existence of a time periodic nontrivial weak solution for a nonlinear parabolic equation involving a fractional p-Laplacian operator modeled as follows:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (-\Delta)_p^s u = h(x,t) & \text{in} \quad Q := \Omega \times (0,T), \\ u(x,0) = u(x,T) & \text{on} \quad \Omega \\ u(x,t) = 0 & \text{on} \quad \partial Q := \left(\mathbb{R}^N \backslash \Omega\right) \times (0,T), \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open set with smooth boundary $\partial\Omega$, $T > 0$ is the period, $s \in (0,1)$, and $2 < p < \frac{N}{s}$. Here, the main operator $(-\Delta)_p^s$ is the fractional p-Laplacian which is non-local operator described on smooth functions by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{B_\varepsilon^c(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} dy, \qquad \text{for } x \in \mathbb{R}^N,$$

where $B_\varepsilon(x) := \left\{y \in \mathbb{R}^N \ : \ |x - y| < \varepsilon\right\}$. In addition, we assume $h \in \mathcal{X}^* := L^{p'}\left(0,T;\left(W_0^{s,p}(\Omega)\right)^*\right)$, with $\mathcal{X} := L^p(0,T;W_0^{s,p}(\Omega))$.

Lately, much attention has been focused on the study of fractional and non-local differential problems. Specifically, the relevance of considering problems similar to (1) is not only based on mathematical objective, but also on their importance in modern applied science such as phase continuum mechanics, fluid

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dynamics, image processing, game theory, transition phenomena, and population dynamics, since they are the standard result of the stochastic stabilisation of Lévy processes, see [5, 24, 33] and and related references. More important, since the revolutionary work of Caffarelli and Silvestre in [15], who proposed the s-harmonic extension to describe the fractional Laplacian operator. In the context of the fractional Laplacian operator, several important results in elliptic problems have been developed. For the analogous elliptic model, see [21, 34] and the related references, [12, 28] for hyperbolic problems, and [27] for the Camassa-Holm system.

On the other hand, regarding the parabolic equation implying a fractional Laplacian operator, the investigation of the anomalous diffusion equation has also attracted great attention during the last few years, due to its appearance in a variety of phenomena in several areas of physics, ecology, biology, geophysics, finance, and many others that can be described as non-Brownian scaling.

A seminal work that we highly recommend is the book by Bisci et al. [13]. It offers a significant and thorough introduction to the study of fractional problems. To elaborate on the issue, we refer back to several earlier studies where the authors examined a special case of the problem (1). We first review some previous results on the parabolic problems (1) with initial conditions u_0.

In [32], by means of the sub-differential approach, Mazăn et al. established the existence and uniqueness of strong solutions for the following diffusion problems implying a nonlocal fractional p-Laplacian operator

$$\frac{\partial u}{\partial t} + (-\Delta)^s_p u = 0, \quad \text{in } \Omega,\ t > 0, \tag{2}$$

with $\Omega \subseteq \mathbb{R}^N,\ N \geq 1,\ s \in (0,1)$ and $p \in (1,\infty)$. In addition, the authors also prove that when $p \neq 2$ and s tends to 1^-, after inserting a normalizing constant, the equation (2) is simplified to the following evolution equation $u_t - \Delta_p u = 0$ implying the p-Laplacian. Similar to the earlier work, Strömqvist in [26] investigated the local regularity properties of subsolutions of non-local parabolic integro-differential problem, for a bounded domain Ω and $p \geq 2$, of the shape

$$\frac{\partial u(x,t)}{\partial t} + \mathcal{P.V.} \int_{R^n} |u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))\mathcal{L}(x,y,t)dy = 0 \qquad \text{in } \Omega \times (t_1, t_2),$$

where $\mathcal{P.V.}$ represents the principal value and the kernel $\mathcal{L}$ checks, for each $s \in (0,1)$ and $\lambda \geq 1$,

$$\frac{\lambda^{-1}}{|x-y|^{n+sp}} \leq \mathcal{L}(x,y,t) \leq \frac{\lambda}{|x-y|^{n+sp}}.$$

In [16], Fu and Pucci considered the following problem, for all $2 < p \leq 2^*_s = 2N/(N-2s)$ and $s \in (0,1), N > 2s$,

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} + (-\Delta)^s u(x,t) = |u(x,t)|^{p-2}u(x,t), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \mathbb{R}^N \backslash \Omega,\ t > 0. \end{cases}$$

By employing the potential well theory, the authors ensured the existence of global weak solutions to the considered problem. Thereafter, they obtained the vacuum isolating and blow-up of strong solutions. As $p = p(x)$ and $s \to 1^-$, problem (1), with an initial data $u_0 \in L^2$, was studied by Hammou [23] by applying the theory of topological degrees. In this direction, we also refer to [18, 22, 25] and references therein for the interested reader.

At the same time, the periodic solutions were concurrently thoroughly researched in the literature. The book [25] by Lions includes a qualitative investigation of periodic solutions to the problem (1) when $s = 1$ is an integer. The author investigated the existence, regularities, and uniqueness of the weak periodic solution to (1) when $f \in L^{p'}\left(0, T; W^{-1,p'}(\Omega)\right)$, with p' is the conjugate exponent of p. Recently, Pucci, Xiang, and Zhang [30] proved, by using some standard techniques, the existence of periodic solutions for similar initial-boundary value problem for fractional p-Laplacian parabolic equations (1) but with a Kirchhoff term. We also refer to [36, 38, 39] for more details.

Inspired by the previous works, and by means of the topological degree theory, we establish the existence of periodic weak solutions to the nonlinear parabolic problems (1) involving the fractional p-Laplacian operator, which, to the best of our knowledge, are not addressed in earlier studies. We reduce this fractional parabolic problem to a new problem ruled by an operator equation of the shape $\mathcal{K}u+\mathcal{B}u = h$, in which $\mathcal{K}$ is a densely defined monotone maximal linear operator, and $\mathcal{B}$ is a demicontinuous bounded map of type (S_+) with regard to the domain of $\mathcal{K}$. We recall that the topological degree theory for perturbations of linear maximal monotone mappings and their applications to a class of parabolic problems was proposed by Berkovits and Mustonen in 1991, see [10]. This method has been employed extensively by various authors to study the nonlinear parabolic problems and has proven to be a very successful tool; we refer to the works [6–9] for more details. For part of the background to this theory and its applications, the reader can consult the articles [1–4, 11, 17].

This paper is structured as follows: In Section 2, we state some necessary preliminary results and give some related lemmas that will be used in the sequel. In Section 3, we establish the main results of this paper is devoted to the proof of the main results.

2. Preliminaries

We begin in this section by presenting the functional framework required to investigate the problem (1), as well as the fundamental definitions and properties of topological degree theory that are relevant to our purpose.

In this work, $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded open set with Lipschitz boundary, $p \in \Big[1, \infty\Big)$ and the fractional exponent $s \in (0, 1)$, we define the fractional Sobolev space $W^{s,p}(\Omega)$ such as

$$W^{s,p}(\Omega) = \Big\{u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega)\Big\},$$

It is equipped with the following norm

$$\|u\|_{s,p} = \Big(\|u\|_p^p + [u]_{s,p}^p\Big)^{\frac{1}{p}},$$

where $\|\cdot\|_p$ is the norm in $L^p(\Omega)$ and $[\cdot]_{s,p}$ is the Gagliardo semi-norm, setting by

$$[u]_{s,p} = \Big(\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dxdy\Big)^{\frac{1}{p}}.$$

Then $W^{s,p}(\Omega)$ is a separable and reflexive Banach space if $1 \leq p < \infty$ and $1 < p < \infty$, respectively(see [18]). To establish the existence of weak periodic solutions for (1), we take the closed linear subspace of $W_0^{s,p}(\Omega)$ given by

$$W_0^{s,p}(\Omega) = \Big\{u \in W^{s,p}(\mathbb{R}^N) \ : \ u = 0 \text{ a.e. } \mathbb{R}^N \backslash \Omega\Big\},$$

with the associated norm expressed by $[\cdot]_{s,p}$ (see [20]). Then $W_0^{s,p}(\Omega)$ is a uniformly convex Banach space (see Lemma 2.4 in [37]). The following theorem summarizes some interesting properties of fractional Sobolev spaces.

Theorem 2.1. *(see [18]) Assume that Ω is a bounded open set with Lipschitz boundary, $s \in (0, 1)$ and $p \geq 1$. Then the following embeddings hold*

(i) if $sp < N$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, p^*)$,

(ii) if $sp = N$, then $W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [1, +\infty)$,

(iii) if $sp > N$, then $W^{s,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.

where p^* is the fractional critical Sobolev exponent, that is

$$p^* := \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N, \\ +\infty & \text{if } sp \geq N. \end{cases} \tag{3}$$

Moreover, the space $W^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$, if $q \in [p; p^*)$.
Following that, we introduce the proposed framework to solve the problem (1). Let $0 < T < \infty$, we regard the functional space

$$\mathcal{X} := L^p(0, T; W_0^{s,p}(\Omega)),$$

that is a separable and reflexive Banach space with the norm

$$|u|_{\mathcal{X}} = \Big(\int_0^T \|u\|^p_{W^{s,p}(\Omega)} dt \Big)^{1/p}.$$

In view of [35], we can clearly establish that the norm $|u|_{\mathcal{X}}$ is equivalent to the following standard norm

$$\|u\|_{\mathcal{X}} = \Big(\int_0^T [u]^p_{s,p} dt \Big)^{1/p}.$$

For reader's convenience, we start by recalling some results and properties from the Berkovits and Mustonen degree theory for demicontinuous operators of generalized (S_+) type in real separable reflexive Banach $\mathcal{Z}$. In what follows, We respectively denote by $\mathcal{Z}^*$ the topological dual of the Banach space $\mathcal{Z}$ with continuous dual pairing $\langle \cdot, \cdot \rangle$ and $\rightharpoonup$ represents the weak convergence. Given a nonempty subset Ω of $\mathcal{Z}$.
Let $\mathcal{A}$ from $\mathcal{Z}$ to $2^{\mathcal{Z}^*}$ be a multi-values mapping. We designate by $Gr(\mathcal{A})$ the graph of $\mathcal{A}$, i.e.

$$Gr(\mathcal{A}) = \Big\{ (u, v) \in \mathcal{Z} \times \mathcal{Z}^* \ : \ v \in \mathcal{A}(u) \Big\}.$$

Definition 2.2. *The multi-values mapping $\mathcal{A}$ is called*

1. *monotone, if for each pair of elements $(\eta_1, \theta_1), (\eta_2, \theta_2)$ in $Gr(\mathcal{A})$, we have the inequality*

$$\langle \theta_1 - \theta_2, \eta_1 - \eta_2 \rangle \geq 0.$$

2. *maximal monotone, if it is monotone and maximal in the sense of graph inclusion among monotone multi-values mappings from $\mathcal{Z}$ to $2^{\mathcal{Z}^*}$. The last clause has an analogous variant in that, for each $(\eta_0, \theta_0) \in \mathcal{Z} \times \mathcal{Z}^*$ for which $\langle \theta_0 - \theta, \eta_0 - \eta \rangle \geq 0$, for all $(\eta, \theta) \in Gr(\mathcal{A})$, we have $(\eta_0, \theta_0) \in Gr(\mathcal{A})$.*

Let $\mathcal{Y}$ be another real Banach space.

Definition 2.3. *A mapping $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{Z} \to \mathcal{Y}$ is said to be*

1. *demicontinuous, if for each sequence $(u_n) \subset \Omega$, $u_n \to u$ implies $\mathcal{B}(u_n) \rightharpoonup \mathcal{B}(u)$.*
2. *of type (S_+), if for any sequence $(u_n) \subset D(\mathcal{B})$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty} \langle \mathcal{B}u_n, u_n - u \rangle \leq 0$, we have $u_n \to u$.*

Let $\mathcal{K} : \mathcal{D}(\mathcal{K}) \subset \mathcal{Z} \to \mathcal{Z}^*$ be a linear maximal monotone map such that $\mathcal{D}(\mathcal{K})$ is dense in $\mathcal{Z}$.
In the following, for each open and bounded subset O on $\mathcal{Z}$, we consider classes of operators :

$$\mathcal{F}_O(\Omega) := \Big\{ \mathcal{K} + \mathcal{B} : \overline{O} \cap \mathcal{D}(\mathcal{K}) \to \mathcal{Z}^* \mid \mathcal{B} \ \text{ is bounded, demicontinuous and of type } (S_+) \text{ with respect to } \mathcal{D}(\mathcal{K}) \text{ from } O \text{ to } \mathcal{Z}^* \Big\},$$

$$\mathcal{H}_O := \Big\{ \mathcal{K} + \mathcal{B}(t) \ : \ \overline{O} \cap D(L) \to \mathcal{Z}^* \mid \mathcal{B}(t) \ \text{ is a bounded homotopy of type } (S_+) \ \text{ with respect to } \mathcal{D}(\mathcal{K}) \ \text{ from } \overline{O} \ \text{ to } \mathcal{Z}^* \Big\}.$$

Remark 2.4. *[10] Remark that the class $\mathcal{H}_O$ contains all affine homotopy*

$$\mathcal{K} + (1-t)\mathcal{B}_1 + t\mathcal{B}_2 \quad \text{with} \quad (\mathcal{K} + \mathcal{B}_i) \in \mathcal{F}_O, \ \ i = 1, 2.$$

The following theorem provides the notion of the Berkovits and Mustonen topological degree for a class of demicontinuous operators satisfying the condition (S_+), which is the main key to the existence proof, for more details see [10].

Theorem 2.5. *Let $\mathcal{K}$ be a linear maximal monotone densely defined map from $\mathcal{D}(\mathcal{K}) \subset \mathcal{Z}$ to $\mathcal{Z}^*$ and*

$$\mathcal{M} = \Big\{(F, G, \psi) \ : \ F \in \mathcal{F}_O, \ O \text{ an open bounded subset in } \mathcal{Z}, \ \psi \notin F\Big(\partial O \cap \mathcal{D}(\mathcal{K})\Big)\Big\}.$$

There is a unique degree function $d \ : \ \mathcal{M} \longrightarrow \mathbb{Z}$ which satisfies the following properties :

1. *(Normalization) $\mathcal{K} + \mathcal{J}$ is a normalising map, where $\mathcal{J}$ is the duality mapping of $\mathcal{Z}$ into $\mathcal{Z}^*$, that is, $d(\mathcal{K} + \mathcal{J}, O, \psi) = 1$, when $\psi \in (\mathcal{K} + \mathcal{J})(O \cap \mathcal{D}(\mathcal{K}))$.*
2. *(Additivity) Let $\mathcal{B} \in \mathcal{F}_O$. If O_1 and O_2 are two disjoint open subsets of O such that $\psi \notin \mathcal{B}\Big((\overline{O}\backslash(O_1 \cup O_2)) \cap D(L)\Big)$ then we have*
$$d(\mathcal{B}, O, \psi) = d(\mathcal{B}, O_1, \psi) + d(\mathcal{B}, O_2, \psi).$$
3. *(Homotopy invariance) If $\mathcal{B}(t) \in \mathcal{H}_O$ and $\psi(t) \notin \mathcal{B}(t)(\partial O \cap \mathcal{D}(\mathcal{K}))$ for every $t \in [0,1]$, where $\psi(t)$ is a continuous curve in $\mathcal{Z}^*$, then*
$$d(\mathcal{B}(t), O, \psi(t)) = \text{ constant}, \qquad \text{for all } t \in [0,1].$$
4. *(Existence) if $d(\mathcal{B}, O, \psi) \neq 0$, then the equation $\mathcal{B}u = \psi$ has a solution in $O \cap \mathcal{D}(\mathcal{K})$.*

Lemma 2.6. *Let $\mathcal{K} + \mathcal{B} \in \mathcal{F}_\mathcal{Z}$ and $\psi \in \mathcal{Z}^*$. Suppose that there is $R > 0$ such as*

$$\langle \mathcal{K}u + \mathcal{B}u - \psi, u \rangle > 0, \tag{4}$$

for each $u \in \partial B_R(0) \cap \mathcal{D}(\mathcal{K})$. Hence

$$(\mathcal{K} + \mathcal{B})(\mathcal{D}(\mathcal{K})) = \mathcal{Z}^*. \tag{5}$$

Proof. Let $\varepsilon > 0$, $\theta \in [0,1]$ and

$$\mathcal{S}_\varepsilon(\theta, u) = \mathcal{K}u + (1-\theta)\mathcal{J}u + \theta(\mathcal{B}u + \varepsilon\mathcal{J}u - \psi).$$

As $0 \in \mathcal{K}(0)$ and applying the boundary condition (4), we have

$$\begin{aligned}
\langle \mathcal{S}_\varepsilon(\theta, u), u \rangle &= \Big\langle \theta(\mathcal{K}u + \mathcal{B}u - \psi, u\rangle + \langle(1-\theta)\mathcal{K}u + (1-\theta+\varepsilon)\mathcal{J}u, u \Big\rangle \\
&\geq \Big\langle (1-\theta)\mathcal{K}u + (1-\theta+\varepsilon)\mathcal{J}u, u \Big\rangle \\
&= (1-\theta)\Big\langle \mathcal{K}u, u \Big\rangle + (1-\theta+\varepsilon)\Big\langle \mathcal{J}u, u \Big\rangle \\
&\geq (1-\theta+\varepsilon)\|u\|^2 = (1-\theta+\varepsilon)R^2 > 0.
\end{aligned}$$

Which means that $0 \notin \mathcal{S}_\varepsilon(\theta, u)$. As $\mathcal{J}$ and $\mathcal{B} + \varepsilon\mathcal{J}$ are bounded, continuous and of type (S_+), $\{\mathcal{S}_\varepsilon(\theta, \cdot)\}_{\theta\in[0,1]}$ is an admissible homotopy. Hence, by using the normalisation and invariance under homotopy, we get

$$d(\mathcal{S}_\varepsilon(\theta, \cdot), B_R(0), 0) = d(\mathcal{K} + \mathcal{J}, B_R(0), 0) = 1.$$

As a result, there is $u_\varepsilon \in \mathcal{D}(\mathcal{K})$ such that $0 \in \mathcal{S}_\varepsilon(\theta, \cdot)$.
If we take $\theta = 1$ and when $\varepsilon \to 0^+$, then we have $\psi = \mathcal{K}u + \mathcal{B}u$ for certain $u \in \mathcal{D}(\mathcal{K})$. As $\psi \in \mathcal{Z}^*$ is of any kind, we deduce that $(\mathcal{K} + \mathcal{B})(\mathcal{D}(\mathcal{K})) = \mathcal{Z}^*$. □

3. Existence result

To show the existence of a weak periodic solution of (1), we use compactness methods. To begin, we convert this nonlinear parabolic problem into a new problem that is governed by an operator equation of the form $\mathcal{K}u + \mathcal{B}u = h$. Then, we apply the theory of topological degrees.

In this spirit, we consider $\mathcal{B} : \mathcal{X} \longrightarrow \mathcal{X}^*$ such that

$$\langle \mathcal{B}u, v\rangle = \int_0^T \int_\Omega \int_\Omega |u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))(v(x,t) - v(y,t))\mathcal{L}(x,y)dxdydt, \tag{6}$$

for all $v \in \mathcal{X}$, where $\mathcal{L}(x,y) = |x-y|^{-N-ps}$.
The main result of this study is presented in the following theorem.

Theorem 3.1. *Suppose that $h \in \mathcal{X}^*$ and $u(x,0) = u(x,T) \in L^2(\Omega)$. Then the problem (1) admits a weak periodic solution $u \in \mathcal{D}(\mathcal{K})$ in the following sense*

$$-\int_Q uv_t dxdt + \langle \mathcal{B}u, v\rangle = \int_Q hvdxdt, \tag{7}$$

for each $v \in \mathcal{X}$.

In order to establish Theorem 3.1, we first required the following technical lemma

Lemma 3.2. *Let $0 < s < 1$ and $2 < p < +\infty$, then the operator $\mathcal{B}$ setting in (6) is*

(i) bounded and demicontinuous

(ii) strictly monotone.

(iii) of type (S_+).

Proof. i)− Using [29], the operator defined by

$$\langle \mathcal{H}u, v\rangle = \int_\Omega \int_\Omega |u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))(v(x,t) - v(y,t))\mathcal{L}(x,y)dxdy, \quad \forall u, v \in W^{s,p}(\Omega)$$

is well defined, bounded, continuous.
Moreover, the form $\mathcal{H}$ induces a Nemytskii operator that invokes the properties given above, i.e. the nonlinear operator $\mathcal{B}$ is bounded, demicontinuous.

ii)− According to Perera et al. [31, Lemma 6.3], it is enough to demonstrate that

$$\langle \mathcal{B}u, v\rangle \leq \|u\|^{p-1}\|v\| \qquad \text{for all } u, v \in \mathcal{X}$$

and the equality applies if and only if $\delta u = \gamma v$ for some $\delta, \gamma \geq 0$, not both null.
Under Hölder's inequality, we obtain

$$\langle \mathcal{B}u, v\rangle \leq \int_0^T \int_\Omega \int_\Omega |u(x,t) - u(y,t)|^{p-1}|v(x,t) - v(y,t)|\mathcal{L}(x,y)dxdydt \leq \|u\|^{p-1}\|v\|$$

It is obvious that the equality is true if $\alpha u = \gamma v$ for any $\delta, \gamma \geq 0$, not both zero. Inversely, if $\langle \mathcal{B}u, v\rangle = \|u\|^{p-1}\|v\|$, equality exists in the two inequalities. Thus, the equality of the second inequality yields

$$\delta|u(x,t) - u(y,t)| = \gamma|v(x,t) - v(y,t)| \quad \text{a.e. in } \mathbb{R}^N \times \mathbb{R}^N \times (0,T)$$

for any $\delta, \gamma \geq 0$, not both null, thus the equality of the first inequality provides

$$\delta(u(x,t) - u(y,t)) = \gamma(v(x,t) - v(y,t)) \quad \text{a.e. in } \mathbb{R}^N \times \mathbb{R}^N \times (0,T).$$

As u and v disappear a.e. in $\mathbb{R}^N\backslash\Omega\times(0,T)$, it results that $\delta u=\gamma v$ a.e. in Q.

iii)− It is yet to be established that the operator $\mathcal{B}$ is of type (S_+).

Let $(u_n)_n$ be a sequence in $D(\mathcal{B})$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{in } \mathcal{X} \\ \limsup\limits_{n\to\infty}\langle \mathcal{B}u_n, u_n-u\rangle \le 0. \end{cases}$$

We want to demonstrate that $u_n \to u$ in $\mathcal{X}$.

From the weak convergence $u_n \rightharpoonup u$, $\limsup\limits_{n\to\infty}\langle \mathcal{B}u_n-\mathcal{B}u, u_n-u\rangle \le 0$ and (ii), we derive

$$\lim_{n\to+\infty}\langle \mathcal{B}u_n, u_n-u\rangle = \lim_{n\to+\infty}\langle \mathcal{B}u_n-\mathcal{B}u, u_n-u\rangle = 0. \tag{8}$$

Thanks to Theorem 2.1 and [25, Theorem 5.1] informs us that $\mathcal{X} \hookrightarrow\hookrightarrow L^p(Q)$. Consequently, there is a subsequence still referred to as (u_n), such that

$$u_n(x,t) \to u(x,t), \text{ a.e. } (x,t)\in Q. \tag{9}$$

then we obtain from the Fatou lemma and (9)

$$\liminf_{n\to+\infty}\int_0^T\int_\Omega\int_\Omega |u_n(x)-u_n(y)|^p\mathcal{L}(x,y)dxdydt \ge \int_0^T\int_\Omega\int_\Omega |u(x)-u(y)|^p\mathcal{L}(x,y)dxdydt, \tag{10}$$

On the other side, By invoking the Young inequality, there is a positive constant C such that

$$\begin{aligned} \langle \mathcal{B}u_n, u_n-u\rangle &= \int_0^T\int_\Omega\int_\Omega |u_n(x,t)-u_n(y,t)|^p\mathcal{L}(x,y)dxdydt \\ &-\int_0^T\int_\Omega\int_\Omega |u_n(x,t)-u_n(y,t)|^{p-2}(u_n(x,t)-u_n(y,t))(u(x,t)-u(y,t))\mathcal{L}(x,y)dxdydt \\ &\ge \int_0^T\int_\Omega\int_\Omega |u_n(x,t)-u_n(y,t)|^p\mathcal{L}(x,y)dxdydt \\ &\quad -\int_0^T\int_\Omega\int_\Omega |u_n(x,t)-u_n(y,t)|^{p-1}|u(x,t)-u(y,t)|\mathcal{L}(x,y)dxdydt \\ &\ge C\int_0^T\int_\Omega\int_\Omega |u_n(x,t)-u_n(y,t)|^p\mathcal{L}(x,y)dxdydt - C\int_0^T\int_\Omega\int_\Omega |u(x,t)-u(y,t)|^p\mathcal{L}(x,y)dxdydt, \end{aligned} \tag{11}$$

combining (10), (8) and (11), we deduce

$$\lim_{n\to+\infty}\int_0^T\int_\Omega\int_\Omega |u_n(x,t)-u_n(y,t)|^p\mathcal{L}(x,y)dxdydt = \int_0^T\int_\Omega\int_\Omega |u(x,t)-u(y,t)|^p\mathcal{L}(x,y)dxdydt. \tag{12}$$

Thanks to (9), (12) and the Brezis-Lieb lemma [14], our conclusion has been established. □

We are now prepared to demonstrate Theorem 3.1.

Proof. [Proof of Theorem 3.1] To exhibit the existence of a weak solution to (1), we wish to use the topological degree methods. To achieve this, we define

$$\mathcal{D}(\mathcal{K}) = \Big\{v\in\mathcal{X} \ : \ v'\in\mathcal{X}^*,\ v(0)=0\Big\},$$

From the density property of $C_c^\infty(Q_T)$ in $\mathcal{X}$ and using the fact that $C_c^\infty(Q_T)\subset\mathcal{D}(\mathcal{K})$, we can infer that $\mathcal{D}(\mathcal{K})$ is dense in $\mathcal{X}$. Let us consider the operator $\mathcal{K}:\mathcal{D}(\mathcal{K})\subset\mathcal{X}\longrightarrow\mathcal{X}^*$ such as

$$\langle \mathcal{K}u, v\rangle = -\int_Q uvdxdt, \qquad \text{for all } u\in\mathcal{D}(\mathcal{K}),\ v\in\mathcal{X}.$$

Thereby, the operator $\mathcal{K}$ is generated by $\partial/\partial t$ by making of the relation

$$\langle \mathcal{K}u, v\rangle = \int_0^T \Big\langle \frac{\partial u(t)}{\partial t}, v(t)\Big\rangle dt, \quad \text{for each } u \in \mathcal{D}(\mathcal{K}),\ v \in \mathcal{X}.$$

The result of [25, Lemma 1.1, p. 313] enables us to conclude that $\mathcal{L}$ is maximal monotone operator. For more information, see, for example [40].
On the other hand, the monotonicity of $\mathcal{K}$ $\Big(\langle \mathcal{K}u, u\rangle \geq 0 \text{ for all } u \in \mathcal{D}(\mathcal{K})\Big)$ ensures that

$$\begin{aligned}\langle \mathcal{K}u + \mathcal{B}u, u\rangle \geq \langle \mathcal{B}u, u\rangle &= \int_0^T \int_\Omega \int_\Omega |u(x,t) - u(y,t)|^{p-2}(u(x,t) - u(y,t))^2 \mathcal{L}(x,y)dxdydt \\ &= \int_0^T \int_\Omega \int_\Omega |u(x,t) - u(y,t)|^p \mathcal{L}(x,y)dxdydt = \|u\|^p_{\mathcal{X}} \end{aligned} \quad (13)$$

for all $u \in \mathcal{X}$.
Thanks to inequality (13) the right hand side goes to infinity as $\|u\|_{\mathcal{X}} \to \infty$, since for each $h \in \mathcal{X}^*$ there exists $R = R(h)$ for which

$$\langle \mathcal{K}u + \mathcal{B}u - h, u\rangle > 0 \qquad \text{for all } u \in B_R(0) \cap \mathcal{D}(\mathcal{K}).$$

In accordance with Lemma 2.6, we deduce the existence of $u \in \mathcal{D}(\mathcal{K})$, a solution to the operator equation $\mathcal{K}u + \mathcal{B}u = h$. This implies the existence of a weak periodic solution to the problem (1). □

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