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New asymptotic stability conditions for nonlinear stochastic systems driven by fractional Brownian motion

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Abstract. This paper is concerned with the asymptotic stability for nonlinear stochastic systems driven by fractional Brownian motion (fBm) with Hurst parameter $H \in (1/2, 1)$. First, some new asymptotic stability conditions are given for nonlinear stochastic systems with fBm by using the characteristic of the mild solution, the fBm, and the semigroup. Then, the results obtained are extended to the linear case, and some asymptotic stability conditions are derived. Furthermore, the methods proposed are utilized to solve the consensus control problem of the stochastic multi-agent systems (SMAS) with fBm. Finally, simulations are provided to illustrate the effectiveness of the proposed theoretical results.

1. Introduction

Many phenomena in the natural world are inevitable and categorized as deterministic phenomena. However, not all events in the objective world are assured to either happen or not happen. In reality, practical systems like population ecosystems, communication networks, economic systems, and industrial control systems necessitate interactions with the external environment, which are susceptible to various environmental noises. Noted that the primary performance to consider is whether the system can reliably maintain its intended motion or operational state, which is stability. Nowadays, numerous results on the stability analysis of various stochastic systems with Brownian motion were obtained [1–5].

Over the past several decades, a large number of statistical data and experiments in real-world problems show that fBm can better model stochastic processes with long-term dependences, such as hydrology [6], climate [7, 8], finance [9, 10], network traffic [11], and so on. In the 1940s, Kolmogorov [12] first proposed the stochastic process with long memory. Subsequently, Mandelbrot [13] defined the fBm $B^H(t)$ with Hurst parameter $H \in (0, 1)$ and studied its properties. However, since fBm with $H \neq 1/2$ is neither a semimartingale nor an independent incremental process, the qualitative analysis for SPDE driven by fBm is generally complicated and challenging. Recently, Duncan et al. [14–18] have established the existence and uniqueness theorem of mild solutions for stochastic differential equations (SDEs) driven by fBm. Then, the stability for SDEs with fBm has been considered in [19–23]. Note that the theory results obtained in [19–23]

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are not related to the asymptotic behavior. Nowadays, Zeng [24] established the Lyapunov techniques for achieving moment exponential stability of SDEs driven by fBm. Zhang [25] and Ma [26] established conditions for achieving exponential stability of neutral SDEs with fBm by using the fractional power A^{β} with $\beta \in (0, 1]$. Furthermore, Yan [27] obtained sufficient conditions for achieving exponential stability of a stochastic delay equation with fBm by using the fractional power A^{α} with $\alpha \in (0, H)$. Duc [28] proved the exponential stability of the stochastic systems with fBm under the strong dissipative assumption. Then, based on the works of Duc [28], some relevant results on the asymptotic stability of SDEs with fBm have been presented [29–31].

In the domain of fractional stochastic differential equations, researchers employ numerical simulations to dissect intricacies. Moghaddam et al. [32] put forth a computational scheme for tackling nonlinear equations featuring delay, while their subsequent work [33] introduced a spline-based method tailored for equations with constant time delay. Investigating the Hurst index in fractional stochastic dynamical systems, Shahnazi-Pour et al. [34] contributed insights into system behavior. Moghaddam et al.'s algorithm [35] specifically addressed nonlocal nonlinear stochastic delayed systems with variable-order fBm, and Shahnazi-Pour et al.'s technique [36] was designed for nonlinear nonlocal stochastic dynamical systems with variable-order fractional Brownian noise. These diverse numerical approaches serve to deepen our comprehension and enhance the practical utility of fractional stochastic models.

Based on the above discussions, this paper aims to give novel sufficient conditions for achieving the asymptotic stability of the stochastic systems driven by fBm. We respectively suppose that the stochastic system affected by multiplicative noises is described by

$$\begin{cases} dx(t) = \lambda x(t)dt + f(t, x(t))dt + \tau(x(t))dB^H(t), \quad t > 0, \\ x(0) = x_0, \end{cases} \tag{1}$$

and the stochastic system affected by additive noises is described by

$$\begin{cases} dx(t) = \lambda x(t)dt + f(t, x(t))dt + g(t)dB^H(t), \quad t > 0, \\ x(0) = x_0, \end{cases} \tag{2}$$

where $x(t) \in \mathbb{R}^n$ represents the state of the system at time t. $\lambda \in \mathbb{R}$ is a constant, $f(\cdot, \cdot) \in \mathcal{L}^1([0, \infty) \times \mathbb{R}^n; \mathbb{R}^n)$, $\tau(\cdot) \in \mathcal{L}^2(\mathbb{R}^n; \mathcal{L}(\mathbb{U}, \mathbb{R}^n))$ and $g(\cdot) \in \mathcal{L}^2([0, \infty); \mathcal{L}(\mathbb{U}, \mathbb{R}^n))$ are continuous differentiable nonlinear functions; $B^H(t)$ is a $\mathbb{U}$-valued Q-fBm with $H \in \left(\frac{1}{2}, 1\right)$. It is noted that achieving asymptotic stability in nonlinear stochastic systems influenced by fBm presents several fundamental challenges. One primary challenge lies in the integral properties of fBm. As a non-Markovian process, integrating fBm involves specialized stochastic integration techniques. This complexity adds intricacies to system modeling and analysis and imposes higher requirements on control strategy design. On another note, the property of fBm causing variance to diverge as time approaches infinity poses a unique challenge. In mathematical terms, this divergence is distinct from traditional stochastic processes. Such behavior can directly impact the asymptotic stability of the system, as the infinite growth of variance may lead to unstable system behavior. The main contributions of this paper include some new asymptotic stability conditions:

(i) Some novel asymptotic stability conditions are given for nonlinear stochastic systems (1) and (2), respectively. Without using the fractional power such as [25–27], the asymptotic stability analysis in this paper is finished by using the characteristic of the mild solution, the fBm, and the semigroup, directly.

(ii) Some asymptotic stability conditions are derived for the linear stochastic systems driven by fBm.

(iii) Sufficient conditions are obtained for achieving the mean-square consensus of the nonlinear SMAS with fBm.

The paper is organized as follows. Some useful preliminaries are provided in Section 2. The new asymptotic stability conditions for nonlinear stochastic systems driven by fBm are derived in Section 3. Then, the mean-square consensus control of the SMAS with fBm is investigated in Section 4. Furthermore, a numerical example is presented in Section 5. Finally, the conclusions are given in Section 6.

Notations: Throughout this paper, denote $\mathbb{R}$ as the set of real numbers, $\mathbb{R}^n$ as the n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ as the set of $n \times m$ dimensional real matrix. Denote $\|\cdot\|$ as the norm. Let

$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space satisfying the usual conditions. Let $\otimes$ be the Kronecker product, $(\cdot)^T$ be the transpose of a real matrix, and $\lambda_{\min}(\cdot)$ ($\lambda_{\max}(\cdot)$) be the minimum (maximum) nonzero eigenvalue of a real symmetric square matrix. Let $\mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P}) = \mathcal{L}^p$ be the space of all random variables $\xi : \Omega \to \mathbb{R}$ such that $\|\xi\|_p = (\mathbf{E}\|\xi\|^p)^{1/p} < +\infty$.

2. Preliminaries

Definition 2.1. *[37] Let Hurst parameter $H \in (0,1)$. A fBm $\{B^H(t), t \geq 0\}$ is a continuous centered Gaussian process on $(\Omega, \mathcal{F}, \mathbb{P})$ with the following conditions:*

$$(i)\ B^H(0) = 0;\ (ii)\ \mathbf{E}[B^H(t)] = 0;\ (iii)\ R_H(t,s) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right).$$

Lemma 2.2. *[15] Let $\{e_n, n \in \mathbb{N}\}$ be a complete orthonormal basis in a separable Hilbert space $\mathbb{U}$ and $\{\beta_n^H(t), n \in \mathbb{N}, t \geq 0\}$ be a sequence of independent, real valued standard fBm with $H \in (1/2, 1)$. For a non-negative self-adjoint, trace class operator Q on $\mathbb{U}$, i.e., $Qe_n = \lambda_n e_n$, $Q = Q^*$ with $\mathrm{Trace}(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$ for all $n \in \mathbb{N}$. The $\mathbb{U}$-valued Q-fBm $B^H(t)$ with $H \in (1/2, 1)$ is defined by*

$$B^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n^H(t)e_n = \sum_{n=1}^{\infty} \beta_n^H(t)Q^{\frac{1}{2}}e_n, \quad t \geq 0.$$

Let $\mathbb{H}$ be a Hilbert space, and $\mathcal{L}_S^2(\mathbb{U}, \mathbb{H}) = \mathcal{L}^2(Q^{\frac{1}{2}}\mathbb{U}, \mathbb{H})$ be the space of Hilbert-Schmidt operators: $Q^{\frac{1}{2}}\mathbb{U} \to \mathbb{H}$ with norm

$$\|\Psi\|_S^2 = \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n}\Psi e_n \right\|^2 = \left\|\Psi Q^{\frac{1}{2}}\right\|^2 = \mathrm{Trace}(\Psi Q \Psi^*), \text{ for any } \Psi \in \mathbb{U}.$$

Lemma 2.3. *[15] Let $H \in \left(\frac{1}{2}, 1\right)$, $\Psi : [0,T] \to \mathcal{L}_S^2(\mathbb{U}, \mathbb{H})$. If $\sum_{n=1}^{\infty} \left\|\Psi(t)Q^{\frac{1}{2}}e_n\right\|$ is uniformly convergent for $t \in [0,T]$, then*

$$\mathbf{E}\left\|\int_{t_1}^{t_2} \Psi(s)dB^H(s)\right\|^2 \leq c(2H^2 - H)(t_2 - t_1)^{2H-1} \int_{t_1}^{t_2} \|\Psi(s)\|_S^2 ds, \quad \forall 0 \leq t_1 < t_2 \leq T, \tag{3}$$

where $c > 0$ is a constant depends on H.

Lemma 2.4. *[39] Let $T > 0$, $r \geq 0$, $p(\cdot)$ be a Borel measurable bounded nonnegative function on $[0,T]$, and $q(\cdot)$ be a nonnegative integrable function on $[0,T]$, if*

$$p(t) \leq r + \int_0^t p(s)q(s)ds, \text{ then } p(t) \leq r\exp\left(\int_0^t q(s)ds\right).$$

Lemma 2.5. *[39] Let $p(\cdot)$ and $q(\cdot)$ be nonnegative real valued continuous functions on $[0,\infty)$. Suppose that the real-valued function $r(t)$ is integrable on every closed and bounded subinterval of $[0,\infty)$. For all $t \geq 0$, if*

$$p(t) \leq r(t) + \int_0^t p(s)q(s)ds, \text{ then } p(t) \leq r(t) + \int_0^t \left[r(s)q(s)\exp\left(\int_s^t q(u)du\right)\right]ds.$$

Lemma 2.6. *[40](Hölder's inequality) Suppose that $x > 1$, $\frac{1}{x} + \frac{1}{y} = 1$. If $p \in \mathcal{L}^x(\Omega)$, $q \in \mathcal{L}^y(\Omega)$, then*

$$\int_\Omega p(s)q(s)ds \leq \left(\int_\Omega |p(s)|^x ds\right)^{\frac{1}{x}} \left(\int_\Omega |q(s)|^y ds\right)^{\frac{1}{y}}.$$

Graph theory. [41] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be a directed graph with N nodes, where $\mathcal{V} = \{1, 2, \ldots, N\}$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and $\mathcal{A} = [a_{ij}]_{N\times N} \in \mathbb{R}^{N\times N}$ respectively represent the set of nodes, the set of edges, and the weighted adjacency matrix. A directed edge from j to i is defined as $(i, j) \in \mathcal{E}$; a_{ij} is defined as the communication quantity between agent i and agent j that satisfies $a_{ij} > 0$ when $(i, j) \in \mathcal{E}$, and $a_{ij} = 0$, otherwise. Let $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$ represents the neighbors set of node i. The Laplacian of $\mathcal{G}$ is defined as $\mathcal{L} = [l_{ij}]_{N\times N} \in \mathbb{R}^{N\times N}$ with $l_{ii} = \sum_{j\neq i} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$. A sequence of edges $(i_k, i_{k-1}), (i_{k-1}, i_{k-2}), \cdots, (i_2, i_1)$ is called a directed path from node i_1 to node i_k, where $i_j \in \mathcal{V}$. A directed spanning tree is a directed tree, where every vertex except on the root vertex has exactly one parent and the root vertex can be connected to any other vertices through paths.

3. Asymptotic stability for stochastic systems driven by fBm

3.1. Stochastic systems with multiplicative noises

Consider the following nonlinear stochastic system:

$$\begin{cases} dx(t) = -\lambda x(t)dt + f(t, x(t))dt + \tau(x(t))dB^H(t), \quad t > 0, \\ x(0) = x_0, \end{cases} \tag{4}$$

where $x(t) \in \mathbb{R}^n$ represents the state of the system at time t. $\lambda > 0$ is a positive constant, $f(\cdot, \cdot) \in \mathcal{L}^1([0, \infty) \times \mathbb{R}^n; \mathbb{R}^n)$ and $\tau(\cdot) \in \mathcal{L}^2(\mathbb{R}^n; \mathcal{L}(\mathbb{U}, \mathbb{R}^n))$ are continuous differentiable nonlinear mapping functions, and $B^H(t)$ is a $\mathbb{U}$-valued Q-fBm with $H \in \left(\frac{1}{2}, 1\right)$.

Assumption 3.1. *$f(t, 0) = 0$ and $\tau(0) = 0$, $\forall t \geq 0$. And, there exists a positive constant $\rho > 0$ such that*

$$\|f(t, a) - f(t, b)\| \vee \|\tau(a) - \tau(b)\| \leq \rho\|a - b\|, \quad \forall a, b \in \mathbb{R}^n.$$

Theorem 3.2. *Under Assumption 3.1, if*

$$\lambda > \left(3c(2H^2 - H)\rho^2\|Q\| + 3\rho^2\right)^{1/2}, \tag{5}$$

then the stochastic system (4) is asymptotic stability in the mean-square sense.

Proof. From Theorem 3.3 in [16], the mild solution of Eq. (4) is

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x(s))ds + \int_0^t S(t-s)\tau(x(s))dB^H(s), \text{ with } S(t) = e^{-\lambda t}. \tag{6}$$

From Eq. (6), then

$$\mathbf{E}\|x(t)\|^2 \leq 3\|S(t)x_0\|^2 + \Xi_1(t) + \Xi_2(t), \tag{7}$$

with

$$\Xi_1(t) = 3\mathbf{E}\left\|\int_0^t S(t-s)f(s, x(s))ds\right\|^2, \;\; \Xi_2(t) = 3\mathbf{E}\left\|\int_0^t S(t-s)\tau(x(s))dB^H(s)\right\|^2.$$

From Lemma 2.6 and Assumption 3.1, then

$$\begin{aligned}\Xi_1(t) &= 3\mathbf{E}\left\|\int_0^t S(t-s)f(s,x(s))ds\right\|^2\\ &\le 3\mathbf{E}\left\|\int_0^t \|S(t-s)\|^{\frac{1}{2}}\cdot\|S(t-s)\|^{\frac{1}{2}}\cdot\|f(s,x(s))\|ds\right\|^2\\ &\le 3\mathbf{E}\left\|\left(\int_0^t \|S(t-s)\|ds\right)^{\frac{1}{2}}\left(\int_0^t \|S(t-s)\|\cdot\|f(s,x(s))\|^2ds\right)^{\frac{1}{2}}\right\|^2\\ &= 3\int_0^t \|S(t-s)\|ds\int_0^t\left[\|S(t-s)\|\mathbf{E}\|f(s,x(s))\|^2\right]ds\\ &\le 3\rho^2\int_0^t \|S(t-s)\|ds\int_0^t\left[\|S(t-s)\|\mathbf{E}\|x(s)\|^2\right]ds\\ &\le 3\rho^2\int_0^t e^{-\lambda(t-s)}ds\int_0^t\left(e^{-\lambda(t-s)}\mathbf{E}\|x(s)\|^2\right)ds.\end{aligned} \tag{8}$$

According to Lemma 2.3 and Assumption 3.1, then

$$\begin{aligned}\Xi_2(t) &= 3\mathbf{E}\left\|\int_0^t S(t-s)\tau(x(s))dB^H(s)\right\|^2\\ &\le 3c(2H^2-H)t^{2H-1}\int_0^t \|S(t-s)\tau(x(s))\|_S^2\,ds\\ &\le c_1t^{2H-1}\int_0^t\left[\|S(t-s)\|^2\,\mathbf{E}\|x(s)\|^2\right]ds\\ &\le c_1t^{2H-1}\int_0^t e^{-2\lambda(t-s)}\mathbf{E}\|x(s)\|^2ds,\end{aligned} \tag{9}$$

where $c_1 = 3c(2H^2-H)\rho^2\|Q\|$. Substituting (8) and (9) into the (7) leads to

$$\begin{aligned}\mathbf{E}\|x(t)\|^2 &\le c_0e^{-\lambda t}+3\rho^2\int_0^t e^{-\lambda(t-s)}ds\int_0^t\left(e^{-\lambda(t-s)}\mathbf{E}\|x(s)\|^2\right)ds+c_1t^{2H-1}\int_0^t e^{-2\lambda(t-s)}\mathbf{E}\|x(s)\|^2ds\\ &= c_0e^{-\lambda t}+3\rho^2\tfrac{1}{\lambda}(1-e^{-\lambda t})\int_0^t\left(e^{-\lambda(t-s)}\mathbf{E}\|x(s)\|^2\right)ds+c_1t^{2H-1}e^{-2\lambda t}\int_0^t\left(e^{2\lambda s}\mathbf{E}\|x(s)\|^2\right)ds\\ &= c_0e^{-\lambda t}+3\rho^2\tfrac{1}{\lambda}(e^{-\lambda t}-e^{-2\lambda t})\int_0^t\left(e^{\lambda s}\mathbf{E}\|x(s)\|^2\right)ds+c_1t^{2H-1}e^{-2\lambda t}\int_0^t\left(e^{2\lambda s}\mathbf{E}\|x(s)\|^2\right)ds\\ &\le c_0e^{-\lambda t}+3\rho^2\tfrac{1}{\lambda}e^{-\lambda t}\int_0^t\left(e^{\lambda s}\mathbf{E}\|x(s)\|^2\right)ds+c_1t^{2H-1}e^{-2\lambda t}\int_0^t\left(e^{2\lambda s}\mathbf{E}\|x(s)\|^2\right)ds\\ &= c_0e^{-\lambda t}+e^{-\lambda t}\int_0^t\left(c_1t^{2H-1}e^{-\lambda t}e^{\lambda s}+c_2\right)\left(e^{\lambda s}\mathbf{E}\|x(s)\|^2\right)ds,\end{aligned}$$

where $c_0 = 3\|x_0\|^2$ and $c_2 = 3\rho^2/\lambda$. Then, one has

$$e^{\lambda t}\mathbf{E}\|x(t)\|^2 \le c_0+\int_0^t\left(c_1t^{2H-1}e^{-\lambda t}e^{\lambda s}+c_2\right)\left(e^{\lambda s}\mathbf{E}\|x(s)\|^2\right)ds.$$

From Lemma 2.4, then

$$\begin{aligned}e^{\lambda t}\mathbf{E}\|x(t)\|^2 &\le c_0\exp\left(\int_0^t\left(c_1t^{2H-1}e^{-\lambda t}e^{\lambda s}+c_2\right)ds\right)\\ &= c_0\exp\left(c_1t^{2H-1}\tfrac{1}{\lambda}(1-e^{-\lambda t})+c_2t\right)\\ &\le c_0\exp\left(c_1t^{2H-1}\tfrac{1}{\lambda}+c_2t\right).\end{aligned}$$

Since $H\in\left(\frac{1}{2},1\right)$, then $2H-1\in(0,1)$. Furthermore, for $0\le t<1$, one has $e^{\lambda t}\mathbf{E}\|x(t)\|^2 \le c_0e^{(c_1/\lambda+c_2t)}$; for $t\ge 1$, one has $e^{\lambda t}\mathbf{E}\|x(t)\|^2\le c_0e^{(c_1/\lambda+c_2)t}$, such that $\mathbf{E}\|x(t)\|^2\le c_0e^{(c_1/\lambda+c_2-\lambda)t}$. When condition (5) is satisfied, we have $c_1/\lambda+c_2-\lambda<0$. Therefore, $\lim_{t\to+\infty}\mathbf{E}\|x(t)\|^2 = 0$.

The proof of Theorem 3.2 is completed.

Remark 3.3. *The proposed stability condition* $\lambda > (3c(2H^2 - H)\rho^2\|Q\| + 3\rho^2)^{1/2}$ *depends on the Lipschitz constant* ρ *of the nonlinearities and the Hurst parameter* H *of the fBm, which reflect real-world systems' dynamics. As H increases from* $1/2$ *to* 1*, the term* $2H^2 - H$ *in the inequality's right-hand side increases, resulting in an overall increase in the square root term. Consequently,* λ *increases with the increase in H.*

Consider the following linear stochastic system:

$$\begin{cases} dx(t) = -\lambda x(t)dt + \tau(x(t))dB^H(t), \quad t > 0, \\ x(0) = x_0. \end{cases} \tag{10}$$

Assumption 3.4. $\tau(0) = 0$, *and there exists a positive constant* $\rho > 0$ *such that*

$$\|\tau(a) - \tau(b)\| \leq \rho\|a - b\|, \quad \forall a, b \in \mathbb{R}^n.$$

Corollary 3.5. *Under Assumption 3.4, if*

$$\lambda > \left(2c(2H^2 - H)\rho^2\|Q\|\right)^{1/2}, \tag{11}$$

then the stochastic system (10) is asymptotic stability in the mean-square sense.

3.2. Stochastic systems with additive noises

Consider the following nonlinear stochastic system:

$$\begin{cases} dx(t) = -\lambda x(t)dt + f(t, x(t))dt + g(t)dB^H(t), \quad t > 0, \\ x(0) = x_0, \end{cases} \tag{12}$$

where $x(t) \in \mathbb{R}^n$ represents the state of the system at time t. $\lambda > 0$ is a positive constant, $f(\cdot, \cdot) \in \mathcal{L}^1([0, \infty) \times \mathbb{R}^n; \mathbb{R}^n)$ is a continuous differentiable nonlinear function, $B^H(t)$ is a $\mathbb{U}$-valued Q-fBm with $H \in \left(\frac{1}{2}, 1\right)$. $g(\cdot) \in \mathcal{L}^2([0, \infty); \mathcal{L}(\mathbb{U}, \mathbb{R}^n))$ represents the noise intensity function, which is a continuous differentiable function.

Assumption 3.6. $f(t, 0) = 0$, $\forall t \geq 0$. *And, there exists a positive constant* $\rho > 0$ *such that*

$$\|f(t, a) - f(t, b)\| \leq \rho\|a - b\|, \quad \forall a, b \in \mathbb{R}^n.$$

Theorem 3.7. *Under Assumption 3.6, if the following condition is satisfied:*

$$\int_0^\infty e^{\lambda s}\|g(s)\|^2 ds < \infty, \tag{13}$$

then the stochastic system (12) is asymptotic stability in the mean-square sense.

Proof. From Theorem 3.3 in [16], the mild solution of Eq. (12) is

$$x(t) = S(t)x_0 + \int_0^t S(t - s)f(s, x(s))ds + \int_0^t S(t - s)g(s)dB^H(s), \text{ with } S(t) = e^{-\lambda t}. \tag{14}$$

From (13) and (14), then

$$\begin{aligned} \mathbf{E}\|x(t)\|^2 &\leq w_0\|S(t)\|^2 + 3\rho^2 \int_0^t \|S(t - s)\|ds \int_0^t \left[\|S(t - s)\| \cdot \mathbf{E}\|x(s)\|^2\right] ds + w_1 t^{2H-1} \int_0^t \left[\|S(t - s)\|^2 \|g(s)\|^2\right] ds \\ &\leq w_0 e^{-\lambda t} + 3\rho^2 \int_0^t e^{-\lambda(t-s)} ds \int_0^t \left(e^{-\lambda(t-s)}\mathbf{E}\|x(s)\|^2\right) ds + w_1 t^{2H-1} \int_0^t e^{-2\lambda(t-s)}\|g(s)\|^2 ds \\ &= w_0 e^{-\lambda t} + 3\rho^2 \tfrac{1}{\lambda}(1 - e^{-\lambda t}) \int_0^t \left(e^{-\lambda(t-s)}\mathbf{E}\|x(s)\|^2\right) ds + w_1 t^{2H-1} e^{-\lambda t} \int_0^t \left(e^{-\lambda(t-s)} e^{\lambda s}\|g(s)\|^2\right) ds \\ &\leq w_0 e^{-\lambda t} + 3\rho^2 \tfrac{1}{\lambda} \int_0^t \left(e^{-\lambda(t-s)}\mathbf{E}\|x(s)\|^2\right) ds + w_1 t^{2H-1} e^{-\lambda t} \int_0^\infty e^{\lambda s}\|g(s)\|^2 ds \\ &= w_0 e^{-\lambda t} + w_2 t^{2H-1} e^{-\lambda t} + w_3 \int_0^t \left(e^{-\lambda(t-s)}\mathbf{E}\|x(s)\|^2\right) ds, \end{aligned}$$

where $w_0 = 3\|x_0\|^2$, $w_1 = 3c(2H^2 - H)\|Q\|$, $w_2 = w_1 \int_0^\infty e^{\lambda s}\|g(s)\|^2 ds$, and $w_3 = 3\rho^2/\lambda$. From Lemma 2.5, then

$$\begin{aligned}\mathbf{E}\|x(t)\|^2 &\le w_0 e^{-\lambda t} + w_2 e^{-\lambda t} t^{2H-1} + w_3 \int_0^t \left[\left(w_1 e^{-\lambda s} + w_2 s^{2H-1} e^{-\lambda s} \right) e^{-\lambda(t-s)} \exp\left(\int_s^t e^{-\lambda(t-u)} du \right) \right] ds. \\ &= w_0 e^{-\lambda t} + w_2 e^{-\lambda t} t^{2H-1} + w_1 w_3 e^{-\lambda t} \int_0^t \left[\exp\left(\int_s^t e^{-\lambda(t-u)} du \right) \right] ds \\ &\quad + w_2 w_3 e^{-\lambda t} \int_0^t \left[s^{2H-1} \exp\left(\int_s^t e^{-\lambda(t-u)} du \right) \right] ds.\end{aligned}$$

Noted that

$$\exp\left(\int_s^t e^{-\lambda(t-u)} du \right) = \exp\left(\frac{1}{\lambda} \left(1 - e^{-\lambda(t-s)} \right) \right) \le e^{1/\lambda}, \quad \forall t \ge s.$$

Then,

$$\mathbf{E}\|x(t)\|^2 \le w_0 e^{-\lambda t} + w_2 e^{-\lambda t} t^{2H-1} + w_1 w_3 e^{1/\lambda} e^{-\lambda t} t + \frac{1}{2H} w_2 w_3 e^{1/\lambda} e^{-\lambda t} t^{2H}.$$

Therefore,

$$\lim_{t \to +\infty} \mathbf{E}\|x(t)\|^2 = 0.$$

The proof of Theorem 3.7 is completed.

Consider the following linear stochastic system:

$$\begin{cases} dx(t) = -\lambda x(t)dt + g(t)dB^H(t), \quad t > 0, \\ x(0) = x_0. \end{cases} \tag{15}$$

Corollary 3.8. *If the following condition is satisfied:*

$$\int_0^\infty e^{\lambda s}\|g(s)\|^2 ds < \infty, \tag{16}$$

then the stochastic system (15) is asymptotic stability in the mean-square sense.

4. Consensus of stochastic multi-agent systems with fBm

Suppose that the dynamics of each agent is described by

$$\begin{cases} d\phi_i(t) = p(t, \phi_i(t))dt + u_i(t)dt + \sigma(\phi_i(t))dB^H(t), \quad t > 0, \ i \in \mathcal{V}, \\ \phi_i(0) = \phi_{i0}, \end{cases} \tag{17}$$

where $\phi_i(t) \in \mathbb{R}^n$ and $u_i(t) \in \mathbb{R}^n$ represents the state and control input of the ith agent at time t, respectively. $p(\cdot, \cdot) \in \mathcal{L}^1([0, \infty) \times \mathbb{R}^n; \mathbb{R}^n)$ and $\sigma(\cdot) \in \mathcal{L}^2(\mathbb{R}^n; \mathcal{L}(\mathbb{U}, \mathbb{R}^n))$ are continuous differentiable nonlinear mapping functions, $B^H(t)$ is a $\mathbb{U}$-valued Q-fBm with $H \in \left(\frac{1}{2}, 1\right)$.

Consider the following distributed controller:

$$u_i(t) = k \sum_{j \in \mathcal{N}_i} a_{ij} \left[\phi_j(t) - \phi_i(t) \right], \quad i \in \mathcal{V}, \tag{18}$$

where $k > 0$ is the control gain to be designed later.

To continue, the following assumptions are necessary.

Assumption 4.1. *$p(t,0) = 0$ and $\sigma(0) = 0$. And, there exists a positive constant $\ell > 0$ such that*

$$\|p(t,a) - p(t,b)\| \vee \|\sigma(a) - \sigma(b)\| \le \ell\|a - b\|, \ \forall a, b \in \mathbb{R}^n.$$

Assumption 4.2. *The directed graph $\mathcal{G}$ has a spanning tree, and the root node cannot receive information from other nodes.*

Under Assumption 4.2, without loss of generality, rearrange the agents so that the root node is the first agent. Denote $\overline{\mathcal{G}} = \left(\overline{\mathcal{V}}, \overline{\mathcal{E}}, \overline{\mathcal{A}}\right)$ with $\overline{\mathcal{V}} = \mathcal{V} \setminus \{1\}$. The Laplacian matrix of graph $\overline{\mathcal{G}}$ is denoted as $\overline{\mathcal{L}} = \left[\bar{l}_{ij}\right]_{(N-1)\times(N-1)} \in \mathbb{R}^{(N-1)\times(N-1)}$.

Theorem 4.3. *Under Assumptions 4.1 and 4.2, with the controller (18), the consensus of the SMAS (17) can be achieved in the mean-square sense, if the following condition is satisfied:*

$$\max\{\Re(\lambda_\iota)\} < -2\sqrt{\bar{c}_1 + 3\mathcal{M}\ell^2}, \tag{19}$$

where λ_ι is the ιth eigenvalue of $-k(\overline{\mathcal{L}} + \mathcal{D})$, $\mathcal{D} = diag\{a_{21}, \dots, a_{N1}\}$, $\bar{c}_1 = 3c\mathcal{M}(2H^2 - H)\ell^2\|Q\|$, and $\mathcal{M} > 0$ is a positive constant which will be given in the following analysis.

Proof. Under Assumption 4.2, one has $u_1(t) = 0$. Then, with the controller (18), we have

$$d\phi_1(t) = p(t, \phi_1(t))dt + \sigma(\phi_1(t))dB^H(t), \tag{20}$$

$$d\phi_i(t) = p(t, \phi_i(t))dt + k\sum_{j\in\mathcal{N}_i} a_{ij}\left[\phi_j(t) - \phi_i(t)\right]dt + \sigma(\phi_i(t))dB^H(t), \quad i \in \overline{\mathcal{V}}. \tag{21}$$

Denote the tracking error as

$$\varepsilon_i(t) = \phi_i(t) - \phi_1(t), \quad i \in \overline{\mathcal{V}}. \tag{22}$$

Let $p(t, \varepsilon_i(t)) = p(t, \phi_i(t)) - p(t, \phi_1(t))$, $\sigma(\varepsilon_i(t)) = \sigma(\phi_i(t)) - \sigma(\phi_1(t))$, for $i \in \overline{\mathcal{V}}$. Then, one has

$$\begin{aligned} d\varepsilon_i(t) &= p(t, \varepsilon_i(t))dt + k\sum_{j\in\mathcal{N}_i\setminus\{1\}} a_{ij}\left[\phi_j(t) - \phi_i(t)\right]dt + ka_{i1}\left(\phi_1(t) - \phi_i(t)\right)dt + \sigma(\varepsilon_i(t))dB^H(t) \\ &= p(t, \varepsilon_i(t))dt + k\sum_{j\in\mathcal{N}_i\setminus\{1\}} a_{ij}\left[\varepsilon_j(t) - \varepsilon_i(t)\right]dt - ka_{i1}\varepsilon_i(t)dt + \sigma(\varepsilon_i(t))dB^H(t). \end{aligned} \tag{23}$$

Let

$$\varepsilon(t) = \left[\varepsilon_2^T(t), \varepsilon_3^T(t), \dots, \varepsilon_N^T(t)\right]^T, \ \Sigma(\varepsilon(t)) = \left[\sigma^T(\varepsilon_2(t)), \sigma^T(\varepsilon_3(t)), \dots, \sigma^T(\varepsilon_N(t))\right]^T,$$
$$P(t, \varepsilon(t)) = \left[p^T(t, \varepsilon_2(t)), p^T(t, \varepsilon_3(t)), \dots, p^T(t, \varepsilon_N(t))\right]^T.$$

Then, the compact form of the tracking error can be written as

$$\begin{cases} d\varepsilon(t) = \Lambda\varepsilon(t)dt + P(t, \varepsilon(t))dt + \Sigma(\varepsilon(t))dB^H(t), \quad t > 0, \\ \varepsilon(0) = \varepsilon_0, \end{cases} \tag{24}$$

where $\Lambda = -k(\overline{\mathcal{L}} + \mathcal{D}) \otimes I_n$. The mild solution of Eq. (24) is

$$\varepsilon(t) = S(t)\varepsilon_0 + \int_0^t S(t-s)P(s, \varepsilon(s))ds + \int_0^t S(t-s)\Sigma(\varepsilon(s))dB^H(s), \ \text{with} \ S(t) = e^{\Lambda t}. \tag{25}$$

From Eq. (25), then

$$\mathbf{E}\|\varepsilon(t)\|^2 \leq 3\|S(t)\varepsilon_0\|^2 + \Psi_1(t) + \Psi_2(t), \tag{26}$$

with

$$\Psi_1(t) = 3\mathbf{E}\left\|\int_0^t S(t-s)P(s,\varepsilon(s))ds\right\|^2, \ \ \Psi_2(t) = 3\mathbf{E}\left\|\int_0^t S(t-s)\Sigma(\varepsilon(s))dB^H(s)\right\|^2.$$

From Lemma 2.6 and Assumption 4.1, then

$$\begin{aligned}
\Psi_1(t) &= 3\mathbf{E}\left\|\int_0^t S(t-s)P(s,\varepsilon(s))ds\right\|^2 \\
&\leq 3\mathbf{E}\left\|\int_0^t \|S(t-s)\|^{\frac{1}{2}}\cdot\|S(t-s)\|^{\frac{1}{2}}\cdot\|P(s,\varepsilon(s))\|ds\right\|^2 \\
&\leq 3\mathbf{E}\left\|\left(\int_0^t \|S(t-s)\|ds\right)^{\frac{1}{2}}\left(\int_0^t \|S(t-s)\|\cdot\|P(s,\varepsilon(s))\|^2ds\right)^{\frac{1}{2}}\right\|^2 \\
&= 3\int_0^t \|S(t-s)\|ds\int_0^t\left[\|S(t-s)\|\mathbf{E}\|P(s,\varepsilon(s))\|^2\right]ds \\
&\leq 3\ell^2\int_0^t \|S(t-s)\|ds\int_0^t\left[\|S(t-s)\|\mathbf{E}\|\varepsilon(s)\|^2\right]ds.
\end{aligned}$$

According to Lemma 2.3 and Assumption 4.1, then

$$\begin{aligned}
\Psi_2(t) &= 3\mathbf{E}\left\|\int_0^t S(t-s)\Sigma(\varepsilon(s))dB^H(s)\right\|^2 \\
&\leq 3c(2H^2-H)\ell^2\|Q\|t^{2H-1}\int_0^t\left[\|S(t-s)\|^2\,\mathbf{E}\|\varepsilon(s)\|^2\right]ds.
\end{aligned}$$

Under Assumption 4.2, all eigenvalues of $-k(\overline{\mathcal{L}}+\mathcal{D})$ have negative real parts, that is $\Re(\lambda_\iota) < 0$, $\forall\iota\in\mathcal{V}$. Noted that there exists a nonsingular matrix P such that $-k(\overline{\mathcal{L}}+\mathcal{D}) = PJP^{-1}$, where J is the Jordan canonical form of $-k(\overline{\mathcal{L}}+\mathcal{D})$, with

$$J = \begin{pmatrix} J_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & J_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & J_r \end{pmatrix}, \text{ and } J_\iota = \begin{pmatrix} \lambda_\iota & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_\iota & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_\iota & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda_\iota \end{pmatrix},$$

where J_ι ($\iota = 1,2,\ldots,r$) is the Jordan block corresponding to the eigenvalue λ_ι ($\iota = 1,2,\ldots,r$). Since $e^{k(\overline{\mathcal{L}}-\mathcal{D})t} = e^{PJP^{-1}t} = Pe^{Jt}P^{-1}$ and

$$D_\iota(t) = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{n_\iota-2}}{(n_\iota-2)!} & \frac{t^{n_\iota-1}}{(n_\iota-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{n_\iota-3}}{(n_\iota-3)!} & \frac{t^{n_\iota-2}}{(n_\iota-2)!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & t \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where n_ι is the multiplicity of the eigenvalue λ_ι. Take $\beta = -\max\{\Re(\lambda_\iota)\}$, then

$$\|e^{k(\overline{\mathcal{L}}-\mathcal{D})t}\| \leq \|P\|\|P^{-1}\|\|e^{Jt}\| \leq \|P\|\|P^{-1}\|\sum_{\iota=1}^{r}\|D_\iota(t)\|e^{-\beta t}. \tag{27}$$

Since $\|D_\iota(t)\|$ is the polynomial of t and $\beta > 0$, then there exists a positive constant $\mathcal{M}$ such that $\|P\|\|P^{-1}\| \sum_{\iota=1}^{r} \|D_\iota(t)\| e^{-\beta t/2} \leq \mathcal{M}$. Denote $\omega = \beta/2$, the $\|S(t)\| = \|e^{\Lambda t}\| \leq \mathcal{M}e^{-\omega t}$. From the above analysis, then

$$\begin{aligned}
\mathbf{E}\|\varepsilon(t)\|^2 &\leq \bar{c}_0 e^{-\omega t} + 3\mathcal{M}\ell^2 \int_0^t e^{-\omega(t-s)} ds \int_0^t \left(e^{-\omega(t-s)}\mathbf{E}\|\varepsilon(s)\|^2\right) ds + \bar{c}_1 t^{2H-1} \int_0^t e^{-2\omega(t-s)}\mathbf{E}\|\varepsilon(s)\|^2 ds \\
&= \bar{c}_0 e^{-\omega t} + 3\mathcal{M}\ell^2 \tfrac{1}{\omega}(1 - e^{-\omega t}) \int_0^t \left(e^{-\omega(t-s)}\mathbf{E}\|\varepsilon(s)\|^2\right) ds + \bar{c}_1 t^{2H-1} e^{-2\omega t} \int_0^t \left(e^{2\omega s}\mathbf{E}\|\varepsilon(s)\|^2\right) ds \\
&\leq \bar{c}_0 e^{-\omega t} + \bar{c}_2 e^{-\omega t} \int_0^t \left(e^{\omega s}\mathbf{E}\|\varepsilon(s)\|^2\right) ds + \bar{c}_1 t^{2H-1} e^{-2\omega t} \int_0^t \left(e^{2\omega s}\mathbf{E}\|\varepsilon(s)\|^2\right) ds,
\end{aligned}$$

where $\bar{c}_0 = 3\mathcal{M}\|\varepsilon_0\|^2$, $\bar{c}_1 = 3c\mathcal{M}(2H^2 - H)\ell^2\|Q\|$, and $\bar{c}_2 = 3\mathcal{M}\ell^2/\omega$. Then, one has

$$e^{\omega t}\mathbf{E}\|\varepsilon(t)\|^2 \leq \bar{c}_0 + \int_0^t \left(\bar{c}_1 t^{2H-1} e^{-\omega t} e^{\omega s} + \bar{c}_2\right)\left(e^{\omega s}\mathbf{E}\|\varepsilon(s)\|^2\right) ds.$$

From Lemma 2.4, then

$$\begin{aligned}
e^{\omega t}\mathbf{E}\|\varepsilon(t)\|^2 &\leq c_0 \exp\left(\int_0^t \left(\bar{c}_1 t^{2H-1} e^{-\omega t} e^{\omega s} + \bar{c}_2\right) ds\right) \\
&= \bar{c}_0 \exp\left(\bar{c}_1 t^{2H-1} \tfrac{1}{\omega}(1 - e^{-\omega t}) + \bar{c}_2 t\right) \\
&\leq \bar{c}_0 \exp\left(\bar{c}_1 t^{2H-1} \tfrac{1}{\omega} + \bar{c}_2 t\right).
\end{aligned}$$

Since $H \in \left(\frac{1}{2}, 1\right)$, then $2H - 1 \in (0, 1)$. Furthermore, for $0 \leq t < 1$, one has $e^{\omega t}\mathbf{E}\|\varepsilon(t)\|^2 \leq \bar{c}_0 e^{(\bar{c}_1/\omega + \bar{c}_2 t)}$; for $t \geq 1$, one has $e^{\omega t}\mathbf{E}\|\varepsilon(t)\|^2 \leq \bar{c}_0 e^{(\bar{c}_1/\omega + \bar{c}_2)t}$, such that $\mathbf{E}\|\varepsilon(t)\|^2 \leq \bar{c}_0 e^{(\bar{c}_1/\omega + \bar{c}_2 - \omega)t}$. When condition (19) is satisfied, we have $\bar{c}_1/\omega + \bar{c}_2 - \omega < 0$. Therefore, $\lim_{t\to+\infty} \mathbf{E}\|\varepsilon(t)\|^2 = 0$, such that

$$\lim_{t\to+\infty} \mathbf{E}\left\|\phi_i(t) - \phi_1(t)\right\|^2 = 0, \quad \forall i \in \overline{\mathcal{V}}.$$

The proof of Theorem 4.3 is completed.

Remark 4.4. *The mathematical framework employed in the paper may be tailored to the distinctive properties of fBm, and its adaptability to alternative stochastic processes requires a thorough assessment of compatibility. Since different stochastic processes exhibit diverse characteristics and the paper's methods are intricately linked to the unique features of fBm, then their direct application to other processes might pose challenges.*

5. Simulation

The Wood-Chan algorithm given in [42] is taken in this paper, whose main idea is to establish a cyclic matrix based on the covariance function of fBm. The sample trajectories of fBm with $H = 0.65, 0.75, 0.85, 0.95$ are depicted in Fig. 1. It can be seen from Fig. 1 that the roughness of the sample trajectory of fBm can intuitively reflect the degree of autocorrelation and the long-term dependence of incremental fBm. In this example, we consider the mean-square consensus control of SMAS described by (17) over the communication topology that is depicted in Fig. 2. Suppose that $n = 3$, $N = 11$, $H = 0.75$, $\phi_i(t) = [\phi_{i1}(t), \phi_{i2}(t), \phi_{i3}(t)]^T$, and

$$p(t, \phi_i(t)) = \begin{bmatrix} \tanh\left(\phi_{i1}(t)\right) \\ \sin\left(\phi_{i2}(t)\right) \\ \cos\left(\phi_{i3}(t)\right) \end{bmatrix}, \quad \sigma(\phi_i(t)) = \begin{bmatrix} \tanh\left(\phi_{i2}(t)\right) \\ \cos\left(\phi_{i1}(t)\right) \\ \sin\left(\phi_{i3}(t)\right) \end{bmatrix}. \tag{28}$$

The initial values are chosen as

$$\phi_1(0) = [1, -1,\ 10]^T, \phi_2(0) = [3, -3, -9]^T, \phi_3(0) = [\ 5, -5,\ 8\ ]^T, \phi_4(0) = [7, -7, -6]^T,$$
$$\phi_5(0) = [9, -9,\ \ 7\ ]^T, \phi_6(0) = [-2, 2, -1]^T, \phi_7(0) = [-4, 4, -2]^T,\ \phi_8(0) = [-6, 6,\ \ 5\ ]^T,$$
$$\phi_9(0) = [-8, 8, -3]^T, \phi_{10}(0) = [-10, 10, 2]^T, \phi_{11}(0) = [-12, 12, 1]^T.$$

The state trajectories of all agents without controller are depicted in Fig. 3. Then, state trajectories of all agents with controller (18) are depicted in Fig. 4. It could see from Fig. 4 that the mean-square consensus of the SMAS (17) is achieved with controller (18), which is consistent with the conclusion of Theorem 4.3.

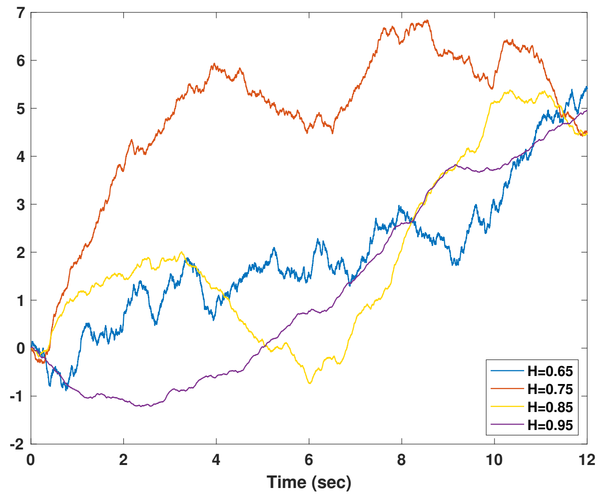


Figure 1: Sample trajectory of fBm with H = 0.65, 0.75, 0.85, 0.95.

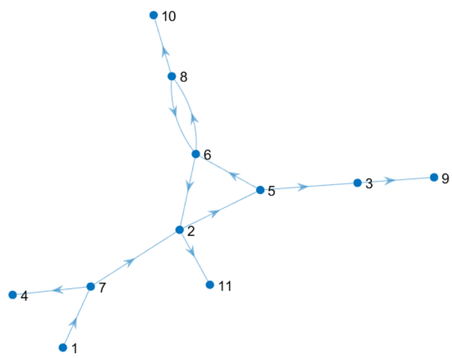
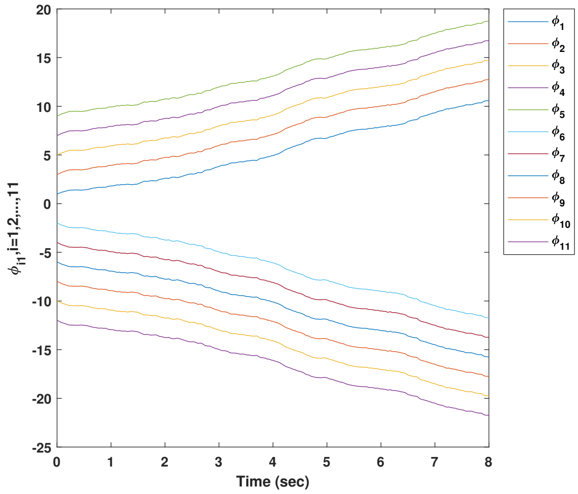
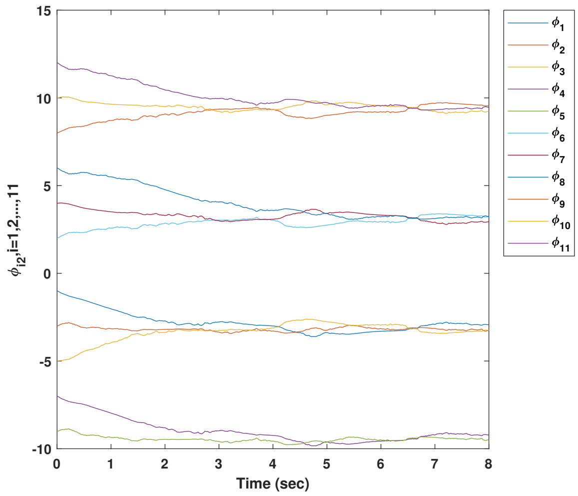


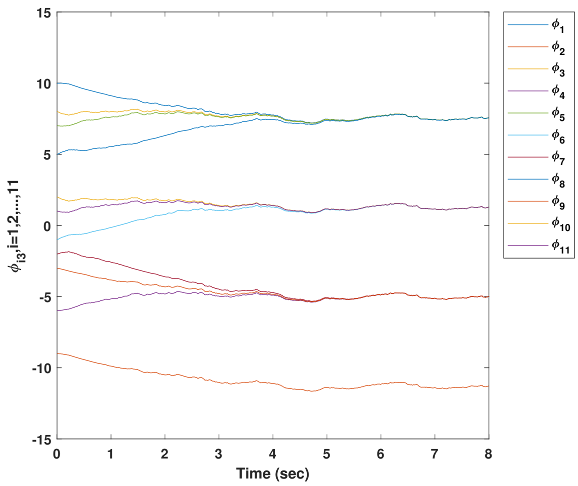
Figure 2: The communication topology.



(a) the first component.

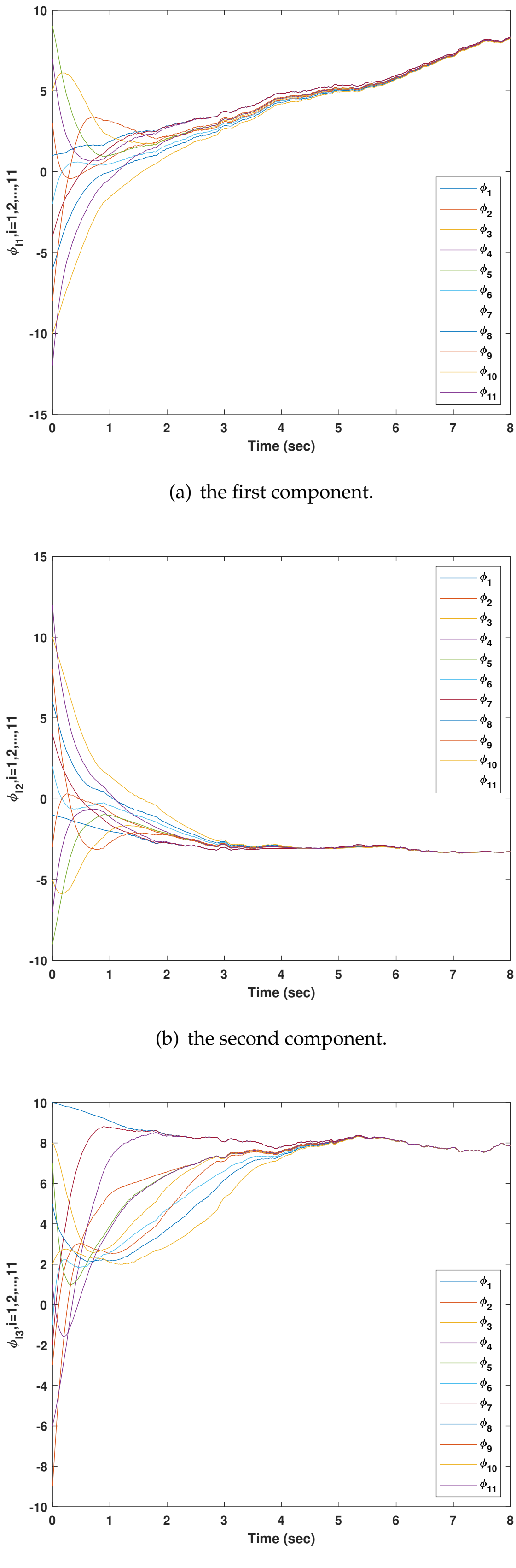


(b) the second component.



(c) the third component.

Figure 3: State trajectories of all agents without controller.



(a) the first component.

(b) the second component.

(c) the third component.

Figure 4: State trajectories of all agents with controller (18).

6. Conclusions

In this paper, novel asymptotic stability conditions were presented for nonlinear stochastic systems driven by fBm with Hurst parameter $H \in (1/2, 1)$. Subsequently, the obtained results were extended to the linear case, and asymptotic stability conditions were derived. Moreover, the proposed methods were employed to address the consensus control problem of the SMAS with fBm. Finally, a numerical example was given to demonstrate the effectiveness of the theoretical results.

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