Filomat 38:19 (2024), 6773–6784
https://doi.org/10.2298/FIL2419773K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Conditional stability and regularized solution of a boundary value problem for a system of mixed type equations in three-dimensional space

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Abstract. In this paper, we study the initial-boundary value problem for the system of second-order mixed-type equations in three-dimensional space. Boundary value problems for equations of mixed type are used in problems of gas dynamics, the theory of infinitesimal bending of surfaces, mathematical biology and other fields. An a priory estimate of the solution is obtained. Theorems of uniqueness and conditional stability are given. A regularized approximate solution is constructed in two cases, which is stable on the set of correctness. Estimates of effectiveness of the regularization method are obtained and formula for regularization parameter is derived.

1. Introduction

This work is devoted to the study of an ill-posed initial-boundary value problem for a system of mixed type equations.

Let $Q_T = \{(x, y, z, t) : (x, y, z) \in \Omega,\ 0 < t < T\}$, $\Omega = \{\, |x| < \pi,\ 0 < y < \pi,\ 0 < z < \pi\}$.

The system of equations of the form

$$\begin{cases} u_{tt} = Lu + a_1 u + b_1 v + f(x, y, z, t), \\ v_{tt} = Lv + a_2 v + b_2 u + g(x, y, z, t), \end{cases} \tag{1}$$

is investigated on the region $Q_T \cap \{x \neq 0\}$, where $Lu \equiv sign(x)u_{xx} + u_{yy} + u_{zz}$, a_1, a_2, b_1, b_2 are some bounded constants, where $(a_1 - a_2)^2 + 4b_1b_2 > 0$, $b_2 \neq 0$.

Let a pair of functions $(u(x, y, z, t),\ v(x, y, z, t))$ satisfy the system of equations (1) on the region $Q_T \cap \{x \neq 0\}$ and the following conditions: initial

$$\left.\begin{array}{l} u|_{t=0} = \varphi_1(x, y, z),\ \ v|_{t=0} = \psi_1(x, y, z), \\ u_t|_{t=0} = \varphi_2(x, y, z),\ \ v_t\big|_{t=0} = \psi_2(x, y, z), \end{array}\right\},\ (x, y, z) \in \bar{\Omega} \tag{2}$$

2020 *Mathematics Subject Classification*. Primary 35R25; Secondary 35M32, 65M30
Keywords. Mixed type equation, ill-posed problem, priory estimate, set of correctness, regularization, estimate of effectiveness
Received: 28 December 2023; Revised: 01 May 2024; Accepted: 17 July 2024
Communicated by Ljubiša D. R. Kočinac
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boundary

$$u|_{\partial\Omega} = 0,\ v|_{\partial\Omega} = 0,\ t \in [0,\ T] \tag{3}$$

and gluing conditions

$$\left.\begin{array}{l} \frac{\partial^i u}{\partial x^i}\Big|_{x=-0} = \frac{\partial^i u}{\partial x^i}\Big|_{x=+0} \\ \frac{\partial^i v}{\partial x^i}\Big|_{x=-0} = \frac{\partial^i v}{\partial x^i}\Big|_{x=+0} \end{array}\right\},\ i = 0, 1.,\ 0 \le y \le \pi,\ 0 \le z \le \pi,\ 0 \le t \le T, \tag{4}$$

where $\varphi_i(x, y, z)$, $\psi_i(x, y, z)$ are given sufficient smooth functions, and they satisfy the matching conditions, $i = 1, 2$, $f(x, y, z, t)$, $g(x, y, z, t)$ are source functions.

The investigate of mixed-type differential equations was first studied in the works of F. Tricomi and S. Gellerstedt. Later, many mathematicians researched various problems for this type of equation. You can get acquainted with these studies through monographs [15], [16], [17] and the works in the list of references cited in them.

Checking to conditional correctness and constructing a regularized approximate solution of ill-posed problems for equations similar to equation (1) are studied in works [2]-[6]. Construction of the approximate solution of mixed integral equations was studied in [9]-[12], [1].

In this paper, the initial-boundary value problem (1)-(4) is studied to well-posedness. Theorems on uniqueness and conditional stability are presented. Then, a regularized approximate solution is constructed on the set of correctness. Estimates of effectiveness of the regularization method are obtained and formula for regularization parameter is derived.

2. Transformation of the problem (1)-(4)

Let us introduce the notation

$$u = \frac{a_1 - \lambda_2}{b_2(\lambda_1 - \lambda_2)}\omega - \frac{a_1 - \lambda_1}{b_2(\lambda_1 - \lambda_2)}\vartheta, \quad v = \frac{1}{(\lambda_1 - \lambda_2)}(\omega - \vartheta), \tag{5}$$

where $\omega(x, y, z, t)$ and $\vartheta(x, y, z, t)$ are new functions, which we define later, λ_1 and λ_2 are the real roots of the quadratic equation

$$\lambda^2 - (a_1 + a_2)\lambda + a_1 a_2 - b_1 b_2 = 0.$$

After transformation, we get the following tasks.

Problem 1. Find a function $\omega(x, y, z, t)$ satisfying the equation

$$\omega_{tt} = L\omega + \lambda_2 w + \bar{f}(x, y, z, t),$$

as well as the following conditions

$$\omega|_{t=0} = \bar{\varphi}_1(x, y, z),\ \ \omega_t|_{t=0} = \bar{\varphi}_2(x, y, z),$$
$$\omega|_{\partial\Omega} = 0,$$
$$\omega|_{x=-0} = \omega|_{x=+0},\ \omega_x|_{x=-0} = \omega_x|_{x=+0}$$

on the domain $Q_T \cap \{x \neq 0\}$, where $\bar{f}(x, y, z, t) = b_2 f + (\lambda_2 - a_1) g$, $\bar{\varphi}_1 = b_2\varphi_1 + (\lambda_2 - a_1)\psi_1$, $\bar{\varphi}_2 = b_2\varphi_2 + (\lambda_2 - a_1)\psi_2$.

Problem 2. Find a function $\vartheta(x, y, z, t)$ satisfying the equation

$$\vartheta_{tt} = L\,\vartheta + \lambda_1\vartheta + \bar{g}(x, y, z, t),$$

as well as the following conditions

$$\vartheta|_{t=0} = \bar{\psi}_1(x, y, z),\ \ \vartheta_t|_{t=0} = \bar{\psi}_2(x, y, z)$$
$$\vartheta|_{\partial\Omega} = 0,$$
$$\vartheta|_{x=-0} = \vartheta|_{x=+0},\ \vartheta_x|_{x=-0} = \vartheta_x|_{x=+0}$$

on the domain $Q_T \cap \{x \neq 0\}$, where $\bar{g}(x, y, z, t) = b_2 f + (\lambda_1 - a_1) g$, $\bar{\psi}_1 = b_2\varphi_1 + (\lambda_1 - a_1)\psi_1$, $\bar{\psi}_2 = b_2\varphi_2 + (\lambda_1 - a_1)\psi_2$.

3. Auxiliary facts

To obtain the main result, we will need in the solutions of the following spectral problem: Find values of ξ such that the problem

$$\begin{aligned} & L\phi(x,y,z) - \xi\phi(x,y,z) = 0,\ (x,y,z) \in \Omega \cap \{x \neq 0\}, \\ & \phi(-\pi,y,z) = \phi(\pi,y,z) = 0,\ y \in [0;\pi],\ z \in [0;\pi], \\ & \phi(x,0,z) = \phi(x,\pi,z) = 0,\ x \in [-\pi;\pi],\ z \in [0;\pi], \\ & \phi(x,y,0) = \phi(x,y,\pi) = 0,\ x \in [-\pi;\pi],\ y \in [0;\pi], \\ & \phi(-0,y,z) = \phi(+0,y,z),\ \phi_x(-0,y,z) = \phi_x(+0,y,z),\ y \in [0;\pi],\ z \in [0;\pi] \end{aligned} \tag{6}$$

has nontrivial solutions.

Let us denote $(u,v) = \int\limits_{\Omega} u\, v\, dxdydz$ the scalar product and the norm $\|u\| = \sqrt{(u,u)}$ in $L_2(\Omega)$.

Using the results of work [14], we can prove that problem (6) has $\bar{\xi}_{k,n,m} = \mu_k^+ - n^2 - m^2$, $\tilde{\xi}_{k,n,m} = \mu_k^- - n^2 - m^2$, $\{\bar{\xi}_{k,n,m}\}_{k,n,m=1}^{\infty}$, $\{\tilde{\xi}_{k,n,m}\}_{k,n,m=1}^{\infty}$ eigenvalues and the corresponding eigenfunctions $\{\bar{\phi}_{k,n,m}\}_{k,n,m=1}^{\infty}$, $\{\tilde{\phi}_{k,n,m}\}_{k,n,m=1}^{\infty}$ of which can be represented as:

$$\begin{aligned} & \bar{\phi}_{k,n,m}(x,y,z) = X_k^+(x) \cdot Y_n(y) \cdot Z_m(z), \\ & \tilde{\phi}_{k,n,m}(x,y,z) = X_k^-(x) \cdot Y_n(y) \cdot Z_m(z), \end{aligned} \tag{7}$$

and they have the property

$$\left(\bar{\phi}_{k,n,m}, \bar{\phi}_{r,s,q}\right) = \begin{cases} 1, & k = r \wedge n = s \wedge m = q, \\ 0, & k \neq r \vee n \neq s, \vee m \neq q, \end{cases},\ \left(\tilde{\phi}_{k,n,m}, \tilde{\phi}_{r,s,q}\right) = \begin{cases} -1, & k = r \wedge n = s \wedge m = q, \\ 0, & k \neq r \vee n \neq s, \vee m \neq q, \end{cases},$$

$$\left(\bar{\phi}_{k,n,m},\ \tilde{\phi}_{r,s,q}\right) = 0,\ k,\ n, m,\ r,\ s, q \in N,$$

where

$$X_k^+(x) = \begin{cases} \frac{\sin\sqrt{\mu_k^+}(x+\pi)}{\sqrt{\pi}\cos\sqrt{\mu_k^+}\pi}, & -\pi \leq x \leq 0, \\ \frac{sh\sqrt{\mu_k^+}(x-\pi)}{\sqrt{\pi}cosh\sqrt{\mu_k^+}\pi}, & 0 < x \leq \pi, \end{cases},\ X_k^-(x) = \begin{cases} \frac{sinh\sqrt{-\mu_k^-}(x+\pi)}{\sqrt{\pi}cosh\sqrt{-\mu_k^-}\pi}, & -\pi \leq x < 0 \\ \frac{\sin\sqrt{-\mu_k^-}(x-\pi)}{\sqrt{\pi}\cos\sqrt{-\mu_k^-}\pi}, & 0 < x \leq \pi, \end{cases},$$

$Y_n(y) = \sqrt{\frac{2}{\pi}}\sin(ny)$, $Z_m(z) = \sqrt{\frac{2}{\pi}}\sin(mz)$. The numbers μ_k^+, $-\mu_k^-$ form non-decreasing sequences and are solutions to the transcendental equation $tg\sqrt{\pm\mu_k^\pm}\pi + th\sqrt{\pm\mu_k^\pm}\pi = 0$.

Solutions of the equation $tg\sqrt{\pm\mu_k^\pm}\pi + th\sqrt{\pm\mu_k^\pm}\pi = 0$ can be easily found by using numerical methods. For $\varepsilon = 10^{-10}$ with an error we calculate $\mu_1^+ \approx 0.56672194089$, $\mu_2^+ \approx 3.06251868904$, $\mu_3^+ \approx 7.56250005484$, $\mu_4^+ \approx 14.06250000014$, $\mu_k^+ \approx \left(k - \frac{1}{4}\right)^2$, $k > 4$, $\mu_k^- = -\mu_k^+$, $k \in N$. Note that $\mu_k^+ = \left(k - \frac{1}{4}\right)^2 + O\left(e^{-\left(k+\frac{1}{2}\right)\pi}\right)$.

Let $\delta_1 = \min\limits_{k,n,m} \left|\bar{\xi}_{k,n,m} + \lambda_1\right|$, $\delta_2 = \min\limits_{k,n,m} \left|\tilde{\xi}_{k,n,m} + \lambda_2\right|$ for any $k, j \in N$.

According to [14], we have

$$\|\omega\|_0^2 = \sum_{k,n,m=1}^{\infty} \left|\left(sign(x)\,\omega,\ \bar{\phi}_{k,n,m}\right)\right|^2 + \sum_{k,n,m=1}^{\infty} \left|\left(sign(x)\omega,\ \tilde{\phi}_{k,n,m}\right)\right|^2. \tag{8}$$

From the results of [14] it follows that the eigenfunctions of the problem (6) form a Riesz basis in H_0 and the norm in the space $L_2(\Omega)$, defined by equality (8), is equivalent to the original one.

A generalized solution of the boundary value Problem 1 is a function $\omega(x,y,z,t)$ belonging to $W_2^{1,2}(Q_T)$ and satisfying the identity

$$\int\limits_{Q_T} \Big(sign(x)\,\omega\, V_{tt} + \omega_x V_x + sign(x)\,\omega_y\, V_y + sign(x)\,\omega_z\, V_z\Big)\, dQ_T =$$

$$\int\limits_{\Omega} sign(x)\, V|_{t=0}\, \omega_t|_{t=0} d\Omega - \int\limits_{\Omega} sign(x)\, V_t|_{t=0}\, \omega|_{t=0} d\Omega + \int\limits_{Q_T} sign(x)\, \bar{f}\, V\, dQ_T \quad (9)$$

for any function $V(x,y,z,t) \in W_2^{2,2}(Q_T)$ satisfying the conditions $V|_{t=T} = 0,\ V_t|_{t=T} = 0,\ V|_{\bar{\Omega}} = 0$.

4. A priory estimate

Lemma 4.1. *For solution to the Problem 1 at $t \in (0,T)$, the inequality*

$$\int\limits_0^t \|\omega_x\|^2 d\tau \le \left(T\left\|\partial_x \bar{\varphi}_1\right\|^2 + \gamma_1\right)^{\theta(t)} \left(\int\limits_0^T \|\omega_x\|^2 dt + \gamma_1\right)^{1-\theta(t)} c(t) \quad (10)$$

holds, where $\gamma_1 = \left(2T^2+3\right)\int\limits_0^T \left\|\bar{f}_x\right\|^2 dt + \beta_1$,

$$\beta_1 = |\lambda_2|\, T\left\|\bar{\varphi}_1\right\|^2 + \left\|\partial_x\bar{\varphi}_1\right\|^2 + (4+|\lambda_2|)\, T\left\|\partial_x^2\bar{\varphi}_1\right\|^2 + T\left\|\partial_y^2\bar{\varphi}_1\right\|^2 + T\left\|\partial_z^2\bar{\varphi}_1\right\|^2 + (2T+1)\left\|\partial_x\bar{\varphi}_2\right\|^2,$$

$\theta(t) = \frac{1-e^{-t}}{1-e^{-T}}$, $c(t) = \exp\left((T+1)\,\frac{(1-e^{-t})T-(1-e^{-T})t}{1-e^{-T}}\right)$.

One can find the proof of the Lemma 4.1 in [3].

Similarly, for solution to the Problem 2 at $t \in (0,T)$ the inequality

$$\int\limits_0^t \|\vartheta_x\|^2 d\tau \le \left(T\left\|\partial_x \bar{\psi}_1\right\|^2 + \gamma_2\right)^{\theta(t)} \left(\int\limits_0^T \|\vartheta_x\|^2 dt + \gamma_2\right)^{1-\theta(t)} c(t) \quad (11)$$

holds, where $\gamma_2 = \left(2T^2+3\right)\int\limits_0^T \left\|\bar{g}_x\right\|^2 dt + \beta_2$,

$$\beta_2 = |\lambda_1|\, T\left\|\bar{\psi}_1\right\|^2 + \left\|\partial_x\bar{\psi}_1\right\|^2 + (4+|\lambda_1|)\, T\left\|\partial_x^2\bar{\psi}_1\right\|^2 + T\left\|\partial_y^2\bar{\psi}_1\right\|^2 + T\left\|\partial_z^2\bar{\psi}_1\right\|^2 + (2T+1)\left\|\partial_x\bar{\psi}_2\right\|^2.$$

Let

$$C_1 = \frac{2\sqrt{2}\pi(a_1-\lambda_2)}{b_2\,(\lambda_1-\lambda_2)},\ C_2 = \frac{2\sqrt{2}\pi(a_1-\lambda_1)}{b_2\,(\lambda_1-\lambda_2)},\ C_3 = \frac{2\sqrt{2}\pi}{(\lambda_1-\lambda_2)}.$$

For any function $u \in W_2^1[-\pi,\pi]$ with $u|_{x=-\pi} = u|_{x=\pi} = 0$ the inequality

$$\int\limits_{-\pi}^{\pi} u^2 dx \le 4\pi^2 \int\limits_{-\pi}^{\pi} u_x^2 dx$$

is true. Then using formula (5) we get

$$\int\limits_0^t \|u\|^2 d\tau \le C_1^2 \int\limits_0^t \|\omega_x\|^2 d\tau + C_2^2 \int\limits_0^t \|\vartheta_x\|^2 d\tau, \quad (12)$$

$$\int_0^t \|v\|^2 d\tau \le C_3^2 \left(\int_0^t \|\omega_x\|^2 d\tau + \int_0^t \|\vartheta_x\|^2 d\tau \right) \tag{13}$$

and

$$\omega = b_2 u + (\lambda_1 - a_1) v, \ \ \vartheta = b_2 u + (\lambda_2 - a_1) v. \tag{14}$$

Using inequalities (12) and (13) from (14) we have

$$\int_0^t \|u\|^2 d\tau \le C_1^2 \left(T \left\| \partial_x \bar{\varphi}_1 \right\|^2 + \gamma_1 \right)^{\theta(t)} \left(\int_0^T \|b_2 u_x + (\lambda_1 - a_1) v_x\|^2 dt + \gamma_1 \right)^{1-\theta(t)} c(t) + \\ C_2^2 \left(T \left\| \partial_x \bar{\psi}_1 \right\|^2 + \gamma_2 \right)^{\theta(t)} \left(\int_0^T \|b_2 u_x + (\lambda_2 - a_1) v_x\|^2 dt + \gamma_2 \right)^{1-\theta(t)} c(t), \tag{15}$$

$$\int_0^t \|v\|^2 d\tau \le C_3^2 c(t) \left(\left(T \left\| \partial_x \bar{\varphi}_1 \right\|^2 + \gamma_1 \right)^{\theta(t)} \left(\int_0^T \|b_2 u_x + (\lambda_1 - a_1) v_x\|^2 dt + \gamma_1 \right)^{1-\theta(t)} + \\ \left(T \left\| \partial_x \bar{\psi}_1 \right\|^2 + \gamma_2 \right)^{\theta(t)} \left(\int_0^T \|b_2 u_x + (\lambda_2 - a_1) v_x\|^2 dt + \gamma_2 \right)^{1-\theta(t)} \right). \tag{16}$$

5. Main results

In the problem (1)-(4) we introduce the correctness set as follows

$$M = \left\{ (u, v) : \left\| u_x(x, y, z, T) \right\| + \left\| v_x(x, y, z, T) \right\| \le \mathfrak{M}, \ \mathfrak{M} < \infty \right\}. \tag{17}$$

Theorem 5.1. *Let the solution of the problem (1)-(4) exist and $(u, v) \in M$, then the solution of the problem is unique.*

Let the pair of functions (u, v) be the solution of the problem (1)-(4) corresponding to the exact dates $\varphi_i(x, y, z)$, $\psi_i(x, y, z)$, $i = 1, 2.$, $f(x, y, z, t)$, $g(x, y, z, t)$, and let the pair of functions $(u_\varepsilon, v_\varepsilon)$ be a solution of the problem (1)-(4) corresponding to the approximate dates $\varphi_{i\varepsilon}(x, y, z)$, $\psi_{i\varepsilon}(x, y, z)$, $i = 1, 2.$, $f_\varepsilon(x, y, z, t)$, $g_\varepsilon(x, y, z, t)$.

Theorem 5.2. *Let the solution of the problem (1)-(4) exist and (u, v), $(u_\varepsilon, v_\varepsilon) \in M$. Let $\left\| \varphi_1 - \varphi_{1\varepsilon} \right\|_{W_2^2(\Omega)} \le \varepsilon$, $\left\| \varphi_2 - \varphi_{2\varepsilon} \right\|_{W_2^{1,0,0}(\Omega)} \le \varepsilon$, $\left\| \psi_1 - \psi_{1\varepsilon} \right\|_{W_2^2(\Omega)} \le \varepsilon$, $\left\| \psi_2 - \psi_{2\varepsilon} \right\|_{W_2^{1,0,0}(\Omega)} \le \varepsilon$, $\max\limits_{t \in [0;T]} \left\| f - f_\varepsilon \right\|_{W_2^{1,0,0}(\Omega)} \le \varepsilon$ and $\max\limits_{t \in [0;T]} \left\| g - g_\varepsilon \right\|_{W_2^{1,0,0}(\Omega)} \le \varepsilon$. Then the inequalities*

$$\int_0^t \|u - u_\varepsilon\|^2 d\tau \le C_1^2 \delta_1(T, \mathfrak{M}, \varepsilon) + C_2^2 \delta_2(T, \mathfrak{M}, \varepsilon),$$

$$\int_0^t \|v - v_\varepsilon\|^2 d\tau \le C_3^2 \left(\delta_1(T, \mathfrak{M}, \varepsilon) + \delta_2(T, \mathfrak{M}, \varepsilon) \right),$$

are valid, where $\delta_1(T,\mathfrak{M},\varepsilon)=\left(\left(T+C_4^2\right)\varepsilon^2\right)^{\theta(t)}\left(4(|b_2|+|\lambda_1-a_1|)^2\mathfrak{M}^2T+C_4^2\varepsilon^2\right)^{1-\theta(t)}c(t)$,
$\delta_2(T,\mathfrak{M},\varepsilon)=\left(\left(T+C_5^2\right)\varepsilon^2\right)^{\theta(t)}\left(4(|b_2|+|\lambda_2-a_1|)^2\mathfrak{M}^2T+C_5^2\varepsilon^2\right)^{1-\theta(t)}c(t)$, $C_4=(|b_2|+|\lambda_1-a_1|)\left(2T^3+(2|\lambda_1|+9)T+2\right)^{\frac{1}{2}}$,
$C_5=(|b_2|+|\lambda_2-a_1|)\left(2T^3+(2|\lambda_2|+9)T+2\right)^{\frac{1}{2}}$.

The proofs of Theorem 1 and Theorem 2 one can obtain using Lemma 1 and inequalities (15), (16).

6. Approximate solution

We consider the construction of an approximate solution of the problem (1)-(4) in following cases.

I. Let $f(x,y,z,t)=0$, $g(x,y,z,t)=0$ and $\varphi_1(x,y,z)=\frac{a_1-\lambda_1}{b_2}\psi_1(x,y,z)$, $\varphi_2(x,y,z)=\frac{a_1-\lambda_2}{b_2}\psi_2(x,y,z)$. One can get $\bar{\varphi}_1(x,y,z)=d\cdot\psi_1(x,y,z)$, $\bar{\varphi}_2(x,y,z)=0$, $\bar{\psi}_1(x,y,z)=0$, $\bar{\psi}_2(x,y,z)=-d\cdot\psi_2(x,y,z)$, where $d=\lambda_1-\lambda_2$.

Let the solution of the problem (1)-(4) exist. Using equation (9), the solution of the problem (1)-(4) can be transformed to

$$u=A_1\,\omega-A_2\,\vartheta,\ \ v=A_3\,(\omega-\vartheta)\,, \tag{18}$$

where

$$\omega=\sum_{k,n,m=1}^{\infty}\bar{\omega}_{k,n,m}(t)\cdot\bar{\phi}_{k,n,m}+\sum_{k,n,m=1}^{\infty}\tilde{\omega}_{k,n,m}(t)\cdot\tilde{\phi}_{k,n,m},$$

$$\vartheta=\sum_{k,n,m=1}^{\infty}\bar{\vartheta}_{k,n,m}(t)\cdot\bar{\phi}_{k,n,m}+\sum_{k,n,m=1}^{\infty}\tilde{\vartheta}_{k,n,m}(t)\cdot\tilde{\phi}_{k,n,m},$$

$$\tilde{\bar{\omega}}_{k,n,m}(t)=\begin{cases}\tilde{\bar{\varphi}}_{k,n,m}\cos\left(\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_1\right|}t\right), & \tilde{\bar{\xi}}_{k,n,m}+\lambda_1<0,\\ \tilde{\bar{\varphi}}_{k,n,m}, & \tilde{\bar{\xi}}_{k,n,m}+\lambda_1=0,\\ \tilde{\bar{\varphi}}_{k,n,m}\cosh\left(\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_1\right|}t\right), & \tilde{\bar{\xi}}_{k,n,m}+\lambda_1>0,\end{cases}$$

$$\tilde{\bar{\vartheta}}_{k,n,m}(t)=\begin{cases}\tilde{\bar{\psi}}_{k,n,m}\sin\left(\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_2\right|}t\right)/\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_2\right|}, & \tilde{\bar{\xi}}_{k,n,m}+\lambda_2<0,\\ \tilde{\bar{\psi}}_{k,n,m}t, & \tilde{\bar{\xi}}_{k,n,m}+\lambda_2=0,\\ \tilde{\bar{\psi}}_{k,n,m}sinh\left(\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_2\right|}t\right)/\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_2\right|}, & \tilde{\bar{\xi}}_{k,n,m}+\lambda_2>0,\end{cases}$$

$\bar{\varphi}_{k,n,m}=d\int\limits_{\Omega}sign(x)\psi_1(x,y,z)\bar{\phi}_{k,n,m}d\Omega$, $\tilde{\varphi}_{k,n,m}=-d\int\limits_{\Omega}sign(x)\psi_1(x,y,z)\tilde{\phi}_{k,n,m}d\Omega$, $\bar{\psi}_{k,n,m}=-d\int\limits_{\Omega}sign(x)\psi_2(x,y,z)\bar{\phi}_{k,n,m}d\Omega$, $\tilde{\psi}_{k,n,m}=d\int\limits_{\Omega}sign(x)\psi_2(x,y,z)\tilde{\phi}_{k,n,m}d\Omega$, $A_1=\frac{a_1-\lambda_2}{b_2\cdot d}$, $A_2=\frac{a_1-\lambda_1}{b_2\cdot d}$, $A_3=\frac{1}{d}$, $d=\lambda_1-\lambda_2$.

Regularized approximate solution $\left(u^N,\,v^N\right)$ we define by the formula:

$$u^N=A_1\omega^N(x,y,z,t)-A_2\vartheta^N(x,y,z,t),\ \ v^N=A_3\omega^N(x,y,z,t)-A_3\vartheta^N(x,y,z,t), \tag{19}$$

where

$$\omega^N=\sum_{k=1}^{N}\sum_{n,m=1}^{\infty}\bar{\omega}_{k,n,m}(t)\cdot\bar{\phi}_{k,n,m}+\sum_{k,n,m=1}^{\infty}\tilde{\omega}_{k,n,m}(t)\cdot\tilde{\phi}_{k,n,m},$$

$$\vartheta^N=\sum_{k=1}^{N}\sum_{n,m=1}^{\infty}\bar{\vartheta}_{k,n,m}(t)\cdot\bar{\phi}_{k,n,m}+\sum_{k,n,m=1}^{\infty}\tilde{\vartheta}_{k,n,m}(t)\cdot\tilde{\phi}_{k,n,m},$$

here N is an integer parameter of regularization. Then, regularized approximate solution with approximate data will be define as

$$u_\varepsilon^N = A_1\omega_\varepsilon^N(x,y,z,t) - A_2\vartheta_\varepsilon^N(x,y,z,t),\ \ v_\varepsilon^N = A_3\omega_\varepsilon^N(x,y,z,t) - A_3\vartheta_\varepsilon^N(x,y,z,t), \tag{20}$$

where

$$\omega_\varepsilon^N = \sum_{k=1}^{N}\sum_{n,m=1}^{\infty} \bar{\omega}_{\varepsilon_{k,n,m}}(t)\cdot\bar{\phi}_{k,n,m} + \sum_{k,n,m=1}^{\infty} \tilde{\omega}_{\varepsilon_{k,n,m}}(t)\cdot\tilde{\phi}_{k,n,m},$$

$$\vartheta^N = \sum_{k=1}^{N}\sum_{n,m=1}^{\infty} \bar{\vartheta}_{\varepsilon_{k,n,m}}(t)\cdot\bar{\phi}_{k,n,m} + \sum_{k,n,m=1}^{\infty} \tilde{\vartheta}_{\varepsilon_{k,n,m}}(t)\cdot\tilde{\phi}_{k,n,m},$$

$$\tilde{\bar{\omega}}_{\varepsilon_{k,n,m}}(t) = \begin{cases} \tilde{\bar{\varphi}}_{\varepsilon_{k,n,m}}\cos\left(\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_1\right|}t\right), & \tilde{\bar{\xi}}_{k,n,m}+\lambda_1<0,\\ \tilde{\bar{\varphi}}_{\varepsilon_{k,n,m}}, & \tilde{\bar{\xi}}_{k,n,m}+\lambda_1=0,\\ \tilde{\bar{\varphi}}_{\varepsilon_{k,n,m}} cosh\left(\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_1\right|}t\right), & \tilde{\bar{\xi}}_{k,n,m}+\lambda_1>0,\end{cases}$$

$$\tilde{\bar{\vartheta}}_{k,n,m}(t) = \begin{cases} \tilde{\bar{\psi}}_{\varepsilon_{k,n,m}}\sin\left(\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_2\right|}t\right)/\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_2\right|}, & \tilde{\bar{\xi}}_{k,n,m}+\lambda_2<0,\\ \tilde{\bar{\psi}}_{\varepsilon_{k,n,m}}t, & \tilde{\bar{\xi}}_{k,n,m}+\lambda_2=0,\\ \tilde{\bar{\psi}}_{\varepsilon_{k,n,m}} sinh\left(\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_2\right|}t\right)/\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_2\right|}, & \tilde{\bar{\xi}}_{k,n,m}+\lambda_2>0,\end{cases}$$

$\bar{\varphi}_{\varepsilon_{k,n,m}} = d\int\limits_\Omega sign(x)\psi_{1\varepsilon}(x,y,z)\bar{\phi}_{k,n,m}d\Omega, \tilde{\varphi}_{\varepsilon_{k,n,m}} = -d\int\limits_\Omega sign(x)\psi_{1\varepsilon}(x,y,z)\tilde{\phi}_{k,n,m}d\Omega, \bar{\psi}_{\varepsilon_{k,n,m}} = -d\int\limits_\Omega sign(x)\psi_{2\varepsilon}(x,y,z)\bar{\phi}_{k,n,m}d\Omega,$ $\tilde{\psi}_{\varepsilon_{k,n,m}} = d\int\limits_\Omega sign(x)\psi_{2\varepsilon}(x,y,z)\tilde{\phi}_{k,n,m}d\Omega$, since $\varphi_{1_\varepsilon}(x,y,z)$, $\varphi_{2_\varepsilon}(x,y,z)$, $\psi_{1_\varepsilon}(x,y,z)$, $\psi_{2_\varepsilon}(x,y,z)$ are approximate dates, and let $\varphi_{1\varepsilon}(x,y,z) = \frac{a_1-\lambda_1}{b_2}\psi_{1\varepsilon}(x,y,z)$, $\varphi_{2\varepsilon}(x,y,z) = \frac{a_1-\lambda_2}{b_2}\psi_{2\varepsilon}(x,y,z)$.

Let $\left\|\varphi_1(x,y,z)-\varphi_{1\varepsilon}(x,y,z)\right\|_0 \le \varepsilon$, $\left\|\varphi_2(x,y,z)-\varphi_{2\varepsilon}(x,y,z)\right\|_0 \le \varepsilon$, $\left\|\psi_1(x,y,z)-\psi_{1\varepsilon}(x,y,z)\right\|_0 \le \varepsilon$, $\left\|\psi_2(x,y,z)-\psi_{2\varepsilon}(x,y,z)\right\|_0 \le \varepsilon$ and $(u,v)\in M$. Then for the norm of the differences between the exact and approximate solutions we have

$$\left\|u-u_\varepsilon^N\right\|_0 \le \left\|u-u^N\right\|_0 + \left\|u^N-u_\varepsilon^N\right\|_0, \tag{21}$$

$$\left\|v-v_\varepsilon^N\right\|_0 \le \left\|v-v^N\right\|_0 + \left\|v^N-v_\varepsilon^N\right\|_0. \tag{22}$$

First, let's estimate the right-hand side of (21). So,

$$\left\|u^N-u_\varepsilon^N\right\|_0 \le A_1\left\|\omega^N-\omega_\varepsilon^N\right\|_0 + A_2\left\|\vartheta^N-\vartheta_\varepsilon^N\right\|_0. \tag{23}$$

Estimating the first expression on the right-hand side of (23) one can get

$$\left\|\omega^N-\omega_\varepsilon^N\right\|_0^2 = \sum_{k=1}^{N}\sum_{n,m=1}^{\infty}\left(\bar{\omega}_{k,n,m}(t)-\bar{\omega}_{\varepsilon_{k,n,m}}(t)\right)^2 + \sum_{k,n.m=1}^{\infty}\left(\tilde{\omega}_{k,n,m}(t)-\tilde{\omega}_{\varepsilon_{k,n,m}}(t)\right)^2 \le$$
$$cosh^2\left(\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_1\right|}t\right)\left[\sum_{k=1}^{N}\sum_{n,m=1}^{\infty}\left(\bar{\varphi}_{k,n,m}-\bar{\varphi}_{\varepsilon_{k,n,m}}\right)^2 + \sum_{k,n,m=1}^{\infty}\left(\tilde{\varphi}_{k,n,m}-\tilde{\varphi}_{\varepsilon_{k,n,m}}\right)^2\right] \le$$
$$cosh^2\left(\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_1\right|}t\right)\varepsilon^2 \le e^{2\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_1\right|}t}\varepsilon^2.$$

Now we evaluate the expression $\left\|\vartheta^N-\vartheta_\varepsilon^N\right\|_0$ in the similar way and we have

$$\left\|\vartheta^N-\vartheta_\varepsilon^N\right\|_0^2 \le {\delta_2}^{-1}sh^2\left(\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_2\right|}t\right)\varepsilon^2 \le {\delta_2}^{-1}e^{2\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_2\right|}t}\varepsilon^2.$$

From (23) we obtain

$$\left\|u^N - u_\varepsilon^N\right\|_0 \leq A_1 e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_1\right|}t}\varepsilon + A_2 \delta_2{}^{-1/2} e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_2\right|}t}\varepsilon. \tag{24}$$

Let us proceed to estimating the first term on the right side of inequality (21)

$$\left\|u - u^N\right\|_0 \leq A_1 \left\|\omega - \omega^N\right\|_0 + A_2 \left\|\vartheta - \vartheta^N\right\|_0$$

provided $(u, v) \in M$. Next we try to estimate the expression

$$\left\|\omega - \omega^N\right\|_0^2 = \sum_{k=N+1}^{\infty} \sum_{n,m=1}^{\infty} \bar{\omega}_{k,n,m}^2(t),$$

under the conditions $\left\|\omega(x, y, z, T)\right\|_0 \leq \mathfrak{M}_1$, $\left\|\vartheta(x, y, z, T)\right\|_0 \leq \mathfrak{M}_2$, where $\mathfrak{M}_1 = 2\pi\left(|b_2| + |\lambda_1 - a_1|\right)\mathfrak{M}$, $\mathfrak{M}_2 = 2\pi\left(|b_2| + |\lambda_2 - a_1|\right)\mathfrak{M}$. Now, to find the conditional extremum, we use the Lagrange multipliers method, and we get

$$\bar{\varphi}_{k,n,m} = \begin{cases} \mathfrak{M}_1 \cdot cosh^{-1}\left(\sqrt{\left|\bar{\xi}_{k,n,m} + \lambda_1\right|}T\right), & k = N+1,\ n = 1,\ m = 1, \\ 0, & k \neq N+1,\ n \neq 1,\ m \neq 1. \end{cases}$$

Then

$$\left\|\omega - \omega^N\right\|_0 \leq \frac{\mathfrak{M}_1 cosh\left(\sqrt{\left|\bar{\xi}_{N+1,1,1} + \lambda_1\right|}t\right)}{cosh\left(\sqrt{\left|\bar{\xi}_{N+1,1,1} + \lambda_1\right|}T\right)} \leq 2\mathfrak{M}_1 e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_1\right|}(t-T)}. \tag{25}$$

By the same method we get

$$\left\|\vartheta - \vartheta^N\right\|_0 \leq \frac{\mathfrak{M}_2 sinh\left(\sqrt{\left|\bar{\xi}_{N+1,1,1} + \lambda_2\right|}t\right)}{sinh\left(\sqrt{\left|\bar{\xi}_{N+1,1,1} + \lambda_2\right|}T\right)} \leq \mathfrak{M}_2 C(T) e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_2\right|}(t-T)}, \tag{26}$$

where $C_2(T) = \left(1 - e^{-2\sqrt{\delta_2}T}\right)^{-1}$. Combining estimates (25) and (26) we have

$$\left\|u - u^N\right\|_0 \leq 2\mathfrak{M}_1 A_1 e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_1\right|}(t-T)} + \mathfrak{M}_2 A_2 C(T) e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_2\right|}(t-T)}. \tag{27}$$

Substituting (27) and (24) into inequality (21) we obtain that

$$\left\|u - u_\varepsilon^N\right\|_0 \leq 2\mathfrak{M}_1 A_1 e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_1\right|}(t-T)} + \mathfrak{M}_2 A_2 C(T) e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_2\right|}(t-T)} + A_1 e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_1\right|}t}\varepsilon + A_2 \delta_2{}^{-1/2} e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_2\right|}t}\varepsilon. \tag{28}$$

Let us estimate inequality (22). Note that for the $\left\|v^N - v_\varepsilon^N\right\|_0$ expression the estimate

$$\left\|v^N - v_\varepsilon^N\right\|_0 \leq A_3 \left\|\omega^N - \omega_\varepsilon^N\right\|_0 + A_3 \left\|\vartheta^N - \vartheta_\varepsilon^N\right\|_0 \leq A_3 \left(e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_1\right|}t} + \delta_2{}^{-1/2} e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_2\right|}t}\right)\varepsilon$$

is true. For $\left\|v - v^N\right\|_0$ we have

$$\left\|v - v^N\right\|_0 \leq 2\mathfrak{M}_1 A_3 e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_1\right|}(t-T)} + \mathfrak{M}_2 A_3 C(T) e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_2\right|}(t-T)}.$$

Finally, we get

$$\left\|v - v_{\varepsilon}^{N}\right\|_{0} \leq 2\mathfrak{M}_1 A_3 e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_1\right|}(t-T)} + \mathfrak{M}_2 A_3 C(T) e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_2\right|}(t-T)} + A_3\left(e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_1\right|}t} + \delta_2^{-1/2} e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_2\right|}t}\right)\varepsilon. \quad (29)$$

We minimize the right side of inequalities (28) and (29) with respect to N and find the corresponding regularization parameters N.

II. Let $\varphi_i(x,y,z) = 0$, $\psi_i(x,y,z) = 0$, $i = 1, 2.$, $f(x,y,z,t) = f(x,y,z)$, $g(x,y,z,t) = g(x,y,z)$. Then $\bar{\varphi}_i(x,y,z) = 0$, $\bar{\psi}_i(x,y,z) = 0$, $i = 1, 2$.

Let the solution of the problem (1) - (4) exist. Then it can be represented in the form

$$u = A_1\omega - A_2\vartheta, \; v = A_3\omega - A_3\vartheta,$$

where

$$\omega = \sum_{k,n,m=1}^{\infty} \bar{\omega}_{k,n,m}(t)\cdot\bar{\phi}_{k,n,m} + \sum_{k,n,m=1}^{\infty} \tilde{\omega}_{k,n,m}(t)\cdot\tilde{\phi}_{k,n,m},$$

$$\vartheta = \sum_{k,n,m=1}^{\infty} \bar{\vartheta}_{k,n,m}(t)\cdot\bar{\phi}_{k,n,m} + \sum_{k,n,m=1}^{\infty} \tilde{\vartheta}_{k,n,m}(t)\cdot\tilde{\phi}_{k,n,m},$$

$$\tilde{\bar{\omega}}_{k,n,m}(t) = \begin{cases} \frac{\tilde{\bar{f}}_{k,n,m}}{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_1\right|}\left(1-\cos\left(\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_1\right|}t\right)\right), & \tilde{\bar{\xi}}_{k,n,m}+\lambda_1 < 0, \\ \tilde{\bar{f}}_{k,n,m} t^2/2, & \tilde{\bar{\xi}}_{k,n,m}+\lambda_1 = 0, \\ \frac{\tilde{\bar{f}}_{k,n,m}}{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_1\right|}\left(cosh\left(\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_1\right|}t\right)-1\right), & \tilde{\bar{\xi}}_{k,n,m}+\lambda_1 > 0, \end{cases}$$

$$\tilde{\bar{\vartheta}}_{k,n,m}(t) = \begin{cases} \frac{\tilde{\bar{g}}_{k,n,m}}{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_2\right|}\left(1-\cos\left(\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_2\right|}t\right)\right), & \tilde{\bar{\xi}}_{k,n,m}+\lambda_2 < 0, \\ \tilde{\bar{g}}_{k,n,m} t^2/2, & \tilde{\bar{\xi}}_{k,n,m}+\lambda_2 = 0, \\ \frac{\tilde{\bar{g}}_{k,n,m}}{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_2\right|}\left(cosh\left(\sqrt{\left|\tilde{\bar{\xi}}_{k,n,m}+\lambda_2\right|}t\right)-1\right), & \tilde{\bar{\xi}}_{k,n,m}+\lambda_2 > 0, \end{cases}$$

$\bar{f}_{k,n,m} = \int\limits_{\Omega} sign(x)\bar{f}(x,y,z)\bar{\phi}_{k,n,m}d\Omega$, $\tilde{f}_{k,n,m} = -\int\limits_{\Omega} sign(x)\bar{f}(x,y,z)\tilde{\phi}_{k,n,m}d\Omega$, $\bar{g}_{k,n,m} = \int\limits_{\Omega} sign(x)\bar{g}(x,y,z)\bar{\phi}_{k,n,m}d\Omega$, $\tilde{g}_{k,n,m} = -\int\limits_{\Omega} sign(x)\bar{g}(x,y,z)\tilde{\phi}_{k,n,m}d\Omega$.

In this case, we also construct a regularized approximate solution $\left(u^N, v^N\right)$ and an approximate solution $\left(u_{\varepsilon}^N, v_{\varepsilon}^N\right)$ based on approximate data in the same way as (19) and (20).

Let $f_{\varepsilon}(x,y,z)$, $g_{\varepsilon}(x,y,z)$ be approximate data. Then we denote $\bar{f}_{\varepsilon}(x,y,z) = b_2 f_{\varepsilon} + (\lambda_2 - a_1)\, g_{\varepsilon}$, $\bar{g}_{\varepsilon}(x,y,z) = b_2 f_{\varepsilon} + (\lambda_1 - a_1)\, g_{\varepsilon}$. Corresponding Fourier coefficients are:

$$\bar{f}_{\varepsilon_{k,n,m}} = \int\limits_{\Omega} sign(x)\bar{f}_{\varepsilon}(x,y,z)\bar{\phi}_{k,n,m}d\Omega, \;\; \tilde{f}_{\varepsilon_{k,n,m}} = -\int\limits_{\Omega} sign(x)\bar{f}_{\varepsilon}(x,y,z)\tilde{\phi}_{k,n,m}d\Omega,$$

$$\bar{g}_{\varepsilon_{k,n,m}} = \int\limits_{\Omega} sign(x)\bar{g}_{\varepsilon}(x,y,z)\bar{\phi}_{k,n,m}d\Omega, \;\; \tilde{g}_{\varepsilon_{k,n,m}} = -\int\limits_{\Omega} sign(x)\bar{g}_{\varepsilon}(x,y,z)\tilde{\phi}_{k,n,m}d\Omega.$$

Let $\left\|f - f_\varepsilon\right\|_0 \le \varepsilon$, $\left\|g - g_\varepsilon\right\|_0 \le \varepsilon$ and $(u, v) \in M$. Then

$$\left\|\omega^N - \omega_\varepsilon^N\right\|_0^2 \le \sum_{k=1}^{N}\sum_{n,m=1}^{\infty}\left(\bar{\omega}_{k,n,m} - \bar{\omega}_{\varepsilon_{k,n,m}}\right)^2 + \sum_{k,n,m=1}^{\infty}\left(\tilde{\omega}_{k,n,m} - \tilde{\omega}_{\varepsilon_{k,n,m}}\right)^2 \le$$

$$\frac{cosh^2\left(\sqrt{\left|\bar{\xi}_{N,1,1} + \lambda_1\right|}t\right)}{\delta_1^2}\left(\sum_{k=1}^{N}\sum_{n,m=1}^{\infty}\left(\bar{f}_{k,n,m} - \bar{f}_{\varepsilon_{k,n,m}}\right)^2 + \sum_{k,n,m=1}^{\infty}\left(\tilde{f}_{k,n,m} - \tilde{f}_{\varepsilon_{k,n,m}}\right)^2\right) \le \delta_1^{-2} e^{2\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_1\right|}t}\varepsilon^2.$$

In deriving this estimate, we used the maximum element replacement. Now, similarly evaluating the expression $\left\|\vartheta^N - \vartheta_\varepsilon^N\right\|_0$ we have

$$\left\|\vartheta^N - \vartheta_\varepsilon^N\right\|_0 \le \delta_2^{-1} e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_2\right|}t}\varepsilon.$$

Substituting the obtained estimates, we get

$$\left\|u^N - u_\varepsilon^N\right\|_0 \le |A_1|\,\delta_1^{-1} e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_1\right|}t}\varepsilon + |A_2|\,\delta_2^{-1} e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_2\right|}t}\varepsilon. \tag{30}$$

Let us move on to estimating $\left\|u - u^N\right\|_0$ under the condition $(u, v) \in M$. Then we come to an estimate of the expression

$$\left\|\omega - \omega^N\right\|_0^2 = \sum_{k=N+1}^{\infty}\sum_{n,m=1}^{\infty}\bar{\omega}_{k,n,m}^2,$$

under the condition $\left\|\omega(x, y, z, T)\right\|_0 \le \mathfrak{M}_1$. Then

$$\left\|\omega - \omega^N\right\|_0^2 \le \sum_{k=N+1}^{\infty}\sum_{n,m=1}^{\infty}\frac{\bar{f}_{k,n,m}^2}{\left|\bar{\bar{\xi}}_{k,n,m} + \lambda_1\right|^2}\left(cosh\left(\sqrt{\left|\bar{\bar{\xi}}_{k,n,m} + \lambda_1\right|}t\right) - 1\right)^2 \le$$

$$\frac{\mathfrak{M}_1^2 cosh^2\left(\sqrt{\left|\bar{\xi}_{N+1,1,1} + \lambda_1\right|}t\right) - 1}{cosh^2\left(\sqrt{\left|\bar{\xi}_{N+1,1,1} + \lambda_1\right|}T\right) - 1} \le C_1^2(T)\mathfrak{M}_1^2 e^{2\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_1\right|}(t-T)}, \tag{31}$$

where $C_1(T) = \left(1 - e^{-2\sqrt{\delta_1}T}\right)^{-1}$. Similarly, evaluating the expressions $\left\|\vartheta - \vartheta^N\right\|_0$ we get

$$\left\|\vartheta - \vartheta^N\right\|_0 \le C_2(T)\mathfrak{M}_2 e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_2\right|}(t-T)}. \tag{32}$$

Combining estimates (31) and (32) we have

$$\left\|u - u^N\right\|_0 \le C_1(T)\,|A_1|\,\mathfrak{M}_1 e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_1\right|}(t-T)} + C_2(T)\,|A_2|\,\mathfrak{M}_2 e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_2\right|}(t-T)}. \tag{33}$$

Substituting (33) and (30) into the inequality

$$\left\|u - u_\varepsilon^N\right\|_0 \le \left\|u - u^N\right\|_0 + \left\|u^N - u_\varepsilon^N\right\|_0,$$

we obtain that

$$\left\|u - u_\varepsilon^N\right\|_0 \le C_1(T)\,|A_1|\,\mathfrak{M}_1 e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_1\right|}(t-T)} + C_2(T)\,|A_2|\,\mathfrak{M}_2 e^{\sqrt{\left|\bar{\xi}_{N+1,1,1}+\lambda_2\right|}(t-T)} +$$

$$|A_1|\,\delta_1^{-1} e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_1\right|}t}\varepsilon + |A_2|\,\delta_2^{-1} e^{\sqrt{\left|\bar{\xi}_{N,1,1}+\lambda_2\right|}t}\varepsilon. \tag{34}$$

Let us estimate the inequality $\left\|v - v_\varepsilon^N\right\|_0$. Note that for the expression $\left\|v^N - v_\varepsilon^N\right\|_0$ the estimate

$$\left\|v^N - v_\varepsilon^N\right\|_0 \le |A_3|\,\delta_1{}^{-1} e^{\sqrt{|\bar{\xi}_{N,1,1}+\lambda_1|}t}\varepsilon + |A_3|\,\delta_2{}^{-1} e^{\sqrt{|\bar{\xi}_{N,1,1}+\lambda_2|}t}\varepsilon$$

is true. For $\left\|v - v^N\right\|_0$ we have

$$\left\|v - v^N\right\|_0 \le C_1(T)\,|A_3|\,\mathfrak{M}_1 e^{\sqrt{|\bar{\xi}_{N+1,1,1}+\lambda_1|}(t-T)} + C_2(T)\,|A_3|\,\mathfrak{M}_2 e^{\sqrt{|\bar{\xi}_{N+1,1,1}+\lambda_2|}(t-T)}.$$

Finally we get

$$\left\|v - v_\varepsilon^N\right\|_0 \le C_1(T)\,|A_3|\,\mathfrak{M}_1 e^{\sqrt{|\bar{\xi}_{N+1,1,1}+\lambda_1|}(t-T)} + C_2(T)\,|A_3|\,\mathfrak{M}_2 e^{\sqrt{|\bar{\xi}_{N+1,1,1}+\lambda_2|}(t-T)} + |A_3|\,\delta_1{}^{-1} e^{\sqrt{|\bar{\xi}_{N,1,1}+\lambda_1|}t}\varepsilon + |A_3|\,\delta_2{}^{-1} e^{\sqrt{|\bar{\xi}_{N,1,1}+\lambda_2|}t}\varepsilon. \quad (35)$$

For this case, we also minimize the right-hand side of inequalities (35) and (34) with respect to N and find the corresponding regularization parameter N.

7. Conclusion

In this paper, the ill-posed problem for the system of second-order mixed-type equations in three-dimensional space was considered. On the base of Tikhonov definitions we have shown that this problem has a unique and conditional stable solution on the set of correctness M. This gives us the ability to construct an approximate solution of the problem using the regularization method. Calculations show that approximate solution are close to exact solution if we choose parameter of regularization from the minimization of the estimate norm of the difference between exact and approximated solutions.

Acknowledgement

The paper has been presented at the conference Modern Problem of the Applied Mathematics and Information Technologies - Al-Khwarizmi 2023, Samarkand, Uzbekistan, September 25-26, 2023. The authors are grateful to the organizers.

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