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On Ostrowski-Mercer type inequalities for twice differentiable convex functions

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Abstract. Based on the Jensen-Mercer inequality proposed by Mercer in 2003, numerous studies have been given by researchers up to the present day. Taking into account the ongoing investigation, we have constructed a new integral equality that produces new Ostrowski-Mercer type inequalities. By using this identity, we obtained new upper bounds for the Ostrowski-Mercer inequalities for twice differentiable convex functions.

1. Introduction

The theory of inequalities has become a rapidly growing discipline with its increasing applications in economics, physics, engineering and other fields, as well as various branches of mathematics. This growth has been instrumental in the emergence of the theory as an independent field of mathematical analysis. The first book on inequalities was the joint work of Hardy, Littlewood and Pólya called "Inequalities" in 1934 [1]. The work was supported by Beckenbach and Bellman's "Inequalities" written in 1961 and Mitrinović's "Inequalities" published in 1970 [2], [3]. Since this date, generalizations, developments and new applications of inequalities have been given by researchers in various branches of mathematics.

The first theorem contains the integral inequality is known in the literature as Ostrowski inequality.

Theorem 1.1. *[4] Let $\Psi : [\eta, \mu] \to \mathbb{R}$ be a differentiable function on (η, μ) with the property that $|\Psi'| \leq M$ for all. Then*

$$\left| \Psi(x) - \frac{1}{\mu - \eta} \int_{\eta}^{\mu} \Psi(t)\, dt \right| \leq M(\mu - \eta) \left[\frac{1}{4} + \left(\frac{x - \frac{\mu+\eta}{2}}{\mu - \eta} \right)^2 \right] \tag{1}$$

for all $x \in [\eta, \mu]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity.

The following Ostrowski type results for absolutely continuous functions hold.

Theorem 1.2. *[5] Let $\Psi : [\eta, \mu] \to \mathbb{R}$ be absolutely continuous on $[\eta, \mu]$. Then for all $x \in [\eta, \mu]$, we have:*

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$$\left|\Psi(x)-\frac{1}{\mu-\eta}\int_{\eta}^{\mu}\Psi(t)\,dt\right| \tag{2}$$

$$\leq \begin{cases} \|\Psi'\|_{\infty}(\mu-\eta)\left[\frac{1}{4}+\left(\frac{x-\frac{\mu+\eta}{2}}{\mu-\eta}\right)^{2}\right] & if\ \ \Psi'\in L_{\infty}[\eta,\mu] \\ \|\Psi'\|_{q}(\mu-\eta)^{\frac{1}{p}}\left[\left(\frac{x-\eta}{\mu-\eta}\right)^{p+1}+\left(\frac{\mu-x}{\mu-\eta}\right)^{p+1}\right]^{\frac{1}{p}} & \begin{array}{c} if\ \ \Psi'\in L_{q}[\eta,\mu] \\ \frac{1}{p}+\frac{1}{q}=1 \\ p>1 \end{array} \\ \|\Psi'\|_{1}\left[\frac{1}{2}+\left|\frac{x-\frac{\mu+\eta}{2}}{\mu-\eta}\right|\right] & \end{cases}$$

where $\|.\|_r$ $(r\in[1,\infty])$ *are the usual Lebesgue norms on* $L_r[\eta,\mu]$, *i.e.,*

$$\|\Psi\|_{\infty}=\sup_{t\in[\eta,\mu]}|\Psi(t)|$$

and

$$\|\Psi\|_{r}=\left(\int_{\eta}^{\mu}|\Psi(t)|^{r}\,dt\right)^{\frac{1}{r}},\ \ r\in[1,\infty).$$

The constants $\frac{1}{4}$, $\frac{1}{(p+1)^{\frac{1}{p}}}$ *and* $\frac{1}{2}$ *respectively are sharp.*

Ostrowski type inequalities provide sharp estimates of error in approximating the value of a function relative to its integral mean. They can be used to obtain a priori error bounds for different quadrature rules when approximating the Riemann integral with different Riemann sums.

Definition 1.3. *[6] Suppose that I be an interval in $\mathbb{R}$. Then $\Psi : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ is said to be convex if*

$$\Psi(\xi\eta+(1-\xi)\mu)\leq\xi\Psi(\eta)+(1-\xi)\Psi(\mu). \tag{3}$$

for all $\eta,\mu\in I$ and $\xi\in[0,1]$.

Definition 1.4. *[7] Suppose that Ψ is a convex function on $[\eta,\mu]$, then*

$$\Psi\left(\sum_{i=1}^{n}\xi_i x_i\right)\leq\left(\sum_{i=1}^{n}\xi_i\Psi(x_i)\right) \tag{4}$$

for all $x_i\in[\eta,\mu]$ and $\xi_i\in[0,1]$, where $i=1,2,...,n$ with $\sum_{i=1}^{n}\xi_i=1$.

In 2003, Mercer inroduced a variant of the Jensen inequality, known as the Jensen–Mercer inequality as follows.

Definition 1.5. *[8] Suppose that Ψ is a convex function on $[\eta,\mu]$, then*

$$\Psi\left(\eta+\mu-\sum_{i=1}^{n}\xi_i x_i\right)\leq\Psi(\eta)+\Psi(\mu)-\sum_{i=1}^{n}\xi_i\Psi(x_i) \tag{5}$$

for all $x_i\in[\eta,\mu]$ and $\xi_i\in[0,1]$, where $i=1,2,...,n$ with $\sum_{i=1}^{n}\xi_i=1$.

In 2013 the following Hermite-Hadamard-Mercer type inequality was introduced by Kian and Moslehian.

Definition 1.6. *[9] Suppose that Ψ is a convex function on $[\eta,\mu]$, then*

$$\begin{aligned} & \Psi\left(\eta+\mu-\frac{x_1+x_2}{2}\right) \\ \leq\ & \Psi(\eta)+\Psi(\mu)-\int_\eta^\mu \Psi(\xi x_1+(1-\xi)x_2)\,d\xi \\ \leq\ & \Psi(\eta)+\Psi(\mu)-\Psi\left(\frac{x_1+x_2}{2}\right) \end{aligned} \tag{6}$$

and

$$\begin{aligned} & \Psi\left(\eta+\mu-\frac{x_1+x_2}{2}\right) \\ \leq\ & \frac{1}{x_2-x_1}\int_{x_1}^{x_2} \Psi(\eta+\mu-\xi)\,d\xi \leq \Psi(\eta)+\Psi(\mu)-\Psi\left(\frac{x_1+x_2}{2}\right) \end{aligned} \tag{7}$$

for all $x_1, x_2 \in [\eta,\mu]$ and $\xi \in [0,1]$.

Generalizations and improvements have been made about the Jensen-Mercer inequality, which has been of interest to many researchers in recent years, and many articles have been published on this subject. Niezgoda et al., Butt et al., Sial et al., Ali et al., obtained Hermite-Hadamard-Mercer type inequalities with the help of different types of convex functions [10], [11], [12], [13]. In addition, Abdeljawad et al., Chu et al., Wang et al., Vivas-Kortez et al., Set et al., gave new bounds for Hermite-Hadamard-Mercer type inequalities by using fractional integrals [14], [15], [16], [17], [18]. Furthermore, as various studies on Mercer-type inequalities are rapidly increasing, researchers interested in the subject can examine articles [19], [20], [21], [22], [23], [24], [25], [26]. The importance of Jensen's inequality and the above-mentioned articles obtained in recent years have prompted us to study this subject.

Next section focuses on establishing a new identity for Jensen-Mercer type inequality involving the twice differentiable functions. Then, by considering this identity and with the help of some fundamental integral inequalities such as Hölder inequality, power-mean inequality, new Ostrowski-Mercer type inequalities are presented.

2. New Ostrowski-Mercer Type Inequalities

A lemma will be given to prove main findings. Based on this lemma, new upper bounds for the Ostrowski-Mercer type inequalities will be obtained by using the Jensen–Mercer inequalities for twice differentiable functions.

Lemma 2.1. *Suppose that $\Psi : [\eta,\mu] \to \mathbb{R}$ be a twice differentiable function on (η,μ) and $\Psi'' \in L[\eta,\mu]$, for all $\lambda \in [x_1,x_2]$, $x_1, x_2 \in [\eta,\mu]$ and $\xi \in [0,1]$, then the following inequality holds:*

$$\begin{aligned} & \frac{(\lambda-x_1)}{2}\int_0^1 \xi^2\Psi''(\lambda+\eta-(\xi x_1+(1-\xi)\lambda))\,d\xi \\ & \frac{(x_2-\lambda)}{2}\int_0^1 \xi^2\Psi''(\lambda+\mu-(\xi x_2+(1-\xi)\lambda))\,d\xi \\ & \frac{(\mu-\eta+x_1-x_2)}{2}\int_0^1 \Psi''(\xi(\lambda+\mu-x_2)+(1-\xi)(\lambda+\eta-x_1))\,d\xi \\ =\ & \left(\frac{1}{(\lambda-x_1)^2}\int_\eta^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\right) \\ & -\left(\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}\right) \end{aligned} \tag{8}$$

Proof. Using the integration by parts, we obtain the equalities

$$\begin{aligned}\Lambda_1 &= \frac{(\lambda-x_1)}{2}\int_0^1 \xi^2\Psi''(\lambda+\eta-(\xi x_1+(1-\xi)\lambda))\,d\xi\\ &= \frac{\Psi'(\lambda+\eta-x_1)}{2}\\ &\quad -\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{1}{(\lambda-x_1)^2}\int_\eta^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa\end{aligned}$$

$$\begin{aligned}\Lambda_2 &= \frac{(x_2-\lambda)}{2}\int_0^1 \xi^2\Psi''(\lambda+\mu-(\xi x_2+(1-\xi)\lambda))\,d\xi\\ &= -\frac{\Psi'(\lambda+\mu-x_2)}{2}\\ &\quad -\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\end{aligned}$$

and

$$\begin{aligned}\Lambda_3 &= \frac{(\mu-\eta+x_1-x_2)}{2}\\ &\quad\times\int_0^1 \Psi''(\xi(\lambda+\mu-x_2)+(1-\xi)(\lambda+\eta-x_1))\,d\xi\\ &= \frac{\Psi'(\lambda+\mu-x_2)-\Psi'(\lambda+\eta-x_1)}{2}.\end{aligned}$$

Adding Λ_1, Λ_2 and Λ_3, we get the required result. □

Theorem 2.2. *Suppose that $\Psi : [\eta,\mu] \to \mathbb{R}$ be a twice differentiable function on (η,μ) and $\Psi'' \in L[\eta,\mu]$, for all $\lambda \in [x_1,x_2]$, $x_1,x_2 \in [\eta,\mu]$, $\xi \in [0,1]$. If $|\Psi''|$ is convex on $[\eta,\mu]$ then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\left(\frac{1}{(\lambda-x_1)^2}\int_\eta^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa\right.\right.\\ &\left.\left.+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\right)\right|\\ \le\ &\frac{(\lambda-x_1)}{2}\left(\frac{|\Psi''(\lambda)|-|\Psi''(x_1)|}{4}+\frac{|\Psi''(\eta)|}{3}\right)\\ &+\frac{(x_2-\lambda)}{2}\left(\frac{|\Psi''(\lambda)|-|\Psi''(x_2)|}{4}+\frac{|\Psi''(\mu)|}{3}\right)\\ &+\frac{(\mu-\eta+x_1-x_2)}{4}\left(|\Psi''(\lambda+\eta-x_1)|+|\Psi''(\lambda+\mu-x_2)|\right).\end{aligned}\tag{9}$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}&\left|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\left(\frac{1}{(\lambda-x_1)^2}\int_\eta^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa\right.\right.\\ &\left.\left.+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\right)\right|\\ \le\ &\frac{(\lambda-x_1)}{2}\int_0^1 \xi^2\left|\Psi''(\lambda+\eta-(\xi x_1+(1-\xi)\lambda))\right|d\xi\end{aligned}$$

$$+\frac{(x_2-\lambda)}{2}\int_0^1 \xi^2\left|\Psi''(\lambda+\mu-(\xi x_2+(1-\xi)\lambda))\right|d\xi$$
$$+\frac{(\mu-\eta+x_1-x_2)}{2}\int_0^1\left|\Psi''(\xi(\lambda+\mu-x_2)+(1-\xi)(\lambda+\eta-x_1))\right|d\xi.$$

By applying Jensen-Mercer inequality with using the convexity of $|\Psi''|$, *we get*

$$\begin{aligned}
&\left|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\left(\frac{1}{(\lambda-x_1)^2}\int_\eta^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa\right.\right.\\
&\left.\left.+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\right)\right|\\
\leq\ &\frac{(\lambda-x_1)}{2}\int_0^1\xi^2\left(|\Psi''(\lambda)|+\left|\Psi''(\eta)\right|-\xi\,|\Psi''(x_1)|-(1-\xi)\,|\Psi''(\lambda)|\right)d\xi\\
&+\frac{(x_2-\lambda)}{2}\int_0^1\xi^2\left(|\Psi''(\lambda)|+\left|\Psi''(\eta)\right|-\xi\,|\Psi''(x_1)|-(1-\xi)\,|\Psi''(\lambda)|\right)d\xi\\
&+\frac{(\mu-\eta+x_1-x_2)}{2}\int_0^1\left(\xi\left|\Psi''(\lambda+\mu-x_2)\right|+(1-\xi)\left|\Psi''(\lambda+\eta-x_1)\right|\right)d\xi
\end{aligned}$$

By calculating the above integrals and simplifying, then the proof is completed. □

Corollary 2.3. *Let all the assumptions of Theorem 2.2 hold and* $x_1=\eta$, $x_2=\mu$, *then we get*

$$\begin{aligned}
&\left|\frac{\Psi(\lambda)(\mu-\eta)}{(\lambda-\eta)(\mu-\lambda)}-\left(\frac{1}{(\lambda-\eta)^2}\int_\eta^\lambda\Psi(\kappa)\,d\kappa+\frac{1}{(\mu-\lambda)^2}\int_\lambda^\mu\Psi(\kappa)\,d\kappa\right)\right|\\
\leq\ &\frac{(\lambda-\eta)}{2}\left(\frac{1}{4}|\Psi''(\lambda)|+\frac{1}{12}\left|\Psi''(\eta)\right|\right)\\
&+\frac{(\mu-\lambda)}{2}\left(\frac{1}{4}|\Psi''(\lambda)|+\frac{1}{12}\left|\Psi''(\mu)\right|\right).
\end{aligned}\tag{10}$$

Corollary 2.4. *Under the same assumptions of Theorem 2.2 with* $|\Psi''|\leq M$, *then we have the following Ostrowski-Mercer type inequality*

$$\begin{aligned}
&\left|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\left(\frac{1}{(\lambda-x_1)^2}\int_\eta^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa\right.\right.\\
&\left.\left.+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\right)\right|\\
\leq\ &\frac{M(2x_1-2x_2+3(\mu-\eta))}{6}.
\end{aligned}\tag{11}$$

Corollary 2.5. *If we choose* $x_1=\eta$, $x_2=\mu$ *in Corollary 2.4, then we have*

$$\begin{aligned}
&\left|\frac{\Psi(\lambda)(\mu-\eta)}{(\lambda-\eta)(\mu-\lambda)}-\left(\frac{1}{(\lambda-\eta)^2}\int_\eta^\lambda\Psi(\kappa)\,d\kappa+\frac{1}{(\mu-\lambda)^2}\int_\lambda^\mu\Psi(\kappa)\,d\kappa\right)\right|\\
\leq\ &\frac{M(\mu-\eta)}{6}.
\end{aligned}\tag{12}$$

Theorem 2.6. *Suppose that* $\Psi:[\eta,\mu]\to\mathbb{R}$ *be a twice differentiable function on* (η,μ) *and* $\Psi''\in L[\eta,\mu]$, *for all* $\lambda\in[x_1,x_2]$, $x_1,x_2\in[\eta,\mu]$, $\xi\in[0,1]$. *If* $|\Psi''|^\rho$ *is convex on* $[\eta,\mu]$ *and* $\rho>1$, $\frac{1}{\rho}+\frac{1}{\varphi}=1$ *then the following inequality holds:*

$$\left|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\left(\frac{1}{(\lambda-x_1)^2}\int_\eta^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa\right.\right.$$

$$
\begin{aligned}
&+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\Bigg)\Bigg| \qquad (13)\\
\leq\ & \frac{(\lambda-x_1)}{2^{\frac{1}{\varphi}+1}(2\rho+1)^{\frac{1}{\rho}}}\left(|\Psi''(\lambda)|^{\varphi}+2\left|\Psi''(\eta)\right|^{\varphi}-|\Psi''(x_1)|^{\varphi}\right)^{\frac{1}{\varphi}}\\
&+\frac{(x_2-\lambda)}{2^{\frac{1}{\varphi}+1}(2\rho+1)^{\frac{1}{\rho}}}\left(|\Psi''(\lambda)|^{\varphi}+2\left|\Psi''(\mu)\right|^{\varphi}-|\Psi''(x_2)|^{\varphi}\right)^{\frac{1}{\varphi}}\\
&+\frac{(\mu-\eta+x_1-x_2)}{2^{\frac{1}{\varphi}+1}}\left(\left|\Psi''(\lambda+\eta-x_1)\right|^{\varphi}+\left|\Psi''(\lambda+\mu-x_2)\right|^{\varphi}\right)^{\frac{1}{\varphi}}.
\end{aligned}
$$

Proof. By Lemma 2.1 and applying Hölder inequality, then we have

$$
\begin{aligned}
&\Bigg|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\Bigg(\frac{1}{(\lambda-x_1)^2}\int_{\eta}^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa\\
&+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\Bigg)\Bigg|\\
\leq\ & \frac{(\lambda-x_1)}{2}\left(\int_0^1\xi^{2\rho}d\xi\right)^{\frac{1}{\rho}}\left(\int_0^1\left|\Psi''(\lambda+\eta-(\xi x_1+(1-\xi)\lambda))\right|^{\varphi}d\xi\right)^{\frac{1}{\varphi}}\\
&+\frac{(x_2-\lambda)}{2}\left(\int_0^1\xi^{2\rho}d\xi\right)^{\frac{1}{\rho}}\left(\int_0^1\left|\Psi''(\lambda+\mu-(\xi x_2+(1-\xi)\lambda))\right|^{\varphi}d\xi\right)^{\frac{1}{\varphi}}\\
&+\frac{(\mu-\eta+x_1-x_2)}{2}\left(\int_0^1 1d\xi\right)^{\frac{1}{\rho}}\\
&\times\left(\int_0^1\left|\Psi''(\xi(\lambda+\mu-x_2)+(1-\xi)(\lambda+\eta-x_1))\right|^{\varphi}d\xi\right)^{\frac{1}{\varphi}}
\end{aligned}
$$

On the other hand, by using Jensen-Mercer inequality, we can write

$$
\begin{aligned}
&\Bigg|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\Bigg(\frac{1}{(\lambda-x_1)^2}\int_{\eta}^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa\\
&+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\Bigg)\Bigg|\\
\leq\ & \frac{(\lambda-x_1)}{2}\left(\frac{1}{2\rho+1}\right)^{\frac{1}{\rho}}\\
&\times\left(\int_0^1\left(|\Psi''(\lambda)|^{\varphi}+\left|\Psi''(\eta)\right|^{\varphi}-\xi|\Psi''(x_1)|^{\varphi}-(1-\xi)|\Psi''(\lambda)|^{\varphi}\right)d\xi\right)^{\frac{1}{\varphi}}\\
&+\frac{(x_2-\lambda)}{2}\left(\frac{1}{2\rho+1}\right)^{\frac{1}{\rho}}\\
&\times\left(\int_0^1\left(|\Psi''(\lambda)|^{\varphi}+\left|\Psi''(\mu)\right|^{\varphi}-\xi|\Psi''(x_2)|^{\varphi}-(1-\xi)|\Psi''(\lambda)|^{\varphi}\right)d\xi\right)^{\frac{1}{\varphi}}\\
&+\frac{(\mu-\eta+x_1-x_2)}{2}
\end{aligned}
$$

$$\times\left(\int_0^1 \left(\xi\left|\Psi''\left(\lambda+\mu-x_2\right)\right|^{\varphi}+(1-\xi)\left|\Psi''\left(\lambda+\eta-x_1\right)\right|^{\varphi}\right)d\xi\right)^{\frac{1}{\varphi}}$$

By a simple computation, we get the desired result. □

Corollary 2.7. *Let all the assumptions of Theorem 2.6 hold and $x_1=\eta$, $x_2=\mu$, then we get*

$$\begin{aligned}&\left|\frac{\Psi(\lambda)(\mu-\eta)}{(\lambda-\eta)(\mu-\lambda)}-\left(\frac{1}{(\lambda-\eta)^2}\int_\eta^\lambda \Psi(\kappa)\,d\kappa+\frac{1}{(\mu-\lambda)^2}\int_\lambda^\mu \Psi(\kappa)\,d\kappa\right)\right|\\ \leq\;& \frac{1}{\left(2^{1+\frac{1}{\varphi}}\right)(2\rho+1)^{\frac{1}{\rho}}}\left((\lambda-\eta)\left(|\Psi''(\lambda)|^{\varphi}+\left|\Psi''(\eta)\right|^{\varphi}\right)^{\frac{1}{\varphi}}\right.\\ &\left.+(\mu-\lambda)\left(|\Psi''(\lambda)|^{\varphi}+\left|\Psi''(\mu)\right|^{\varphi}\right)^{\frac{1}{\varphi}}\right).\end{aligned}\tag{14}$$

Corollary 2.8. *Under the same assumptions of Theorem 2.6 with $|\Psi''|\leq M$, then we have the following Ostrowski-Mercer type inequality*

$$\begin{aligned}&\left|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\left(\frac{1}{(\lambda-x_1)^2}\int_\eta^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa\right.\right.\\ &\left.\left.+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\right)\right|\\ \leq\;& M\left(\frac{1}{2\rho+1}\right)^{\frac{1}{\rho}}\frac{(x_2-x_1)}{2}+M\frac{(\mu-\eta+x_1-x_2)}{2}.\end{aligned}\tag{15}$$

Corollary 2.9. *If we choose $x_1=\eta$, $x_2=\mu$ in Corollary 2.8, then we have*

$$\begin{aligned}&\left|\frac{\Psi(\lambda)(\mu-\eta)}{(\lambda-\eta)(\mu-\lambda)}-\left(\frac{1}{(\lambda-\eta)^2}\int_\eta^\lambda \Psi(\kappa)\,d\kappa+\frac{1}{(\mu-\lambda)^2}\int_\lambda^\mu \Psi(\kappa)\,d\kappa\right)\right|\\ \leq\;& M\left(\frac{1}{2\rho+1}\right)^{\frac{1}{\rho}}\frac{(\mu-\eta)}{2}.\end{aligned}\tag{16}$$

Theorem 2.10. *Suppose that $\Psi:[\eta,\mu]\to\mathbb{R}$ be a twice differentiable function on (η,μ) and $\Psi''\in L[\eta,\mu]$, for all $\lambda\in[x_1,x_2]$, $x_1,x_2\in[\eta,\mu]$, $\xi\in[0,1]$. If $|\Psi''|^{\rho}$ is convex on $[\eta,\mu]$ and $\rho\geq 1$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\left(\frac{1}{(\lambda-x_1)^2}\int_\eta^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa\right.\right.\\ &\left.\left.+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\right)\right|\\ \leq\;& \frac{(\lambda-x_1)}{2.12^{\frac{1}{\rho}}}\left(\frac{1}{3}\right)^{1-\frac{1}{\rho}}\left(3|\Psi''(\lambda)|^{\rho}+4\left|\Psi''(\eta)\right|^{\varphi}-3|\Psi''(x_1)|^{\rho}\right)^{\frac{1}{\rho}}\\ &+\frac{(x_2-\mu)}{2.12^{\frac{1}{\rho}}}\left(\frac{1}{3}\right)^{1-\frac{1}{\rho}}\left(3|\Psi''(\lambda)|^{\rho}+4\left|\Psi''(\mu)\right|^{\varphi}-3|\Psi''(x_2)|^{\rho}\right)^{\frac{1}{\rho}}\\ &+\frac{(\mu-\eta+x_1-x_2)}{2^{\frac{1}{\rho}+1}}\left(\left|\Psi''(\lambda+\eta-x_1)\right|^{\rho}+\left|\Psi''(\lambda+\mu-x_2)\right|^{\rho}\right)^{\frac{1}{\rho}}.\end{aligned}\tag{17}$$

Proof. Taking into account Lemma 2.1 and power mean inequality, we can write

$$\left|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\left(\frac{1}{(\lambda-x_1)^2}\int_\eta^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa\right.\right.$$

$$
\begin{aligned}
&+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\Bigg)\Bigg| \\
\leq\ & \frac{(\lambda-x_1)}{2}\left(\int_0^1 \xi^2 d\xi\right)^{1-\frac{1}{\rho}}\left(\int_0^1 \xi^2\left|\Psi''(\lambda+\eta-(\xi x_1+(1-\xi)\lambda))\right|^{\rho}d\xi\right)^{\frac{1}{\rho}} \\
&+\frac{(x_2-\lambda)}{2}\left(\int_0^1 \xi^2 d\xi\right)^{1-\frac{1}{\rho}}\left(\int_0^1 \xi^2\left|\Psi''(\lambda+\mu-(\xi x_2+(1-\xi)\lambda))\right|^{\rho}d\xi\right)^{\frac{1}{\rho}} \\
&+\frac{(\mu-\eta+x_1-x_2)}{2}\left(\int_0^1 1d\xi\right)^{1-\frac{1}{\rho}} \\
&\times\left(\int_0^1\left|\Psi''(\xi(\lambda+\mu-x_2)+(1-\xi)(\lambda+\eta-x_1))\right|^{\rho}d\xi\right)^{\frac{1}{\rho}}
\end{aligned}
$$

By means of Jensen-Mercer inequality, we have

$$
\begin{aligned}
&\Bigg|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\Bigg(\frac{1}{(\lambda-x_1)^2}\int_{\eta}^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa \\
&+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\Bigg)\Bigg| \\
\leq\ & \frac{(\lambda-x_1)}{2}\left(\frac{1}{3}\right)^{1-\frac{1}{\rho}} \\
&\times\left(\int_0^1 \xi^2\left(|\Psi''(\lambda)|^{\rho}+\left|\Psi''(\eta)\right|^{\rho}-\xi|\Psi''(x_1)|^{\rho}-(1-\xi)|\Psi''(\lambda)|^{\rho}\right)d\xi\right)^{\frac{1}{\rho}} \\
&+\frac{(x_2-\mu)}{2}\left(\frac{1}{3}\right)^{1-\frac{1}{\rho}} \\
&\times\left(\int_0^1 \xi^2\left(|\Psi''(\lambda)|^{\rho}+\left|\Psi''(\mu)\right|^{\rho}-\xi|\Psi''(x_2)|^{\rho}-(1-\xi)|\Psi''(\lambda)|^{\rho}\right)d\xi\right)^{\frac{1}{\rho}} \\
&+\frac{(\mu-\eta+x_1-x_2)}{2^{1+\frac{1}{\rho}}}\left(\left|\Psi''(\lambda+\eta-x_1)\right|^{\rho}+\left|\Psi''(\lambda+\mu-x_2)\right|^{\rho}\right)^{\frac{1}{\rho}}.
\end{aligned}
$$

By computing the above integrals, the statement is obtained. □

Corollary 2.11. *Under the same assumptions of Theorem 2.10 with $x_1=\eta$, $x_2=\mu$, then we obtain*

$$
\begin{aligned}
&\left|\frac{\Psi(\lambda)(\mu-\eta)}{(\lambda-\eta)(\mu-\lambda)}-\left(\frac{1}{(\lambda-\eta)^2}\int_{\eta}^{\lambda}\Psi(\kappa)\,d\kappa+\frac{1}{(\mu-\lambda)^2}\int_{\lambda}^{\mu}\Psi(\kappa)\,d\kappa\right)\right| \\
\leq\ & \frac{(\lambda-\eta)}{2.12^{\frac{1}{\rho}}}\left(\frac{1}{3}\right)^{1-\frac{1}{\rho}}\left(3|\Psi''(\lambda)|^{\rho}+\left|\Psi''(\eta)\right|^{\rho}\right)^{\frac{1}{\rho}} \\
&+\frac{(\mu-\lambda)}{2.12^{\frac{1}{\rho}}}\left(\frac{1}{3}\right)^{1-\frac{1}{\rho}}\left(3|\Psi''(\lambda)|^{\rho}+\left|\Psi''(\mu)\right|^{\rho}\right)^{\frac{1}{\rho}}.
\end{aligned} \tag{18}
$$

Corollary 2.12. *Let all the assumptions of Theorem 2.10 hold and $|\Psi''|\leq M$, then we get the following Ostrowski-Mercer type inequality*

$$
\left|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\left(\frac{1}{(\lambda-x_1)^2}\int_{\eta}^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa\right.\right.
$$

$$+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\Bigg)\Bigg| \tag{19}$$
$$\leq \frac{2M(x_1-x_2)+3M(\mu-\eta)}{6}.$$

Corollary 2.13. *If we take $x_1=\eta$, $x_2=\mu$ in Corollary 2.12, then it reduces the following inequality*

$$\left|\frac{\Psi(\lambda)(\mu-\eta)}{(\lambda-\eta)(\mu-\lambda)}-\left(\frac{1}{(\lambda-\eta)^2}\int_{\eta}^{\lambda}\Psi(\kappa)\,d\kappa+\frac{1}{(\mu-\lambda)^2}\int_{\lambda}^{\mu}\Psi(\kappa)\,d\kappa\right)\right|$$
$$\leq \frac{M((\lambda-\eta)+(\mu-\lambda))}{6}. \tag{20}$$

Theorem 2.14. *Suppose that $\Psi:[\eta,\mu]\to\mathbb{R}$ be a twice differentiable function on (η,μ) and $\Psi''\in L[\eta,\mu]$, for all $\lambda\in[x_1,x_2]$, $x_1,x_2\in[\eta,\mu]$, $\xi\in[0,1]$. If $|\Psi''|^{\rho}$ is concave on $[\eta,\mu]$ and $\rho>1$, $\frac{1}{\rho}+\frac{1}{\varphi}=1$ then the following inequality holds:*

$$\Bigg|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\Bigg(\frac{1}{(\lambda-x_1)^2}\int_{\eta}^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa$$
$$+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\Bigg)\Bigg| \tag{21}$$
$$\leq \frac{(\lambda-x_1)}{2}\left(\frac{1}{1+2\rho}\right)^{\frac{1}{\rho}}\left|\Psi''\left(\frac{2\eta+\lambda-x_1}{2}\right)\right|$$
$$+\frac{(x_2-\lambda)}{2}\left(\frac{1}{1+2\rho}\right)^{\frac{1}{\rho}}\left|\Psi''\left(\frac{2\mu+\lambda-x_2}{2}\right)\right|$$
$$+\frac{(\mu-\eta+x_1-x_2)}{2}\left|\Psi''\left(\frac{2\lambda+\eta+\mu-(x_1+x_2)}{2}\right)\right|.$$

Proof. By applying Hölder inequality and Lemma 2.1

$$\Bigg|\frac{\Psi(\lambda+\eta-x_1)}{(\lambda-x_1)}+\frac{\Psi(\lambda+\mu-x_2)}{(x_2-\lambda)}-\Bigg(\frac{1}{(\lambda-x_1)^2}\int_{\eta}^{\lambda+\eta-x_1}\Psi(\kappa)\,d\kappa$$
$$+\frac{1}{(x_2-\lambda)^2}\int_{\lambda+\mu-x_2}^{\mu}\Psi(\kappa)\,d\kappa\Bigg)\Bigg| \tag{22}$$
$$\leq \frac{(\lambda-x_1)}{2}\left(\int_0^1\xi^{2\rho}d\xi\right)^{\frac{1}{\rho}}\left(\int_0^1\left|\Psi''(\lambda+\eta-(\xi x_1+(1-\xi)\lambda))\right|^{\varphi}d\xi\right)^{\frac{1}{\varphi}}$$
$$+\frac{(x_2-\lambda)}{2}\left(\int_0^1\xi^{2\rho}d\xi\right)^{\frac{1}{\rho}}\left(\int_0^1\left|\Psi''(\lambda+\mu-(\xi x_2+(1-\xi)\lambda))\right|^{\varphi}d\xi\right)^{\frac{1}{\varphi}}$$
$$+\frac{(\mu-\eta+x_1-x_2)}{2}\left(\int_0^1 1d\xi\right)^{\frac{1}{\rho}}$$
$$\times\left(\int_0^1\left|\Psi''(\xi(\lambda+\mu-x_2)+(1-\xi)(\lambda+\eta-x_1))\right|^{\varphi}d\xi\right)^{\frac{1}{\varphi}}$$

By using concavity of $|\Psi''|^{\rho}$, we obtain

$$\int_0^1\left|\Psi''(\lambda+\eta-(\xi x_1+(1-\xi)\lambda))\right|^{\varphi}d\xi \leq \left|\Psi''\left(\frac{2\eta+\lambda-x_1}{2}\right)\right|^{\varphi} \tag{23}$$

$$\int_0^1 \left|\Psi''\left(\lambda+\mu-\left(\xi x_2+(1-\xi)\lambda\right)\right)\right|^{\varphi} d\xi \quad \leq \quad \left|\Psi''\left(\frac{2\mu+\lambda-x_2}{2}\right)\right|^{\varphi}$$

and

$$\begin{aligned}&\int_0^1 \left|\Psi''\left(\xi\left(\lambda+\mu-x_2\right)+(1-\xi)\left(\lambda+\eta-x_1\right)\right)\right|^{\varphi} d\xi \\ \leq\ & \left|\Psi''\left(\frac{2\lambda+\eta+\mu-(x_1+x_2)}{2}\right)\right|^{\varphi}\end{aligned}$$

Substituting 23 into 22, the proof is completed. □

Corollary 2.15. *Under the same assumptions of Theorem 2.14 with $x_1=\eta$, $x_2=\mu$, then we have*

$$\begin{aligned}&\left|\frac{\Psi(\lambda)(\mu-\eta)}{(\lambda-\eta)(\mu-\lambda)}\right. \\ &\left.-\left(\frac{1}{(\lambda-\eta)^2}\int_\eta^\lambda \Psi(\kappa)\,d\kappa+\frac{1}{(\mu-\lambda)^2}\int_\lambda^\mu \Psi(\kappa)\,d\kappa\right)\right| \\ \leq\ & \frac{(\lambda-\eta)}{2}\left(\frac{1}{1+2\rho}\right)^{\frac{1}{\rho}}\left|\Psi''\left(\frac{\eta+\lambda}{2}\right)\right| \\ &+\frac{(\mu-\lambda)}{2}\left(\frac{1}{1+2\rho}\right)^{\frac{1}{\rho}}\left|\Psi''\left(\frac{\mu+\lambda}{2}\right)\right|.\end{aligned} \tag{24}$$

3. Conclusion

New upper bounds for the Ostrowski-Mercer type inequalities are obtained by using the Jensen-Mercer inequalities for differentiable functions. To prove the main findings, a new lemma is established and Hölder's inequality and power-mean inequality are used. Using the appropriate fractional integral operators or different order derivative functions, these methods can be followed to develop further results for other classes related to convex functions. The results obtained are useful for different type inequalities and can be an inspiration for researchers working on this subject.

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