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Parseval-Goldstein type theorems for the Kontorovich-Lebedev transform and the Mehler-Fock transform of general order

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Abstract. This paper is dedicated to the pursuit of Parseval-Goldstein type relations for the Kontorovich-Lebedev transform and the Mehler-Fock transform of general order.

1. Introduction and Preliminaries

The Kontorovich-Lebedev transform, introduced by M. I. Kontorovich and N. N. Lebedev in 1938, is a specific form of index integral transformation. It employs the Macdonald function, a modified Bessel function of the second kind, with a purely imaginary index as its kernel. This transform has been effectively applied to address boundary value problems in fields such as diffraction theory and electrodynamics, as evidenced by references [9, 10]. Multiple authors have presented a range of definitions for the Kontorovich-Lebedev transform. For a comprehensive understanding, one refer to the following references [6, 7, 13, 15, 16, 19–21, 24–26]. In [18], there is a comprehensive and systematic examination of various general families of integral transforms with kernels involving Bessel, Whittaker and other special functions.

The Kontorovich-Lebedev transform of a suitable complex-valued function f defined in $\mathbb{R}_+$ is given by [21]

$$(\mathcal{F} f)(\tau) = \int_0^\infty f(x) K_{i\tau}(x) dx, \ \tau > 0, \tag{1.1}$$

where $K_{i\tau}(x)$ is the modified Bessel function of the third kind (or the Macdonald function) defined by [2, Chapter 7]

$$K_{i\tau}(x) = \frac{\pi}{2\sin(i\tau\pi)} \left[I_{-i\tau}(x) - I_{i\tau}(x) \right],$$

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where

$$I_{i\tau}(x) = \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^{i\tau+2n}}{n!\Gamma(i\tau+n+1)}.$$

We first recall from [2, p. 82, Entry 21] that

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh u}\cos(\tau u)du,\ x>0,\ \tau>0,$$

which readily yields the following inequality

$$|K_{i\tau}(x)| \leq \int_0^\infty e^{-x\cosh u}du = K_0(x). \tag{1.2}$$

On the other hand, from [27, p. 172, Entry 3] and [27, p. 173, Entry 4] we have that

$$K_0(x) \sim -\ln\left(\frac{Cx}{2}\right) \quad \text{as} \quad x \to +0$$

$$K_0(x) \sim \frac{\sqrt{\pi}}{2}\frac{e^{-x}}{\sqrt{x}} \quad \text{as} \quad x \to +\infty,$$

where $C = e^\gamma$ and $\gamma = 0.5772\cdots$ is the Euler-Mascheroni constant. Observe that K_0 is unbounded in $(0,\infty)$. Set

$$(\mathcal{F}^* g)(x) = \int_0^\infty g(\tau)K_{i\tau}(x)d\tau,\ x>0. \tag{1.3}$$

For $g \in L^1(\mathbb{R}_+)$ the integral (1.3) converges for each $x > 0$ since $|(\mathcal{F}^*g)(x)| \leq \int_0^\infty |g(\tau)|d\tau \cdot K_0(x) < \infty$.
Now observe that for each $\tau > 0$

$$|(\mathcal{F}f)(\tau)| \leq \int_0^\infty |f(x)|K_0(x)dx < \infty,$$

then for each $f \in L^1(\mathbb{R}_+, K_0(x)dx)$ the integral (1.1) converges for each $\tau > 0$.
The concept of the Mehler-Fock transform originated from the pioneering work of F. G. Mehler [12] and V. A. Fock [5]. It was subsequently developed into an independent integral transform, and its applications extended to address a wide range of mathematical physics problems. Numerous authors have conducted comprehensive investigations of this transform. For detailed study, one can refer to the following references, amongst others [6–8, 11, 16, 17, 22, 23].

We also consider the Mehler-Fock transform of general order $\mu \in \mathbb{C}$ of a suitable complex-valued function f defined in $\mathbb{R}_+$ is given by [21]

$$(\mathcal{F}_\mu f)(\tau) = \int_0^\infty f(x)P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x)dx,\ \tau>0,\ \Re(\mu) > \frac{-1}{2}, \tag{1.4}$$

where $P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x)$ is the associated Legendre function of the first kind [1, Chapter 3].
We next recall the following integral representation [1, p. 156, Entry 7].

$$P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x) = \frac{\sinh x}{2^\mu\sqrt{\pi}\Gamma(\mu+\frac{1}{2})}\int_0^\pi (\cosh x + \sinh x\cos u)^{-\frac{1}{2}+i\tau-\mu}(\sin u)^{2\mu}du,\ x>0,\ \tau>0,\ \Re(\mu) > \frac{-1}{2}.$$

It follows that

$$\begin{aligned} |P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x)| &\leq \frac{\sinh x}{2^{\Re(\mu)}\sqrt{\pi}\left|\Gamma(\mu+\frac{1}{2})\right|}\int_0^\pi (\cosh x + \sinh x\cos u)^{-\frac{1}{2}-\Re(\mu)}(\sin u)^{2\Re(\mu)}du, \\ &= \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|}P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x),\ x>0,\ \tau>0,\ \Re(\mu) > \frac{-1}{2}. \end{aligned} \tag{1.5}$$

where [14, p. 171, Entry 12.08]

$$P^{-\mu}_{-\frac{1}{2}}(\cosh x) \sim \frac{x^{\mu}}{2^{\mu}\Gamma(\mu+1)} \quad \text{as} \quad x \to +0$$
$$P^{-\mu}_{-\frac{1}{2}}(\cosh x) \sim \frac{2}{\sqrt{\pi}\Gamma(\mu+\frac{1}{2})} x e^{\frac{-x}{2}} \quad \text{as} \quad x \to +\infty,$$

observe that for $\Re(\mu) \geq 0$, $P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)$ is bounded and so

$$\begin{aligned} |P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x)| &\leq \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|} P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x) \\ &\leq M_{\mu}, \text{ for some } M_{\mu} > 0. \end{aligned} \tag{1.6}$$

Set

$$(\mathcal{F}^{*}_{\mu} g)(x) = \int_0^{\infty} g(\tau) P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x) dx, \ x > 0, \ \Re(\mu) > \frac{-1}{2}. \tag{1.7}$$

For $g \in L^1(\mathbb{R}_+)$ the integral (1.7) converges for each $x > 0$ since $\left|\left(\mathcal{F}^{*}_{\mu} g\right)(x)\right| \leq \int_0^{\infty} |g(\tau)| d\tau \cdot P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x) < \infty$. Now, we consider the space $L^1\left(\mathbb{R}_+, P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)dx\right)$. Observe that for $f \in L^1\left(\mathbb{R}_+, P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)dx\right)$ the integral (1.4) converges for each $\tau > 0$ since $\left|(\mathcal{F}_{\mu} f)(\tau)\right| \leq \int_0^{\infty} P^{-\Re(\mu)}_{-\frac{1}{2}+i\tau}(\cosh x)|f(x)|dx < \infty$. Also observe that for $\Re(\mu) \geq 0$: $L^1(\mathbb{R}_+) \subseteq L^1\left(\mathbb{R}_+, P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)dx\right)$ since $P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)$ is bounded. Furthermore it is a proper subset as the function $f(x) = x^r \notin L^1(\mathbb{R}_+)$ for any r. However $f(x) = x^r \in L^1\left(\mathbb{R}_+, P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)dx\right)$ for $r > -1 - \Re(\mu)$.
The $C^k_c(\mathbb{R}_+)$, $k \in \mathbb{N}$, denotes as it is usual the space of compactly supported functions on $\mathbb{R}_+$ which are k-times differentiable with continuity.
The Parseval-Goldstein relations for integral transforms establish a connection between the norm in the original domain and its transformed counterpart [3, 4]. The present article deals with the study of Parseval-Goldstein type relations for the Kontorovich-Lebedev transform and the Mehler-Fock transform of general order.

The content of this article is as follows: Section 1 is concerned with definitions and useful results which are used in the entire sequel. Section 2 deals with continuity features over Lebesgue spaces of the Kontorovich-Lebedev transform and its adjoint transform and Parseval-Goldstein type relations for the Kontorovich-Lebedev transform. Section 3 focuses on exploring continuity properties within Lebesgue spaces of both the Mehler-Fock transform of general order and its adjoint transform, as well as delving into Parseval-Goldstein type relationships associated with the Mehler-Fock transform of general order. Section 4 gives concluding notes.

2. Parseval-Goldstein type relations for the Kontorovich-Lebedev transform

2.1. The $\mathcal{F}$ transform over the spaces $L^1(\mathbb{R}_+, K_0(x)dx)$

Proposition 2.1. *The Kontorovich-Lebedev transform $\mathcal{F}$ given by* (1.1) *is a bounded linear operator from $L^1(\mathbb{R}_+, K_0(x)dx)$ into $L^{\infty}(\mathbb{R}_+)$. If $f \in L^1(\mathbb{R}_+, K_0(x)dx)$ then*

$$\|\mathcal{F} f\|_{L^{\infty}(\mathbb{R}_+)} \leq \|f\|_{L^1(\mathbb{R}_+, K_0(x)dx)},$$

and $\mathcal{F} f$ is a continuous function on $\mathbb{R}_+$. Moreover, the Kontorovich-Lebedev transform $\mathcal{F}$ is a continuous map from $L^1(\mathbb{R}_+, K_0(x)dx)$ to the Banach space of bounded continuous functions on $\mathbb{R}_+$.

Proof. Let $\tau_0 > 0$ be arbitrary. Since the map $\tau \to K_{i\tau}(x)$ is continuous for each fixed $x > 0$, we have

$$K_{i\tau}(x) \to K_{i\tau_0}(x) \text{ as } \tau \to \tau_0.$$

Further, we have that $\left|K_{i\tau}(x) - K_{i\tau_0}(x)\right| |f(x)|$ is dominated by the integrable function $2K_0(x)|f(x)|$. Therefore, by using dominated convergence theorem, we get

$$\left|(\mathcal{F} f)(\tau) - (\mathcal{F} f)(\tau_0)\right| \leq \int_0^\infty \left|K_{i\tau}(x) - K_{i\tau_0}(x)\right| |f(x)|dx \to 0, \text{ as } \tau \to \tau_0.$$

Thus, $\mathcal{F} f$ is a continuous function on $\mathbb{R}_+$.
Since for each $\tau > 0$

$$\begin{aligned} \left|(\mathcal{F} f)(\tau)\right| &\leq \int_0^\infty |K_{i\tau}(x)|\,|f(x)|dx \\ &\leq \int_0^\infty K_0(x)|f(x)|dx = \|f\|_{L^1(\mathbb{R}_+, K_0(x)dx)}, \end{aligned} \tag{2.1}$$

one has that $\mathcal{F} f$ is a bounded function.
The linearity of the integral operator implies that the $\mathcal{F}$ transform is linear. Also from (2.1) we get that $\|\mathcal{F} f\|_{L^\infty(\mathbb{R}_+)} \leq \|f\|_{L^1(\mathbb{R}_+, K_0(x)dx)}$ and hence $\mathcal{F} : L^1(\mathbb{R}_+, K_0(x)dx) \to L^\infty(\mathbb{R}_+)$ is a continuous linear map. □

Proposition 2.2. *The Kontorovich-Lebedev transform given by* (1.1) *is a bounded linear operator from* $L^1(\mathbb{R}_+, K_0(x)dx)$ *into* $L^q(\mathbb{R}_+, w(x)dx)$, $0 < q < \infty$, *when* $w > 0$ *a.e. on* $\mathbb{R}_+$ *and* $\int_0^\infty w(x)dx < \infty$.

Proof. Observe that from (2.1) for each $\tau > 0$

$$\begin{aligned} |(\mathcal{F} f)(\tau)| &\leq \int_0^\infty |K_{i\tau}(x)|\,|f(x)|dx \\ &\leq \int_0^\infty K_0(x)|f(x)|dx = \|f\|_{L^1(\mathbb{R}_+, K_0(x)dx)}. \end{aligned}$$

Then, for $0 < q < \infty$, one has

$$\left(\int_0^\infty |(\mathcal{F} f)(x)|^q w(x)dx\right)^{\frac{1}{q}} \leq \|f\|_{L^1(\mathbb{R}_+, K_0(x)dx)} \left(\int_0^\infty w(x)dx\right)^{\frac{1}{q}} < \infty.$$

□

Remark 2.3. *Examples of weights* w *for Proposition 2.2 are:*

(i) $w(x) = (1 + x)^r$, *for* $r < -1$.
(ii) $w(x) = e^{rx}$, *for* $r < 0$.

2.2. *The transform* $\mathcal{F}^*$ *over the spaces* $L^1(\mathbb{R}_+)$

Proposition 2.4. *The* $\mathcal{F}^*$ *given by* (1.3) *is a bounded linear operator from* $L^1(\mathbb{R}_+)$ *into* $L^q(\mathbb{R}_+, w(x)dx)$, $0 < q < \infty$, *when* $w > 0$ *a.e. on* $\mathbb{R}_+$ *and* $K_0(x) \in L^q(\mathbb{R}_+, w(x)dx)$.

Proof. Observe that for each $x > 0$

$$\begin{aligned} \left|(\mathcal{F}^* f)(x)\right| &\leq \int_0^\infty |K_{i\tau}(x)|\,|f(\tau)|d\tau \\ &\leq K_0(x) \int_0^\infty |f(\tau)|d\tau. \end{aligned}$$

Then, for $0 < q < \infty$, one has

$$\left(\int_0^\infty \left|(\mathcal{F}^* f)(x)\right|^q w(x)dx\right)^{\frac{1}{q}} \quad \leq \quad \|f\|_{L^1(\mathbb{R}_+)}\left(\int_0^\infty (K_0(x))^q\, w(x)dx\right)^{\frac{1}{q}} < \infty.$$

□

Remark 2.5. *Examples of weights w for Proposition 2.4 are:*

(i) $w(x) = x^r$, *for* $r > -1$.
(ii) $w(x) = (1 + x)^r$, *for all* r.
(iii) $w(x) = e^{rx}$, *for* $r < q$; *and* $r = q$ *being* $q > 2$.

2.3. Parseval-Goldstein type theorems

Theorem 2.6. *If $f \in L^1(\mathbb{R}_+, K_0(x)dx)$ and $g \in L^1(\mathbb{R}_+)$, then the following Parseval-Goldstein type relation holds*

$$\int_0^\infty (\mathcal{F} f)(x)g(x)dx = \int_0^\infty f(x)(\mathcal{F}^* g)(x)dx. \tag{2.2}$$

Proof. In fact, for each $\tau > 0$

$$|(\mathcal{F} f)(\tau)| \leq \|f\|_{L^1(\mathbb{R}_+,K_0(x)dx)}.$$

Therefore,

$$\int_0^\infty |(\mathcal{F} f)(\tau)||g(\tau)|d\tau \leq \|f\|_{L^1(\mathbb{R}_+,K_0(x)dx)}\|g\|_{L^1(\mathbb{R}_+)}.$$

Also, for each $x > 0$

$$|(\mathcal{F}^* g)(x)| \leq \int_0^\infty |K_{i\tau}(x)|\,|g(\tau)|d\tau \leq K_0(x)\|g\|_{L^1(\mathbb{R}_+)}.$$

Then

$$\begin{aligned}\int_0^\infty |f(x)||(\mathcal{F}^* g)(x)|dx &\leq \int_0^\infty |f(x)|K_0(x)dx\|g\|_{L^1(\mathbb{R}_+)}\\ &= \|f\|_{L^1(\mathbb{R}_+,K_0(x)dx)}\|g\|_{L^1(\mathbb{R}_+)}.\end{aligned}$$

Thus, by using Fubini's theorem one obtains the relation (2.2). □

Remark 2.7. *From this result the transform $\mathcal{F}^*$ becomes the adjoint of the Kontorovich-Lebedev transform $\mathcal{F}$ over $L^1(\mathbb{R}_+, K_0(x)dx)$.*

Denote

$$A_x = xD_xxD_x - x^2 = x^2D_x^2 + xD_x - x^2. \tag{2.3}$$

and

$$A_x' = D_xxD_xx - x^2 = x^2D_x^2 + 3xD_x + (1 - x^2). \tag{2.4}$$

One has, for $k \in \mathbb{N}$,

$$A_x^k(K_{i\tau}(x)) = (-1)^k\tau^{2k}K_{i\tau}(x), \tag{2.5}$$

and so, for $f \in C_c^{2k}(\mathbb{R}_+)$, $k \in \mathbb{N}$,

$$\left(\mathcal{F}\left({A_x'}^k f\right)\right)(\tau) = (-1)^k\tau^{2k}(\mathcal{F} f)(\tau),\ \tau > 0. \tag{2.6}$$

Theorem 2.8. *If $f \in C_c^{2k}(\mathbb{R}_+)$, $k \in \mathbb{N}$, and $g \in L^1(\mathbb{R}_+)$, then the following Parseval-Goldstein type relation holds*

$$(-1)^k \int_0^\infty (\mathcal{F} f)(x) g(x) x^{2k} dx = \int_0^\infty (A_x^{'k} f)(x)(\mathcal{F}^* g)(x) dx. \tag{2.7}$$

Proof. For $f \in C_c^2(\mathbb{R}_+)$, then f and $A_x' f \in L^1(\mathbb{R}_+, K_0(x)dx)$. Also for $\tau > 0$,

$$\begin{aligned} \left(\mathcal{F}\left(A_x' f\right)\right)(\tau) &= \int_0^\infty (A_x' f)(x) K_{i\tau}(x) dx \\ &= \int_0^\infty f(x) \left(A_x(K_{i\tau}(x))\right)(x) dx \\ &= -\tau^2 \int_0^\infty f(x) K_{i\tau}(x) dx \\ &= -\tau^2 (\mathcal{F} f)(\tau). \end{aligned}$$

Then for $f \in C_c^2(\mathbb{R}_+)$ and $g \in L^1(\mathbb{R}_+)$ and using Theorem 2.6 above, one has

$$\int_0^\infty \left(\mathcal{F}\left(A_x' f\right)\right)(\tau) g(\tau) d\tau = \int_0^\infty (A_x' f)(x)(\mathcal{F}^* g)(x) dx.$$

Thus

$$-\int_0^\infty \tau^2 (\mathcal{F} f)(\tau) g(\tau) d\tau = \int_0^\infty (A_x' f)(x)(\mathcal{F}^* g)(x) dx.$$

Also, in general, for $f \in C_c^{2k}(\mathbb{R}_+)$, $k \in \mathbb{N}$, and $g \in L^1(\mathbb{R}_+)$ one obtains

$$(-1)^k \int_0^\infty (\mathcal{F} f)(\tau) g(\tau) \tau^{2k} d\tau = \int_0^\infty (A_x^{'k} f)(x)(\mathcal{F}^* g)(x) dx.$$

□

3. Parseval-Goldstein type relations for the Mehler-Fock transform of general order

3.1. The $\mathcal{F}_\mu$ transform over the spaces $L^1(\mathbb{R}_+, P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)dx)$

Proposition 3.1. *Set $\Re(\mu) > \frac{-1}{2}$. The Mehler-Fock transform $\mathcal{F}_\mu$ given by* (1.4) *is a bounded linear operator from $L^1\left(\mathbb{R}_+, P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)dx\right)$ into $L^\infty(\mathbb{R}_+)$. If $f \in L^1\left(\mathbb{R}_+, P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)dx\right)$ then*

$$\|\mathcal{F}_\mu f\|_{L^\infty(\mathbb{R}_+)} \leq \frac{\Gamma(\Re(\mu) + \frac{1}{2})}{\left|\Gamma(\mu + \frac{1}{2})\right|} \|f\|_{L^1\left(\mathbb{R}_+, P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)dx\right)}$$

and $\mathcal{F}_\mu f$ is a continuous function on $\mathbb{R}_+$. Moreover, the Mehler-Fock transform $\mathcal{F}_\mu$ is a continuous map from $L^1\left(\mathbb{R}_+, P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)dx\right)$ to the Banach space of bounded continuous functions on $\mathbb{R}_+$.

Proof. Let $\tau_0 > 0$ be arbitrary. Since the map $\tau \to P_{-\frac{1}{2}+i\tau}^{-\mu}(\cosh x)$ is continuous for each fixed $x > 0$, we have

$$P_{-\frac{1}{2}+i\tau}^{-\mu}(\cosh x) \to P_{-\frac{1}{2}+i\tau_0}^{-\mu}(\cosh x) \text{ as } \tau \to \tau_0.$$

Further, we have that $\left|P_{-\frac{1}{2}+i\tau}^{-\mu}(\cosh x) - P_{-\frac{1}{2}+i\tau_0}^{-\mu}(\cosh x)\right| |f(x)|$ is dominated by the integrable function

$$2\frac{\Gamma(\Re(\mu) + \frac{1}{2})}{\left|\Gamma(\mu + \frac{1}{2})\right|} P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)|f(x)|.$$

Therefore, by using dominated convergence theorem, we get

$$\left|(\mathcal{F}_\mu f)(\tau) - (\mathcal{F}_\mu f)(\tau_0)\right| \leq \int_0^\infty \left|P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x) - P^{-\mu}_{-\frac{1}{2}+i\tau_0}(\cosh x)\right| |f(x)|dx \to 0,\ as\ \tau \to \tau_0.$$

Thus, $\mathcal{F}_\mu f$ is a continuous function on $\mathbb{R}_+$.
Since for each $\tau > 0$

$$\begin{aligned} \left|(\mathcal{F}_\mu f)(\tau)\right| &\leq \int_0^\infty \left|P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x)\right| |f(x)|dx \\ &\leq \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|} \int_0^\infty P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)|f(x)|dx \\ &= \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|} \|f\|_{L^1\left(\mathbb{R}_+, P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)dx\right)}, \end{aligned} \tag{3.1}$$

one has that $\mathcal{F}_\mu f$ is a bounded function.
The linearity of the integral operator implies that the $\mathcal{F}_\mu$ transform is linear. Also from (3.1) we get that $\|\mathcal{F}_\mu f\|_{L^\infty(\mathbb{R}_+)} \leq \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{|\Gamma(\mu+\frac{1}{2})|}\|f\|_{L^1\left(\mathbb{R}_+, P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)dx\right)}$ and hence $\mathcal{F}_\mu : L^1\left(\mathbb{R}_+, P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)dx\right) \to L^\infty(\mathbb{R}_+)$ is a continuous linear map. □

Proposition 3.2. *Set* $\Re(\mu) > \frac{-1}{2}$. *The Mehler-Fock transform* $\mathcal{F}_\mu$ *given by* (1.4) *is a bounded linear operator from* $L^1\left(\mathbb{R}_+, P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)dx\right)$ *into* $L^q(\mathbb{R}_+, w(x)dx)$, $0 < q < \infty$, *when* $w > 0$ *a.e. on* $\mathbb{R}_+$ *and* $\int_0^\infty w(x)dx < \infty$.

Proof. Observe that from (3.1) for each $\tau > 0$

$$\begin{aligned} |(\mathcal{F}_\mu f)(\tau)| &\leq \int_0^\infty \left|P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x)\right| |f(x)|dx \\ &\leq \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|} \int_0^\infty P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)|f(x)|dx \\ &= \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|} \|f\|_{L^1\left(\mathbb{R}_+, P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)dx\right)}. \end{aligned}$$

Then, for $0 < q < \infty$, one has

$$\left(\int_0^\infty |(\mathcal{F}_\mu f)(x)|^q w(x)dx\right)^{\frac{1}{q}} \leq \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|} \|f\|_{L^1\left(\mathbb{R}_+, P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)dx\right)} \left(\int_0^\infty w(x)dx\right)^{\frac{1}{q}} < \infty.$$

□

Remark 3.3. *Examples of weights* w *for Proposition 3.2 are:*

(i) $w(x) = (1+x)^r$, *for* $r < -1$.
(ii) $w(x) = e^{rx}$, *for* $r < 0$.

3.2. The transform $\mathcal{F}_\mu^*$ over the spaces $L^1(\mathbb{R}_+)$

Proposition 3.4. *Set $\Re(\mu) \geq 0$. The $\mathcal{F}_\mu^*$ given by (1.7) is a bounded linear operator from $L^1(\mathbb{R}_+)$ into $L^\infty(\mathbb{R}_+)$. If $f \in L^1(\mathbb{R}_+)$ then*

$$\left\|\mathcal{F}_\mu^* f\right\|_{L^\infty(\mathbb{R}_+)} \leq M_\mu \|f\|_{L^1(\mathbb{R}_+)}, \text{ for some } M_\mu > 0,$$

and $\mathcal{F}_\mu^* f$ is a continuous function on $\mathbb{R}_+$. Moreover, the $\mathcal{F}_\mu^*$ is a continuous map from $L^1(\mathbb{R}_+)$ to the Banach space of bounded continuous functions on $\mathbb{R}_+$.

Proof. Let $x_0 > 0$ be arbitrary. Since the map $x \to P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x)$ is continuous for each fixed $\tau > 0$, we have

$$P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x) \to P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x_0) \text{ as } x \to x_0.$$

Further, from (1.6) and being $\Re(\mu) \geq 0$ we have that:
$\left|P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x) - P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x_0)\right| |f(\tau)|$ is dominated by the integrable function $2M_\mu|f(\tau)|$, for some $M_\mu > 0$. Therefore, by using dominated convergence theorem, we get

$$\left|\left(\mathcal{F}_\mu^* f\right)(x) - \left(\mathcal{F}_\mu^* f\right)(x_0)\right| \leq \int_0^\infty \left|P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x) - P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x_0)\right| |f(\tau)| d\tau \to 0, \text{ as } x \to x_0.$$

Thus, $\mathcal{F}_\mu^* f$ is a continuous function on $\mathbb{R}_+$.
Since for each $x > 0$

$$\begin{aligned} \left|\left(\mathcal{F}_\mu^* f\right)(x)\right| &\leq \int_0^\infty \left|P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x)\right| |f(\tau)| d\tau \\ &\leq M_\mu \int_0^\infty |f(\tau)| d\tau \\ &\leq M_\mu \|f\|_{L^1(\mathbb{R}_+)}, \end{aligned} \tag{3.2}$$

one has that $\mathcal{F}_\mu^* f$ is a bounded function.
The linearity of the integral operator implies that the $\mathcal{F}_\mu^*$ is linear. Also from (3.2) we get that $\left\|\mathcal{F}_\mu^* f\right\|_{L^\infty(\mathbb{R}_+)} \leq M_\mu \|f\|_{L^1(\mathbb{R}_+)}$ and hence $\mathcal{F}_\mu^* : L^1(\mathbb{R}_+) \to L^\infty(\mathbb{R}_+)$ is a continuous linear map. □

Proposition 3.5. *Set $\Re(\mu) > \frac{-1}{2}$. The $\mathcal{F}_\mu^*$ given by (1.7) is a bounded linear operator from $L^1(\mathbb{R}_+)$ into $L^q(\mathbb{R}_+, w(x)dx)$, $0 < q < \infty$, when $w > 0$ a.e. on $\mathbb{R}_+$ and $P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x) \in L^q(\mathbb{R}_+, w(x)dx)$.*

Proof. Observe that for each $x > 0$

$$\begin{aligned} \left|\left(\mathcal{F}_\mu^* f\right)(x)\right| &\leq \int_0^\infty \left|P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x)\right| |f(\tau)| d\tau \\ &\leq \frac{\Gamma(\Re(\mu) + \frac{1}{2})}{\left|\Gamma(\mu + \frac{1}{2})\right|} P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x) \int_0^\infty |f(\tau)| d\tau. \end{aligned}$$

Then, for $0 < q < \infty$, one has

$$\left(\int_0^\infty \left|(\mathcal{F}_\mu^* f)(x)\right|^q w(x)dx\right)^{\frac{1}{q}} \leq \frac{\Gamma(\Re(\mu) + \frac{1}{2})}{\left|\Gamma(\mu + \frac{1}{2})\right|} \|f\|_{L^1(\mathbb{R}_+)} \left(\int_0^\infty \left(P^{-\Re(\mu)}_{-\frac{1}{2}}(\cosh x)\right)^q w(x)dx\right)^{\frac{1}{q}} < \infty.$$

□

Remark 3.6. *Examples of weights w for Proposition 3.5 are:*

(i) $w(x) = x^r$, *for* $r > -1 - q\Re(\mu)$.

(ii) $w(x) = (1+x)^r$, *for all* r *and* $\Re(\mu) > \frac{-1}{q}$.

(iii) $w(x) = e^{rx}$, *for* $r < \frac{q}{2}$ *and* $\Re(\mu) > \frac{-1}{q}$.

3.3. Parseval-Goldstein type theorems

Theorem 3.7. *Set* $\Re(\mu) > \frac{-1}{2}$. *If* $f \in L^1\left(\mathbb{R}_+, P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)dx\right)$ *and* $g \in L^1(\mathbb{R}_+)$, *then the following Parseval-Goldstein type relation holds*

$$\int_0^\infty (\mathcal{F}_\mu f)(x)g(x)dx = \int_0^\infty f(x)(\mathcal{F}_\mu^* g)(x)dx. \tag{3.3}$$

Proof. In fact, for each $\tau > 0$

$$|(\mathcal{F}_\mu f)(\tau)| \le \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|}\|f\|_{L^1\left(\mathbb{R}_+, P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)dx\right)}.$$

Therefore,

$$\int_0^\infty |(\mathcal{F}_\mu f)(\tau)||g(\tau)|d\tau \le \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|}\|f\|_{L^1\left(\mathbb{R}_+, P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)dx\right)}\|g\|_{L^1(\mathbb{R}_+)}.$$

Also, for each $x > 0$

$$|(\mathcal{F}_\mu^* g)(x)| \le \int_0^\infty \left|P_{-\frac{1}{2}+i\tau}^{-\mu}(\cosh x)\right||g(\tau)|d\tau \le \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|}P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)\|g\|_{L^1(\mathbb{R}_+)}.$$

Then

$$\begin{aligned}\int_0^\infty |f(x)||(\mathcal{F}_\mu^* g)(x)|dx &\le \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|}\int_0^\infty |f(x)|P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)dx\|g\|_{L^1(\mathbb{R}_+)} \\ &= \frac{\Gamma(\Re(\mu)+\frac{1}{2})}{\left|\Gamma(\mu+\frac{1}{2})\right|}\|f\|_{L^1\left(\mathbb{R}_+, P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)dx\right)}\|g\|_{L^1(\mathbb{R}_+)}.\end{aligned}$$

Thus, by using Fubini's theorem one obtains the relation (3.3). □

Remark 3.8. *From this result the transform* $\mathcal{F}_\mu^*$ *becomes the adjoint of the Mehler-Fock transform* $\mathcal{F}_\mu$ *over* $L^1\left(\mathbb{R}_+, P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x)dx\right)$.

Denote

$$B_x = (\sinh x)^{-\mu-1}D_x(\sinh x)^{2\mu+1}D_x(\sinh x)^{-\mu}. \tag{3.4}$$

and

$$B_x^{'} = (\sinh x)^{-\mu}D_x(\sinh x)^{2\mu+1}D_x(\sinh x)^{-\mu-1}. \tag{3.5}$$

One has, for $k \in \mathbb{N}$,

$$B_x^k\left(P_{-\frac{1}{2}+i\tau}^{-\mu}(\cosh x)\right) = (-1)^k\left(\left(\mu+\frac{1}{2}\right)^2+\tau^2\right)^k P_{-\frac{1}{2}+i\tau}^{-\mu}(\cosh x), \tag{3.6}$$

and so, for $f \in C_c^{2k}(\mathbb{R}_+)$, $k \in \mathbb{N}$,

$$\left(\mathcal{F}_\mu\left({B'_x}^k f\right)\right)(\tau) = (-1)^k \left(\left(\mu + \frac{1}{2}\right)^2 + \tau^2\right)^k (\mathcal{F}_\mu f)(\tau), \ \tau > 0. \tag{3.7}$$

Theorem 3.9. *Set $\Re(\mu) > \frac{-1}{2}$. If $f \in C_c^{2k}(\mathbb{R}_+)$, $k \in \mathbb{N}$, and $g \in L^1(\mathbb{R}_+)$, then the following Parseval-Goldstein type relation holds*

$$(-1)^k \int_0^\infty (\mathcal{F}_\mu f)(x) g(x) \left(\left(\mu + \frac{1}{2}\right)^2 + x^2\right)^k dx = \int_0^\infty ({B'_x}^k f)(x)(\mathcal{F}_\mu^* g)(x) dx. \tag{3.8}$$

Proof. For $f \in C_c^2(\mathbb{R}_+)$, then f and $B'_x f \in L^1\left(\mathbb{R}_+, P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x) dx\right)$. Also for $\tau > 0$,

$$\begin{aligned}
\left(\mathcal{F}_\mu\left(B'_x f\right)\right)(\tau) &= \int_0^\infty (B'_x f)(x) P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x) dx \\
&= \int_0^\infty f(x) \left(B_x\left(P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x)\right)\right)(x) dx \\
&= -\left(\left(\mu + \frac{1}{2}\right)^2 + \tau^2\right) \int_0^\infty f(x) P^{-\mu}_{-\frac{1}{2}+i\tau}(\cosh x) dx \\
&= -\left(\left(\mu + \frac{1}{2}\right)^2 + \tau^2\right) (\mathcal{F}_\mu f)(\tau).
\end{aligned}$$

Then for $f \in C_c^2(\mathbb{R}_+)$ and $g \in L^1(\mathbb{R}_+)$ and using Theorem 3.7 above, one has

$$\int_0^\infty \left(\mathcal{F}_\mu\left(B'_x f\right)\right)(\tau) g(\tau) d\tau = \int_0^\infty (B'_x f)(x)(\mathcal{F}_\mu^* g)(x) dx.$$

Thus

$$-\int_0^\infty \left(\left(\mu + \frac{1}{2}\right)^2 + \tau^2\right) (\mathcal{F}_\mu f)(\tau) g(\tau) d\tau = \int_0^\infty (B'_x f)(x)(\mathcal{F}_\mu^* g)(x) dx.$$

Also, in general, for $f \in C_c^{2k}(\mathbb{R}_+)$, $k \in \mathbb{N}$, and $g \in L^1(\mathbb{R}_+)$ one obtains

$$(-1)^k \int_0^\infty (\mathcal{F}_\mu f)(\tau) g(\tau) \left(\left(\mu + \frac{1}{2}\right)^2 + \tau^2\right)^k d\tau = \int_0^\infty ({B'_x}^k f)(x)(\mathcal{F}_\mu^* g)(x) dx.$$

□

Remark 3.10. *For the case when $\Re(\mu) \geq 0$ and one considers the space $L^1(\mathbb{R}_+)$, which is a proper subset of $L^1\left(\mathbb{R}_+, P_{-\frac{1}{2}}^{-\Re(\mu)}(\cosh x) dx\right)$, the results of the Proposition 3.1, Proposition 3.4 and Theorem 3.7 agree with those results obtained in Theorem 3.1, Theorem 6.1 and Theorem 6.2 of [8], respectively.*

4. Conclusions

The present research article extensively investigates continuity properties over Lebesgue spaces for the Kontorovich-Lebedev transform and the Mehler-Fock transform of general order, including their adjoints. Emphasizing Parseval-Goldstein relations, the study reveals energy-preserving traits and inter-domain consistency. This significant analysis contributes to understanding the fundamental properties and applications of these integral transforms in mathematical analysis. The findings presented in this article open the door to the study of numerous other integral transforms.

Declarations

Note: The manuscript has no associated data.
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