Filomat 38:2 (2024), 505–521
https://doi.org/10.2298/FIL2402505I

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Higher order class of finite difference method for time-fractional Liouville-Caputo and space-Riesz fractional diffusion equation

Safar Irandoust-Pakchin[a], Somaiyeh Abdi-Mazraeh[a], Iraj Fahimi-Khalilabad[a]

[a] *Department of Applied Mathematics, Faculty of Mathematics, Statistics and Computer Sciences, University of Tabriz, Tabriz, Iran*

Abstract. In this paper, a class of finite difference method (FDM) is designed for solving the time-fractional Liouville-Caputo and space-Riesz fractional diffusion equation. For this purpose, the fractional linear barycentric rational interpolation method (FLBRI) is adopted to discretize the Liouville-Caputo derivative in the time direction as well as the second order revised fractional backward difference formulae 2 (RFBDF2) is employed in the space direction. The energy method is used to prove unconditionally stability and convergence analysis of the proposed method. Eventually, it is concluded that the proposed method is convergent with the order $O(h_t^{\gamma} + h_x^2)$, where h_t and h_x are the temporal and the spatial step sizes respectively, and $1 \leq \gamma \leq 7$ is the order of accuracy in the time direction. Finally, the presented numerical experiment confirms the theoretical analysis, the high accuracy and efficiency of the offered method.

1. Introduction

In recent decades, many researchers are enthusiastic about fractional differential equations (FDE)s due to its significant role in various fields of science such as control, electromagnetism, biophysics, physics, mathematics, mechanics, signal and image processing, blood flow phenomena and etc [1, 2, 4, 5, 14, 23–25, 29]. Since obtaining the analytical solution for the fractional differential equations often is difficult [7, 11, 25, 30–32, 39] and sometimes even impossible, therefore trying to earn the numerical solution for them can be valuable. One of the most important part of FDEs is fractional partial differential equations (FPDE)s. In this regard, there are several numerical schemes to solve them, for instance, the FDM [6, 22, 40–43, 47], finite element method [18, 20, 26], fractional order of linear multistep methods(LMM)s (recalled FLMMs)[21, 46], L1 method [19, 44] and so on [13, 33–37].

In the same direction, the fractional backward difference formulae (FBDF)s are the one of the most popular methods which have been utilized for solving FPDEs. The FBDFs have proper characteristics such as good stability properties, satisfactory accuracy, and smaller computational costs [16, 27, 28]. The main problems about these methods are unstability and less accuracy of them when the order of fractional differentiation lies in $(1, 2]$ [17, 22]. In order to overcome to these problems, Li et al. [8] have introduced RFBDF2 by constructing new generating functions.

2020 *Mathematics Subject Classification*. 65Nxx; 65N12; 65N22.

Keywords. Liouville-Caputo derivative; Riesz derivative; fractional diffusion equation; barycentric interpolation; the energy method.

Received: 19 January 2023; Revised: 28 June 2023; Accepted: 06 August 2023

Communicated by Hari M. Srivastava

The work of second author was supported by the University of Tabriz, Iran under Grant No. [436].

Email addresses: s.irandoust@tabrizu.ac.ir (Safar Irandoust-Pakchin), s.abdi_m@tabrizu.ac.ir (Somaiyeh Abdi-Mazraeh), i.fahimi@tabrizu.ac.ir (Iraj Fahimi-Khalilabad)

Furthermore, Irandoust-pakchin et al. [12] have just developed FLBRI by extending the linear barycentric rational interpolation (LBRI) scheme in the fractional form for solving FDEs. Recently, Fahimi-khalilabad et al. [9] have used FLBRI for Liouville-Caputo type in time direction and central difference method in space direction for solving the time-fractional sub-diffusion equation.

In this work, a higher order of FDM based on FLBRIs in the temporal-direction and RFBDF2 in the spatial-direction is developed for solving the following time-fractional Liouville-Caputo and Riesz-space fractional diffusion equations as

$$\begin{cases} {}_{LC}D^{\beta}_{0,t}v(x,t) = k_\alpha \frac{\partial^\alpha v(x,t)}{\partial |x|^\alpha} + f(x,t), \quad (x,t) \in (0,L)\times(0,T], \quad \beta\in(0,1), \quad \alpha \in (1,2], \\ v(x,0) = \varphi(x), \quad x\in[0,L], \\ v(0,t) = v(L,t) = 0, \quad t \in [0,T], \end{cases} \tag{1.1}$$

where $k_\alpha > 0$ is the diffusion coefficient and ${}_{C}D^{\beta}_{0,t}v(x,t)$ denotes the Liouville-Caputo fractional derivative with respect to t by [25] as follows

$$ {}_{LC}D^{\beta}_{0,t}v(x,t) = \frac{1}{\Gamma(1-\beta)}\int_0^t (t-\xi)^{-\beta}\frac{\partial v(x,\xi)}{\partial \xi}d\xi, \quad \beta\in(0,1). \tag{1.2}$$

Also, $\dfrac{\partial^\alpha v(x,t)}{\partial |x|^\alpha}$ denotes Riesz fractional derivative with respect to x which is defined below [14],

$$\frac{\partial^\alpha v(x,t)}{\partial |x|^\alpha} = \sigma_\alpha \left({}_{RL}D^{\alpha}_{a,x} + {}_{RL}D^{\alpha}_{x,b}\right) v(x,t), \tag{1.3}$$

where coefficient $\sigma_\alpha = -\dfrac{1}{2cos(\frac{\pi}{2}\alpha)}$. Furthermore, ${}_{RL}D^{\alpha}_{a,x}$ and ${}_{RL}D^{\alpha}_{x,b}$ are the left and right Riemann-Liouville derivatives of order α defined by [29]

$$ {}_{RL}D^{\alpha}_{a,x}v(x,t) = \begin{cases} \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_a^x (x-s)^{1-\alpha}v(s,t)ds, \ \alpha\in(1,2), \\ \\ \frac{\partial^2 v(x,t)}{\partial x^2}, \quad \alpha = 2, \end{cases} \tag{1.4}$$

and

$$ {}_{RL}D^{\alpha}_{x,b}v(x,t) = \begin{cases} \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^b (s-x)^{1-\alpha}v(s,t)ds, \ \alpha\in(1,2), \\ \\ \frac{\partial^2 v(x,t)}{\partial x^2}, \quad \alpha = 2. \end{cases} \tag{1.5}$$

This paper is formed as: The 2, FLBRI and the relation between the FLBRI and the FLMMs is expressed in section 2. In section 3, the coefficients of the RFBDFs especially RFBDF2 method introduced for approximation of Riesz derivatives. In Section 4, the implementation of the proposed method is presented for solving time-fractional Liouville-Caputo and Riesz-space fractional diffusion equations. Section 5 contains the investigation of the stability analysis and the convergence order of the new scheme. In section 6, for confirming the theoretical analysis, a numerical experiment is performed. A brief conclusion, in the last section, is presented.

2. FLBRI and its relations with FLMMs

At the first of this part, LBRI weights are defined and the relations between linear multistep methods (LMM)s and LBRI are clearly expressed. Then, the FLBRI weights and their relations with FLMMs are brought up and using them, the temporal direction of Eq. (1.1) is approximated in the Section 4.

The new efficient and high accurate version of the interpolate polynomials based on the Lagrange polynomials in the distinct nods $(\tau_i, f_i),\ i = 0, \dots, n$ of the interval $[a, b]$ is introduced by Floter et.al [10] as the following form

$$P_n(\tau) = \frac{\sum_{i=0}^{n-m} N_i(\tau)s_i(\tau)}{\sum_{i=0}^{n-m} N_i(\tau)}, \qquad n \geq m, \tag{2.6}$$

where $s_i(\tau)$ is an interpolating polynomial for $m + 1$ values $f_i, f_{i+1}, ..., f_{i+m}$ with at most degree m and $N_i(\tau) = \frac{(-1)^i}{\prod_{k=i}^{i+m}(\tau-\tau_k)}$, $i = 0, \dots, n-m$. If we use the nodes with equal distances, the polynomials s_i can be written in Lagrange form and the barycentric weights are defined as the following form

$$\lambda_i = \frac{(-1)^{i-m}}{2^m} \sum_{k\in A_i} \binom{m}{i-k}, \qquad A_i = \{k \in \{0, 1, ..., i-m\}, k-m \leq r \leq i\}. \tag{2.7}$$

Note that the order of these interpolating polynomials for the smooth function f is $O(h_\tau^{m+1})$ where $h_\tau = \max_{0\leq k\leq n-1} |\tau_{k+1} - \tau_k|$. To reveal the relationship between LBRI and LMMs the approximation of integer derivative for the function f using Floater-Hormann family of LBRI can be defined as [3]

$$f'(\tau_{j-n+k}) \approx \frac{1}{h_\tau} \sum_{i=0}^{n} \psi_{k,i} f(\tau_{j-n+i}), \ \ k = 0, \dots, n, \tag{2.8}$$

where $n,\ j \in \mathbb{N}$, $n \leq j$ and

$$\psi_{k,i} = \begin{cases} \frac{\lambda_i}{(k-i)\lambda_k}, & i \neq k, \\ -\sum_{t=0,\ t\neq k}^{n} \psi_{k,t}, & i = k. \end{cases} \tag{2.9}$$

where $\tau_{j-n}, \tau_{j-n+1}, ..., \tau_j$ for uniform grid $\tau_0 < \tau_1 < \ < \tau_l = T$ are interior nodes of interval $[\tau_0, \tau_l]$ with the constant step size $h_\tau = \tau_{j+1} - \tau_j,\ j = 0, 1, \dots, l-1$. If $f \in C^{m+2}[\tau_0, T]$, the convergence order of (2.8) will be m. When $n - m$ is odd, the order increases to $m + 1$ (for more details see [15]).

Applying Eq.(2.8) for the following initial value problems (IVP)s

$$\begin{aligned} &y'(\tau) = V(\tau, y), \quad \tau \in [\tau_0, T], \\ &y(0) = y_0, \end{aligned} \tag{2.10}$$

one can obtain the LMMs as the following LBRI form

$$\sum_{r=0}^{n} \psi_{n,r} y_{j-n+r} = h_\tau V(\tau_j, y(\tau_j)), \quad j = n, n+1, \dots, l, \tag{2.11}$$

with generating function

$$W(\tau) = \frac{\psi(1/\tau)}{\varphi(1/\tau)}, \tag{2.12}$$

where

$$\varphi(\tau) = \psi_{n,0} + \psi_{n,1}\tau + \ldots + \psi_{n,n}\tau^n, \quad \psi(\tau) = \tau^n. \tag{2.13}$$

The starting nodes $y_1,\ y_2,\ \ldots,\ y_{l-1}$ can be calculated by other proper numerical methods [3]. The order of convergence is m when $n-m$ is even and $m+1$ when $n-m$ is odd. If $m=n$ and $m=n+1$ are chosen the LMM with LBRI is the same with BDF of order m and m + 1, respectively ([3]).

Consider the fractional IVP as [12]

$$\begin{aligned}&{}_{LC}D_{0,t}^{\beta}y(\tau)=V(\tau,y),\quad 0<\beta<1,\quad \tau\in[\tau_0,T],\\&y(0)=y_0,\end{aligned}\tag{2.14}$$

Similarly, for generalizing LMMs based on LBRI in (2.10) to FLMMs based on FLBRI in (2.14), the first and second characteristic functions $\varphi(\tau)$ and $\psi(\tau)$ must be created. Thus the generating functions based on FLBRI are defined as

$$W^{(-\beta)}(\tau)=\left(\frac{\psi(1/\tau)}{\varphi(1/\tau)}\right)^{\beta}=(\psi_{n,n}+\psi_{n,n-1}\tau+\ldots+\psi_{n,0}\tau^{n})^{\beta}=\sum_{r=0}^{\infty}w_r^{(-\beta)}\tau^r.\tag{2.15}$$

Using Miller's recurrence [45] in (2.15), the FLBRI weights are obtained as follows:

$$\begin{aligned}&w_i^{(-\beta)}=(\psi_{n,n})^{\beta}w_i,\quad i=0,1,...,\\&w_0=1,\\&w_i=(\xi_i-1)\frac{\psi_{n,n-1}}{\psi_{n,n}}w_{i-1}+(2\xi_i-1)\frac{\psi_{n,n-2}}{\psi_{n,n}}w_{i-2}+...+(n\xi_i-1)\frac{\psi_{n,0}}{\psi_{n,n}}w_{i-n},\end{aligned}\tag{2.16}$$

Using the relation between the Riemann–Liouville and Liouville-Caputo derivatives in (2.14), it can be achieved that

$$\begin{aligned}{}_{LC}D_{0,t}^{\beta}y(\tau_r)&={}_{RL}\,D_{0,t}^{\beta}(y(\tau_r)-y_0),\\&={}_{RL}\,D_{0,t}^{\beta}y(\tau_r)-\frac{\tau_r^{-\beta}}{\Gamma(1-\beta)}y_0=V\Big(\tau_r,y(\tau_r)\Big)\quad 0<\beta<1.\end{aligned}\tag{2.17}$$

Assuming $\bar{b}_r=\frac{r^{-\beta}}{\Gamma(1-\beta)},\ y_{r-k}=y(\tau_r-kh_\tau),\ V_r=V\Big(\tau_r,y(\tau_r)\Big)$ and using (2.19), the following relation would be obtained

$$\sum_{k=0}^{r}w_k{}^{(-\beta)}y_{r-k}-\bar{b}_r y_0=h_\tau^{\upsilon}V_r,\tag{2.18}$$

where $w_k^{(-\beta)}, k=0,\ 1,\ldots,$ are defined in equation (2.16).

Lemma 2.1. *[12] The coefficients $w_i^{(-\beta)},\ \ell=0,\ 1,\ ...$ have the following properties:*
(i) *The monotonicity of the coefficients $w_k^{(-\beta)},\ k=p,\ p+1,\ ...,\ p\in\mathbb{Z}$ for all case $(s,m),\ s\le 20,\ m\le 6$ holds.*

For example, according to Figure 1, after $p\ge 23$ when $(s,m)=(20,m),\ m\le 6$, the monotonicity of $w_k^{(-\beta)}$ holds (for more details see [12]).

(ii) $w_0^{(-\beta)}>0,\ \ w_k^{(-\beta)}<0,\ k=p,\ p+1,\ ...,\ p\in\mathbb{Z}.$

(iii) *The Riemann-Liouville (R-L) fractional derivative is approximated by the FLBRI at $\tau=\tau_r$ as*

$${}_{RL}D_{0,t}^{\beta}y(\tau_r)=h_\tau^{-\beta}\sum_{k=0}^{r}w_k^{(-\beta)}y(\tau_r-kh_\tau)+O(h^p),\tag{2.19}$$

providing $y^{(j)}(\tau_0)=0,\ \ j=0,\ 1,\ \ldots,\ p-1,\ 1\le p<7,\ p\in\mathbb{Z}$ and it has the order $m+1$ and m when $s-m$ is odd and $s-m$ is even, respectively.

3. RFBDFs

In this section, the coefficients of the RFBDFs especially RFBDF2 are presented which require for the next section to approximate the spatial direction.

Generally, the p-th order ($p = 1, \ldots, 6$) approximation of R–L fractional derivative is defined by Lubich for smooth function g for $1 < \alpha < 2$ as [21]

$$ {}_{RL}D^{\beta}_{0,t}g(x_r) = h_x^{-\alpha}\sum_{k=0}^{r} w^{(-\alpha)}_{p,k} g(x_r - kh_x) + O(h_x^p), \tag{3.20}$$

where the generating functions are defined as

$$ W_p(x) = \Big(\sum_{l=0}^{p}\frac{1}{l}(1-x)^l\Big)^{\alpha} = \sum_{l=0}^{\infty} w^{(-\alpha)}_{p,l}x^l, \ \ |x| < 1, \tag{3.21}$$

providing $g^{(l)}(l_0) = 0, \ l = 0, 1, \ldots, p-1, 1 \le p < 7, p \in \mathbb{Z}$.

The application of (3.20) to the spatial FDEs with the Riemann– Liouville derivatives (or Riesz derivatives) is also unstable (see [8]). To overcome this problem, using shifted Lubich's numerical differential formula, it can be derived that

$$ {}_{RL}D^{\alpha}_{a,x}g(x) = h_x^{-\alpha}\sum_{k=0}^{\lfloor\frac{x-a}{h_x}\rfloor+1} w^{(-\alpha)}_{p,k} g(x-(k-1)h_x) + O(h_x), \ \ p = 1, 2, ..., 6. \tag{3.22}$$

Considering Eq.(3.22), it is clear that only 1st-order accuracy obtains. Recently, Ding et.al [8] proposed the following shifted and modified FBDFs of Lubich's numerical differential formula which not only has the p order of accuracy but also has the stability properties. In this paper, their method is used in a particular case involving FBDF2 for solving spatial direction in Eq.(1.1). In this case, the generating function is defined as

$$ \widetilde{\omega}_2(x) = \left(\frac{3\alpha-2}{2\alpha} - \frac{3(\alpha-1)}{\alpha}x + \frac{\alpha-2}{2\alpha}x^2\right)^{\alpha} = \sum_{\ell=0}^{\infty}\kappa^{(\alpha)}_{2,\ell}x^{\ell}, \quad |x| < 1. \tag{3.23}$$

where

$$\begin{cases} \kappa^{(\alpha)}_{2,0} = \left(\frac{3\alpha-2}{2\alpha}\right)^{\alpha}, \\ \kappa^{(\alpha)}_{2,1} = \frac{4\alpha(1-\alpha)}{3\alpha-2}\kappa^{(\alpha)}_{2,0}, \\ \kappa^{(\alpha)}_{2,\ell} = \frac{1}{\ell(3\alpha-2)}\left[4(1-\alpha)(\alpha-\ell+1)\kappa^{(\alpha)}_{2,\ell-1} + (\alpha-2)(2\alpha-\ell+2)\kappa^{(\alpha)}_{2,\ell-2}\right], \ \ l \ge 2. \end{cases} \tag{3.24}$$

Remark 3.1. *For the right Remman-Liouville derivative, the following approximation holds*

$$ {}_{RL}D^{\alpha}_{x,+\infty}g(x) = \ {}^{R}\Upsilon^{\alpha}_2(x) + O(h_x^2),$$

where

$$ {}^{R}\Upsilon^{\alpha}_2(x) = h^{-\alpha}\sum_{k=0}^{\infty}\kappa^{(-\alpha)}_{2,k}g(x+(k-1)h_x).$$

Also, supposing that $g(\tau)$ is defined on $[a, b]$ and $g(a) = g(b) = 0$, it can be result that

$$ {}_{RL}D^{\alpha}_{a,x}g(x) = {}_{RL}D^{\alpha}_{-\infty,x}g(x) = \ {}^{L}\Upsilon^{\alpha}_2(x) + O(h_x^2) = \ {}^{L}\Lambda^{\alpha}_2(x) + O(h_x^2) \tag{3.25}$$

and

$$ {}_{RL}D^{\alpha}_{x,b}g(x) = {}_{RL}D^{\alpha}_{x,+\infty}g(x) = {}^{R}\Upsilon^{\alpha}_{2}(x) + O(h_x^2) = {}^{R}\Lambda^{\alpha}_{2}(x) + O(h_x^2), \tag{3.26} $$

where

$$ {}^{L}\Lambda^{\alpha}_{2}g(x) = h_x^{-\alpha} \sum_{k=0}^{\lfloor \frac{x-a}{h_x} \rfloor + 1} \kappa^{(-\alpha)}_{2,k} g(x-(k-1)h_x), $$

and

$$ {}^{R}\Lambda^{\alpha}_{2}g(x) = h_x^{-\alpha} \sum_{k=0}^{\lfloor \frac{b-x}{h_x} \rfloor + 1} \kappa^{(-\alpha)}_{2,k} g(x+(k-1)h_x). $$

Combining Eqs.(1.3), (3.25) and (3.26), the 2nd-order difference scheme can be obtained as

$$ \frac{\partial^{\alpha} g(x)}{\partial |x|^{\alpha}} = \sigma_{\alpha}\left({}^{L}\Lambda^{\alpha}_{2}g(x) + {}^{R}\Lambda^{\alpha}_{2}g(x)\right) + O(h_x^2). \tag{3.27} $$

4. Establishment of the method

In this section, the offered method is implemented for solving the time-fractional Liouville-Caputo and Riesz-space fractional diffusion equations (1.1). let $t_n = nh_t$, $(n = 0,\ 1,\ ...,\ N)$, and $x_i = ih_x$, $(i = 0,\ 1,\ ...,\ M)$, where $h_t = \dfrac{T}{N}$ and $h_x = \dfrac{L}{M}$ are time and space mesh sizes, respectively. For the numerical approximation of Liouville-Caputo fractional derivative in temporal direction of function $\upsilon(x,t)$ at $t_n = nh_t$, using Eqs.(2.17)–(2.18), one can write

$$ \begin{aligned} {}_{LC}D^{\beta}_{0,t}\upsilon(x_i,t_n) &= {}_{RL}D^{\beta}_{0,t}\left(\upsilon(x_i,t_n) - \upsilon(x_i,t_0)\right) \\ &= h_t^{-\beta} \sum_{k=0}^{n} w^{(-\beta)}_{n-k}\left(\upsilon(x_i,t_k) - \upsilon(x_i,t_0)\right) + O(h_t^{\gamma}), \end{aligned} \tag{4.28} $$

where $0 < \beta < 1,\ 1 \le \gamma \le 7$. Afterward, the following second - order formula is used for the numerical approximation of Riesz-space fractional [8]

$$ \frac{\partial^{\alpha} \upsilon(x_i,t_n)}{\partial |x|^{\alpha}} = \frac{-1}{2cos(\frac{\pi\alpha}{2})}\left({}^{L}\Lambda^{\alpha}_{x} + {}^{R}\Lambda^{\alpha}_{x}\right)\upsilon(x_i,t_n) + O(h_x^2), \tag{4.29} $$

where

$$ {}^{L}\Lambda^{\alpha}_{x}\upsilon(x_i,t_n) = \frac{1}{h_x^{\alpha}} \sum_{\ell=0}^{j+1} \kappa_{2,\ell}\upsilon(x_i - (\ell-1)h_x, t_n), \tag{4.30} $$

and

$$ {}^{R}\Lambda^{\alpha}_{x}\upsilon(x_i,t_n) = \frac{1}{h_x^{\alpha}} \sum_{\ell=0}^{M-j+1} \kappa_{2,\ell}\upsilon(x_i + (\ell-1)h_x, t_n), \tag{4.31} $$

In order to summarize, it can be defined that

$$ \delta^{\alpha}_{x} = \frac{-1}{2cos(\frac{\pi\alpha}{2})}\left({}^{L}\Lambda^{\alpha}_{x} + {}^{R}\Lambda^{\alpha}_{x}\right). \tag{4.32} $$

Therefore, equation (4.29) is rewritten as follows

$$\frac{\partial^\alpha v(x_i,t_n)}{\partial|x|^\alpha} = \delta_x^\alpha v(x_i,t_n) + O(h_x^2), \tag{4.33}$$

Next, substituting (4.28) and (4.33) into (1.1), one obtain

$$h_t^{-\beta}\sum_{k=0}^{n} w_{n-k}^{(-\beta)}\left(v(x_i,t_k) - v(x_i,t_0)\right) = k_\alpha \delta_x^\alpha v(x_i,t_n) + f(x_i,t_n) + O(h_t^\gamma + h_x^2),$$
$$\Rightarrow \sum_{k=0}^{n} w_{n-k}^{(-\beta)} v(x_i,t_k) - \sum_{k=0}^{n} w_{n-k}^{(-\beta)} v(x_i,t_0) = k_\alpha h_t^\beta \delta_x^\alpha v(x_i,t_n) + h_t^\beta f(x_i,t_n) + r_i^n,$$

where $r_i^n = O(h_t^{\gamma+\beta} + h_t^\beta h_x^2)$. Now, we have

$$\begin{aligned} w_0^{(-\beta)} v(x_i,t_n) &= -\sum_{k=0}^{n-1} w_{n-k}^{(-\beta)} v(x_i,t_k) + \sum_{k=0}^{n} w_{n-k}^{(-\beta)} v(x_i,t_0) \\ &\quad + k_\alpha h_t^\beta \delta_x^\alpha v(x_i,t_n) + h_t^\beta f(x_i,t_n) + r_i^n, \\ \Rightarrow v(x_i,t_n) &= -\sum_{k=0}^{n-1} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} v(x_i,t_k) + \sum_{k=0}^{n} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} v(x_i,t_0) \\ &\quad + \frac{k_\alpha h_t^\beta}{w_0^{(-\beta)}} \delta_x^\alpha v(x_i,t_n) + \frac{h_t^\beta}{w_0^{(-\beta)}} f(x_i,t_n) + R_i^n, \end{aligned} \tag{4.34}$$

where $R_i^n = \frac{r_i^n}{w_0^{(-\beta)}}$ and there exists a constant σ such that

$$|R_i^n| \le \sigma\left(h_t^\gamma + h_x^2\right),\ 0 \le i \le M,\ 1 \le n \le N. \tag{4.35}$$

Finally, omitting R_i^n in (4.34) and assuming $v(x_i,t_n) = v_i^n$, one can write

$$\begin{cases} v_i^n = -\sum\limits_{k=0}^{n-1} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} v_i^k + \sum\limits_{k=0}^{n} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} v_i^0 + \frac{k_\alpha h_t^\beta}{w_0^{(-\beta)}} \delta_x^\alpha v_i^n + \frac{h_t^\beta}{w_0^{(-\beta)}} f_i^n, \\ v_i^0 = \varphi(x_i), \quad 0 \le i \le M, \\ v_0^n = v_M^n = 0, \quad 1 \le n \le N. \end{cases} \tag{4.36}$$

Denote by $v^n = (v_1^n, v_2^n, ..., v_{M-1}^n)^T$, $f^n = (f_1^n, f_2^n, ..., f_{M-1}^n)^T$ and $\varphi = (\varphi(x_1), \varphi(x_2), ..., \varphi(x_{M-1}))^T$, then the vector representation of the equation (4.36) can be expressed in the following form

$$\begin{cases} v^n = -\sum\limits_{k=0}^{n-1} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} v^k + \sum\limits_{k=0}^{n} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} v^0 + \frac{k_\alpha h_t^\beta}{w_0^{(-\beta)}} \delta_x^\alpha v^n + \frac{h_t^\beta}{w_0^{(-\beta)}} f^n, \\ v^0 = \varphi, \end{cases} \tag{4.37}$$

where $0 < \beta < 1$ and $1 < \alpha \le 2$.

5. Analysis of Stability properties and order of convergency

In this section, to investigate the stability properties and order of convergency of the new scheme, first, some definitions and lemmas are presented.

Definition 5.1. *Let $v = (v_0, v_1, \dots, v_M)^T$, $v_0 = v_M = 0$ and $v^n = (v_0^n, v_1^n, \dots, v_M^n)^T$. Denote the inner product $(.,.)_M$ and the norm $\|.\|_M$ as*

$$(v, \omega)_M = \sum_{\ell=1}^{M-1} v_\ell \omega_\ell, \quad v,\ \omega \in \mathbb{R}^{(M+1)\times 1} \tag{5.38}$$

and

$$\|v\|_M = \sqrt{(v, v)_M}. \tag{5.39}$$

Lemma 5.2. *[8] Let the operator δ_x^α is defined by* (4.32), *then the following inequality*

$$(\delta_x^\alpha v, v) \leq 0, \tag{5.40}$$

holds for all $\alpha \in (1, 2]$.

Theorem 5.3. *The FDM* (4.37) *is unconditionally stable.*

Proof. Taking the inner product of the equation (4.37) with v^n, it is obtained that

$$(v^n, v^n) = -\sum_{k=0}^{n-1} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} (v^k, v^n) + \sum_{k=0}^{n} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} (v^0, v^n) + \frac{k_\alpha h_t^\beta}{w_0^{(-\beta)}} (\delta_x^\alpha v^n, v^n) + \frac{h_t^\beta}{w_0^{(-\beta)}} (f^n, v^n), \tag{5.41}$$

based on the Cauchy-Schwartz inequality and Lemma 2.1 and Lemma 5.2, it can be written that

$$\begin{aligned} \|v^n\|^2 \leq & -\sum_{k=0}^{n-p} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} \|v^k\| \|v^n\| + \sum_{k=n-p+1}^{n-1} \frac{|w_{n-k}^{(-\beta)}|}{w_0^{(-\beta)}} \|v^k\| \|v^n\| + \sum_{k=0}^{n-p} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} \|v^0\| \|v^n\| \\ & + \sum_{k=n-p+1}^{n} \frac{|w_{n-k}^{(-\beta)}|}{w_0^{(-\beta)}} \|v^0\| \|v^n\| + \frac{h_t^\beta}{w_0^{(-\beta)}} \|f^n\| \|v^n\|. \end{aligned} \tag{5.42}$$

Also, according to Lemma 2.1 and removing $\|v^n\|$ from both sides of Eq.(5.42), one can result that

$$\begin{aligned} \|v^n\| \leq & -\sum_{k=0}^{n-p} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} \|v^k\| + \sum_{k=n-p+1}^{n-1} \frac{|w_{n-k}^{(-\beta)}|}{w_0^{(-\beta)}} \|v^k\| + \sum_{k=0}^{n-p} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} \|v^0\| \\ & + \sum_{k=n-p+1}^{n} \frac{|w_{n-k}^{(-\beta)}|}{w_0^{(-\beta)}} \|v^0\| + \frac{h_t^\beta}{w_0^{(-\beta)}} \|f^n\|. \end{aligned} \tag{5.43}$$

Assuming

$$\theta_p = \sum_{k=n-p+1}^{n-1} \frac{|w_{n-k}^{(-\beta)}|}{w_0^{(-\beta)}} \|v^k\| + \sum_{k=n-p+1}^{n-1} \frac{|w_{n-k}^{(-\beta)}|}{w_0^{(-\beta)}} \|\varphi\|, \tag{5.44}$$

from Eq.(5.43), it can be derived that

$$\|v^n\| \leq \theta_p - \sum_{k=0}^{n-p} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} \|v^k\| + \sum_{k=0}^{n-p} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} \|\varphi\| + \|\varphi\| + \frac{h_t^\beta}{w_0^{(-\beta)}} \|f^n\|. \tag{5.45}$$

Now, it can be asserted that

$$\|v^n\| \leq \Upsilon,\ 1 \leq n \leq N, \tag{5.46}$$

where

$$\Upsilon = \max_p(\theta_p) + \|\varphi\| + \frac{h_t^\beta}{w_0^{(-\beta)}} \max_{1\le n\le N} \|f^n\|. \tag{5.47}$$

For proving this assertion, the mathematical induction is used.
For $n = 1$, Eq.(5.45) can be expressed as

$$\|v^1\| \le \theta_1 + \|\varphi\| + \frac{h_t^\beta}{w_0^{(-\beta)}}\|f^1\| \le \Upsilon. \tag{5.48}$$

Suppose that $\|v^n\| \le \Upsilon,\ n = 1,\ 2,\ ,...,\ m-1$. For $n = m$, one can write

$$\begin{aligned}
\|v^m\| &\le \theta_p - \sum_{k=0}^{m-p} \frac{w_{m-k}^{(-\beta)}}{w_0^{(-\beta)}}\|v^k\| + \sum_{k=0}^{m-p} \frac{w_{m-k}^{(-\beta)}}{w_0^{(-\beta)}}\|\varphi\| + \|\varphi\| + \frac{h_t^\beta}{w_0^{(-\beta)}}\|f^m\|,\\
\|v^m\| &\le \theta_p - \sum_{k=0}^{m-p} \frac{w_{m-k}^{(-\beta)}}{w_0^{(-\beta)}}\Upsilon + \sum_{k=0}^{m-p} \frac{w_{m-k}^{(-\beta)}}{w_0^{(-\beta)}}\Upsilon + \|\varphi\| + \frac{h_t^\beta}{w_0^{(-\beta)}}\|f^m\|,\\
&\le \theta_p + \|\varphi\| + \frac{h_t^\beta}{w_0^{(-\beta)}}\|f^m\| \le \Upsilon,
\end{aligned} \tag{5.49}$$

According to energy method, the proof is finished. □

Result 1: According the Theorem 5.3, one can conclude that (4.37) is convergent.

Theorem 5.4. *Let $v(x_i, t_n)$ and v_i^n are the exact solutions of the equation (1.1) and FDM (4.36), respectively for $i = 0, 1, \ldots, M,\ n = 0, 1, \ldots, N-p$, (the number p is defined in Lemma 2.1). In addition, $v(x_i, t_n) = v_i^n = 0$ for $i = 0, 1, \ldots, M,\ n = N-p+1, N-p+2 \ldots, N$. Define $e_i^n = v(x_i, t_n) - v_i^n$, $\mathbf{e}^n = (e_0^n, e_1^n, ..., e_M^n)^T$, $R_i^n = O\left(h_t^\gamma + h_x^2\right)$ and $\mathbf{R}^n = (R_0^n,\ R_1^n,\ R_2^n,\ ...,\ R_M^n)^T$. Then there exists a positive constant C independent of $n,\ h_t,\ \omega_0^{(-\alpha)}$ and h_x such that*

$$\|\mathbf{e}^n\| \le C(h_t^\gamma + h_x^2), \quad 1 \le \gamma \le 7,$$

therefore, the rate of convergence order of new scheme (4.37) *is $O\left(h_t^\gamma + h_x^2\right)$.*

Proof. From equation (4.34), (4.36) one can write

$$\begin{aligned}
v(x_i, t_n) = &-\sum_{k=0}^{n-p} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} v(x_i, t_k) + \sum_{k=0}^{n-p} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} v(x_i, t_0)\\
&+ \frac{k_\alpha h_t^\beta}{w_0^{(-\beta)}} \delta_x^\alpha v(x_i, t_n) + \frac{h_t^\beta}{w_0^{(-\beta)}} f(x_i, t_k) + R_i^n
\end{aligned} \tag{5.50}$$

and

$$v_i^n = -\sum_{k=0}^{n-p} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} v_i^k + \sum_{k=0}^{n-p} \frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}} v_i^0 + \frac{k_\alpha h_t^\beta}{w_0^{(-\beta)}} \delta_x^\alpha v_i^n + \frac{h_t^\beta}{w_0^{(-\beta)}} f_i^n + R_i^n. \tag{5.51}$$

Subtracting (5.51) form (5.50), one can derive

$$v(x_i,t_n)-v_i^n=-\sum_{k=0}^{n-p}\frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}}(v(x_i,t_k)-v_i^k)+\sum_{k=0}^{n-p}\frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}}(v(x_i,t_0)-v_i^0)+\frac{k_\alpha h_t^\beta}{w_0^{(-\beta)}}\delta_x^\alpha(v(x_i,t_n)-v_i^n)+R_i^n. \tag{5.52}$$

Assuming $e_i^n=v(x_i,t_n)-v_i^n$ and using (5.52), it can be obtained

$$e_i^n=-\sum_{k=0}^{n-p}\frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}}e_i^k+\sum_{k=0}^{n-p}\frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}}e_i^0+\frac{k_\alpha h_t^\beta}{w_0^{(-\beta)}}\delta_x^\alpha e_i^n+R_i^n. \tag{5.53}$$

Define $\mathbf{e}^n=(e_0^n,e_1^n,...,e_M^n)^T$and $\mathbf{R}^n=(R_0^n,\ R_1^n,\ R_2^n,\ ...,\ R_M^n)^T$. Then the vector representation of the equation (5.53) can be expressed as

$$\mathbf{e}^n=-\sum_{k=0}^{n-p}\frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}}\mathbf{e}^k+\sum_{k=0}^{n-p}\frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}}\mathbf{e}^0+\frac{k_\alpha h_t^\beta}{w_0^{(-\beta)}}\delta_x^\alpha\mathbf{e}^n+\mathbf{R}^n. \tag{5.54}$$

Using the inner product for the equation (5.54) by $\mathbf{e}^n$, one can write

$$(\mathbf{e}^n,\mathbf{e}^n)=-\sum_{k=0}^{n-p}\frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}}(\mathbf{e}^k,\mathbf{e}^n)+\sum_{k=0}^{n-p}\frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}}(\mathbf{e}^0,\mathbf{e}^n)+\frac{k_\alpha h_t^\beta}{w_0^{(-\beta)}}(\delta_x^\alpha\mathbf{e}^n,\mathbf{e}^n)+(\mathbf{R}^n,\mathbf{e}^n), \tag{5.55}$$

based on the Cauchy-Schwartz inequality and Lemma 5.2, it is resulted that

$$\|\mathbf{e}^n\|^2\le-\sum_{k=0}^{n-p}\frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}}\|\mathbf{e}^k\|\|\mathbf{e}^n\|+\sum_{k=0}^{n-p}\frac{|w_{n-k}^{(-\beta)}|}{w_0^{(-\beta)}}\|\mathbf{e}^0\|\|\mathbf{e}^n\|+\|\mathbf{R}^n\|\|\mathbf{e}^n\|. \tag{5.56}$$

Simplifying the Eq.(5.56), it is derived that

$$\|\mathbf{e}^n\|\le-\sum_{k=0}^{n-p}\frac{w_{n-k}^{(-\beta)}}{w_0^{(-\beta)}}\|\mathbf{e}^k\|+\sum_{k=0}^{n-p}\frac{|w_{n-k}^{(-\beta)}|}{w_0^{(-\beta)}}\|\mathbf{e}^0\|+\|\mathbf{R}^n\|. \tag{5.57}$$

Now it can be asserted that

$$\|\mathbf{e}^n\|\le\max_{p\le n\le N}\|\mathbf{R}^n\|,\ p\le n\le N. \tag{5.58}$$

For proving this assertion, the mathematical induction on n is used.

For $n=p$, Eq.(5.57) can be expressed as

$$\|\mathbf{e}^p\|\le\ \|\mathbf{R}^p\|\le\max_{p\le n\le N}\|\mathbf{R}^n\|. \tag{5.59}$$

Suppose that $\|\mathbf{e}^n\| \leq \max_{p\leq n\leq N} \|\mathbf{R}^n\|,\ n = p,\ p+1,\ ,...,\ m-1$. For $n = m$, one has

$$\begin{aligned} \|\mathbf{e}^m\| &\leq -\sum_{k=0}^{m-p} \frac{w_{m-k}^{(-\beta)}}{w_0^{(-\beta)}} \|\mathbf{e}^k\| + \sum_{k=0}^{m-p} \frac{w_{m-k}^{(-\beta)}}{w_0^{(-\beta)}} \|\mathbf{e}^0\| + \|\mathbf{R}^m\|, \\ &\leq -\sum_{k=0}^{m-p} \frac{w_{m-k}^{(-\beta)}}{w_0^{(-\beta)}} (\max_{p\leq n\leq N} \|\mathbf{R}^n\|) + \sum_{k=0}^{m-p} \frac{w_{m-k}^{(-\beta)}}{w_0^{(-\beta)}} (\max_{p\leq n\leq N} \|\mathbf{R}^n\|) \\ &\quad + \|\mathbf{R}^m\| \leq \|\mathbf{R}^m\| \leq \max_{p\leq n\leq N} \|\mathbf{R}^n\|, \end{aligned} \tag{5.60}$$

From the Eq.(4.35), it can be obtained that

$$\begin{aligned} \max_{p\leq n\leq N} \|\mathbf{R}^n\| &= \sigma(h_t^\gamma + h_x^2) \sqrt{\underbrace{1+1+\ldots+1}_{N-p+1}} \\ &= \sigma(h_t^\gamma + h_x^2) \sqrt{N-p+1} = C(h_t^\gamma + h_x^2), \end{aligned} \tag{5.61}$$

where $C = \sigma\sqrt{N-p+1}$. Finally, it is resulted that

$$\|\mathbf{e}^n\| \leq C(h_t^\gamma + h_x^2),\ p \leq n \leq N,\ 1 \leq \gamma \leq 7, \tag{5.62}$$

the proof is completed. □

6. Numerical example

In this part, the results of numerical experiments are reported to illustrate the effectiveness of the proposed method. The numerical experiments are performed on a computer Intel(R) Pentium(R) CPU G2030 @ 3.00GHz 3.00GHz 12.00GB RAM by running some codes written in MATLAB 2020 software. In the following tables, the maximum of L_2 error of the proposed method and exact solution are calculated as follows

$$E_\infty(h_x, h_t) = \max_{0\leq n\leq n_T} \|\mathbf{e}^n\|_N = \max_{0\leq n\leq n_T} \sqrt{h_t \sum_{j=0}^{N-1} (v(x_j, t_n) - v_j^n)^2}. \tag{6.63}$$

Consider the following equation.

$$\begin{cases} {}_{LC}D_{0,t}^{\beta} v(x,t) = \frac{\partial^\alpha v(x,t)}{\partial |x|^\alpha} + f(x,t), & (x,t) \in (0,L) \times (0,T], \quad \beta \in (0,1), \quad \alpha \in (1,2], \\ v(x,0) = 0, \quad x \in [0,L], \\ v(0,t) = v(L,t) = 0, \quad t \in [0,T], \end{cases} \tag{6.64}$$

where

$$\begin{aligned} f(x,t) = &\frac{\Gamma(\alpha+\beta+9)}{\Gamma(\beta+9)} t^{\beta+8} x^4 (1-x)^4 + \frac{t^{\alpha+\beta+8}}{2\cos(\frac{\pi\beta}{2})} \sum_{\ell=0}^{4} (-1)^\ell \frac{4!\,(4+\ell)!}{\ell!(4-\ell)!\,\Gamma(5+\ell-\beta)} \\ &\times [x^{4+\ell-\beta} + (1-x)^{4+\ell-\beta}]. \end{aligned} \tag{6.65}$$

The exact solution of the problem is $v(x,t) = t^{\alpha+\beta+8} x^4 (1-x)^4$. In all tables, maximum errors L_2 and computational costs are listed for the proposed method. Furthermore, these tables illustrate the numerical convergence orders of the proposed method in the temporal and spatial directions, respectively.
Suppose $E_\infty(h_x, h_t) = O(h_x^p + h_t^\gamma)$. If h_x is sufficiently small, it can be written that

$$E_\infty(h_x, h_t) \approx c_1 h_t^\gamma,\ \ E_\infty(h_x, \frac{h_t}{k}) \approx c_1 (\frac{h_t}{k})^\gamma \Rightarrow k^\gamma \approx \frac{E_\infty(h_x, h_t)}{E_\infty(h_x, \frac{h_t}{k})} \quad \Rightarrow \gamma \approx \log_k (\frac{E_\infty(h_x, h_t)}{E_\infty(h_x, \frac{h_t}{k})}). \tag{6.66}$$

The numerical results are reported for the fixed values $\alpha = 0.25$ and $\beta = 1.5$ in Table 1. This table shows that the convergence rate of new scheme for $(m, s) = (4, 1)$, $s = 2$ and $m \in \{4, 16, 18\}$ are approximately second order in the temporal direction. Assuming $\alpha = 0.25$ and $\beta = 1.5$, the results are reported for $(m, s) = (3, 2)$, $s = 3$ and $m \in \{13, 15, 19\}$ for Table 2. This table illustrates that the convergence rate of the new scheme is around third order in the temporal direction.

In Table 3, for $(m, s) = (12, 3)$, $\beta = 1.5$ and $\alpha \in \{0.25, 0.45, 0.65, 0.85\}$, the results demonstrate that the rate of convergence order of the proposed method is approximately fourth order in the temporal direction.

Considering Table 4, the convergence rate of the new scheme is around fifth order in the temporal direction, for $\alpha = 0.5,\ \beta = 1.5,\ s \in \{4, 5\}$ and $m \in \{5, 7, 19\}$.

The results in Table 5 show that for $\alpha = 0.85,\ \beta = 1.5\ s \in \{5, 6\}$ and $m \in \{6, 8, 18\}$, the rate of convergence order of the new scheme is approximately sixth order in the temporal direction.

Table 6 demonstrates the rate of convergence order of our method for $\alpha = 0.85,\ \beta = 1.5,\ m \in \{7, 9, 11\}$ and $s = 6$ is near seventh order in the temporal direction.

Table 7 shows the convergence rat of new scheme for $\beta = 1.5,\ \alpha \in \{0.25, 0.45, 0.4, 0.50\}$ and $(s, m) \in \{(4, 1), (8, 2), (9, 3), (12, 4)\}$ is around second order in the spatial direction.

The numerical results indicate that the rate of convergence order of new method for $\alpha = 0.5,\ \beta \in \{1.9, 1.95, 1.4\}$ and $(s, m) \in \{(7, 5), (12, 6), (19, 6)\}$ is approximately second order in the spatial direction for Table 8.

7. Conclusion

In this work, an efficient numerical scheme based on fractional linear barycentric rational interpolation, and modified fractional backward difference formulae, have been established for time-fractional Liouville-Caputo and Riesz-space fractional diffusion equations. The unconditional stability of this method has been proven by the energy method. Also, it has been shown that the new scheme is convergent using local truncation error. Finally, the numerical experiments and the theoretical results have confirmed that the scheme is convergent with order $O(\tau^{\gamma} + h^2)$ where $1 \le \gamma \le 7$.

References

[1] M.I. Abbas, M.A. Ragusa, *On the hybrid fractional differential equations with fractional proportional derivatives of a function with respect to a certain function*, Symmetry. (2021) 13:264. https://doi.org/10.3390/sym13020264.

[2] M.I. Abbas, M.A. Ragusa, *Solvability of Langevin equations with two Hadamard fractional derivatives via Mittag-Leffler functions*, Applicable Analysis,(2020) **99**:1-19. https://doi.org/10.1080/00036811.2020.1839645

[3] A. Abdi, S.A. Hosseini, H. Podhaisky,*Adaptive linear barycentric rational finite differences method for stif ODEs*, Journal of Computational and Applied Mathematics, (2019) **357**:204–214. https://doi.org/10.1016/j.cam.2019.02.034

[4] A.O. Akdemir, S.I. Butt, M. Nadeem, M.A. Ragusa, *New general variants of Chebyshev type inequalities via generalized fractional integral operators*, Mathematics. (2021) **9**:122. https://doi.org/10.3390/math9020122.

[5] D. Baleanu, K. Diethelm, E. Scalas, J. Trujillo, *Fractional Calculus Models and Numerical Methods, Series on Complexity, Nonlinearity and Chaos*, World Scientific; 2012.

[6] B. Baeumer, M. Kovacs, M.M. Meerschaert, *Numerical solutions for fractional reaction–diffusion equations*, Computers and Mathematics with Applications. (2008) **55**:2212–2226. https://doi.org/10.1016/j.camwa.2007.11.012

[7] J. Chen, F. Liu, V. Anh, *Analytical solution for the timefractional telegraph equation by the method of separating variables*, Journal of Mathematical Analysis and Applications, (2008) **338**:1364–1377. https://doi.org/10.1016/j.jmaa.2007.06.023

[8] H.F. Ding, C.P. Li, *High–order numerical algorithms for Riesz derivatives via constructing new generating functions*, Journal of Scientific Computing, (2017) **71**:759–784. https://doi.org/10.1007/s10915-016-0317-3

[9] I. Fahimi-khalilabad, S. Irandoust-pakchin, S. Abdi-mazraeh, *High-order finite difference method based on linear barycentric rational interpolation for Caputo type sub-diffusion equation*, Mathematics and Computers in Simulation, (2022) **199**:60-80. https://doi.org/10.1016/j.matcom.2022.03.008

[10] M.S. Floater, K. Hormann, *Barycentric rational interpolation with no poles and high rates of approximation*, Numerische Mathematik. (2007) **107**:315–331. https://doi.org/10.1007/s00211-007-0093-y

[11] F. Huang, B. Guo, *General solutions to a class of time fractional partial differential equations*, Applied Mathematics and Mechanics, (2010) **31**:815–826. https://doi.org/10.1007/s10483-010-1316-9

[12] S. Irandoust-pakchin, S. Abdi-mazraeh, H. Kheiri, *Construction of new generating function based on linear barycentric rational interpolation for numerical solution of fractional differential equations*, Journal of Computational and Applied Mathematics, **375** (2020), 112799. https://doi.org/10.1016/j.cam.2020.112799

[13] S. Irandoust-pakchin, M. Dehghan, S. Abdi-mazraeh, M. Lakestani, *Numerical solution for a class of fractional convection diffusion equations using the flatlet oblique multiwavelets*, Journal of Vibration and Control, **20**(6), (2014), 913-924. doi:10.1177/1077546312470473

[14] A.A. Kilbas, H.M. Srivastava, Trujillo JJ, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[15] G. Klein,*Applications of linear barycentric rational interpolation* (Ph.D. thesis), University of Fribourg, 2012.

[16] C.P. Li, H.F Ding,*Higher order finite difference method for the reaction and anomalous-diffusion equation*, Applied Mathematical Modelling, **38** (15 16),(2014), 3802–3821. https://doi.org/10.1016/j.apm.2013.12.002

[17] C.P. Li, F.H. Zeng,*Numerical methods for fractional calculus*. Chapman and Hall/CRC, 2015.

[18] C.P. Li, F.H. Zeng,*Finite element methods for fractional differential equation*, Recent Advances in Applied Nonlinear Dynamics with Numerical Analysis, World Scientific, Singapore, (2013) 49-68. https://doi.org/10.1142/9789814436465_0003

[19] C.P. Li, M. Cai,*Theory and Numerical Approximations of Fractional Integrals and Derivatives*, SIAM, Philadelphia, 2019.

[20] C.P. Li, Z.G. Zhao, Y.Q. Chen,*Numerical approximation of nonlinear fractional differential equations with subdiffusion and superdiffusion*, Computers and Mathematics with Applications, (2011) **62**:855–875. https://doi.org/10.1016/j.camwa.2011.02.045

[21] C. Lubich,*Discretized fractional calculus*, SIAM Journal on Mathematical Analysis,(1986)**17**:704-719.https://doi.org/10.1137/0517050

[22] M.M. Meerschaert, C. Tadjeran,*Finite difference approximations for fractional advection dispersion flow equations*, Journal of Computational and Applied Mathematics, (2004) **172**:65–77. https://doi.org/10.1016/j.cam.2004.01.033

[23] K.S. Miller, B. Ross,*An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley, New York, 1993.

[24] K.B. Oldham, J. Spanier,*The Fractional Calculus*, Academic Press, New York, 1974.

[25] I. Podlubny,*Fractional Differential Equations*, Academic Press, San Diego, 1999.

[26] J.P. Roop,*Numerical approximation of a one–dimensional space fractional advection-dispersion equation with boundary layer*, Computers and Mathematics with Applications, (2008) **56**:1808–1819. https://doi.org/10.1016/j.camwa.2008.04.025

[27] M.S. Heris, M. Javidi,*On Fractional Backward Differential Formulas Methods for Fractional Differential Equations with Delay*, International Journal of Applied and Computational Mathematics, **4**(72) (2018). https://doi.org/10.1007/s40819-018-0493-y

[28] M.S. Heris, M. Javidi,*On FBDF5 Method for Delay Differential Equations of Fractional Order with Periodic and Anti-Periodic Conditions*, Mediterranean Journal of Mathematics, **14**(134) (2017). https://doi.org/10.1007/s00009-017-0932-8

[29] S.G. Samko, A.A. Kilbas, O.I. Marichev,*Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach Science Publishers, 1993.

[30] H.M Srivastava, *An Introductory Overview of Fractional-Calculus Operators Based Upon the Fox-Wright and Related Higher Transcendental Functions*, J. Adv. Engrg. Comput. **5** (2021), 135–166.

[31] H.M Srivastava, *Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations*, J. Nonlinear Convex Anal. **22** (2021), 1501–1520.

[32] H.M Srivastava, *Fractional-Order Derivatives and Integrals: Introductory Overview and Recent Developments*, KYUNGPOOK Math. J. **60**(2020), 73–116. https://doi.org/10.5666/KMJ.2020.60.1.73

[33] H.M Srivastava, M. Izadi, N. Okhovati, *Viscous splitting finite differtence schemes to convection-diffusion equations with discontinuos coefficient*, Appl. Anal. Optim. **6** (2022), 313–328.

[34] H.M Srivastava, M. Izadi, *The Rothe-Newton approach to simulate the variable coefficient convection-diffusion equations*, J. Mahani Math. Res. Cent., **11** (2022), 141–157.

[35] M. Izadi, H.M Srivastava, *An optimized second order numerical scheme applied to the non-linear Fisher's reaction-diffusion equation*, J. Interdisciplinary Math., **25** (2022), 471–492, http://dx.doi.org/10.1080/09720502.2021.1930662

[36] V.M. Tripathi, H.M. Srivastava, H. Singh, C. Swarup, S. Aggarwal, *Mathematical analysis of non-isothermal reactiondiffusion models arising in spherical catalyst and spherical biocatalyst*, Appl. Sci., **11** (2021), Article ID 10423,1–14.https://doi.org/10.3390/ app112110423

[37] H. M. Srivastava, H.I. Abdel-Gawad, Kh.M. Saad *Oscillatory states and patterns formation in a two-cell cubic autocatalytic reaction-diffusion model subjected to the Dirichlet conditions*, Discrete Continuous Dyn. Syst. - S 4 (2021), 3785-3801. http://dx.doi.org/10.3934/dcdss.2020433

[38] H. M. Srivastava, H. Ahmad, I. Ahmad, P. Thounthong, M. N. Khan *Numerical simulation of three-dimensional fractional-order convection-diffusion PDEs by a local meshless method*, Thermal Sci., **25** (1A) (2021), 347–358.

[39] N.T. Shawagfeh,*Analytical approximate solutions for nonlinear fractional differential equations*, Applied Mathematics and Computation, (2002) **131**:517-529. https://doi.org/10.1016/S0096-3003(01)00167-9

[40] E. Sousa,*Numerical approximations for fractional diffusion equations via splines*, Computers and Mathematics with Applications, (2011), **62**:938–944. https://doi.org/10.1016/j.camwa.2011.04.015

[41] E. Sousa,*Finite difference approximations for a fractional advection diffusion problem*, Journal of Computational Physics, (2009) 228:4038–4054. https://doi.org/10.1016/j.cam.2004.01.033

[42] L. Su, W. Wang, H. Wang,*A characteristic finite difference method for the transient fractional convection–diffusion equations*, Applied Numerical Mathematics, (2011) **61**:946–960. https://doi.org/10.1016/j.apnum.2011.02.007

[43] Q.Q. Yang, F. Liu, I. Turner,*Numerical methods for fractional partial differential equations with Riesz space fractional derivatives*, Applied Mathematical Modelling, (2010) **34**:200–218. https://doi.org/10.1016/j.apm.2009.04.006

[44] Z. Yuxin, H.F. Ding,*Numerical algorithm for the time–Caputo and space–Riesz fractional diffusion equation*, Communications on Applied Mathematics and Computation, (2020) **2**:57–72. https://doi.org/10.1007/s42967-019-00032-x

[45] D. Zeilberger,*The jcp miller recurrence for exponentiating a polynomial, and its q-analog**, Journal of Difference Equations and Applications, (1995)**1**:57–60. https://doi.org/10.1080/10236199508808006

[46] F.H. Zeng, C.P. Li, F. Liu, and I. Turner,*The use of finite difference/element approaches for solving the time-fractional subdiffusion equation*, SIAM J. Sci. Comput., **35**(6): A2976-A3000, (2013). https://doi.org/10.1137/130910865

[47] P. Zhuang, F. Liu, V. Anh, I. Turner,*New solution and analytical techniques of the implicit numerical method for the anomalous subdiffusion equation*, SIAM Journal on Numerical Analysis. (2008) **46**:1079–1095. https://doi.org/10.1137/060673114

Table 1: Maximum L_2 error, rate of convergence oreder in time direction for FLBRI of order 2, $h_x = \frac{1}{500}$, $\alpha = 0.25$, $\beta = 1.5$ and run time.

(m, s)	h_t	Maximum L_2 error	Time convergence orders	Cpu time(s)
(4, 1)	$\frac{1}{4}$	$0.1005e-02$	–	0.17
	$\frac{1}{8}$	$3.2599e-04$	1.6249	0.26
	$\frac{1}{16}$	$9.0579e-05$	1.8476	0.41
	$\frac{1}{32}$	$2.1282e-05$	2.0895	0.82
(4, 2)	$\frac{1}{4}$	$7.2425e-04$	–	0.23
	$\frac{1}{8}$	$1.8988e-04$	1.9314	0.26
	$\frac{1}{16}$	$4.7183e-05$	2.0088	0.37
	$\frac{1}{32}$	$1.0763e-05$	2.1321	0.83
(16, 2)	$\frac{1}{32}$	$7.2458e-06$	–	0.84
	$\frac{1}{64}$	$1.5985e-06$	2.1804	1.46
	$\frac{1}{128}$	$3.7200e-07$	2.1034	2.82
	$\frac{1}{256}$	$1.0303e-07$	1.8528	5.32
(18, 2)	$\frac{1}{2}$	$2.1854e-03$	–	0.10
	$\frac{1}{4}$	$7.0334e-04$	1.6356	0.19
	$\frac{1}{8}$	$1.6978e-04$	2.0505	0.27
	$\frac{1}{16}$	$3.5438e-05$	2.2603	0.61
	$\frac{1}{32}$	$7.0413e-06$	2.3314	0.74
	$\frac{1}{64}$	$1.5005e-06$	2.2304	1.36

Table 2: Maximum L_2 error, rate of convergence order in time direction for FLBRI of order 3, $h_x = \frac{1}{500}$, $\alpha = 0.25$, $\beta = 1.5$ and run time.

(m, s)	h_t	Maximum L_2 error	Time convergence orders	Cpu time(s)
(3, 2)	$\frac{1}{8}$	$1.2917e-04$	–	0.24
	$\frac{1}{16}$	$1.8224e-05$	2.8254	0.66
	$\frac{1}{32}$	$2.0993e-06$	3.1179	0.67
	$\frac{1}{64}$	$2.6368e-07$	2.9930	1.63
(13, 3)	$\frac{1}{8}$	$8.6315e-05$	–	0.23
	$\frac{1}{16}$	$1.0861e-05$	2.9905	0.38
	$\frac{1}{32}$	$1.3125e-06$	3.0487	0.82
	$\frac{1}{64}$	$2.1364e-07$	2.6191	1.34
(15, 3)	$\frac{1}{8}$	$8.6254e-05$	–	2.50
	$\frac{1}{16}$	$1.0781e-05$	3.0001	0.49
	$\frac{1}{32}$	$1.2648e-06$	3.0915	0.73
	$\frac{1}{64}$	$2.0787e-07$	2.6052	1.51
(19, 3)	$\frac{1}{8}$	$8.6208e-05$	–	0.22
	$\frac{1}{16}$	$1.0722e-05$	3.0073	0.38
	$\frac{1}{32}$	$1.2114e-06$	3.1457	0.75
	$\frac{1}{64}$	$1.9906e-07$	2.6054	1.44

Table 3: Maximum L_2 error, rate of convergence order in time direction for FLBRI of order 4, $(s,m) = (12,3)$, $h_x = \frac{1}{500}$, $\beta = 1.5$, various values of α and run time.

α	h_t	Maximum L_2 error	Time convergence orders	Cpu time(s)
0.25	$\frac{1}{4}$	$3.8932e-04$	–	0.08
	$\frac{1}{8}$	$4.4317e-05$	3.1350	0.15
	$\frac{1}{16}$	$3.2901e-06$	3.7516	0.27
	$\frac{1}{32}$	$3.0411e-07$	3.4355	0.55
0.45	$\frac{1}{16}$	$8.3277e-06$	–	0.26
	$\frac{1}{20}$	$3.5060e-06$	3.8769	0.33
	$\frac{1}{24}$	$1.7433e-06$	3.8321	0.39
	$\frac{1}{28}$	$9.7849e-07$	3.7466	0.47
0.65	$\frac{1}{6}$	$5.8965e-04$	–	0.11
	$\frac{1}{8}$	$2.2461e-04$	3.3549	0.15
	$\frac{1}{16}$	$1.6615e-05$	3.7568	0.34
	$\frac{1}{32}$	$1.1349e-06$	3.8719	0.56
0.85	$\frac{1}{8}$	$3.8934e-04$	–	0.15
	$\frac{1}{16}$	$2.9073e-05$	3.7433	0.30
	$\frac{1}{32}$	$1.9463e-06$	3.9009	0.53
	$\frac{1}{64}$	$1.6998e-07$	3.5173	1.07

Table 4: Maximum L_2 error, rate of convergence order in time direction for FLBRI of order 5, various values of (m,s), $h_x = \frac{1}{1500}$, $\alpha = 0.5$, $\beta = 1.5$ and run time.

(m,s)	h_t	Maximum L_2 error	Time convergence orders	Cpu time(s)
$(5,4)$	$\frac{1}{8}$	$1.8170e-04$	–	1.36
	$\frac{1}{16}$	$8.1474e-06$	4.4791	2.74
	$\frac{1}{24}$	$1.1042e-06$	4.9290	3.74
	$\frac{1}{32}$	$2.6064e-07$	5.0186	5.19
	$\frac{1}{40}$	$8.9648e-08$	4.7828	6.44
$(7,4)$	$\frac{1}{8}$	$2.2541e-04$	–	1.09
	$\frac{1}{16}$	$1.2615e-05$	4.1594	1.99
	$\frac{1}{24}$	$1.8557e-06$	4.7269	3.07
	$\frac{1}{32}$	$4.5091e-07$	4.9178	3.85
	$\frac{1}{40}$	$1.5130e-07$	4.8936	5.09
$(7,5)$	$\frac{1}{8}$	$1.3278e-04$	–	0.98
	$\frac{1}{16}$	$6.5351e-06$	4.3447	1.77
	$\frac{1}{24}$	$9.4586e-07$	4.7670	2.97
	$\frac{1}{32}$	$2.3249e-07$	4.8779	3.40
	$\frac{1}{40}$	$8.2752e-08$	4.6292	4.28
$(19,5)$	$\frac{1}{8}$	$1.1938e-04$	–	1.05
	$\frac{1}{16}$	$4.6691e-06$	4.6763	2.16
	$\frac{1}{24}$	$6.2365e-07$	4.9650	3.00
	$\frac{1}{32}$	$1.5619e-07$	4.8126	3.94
	$\frac{1}{40}$	$6.1039e-08$	4.2106	4.25

Table 5: Maximum L_2 error, rate of convergence order in time direction for FLBRI of order 6, various values of (m,s), $h_x = \frac{1}{1500}$, $\alpha = 0.85$, $\beta = 1.5$ and run time.

(m,s)	h_t	Maximum L_2 error	Time convergence orders	Cpu time(s)
(6, 5)	$\frac{1}{8}$	$8.9339e-05$	–	0.93
	$\frac{1}{16}$	$2.0680e-06$	5.4330	1.75
	$\frac{1}{24}$	$1.9649e-07$	5.8050	2.66
	$\frac{1}{28}$	$8.6520e-08$	5.3211	3.05
(8, 5)	$\frac{1}{8}$	$1.1374e-04$	–	0.95
	$\frac{1}{16}$	$3.3117e-06$	5.1020	1.76
	$\frac{1}{24}$	$3.3060e-07$	5.6832	2.62
	$\frac{1}{32}$	$7.1389e-08$	5.3279	5.21
(8, 6)	$\frac{1}{8}$	$6.4822e-05$	–	1.67
	$\frac{1}{16}$	$1.6997e-06$	5.2531	2.47
	$\frac{1}{24}$	$1.7378e-07$	5.6242	3.90
	$\frac{1}{28}$	$7.9301e-08$	5.0894	4.09
(18, 6)	$\frac{1}{10}$	$1.7181e-05$	–	1.78
	$\frac{1}{15}$	$1.6998e-06$	5.7053	2.42
	$\frac{1}{20}$	$3.1822e-07$	5.8243	3.53
	$\frac{1}{25}$	$9.6140e-08$	5.3639	3.81

Table 6: Maximum L_2 error, rate of convergence order in time direction for FLBRI of order 7, various values of (m,s), $h_x = \frac{1}{1000}$, $\alpha = 0.85$, $\beta = 1.5$ and run time.

(m,s)	h_t	Maximum L_2 error	Time convergence orders	Cpu time(s)
(7, 6)	$\frac{1}{8}$	$1.0421e-04$	–	0.60
	$\frac{1}{12}$	$8.3192e-06$	6.2345	0.89
	$\frac{1}{16}$	$1.2254e-06$	6.6578	1.56
	$\frac{1}{20}$	$2.7879e-07$	6.6348	1.84
(9, 6)	$\frac{1}{12}$	$1.2753e-05$	–	1.33
	$\frac{1}{16}$	$2.0186e-06$	6.4078	1.14
	$\frac{1}{20}$	$4.6457e-07$	6.5833	1.52
	$\frac{1}{24}$	$1.5054e-07$	6.1807	1.85
(11, 6)	$\frac{1}{12}$	$1.4771e-05$	–	1.60
	$\frac{1}{16}$	$2.5043e-06$	6.1688	1.13
	$\frac{1}{20}$	$5.9598e-07$	6.4334	1.58
	$\frac{1}{24}$	$1.9041e-07$	6.2584	2.23

Table 7: Maximum L_2 error and second order of convergence in space direction of FLBRI for $h_t = \frac{1}{1000}$, different values of (s, m), $\beta = 1.5$, various values of α and run time.

(s, m)	α	h_x	Maximum L_2 error	Space convergence orders	Cpu time(s)
$(4, 1)$	0.25	$\frac{1}{4}$	$1.8015e-06$	–	0.16
		$\frac{1}{8}$	$1.2671e-05$	1.2111	0.21
		$\frac{1}{16}$	$4.9212e-06$	1.3645	0.38
		$\frac{1}{32}$	$1.8015e-06$	1.4498	0.76
$(8, 2)$	0.45	$\frac{1}{4}$	$2.7493e-05$	–	0.14
		$\frac{1}{8}$	$1.2029e-05$	1.1925	0.21
		$\frac{1}{16}$	$4.7130e-06$	1.3518	0.39
		$\frac{1}{32}$	$1.7299e-06$	1.4459	0.76
$(9, 3)$	0.40	$\frac{1}{4}$	$2.8015e-05$	–	0.14
		$\frac{1}{8}$	$1.2209e-05$	1.1982	0.24
		$\frac{1}{16}$	$4.7715e-06$	1.3555	0.38
		$\frac{1}{32}$	$1.7498e-06$	1.4472	0.85
$(12, 4)$	0.50	$\frac{1}{4}$	$2.6926e-05$	–	0.13
		$\frac{1}{8}$	$1.1834e-05$	1.1860	0.19
		$\frac{1}{16}$	$4.6487e-06$	1.3481	0.41
		$\frac{1}{32}$	$1.7073e-06$	1.4451	0.79

Table 8: Maximum L_2 error, second order of convergence in space direction of FLBRI for $h_t = \frac{1}{1000}$, different values of (s, m), various values of β, $\alpha = 0.5$ and run time.

(s, m)	β	h_x	Maximum L_2 error	Space convergence orders	Cpu time(s)
$(7, 5)$	1.9	$\frac{1}{8}$	$9.0479e-06$	–	0.21
		$\frac{1}{16}$	$3.3325e-06$	1.4410	0.39
		$\frac{1}{32}$	$1.1760e-06$	1.5027	0.83
		$\frac{1}{64}$	$4.1558e-07$	1.5007	1.53
$(12, 6)$	1.95	$\frac{1}{8}$	$8.3326e-06$	–	0.22
		$\frac{1}{16}$	$3.0534e-06$	1.4483	0.38
		$\frac{1}{32}$	$1.0726e-06$	1.5093	0.73
		$\frac{1}{64}$	$3.7827e-07$	1.5036	1.49
$(19, 6)$	1.4	$\frac{1}{8}$	$1.1623e-05$	–	0.21
		$\frac{1}{16}$	$4.6876e-06$	1.3102	0.38
		$\frac{1}{32}$	$1.7410e-06$	1.4290	0.73
		$\frac{1}{64}$	$6.3086e-07$	1.4645	1.50
		$\frac{1}{69}$	$5.6456e-07$	1.4760	1.78
		$\frac{1}{74}$	$5.0912e-07$	1.4776	1.93
		$\frac{1}{94}$	$3.5726e-07$	1.4807	2.41
		$\frac{1}{114}$	$2.6830e-07$	1.4843	2.88