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Korovkin-type theorems via some modes of convergence

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Abstract. In this study, we investigate the Korovkin-type theorems depending upon some type of convergence such as alpha convergence, semi-alpha convergence and the notion of exhaustiveness. Since it is known that the convergence types mentioned above are between point-wise convergence and uniform convergence, it will be observed that the conditions can be alleviated in the Korovkin theorem.

1. Background and Brief History

Unarguably, as well as the convergence of a sequence of functions in ordinary means has an important role, it has various versions of convergence of the sequence of functions that have important applications. The concept of continuous convergence is one of them. The concept of "continuous convergence" (known as alpha convergence in recent years) was first used in the paper of R. Caurant [11] in 1914. Although it was defined and its properties were examined in the paper of H. Hahn [19] in 1921, he stated in his paper that different versions of this concept appeared in studies by Weierstrass and P. Du Bois-Reymond in 19th century. C. Carathéodory mentioned the concept of continuous convergence in his study [9] published in 1929. In 1955, H. Schaefer [26] proved a relationship between the concepts of continuous convergence and local uniform convergence. In 1957, K. Iseki [20] showed that continuous convergence is equivalent to uniform convergence for continuous function sequences on sets with some topological properties. Later, while studies in this area became more rare, different types of convergence of function sequences were defined and their properties were examined in the last quarter century. In 2003, Das and Papanastassiou [13] defined and examined new kind of convergence of real-valued function sequences called (alpha uniform equal, alpha strong uniform equal and alpha equal). Via using this definition they obtained a characterization of the compact metric space.

In addition to alpha convergence, the concepts of exhaustiveness and semi-alpha convergence and the relationships between these concepts are among the problems studied in recent years. In [13], Das and Papanastassiou studied the connections between alpha convergence and some other types of convergences for the sequence of functions. Moreover, Gregoriades and Papanastassiou [18] introduced the notion of exhaustiveness, and established some relations between alpha convergence and exhaustiveness for the sequences of functions. In 2020, Papanastassiou [25] came up with a new perspective in the name of alpha convergence, exhaustiveness and uniform convergence namely semi alpha convergence, semi exhaustiveness and semi uniform convergence for the sequence of functions, respectively.

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One aspect of the studies on convergence concepts is the adaptation of different kinds of concepts (such as statistical convergence and ideal convergence) to existing convergence concepts. In 2010, Papachristodoulos, Papanastassiou and Wilczynski [24] introduced the notions of I-alpha convergence and I-exhaustiveness. Later in [17], Ghosh establish some relationship between these concepts with some well-established concepts such as I*-alpha convergence and I*-exhaustiveness of sequences of metric functions. Later in 2012, Caserta and Kocinac [10] extended the notions of alpha convergence and exhaustiveness to statistical alpha convergence and statistical exhaustiveness respectively. Later in 2021 Das and Ghosh [12] studied the statistical versions of convergence of sequence of functions which introduced by Papanastasiou in 2020.

One of the most important theorems of constructive approximation theory is the Korovkin theorem [22]. While the original theorem was given according to the concept of uniform convergence, in recent years it has been given according to many different concepts of convergence and summability methods. A classical paper can be found in [16] for Korovkin type theorems in the sense statistical convergence. Then in [6] Braha et al. introduced a new weighted statistical convergence and based upon this definition, they prove some Korovkin type theorems. In [8], it has been proposed a new weighted statistical convergence by applying the Nörlund–Cesáro summability method. Based upon this definition, it has been proved a kind of the Korovkin type theorem. Introducing the notion of weighted Nörlund-Euler A-statistical convergence and its application to Korovkin-type theorems are investigated in [28]. In [29] and [30], Srivastava et al. established some statistical versions of new approximation of Korovin-type theorems for martingale sequences of positive linear operators. In [27] Srivastava et al. introduced the ideas of deferred weighted statistical Riemann integrability and statistical deferred weighted Riemann summability for sequences of functions. Then they stated and proved two Korovkin-type approximation theorems involving algebraic test functions by using their proposed concepts and methodologies. There have been also several studies on Korovkin-type theorems related to convergence associated with summability methods, statistical convergence and filter convergence (see [2], [4], [5], [7], [14], [15], [16], [21], [23], [32], and references therein).

In this paper we deal with Korovkin theorems depending upon the kind of convergences such as alpha convergence, semi-alpha convergence and notation of exhaustiveness. Since it is known that the convergence types mentioned above are between point-wise convergence and uniform convergence, it will be observed that the conditions can be alleviated in the Korovkin theorem.

2. Definitions and Auxiliary Results

Let (X, d) and (Y, ρ) be metric spaces, (f_n) be a sequence of functions from X to Y and f be a function from X to Y. Let us recall the definitions of alpha convergence, semi-alpha convergence and exhaustiveness.

Definition 2.1. *[13] The sequence (f_n) alpha converges to f, if for every $x \in X$ and for every sequence (x_n) of points of X converging to x, the sequence $(f_n(x_n))$ converges to $f(x)$.*

The notation $f_n \xrightarrow{\alpha} f$ will be used for alpha convergence of the sequence (f_n) to f. It is proved in [3] that the alpha convergence of the sequence (f_n) at $x_0 \in X$ to f is equivalent with the following condition:

$$\forall \varepsilon > 0, \exists \delta > 0, \exists n_0 \in \mathbb{N} : x \in B_d(x_0, \delta),\ \forall n \geq n_0 \implies \rho(f_n(x), f(x_0)) < \varepsilon$$

where $B_d(x_0, \delta)$ is the ball with radius δ centered at x_0 according to the metric d.

Proposition 2.2. *[18] If $(f_n) \xrightarrow{\alpha} f$ then $(f_{n_k}) \xrightarrow{\alpha} f$ for any strictly increasing sequence of positive integers (n_k).*

Definition 2.3. *[18] The sequence (f_n) is called exhausitive at $x_0 \in X$, if for every $\varepsilon > 0$ there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $x \in B_d(x_0, \delta)$ and all $n \geq n_0$ we have that $\rho(f_n(x), f_n(x_0)) < \varepsilon$.*

Definition 2.4. *The sequence (f_n) is called uniformly exhausitive on X if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and for all $x, y \in X$ that satisfy $d(x, y) < \delta$ implies $\rho\left(f_n(x), f_n(y)\right) < \varepsilon$.*

Definition 2.5. *The sequence (f_n) is called almost uniformly bounded on X if there exists $n_0 \in \mathbb{N}$ and $M > 0$ such that $\rho(f_n(x), 0) \leq M$ for all $n \geq n_0$ and all $x \in X$.*

Remark 2.6. *It is clear that the uniform boundedness of a sequence implies nearly uniformly boundedness. The inverse of this assertion is not true. For example, for $f_n : (1, \infty) \to \mathbb{R}$, $f_n(x) = x^{2n-n^2}$, the sequence (f_n) is not uniformly bounded, but almost uniformly bounded.*

Definition 2.7. *The sequence (f_n) is called locally almost uniformly bounded on X, if for all $x \in X$, there exists $\delta > 0$ such that the sequence (f_n) is almost uniformly bounded on $B_d(x, \delta)$.*

Proposition 2.8. *If the sequence (f_n) is exhaustive at x_0 and $(f_n(x_0))$ is bounded then (f_n) is almost uniformly bounded in a neigborhood at x_0.*

Proof. By boundedness of the sequence $(f_n(x_0))$, there exists a number $M > 0$ such that $\rho(f_n(x_0), 0) \leq M$ for all $n \in \mathbb{N}$. From exhaustiveness of the sequence (f_n) at x_0, there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $x \in B_d(x_0, \delta)$ we have $\rho(f_n(x), f_n(x_0)) < 1$. Since $\rho(f_n(x), 0) \leq 1 + \rho(f_n(x_0), 0) \leq 1 + M$ for all $n \geq n_0$ and all $x \in B_d(x_0, \delta)$, we get the desired result. □

Corollary 2.9. *Let the sequence (f_n) is exhaustive and pointwise bounded on X then (f_n) is locally almost uniformly bounded on X.*

Definition 2.10. *[25] Let $x_0 \in X$. The sequence (f_n) semi-alpha converges to f at x_0, it is denoted by $f_n \xrightarrow{semi-\alpha} f$, if*

1. *$f_n(x_0) \to f(x_0)$.*
2. *For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $m \geq n$ and $\rho(f_m(x), f(x_0)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$.*

A sequence (f_n) has the *semi-alpha property* with respect to f iff (f_n) satisfies the second condition of Definition 2.10. The semi-alpha property can be written as in the proposition given below.

Proposition 2.11. *Let $x_0 \in X$. The sequence (f_n) semi-alpha converges to f at x_0 iff*

1. *$f_n(x_0) \to f(x_0)$*
2. *There exists a strictly increasing sequence of positive integers (n_k) such that (f_{n_k}) is alpha convergent to f at x_0.*

Proof. Let $\varepsilon > 0$ and $x_0 \in \mathbb{R}$ are given. Assume that (f_n) converges to f and there exists a strictly increasing sequence of positive integers (n_k) such that (f_{n_k}) is alpha-convergent to f at x_0. From alpha-convergency there exsists $\delta > 0$ and $k^* \in \mathbb{N}$ such that for all $k \geq k^*$ and for all $x \in B(x_0, \delta)$ we have $\rho(f_{n_k}(x), f(x_0)) < \varepsilon$. Since $n_k \geq n_{k^*} \geq k^*$ so that $n_{k^*+n} \geq k^* + n > n$ for all $k \geq k^*$, then if we choose $m = n_{k^*+n}$ for all $n \in \mathbb{N}$ then for all $x \in B(x_0, \delta)$ we have

$$\rho(f_m(x), f(x_0)) = \rho(f_{n_{k^*+n}}(x), f(x_0)) < \varepsilon.$$

Now, assume that the sequence (f_n) semi-alpha converges to f at x_0. From here we construct the desired subsequence (n_k) as follows: From the second condition of Definition 2.10, there exists $n_1 \geq 1$ such that $\rho(f_{n_1}(x), f(x_0)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. Similarly, there exists $n_2 \geq n_1 + 1$ such that $\rho(f_{n_2}(x), f(x_0)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. If it continues in this way, there exists $n_k \geq n_{k-1} + 1$ such that $\rho(f_{n_k}(x), f(x_0)) < \varepsilon$ for all $x \in B_d(x_0, \delta)$. Consequently, we get a strictly increasing sequence of positive integers (n_k) such that $f_{n_k} \xrightarrow{\alpha} f$. □

With the motivation given by the second feature in the proposition, a subtype of semi-alpha convergence can be defined with the help of natural density. Let's first remind the definition of natural density: For $A \subseteq \mathbb{N}$, we denote the natural density of A by

$$d(A) = \lim_{n\to\infty} \frac{|\{k \in A : k \leq n\}|}{n}$$

if the limit exists, where $|A|$ denotes of the cardinality of the finite set A. It is well known that if $d(A_1) = d(A_2) = 1$ for $A_1, A_2 \subset \mathbb{N}$ then $d(A_1 \cap A_2) = 1$.

Definition 2.12. *It is called that the sequence (f_n) has densely semi-alpha property with respect to f at $x_0 \in X$ if there exists a strictly increasing sequence of positive integers (n_k), with $d(\{n_k\}) = 1$, such that (f_{n_k}) is alpha convergent to f at x_0. If the sequence (f_n) has densely semi-alpha property with respect to f at $x_0 \in X$ and $f_n(x_0) \to f(x_0)$, then it is called the sequence (f_n) densely semi-alpha conerverges to f at $x_0 \in X$.*

It's clear that if a function sequence has densely semi-alpha property then it has semi-alpha property. Reverse implication could not be true. For example the sequence (f_n) defined by

$$f_n : [0,1] \to \mathbb{R} \qquad f_n(x) = \begin{cases} \frac{1}{n}, & n \text{ is even} \\ 1, & n \text{ is odd} \end{cases}$$

has semi-alpha property with respect to zero function but does not have densely semi-alpha property.

Let $C(X)$ denote the space of real valued continuous functions and $B(X)$ denote the space of real valued bounded functions on the metric space (X, ρ). We will deal with the positive and linear operators defined on these spaces. The positivity of an L operator defined on these spaces will be understood as the fact that the $L(f)$ function is also positive for every positive function f. Let be $e_k(x) = x^k$ for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $x \in \mathbb{R}$. For $X = [a, b]$, let us give Korovkin's theorem to deal with an approximation property of the sequences of positive and linear operators on $C(X)$:

Theorem 2.13. *[22] Let (L_n) be a sequence of positive linear operators on $C[a, b]$. If the sequence $L_n(e_k)$ converges uniformly to e_k on $[a, b]$, for $k = 0, 1, 2$ then the sequence $L_n(f)$ converges uniformly to f on $[a, b]$ for all $f \in C[a, b]$.*

In the next section, we deal with Korovkin-type theorems depending upon the kind of convergences such as alpha convergence, semi-alpha convergence and exhaustiveness.

3. Korovkin-Type Theorems

Let (X, ρ) be a metric space for a bounded set $X \subset \mathbb{R}$ and $C_b(X)$ be the space of real valued, bounded and continuous functions on the metric space (X, ρ). For every $x \in X$ denote by $B(x; \delta)$, the set $\{y \in X : \rho(y, x) < \delta\}$ and by ρ_x the function $\rho_x(y) = \rho(x, y)$, $(y \in X)$. It is clear that $\rho_x \in C_b(X)$. In [1], Altomare extend the more general form of Korovkin's theorem to the settings of metric spaces. Using similar method, we give the Korovkin-type theorem based on the concept of alpha convergence.

Theorem 3.1. *Let (L_n) be a sequence of positive linear operators on $C_b(X)$. If $L_n(e_0) \xrightarrow{\alpha} e_0$ and $L_n(\rho_x^r)$ alpha converges to 0 at x for all $x \in X$ and for some $r > 0$, then $L_n(f) \xrightarrow{\alpha} f$ for all $f \in C_b(X)$.*

Proof. Let $f \in C_b(X)$, $x_0 \in X$ and (x_n) be a sequence such that $x_n \to x_0$. Let $\varepsilon > 0$. By the continuity of f at x_0, there exists $\delta > 0$ such that

$$\left|f(t) - f(x_0)\right| < \varepsilon$$

holds for all $t \in X$ that satisfies $\rho(x_0, t) < \delta$. On the other hand, in the case $\rho(x_0, t) \geq \delta$, we have

$$\left|f(t) - f(x_0)\right| \leq 2 \sup_{x \in X} |f(x)| \leq \frac{2M}{\delta} \rho(x_0, t),$$

where $M := \sup_{x \in X} |f(x)|$. Let $r > 0$. From the discussion above, we can write the following inequality on X:

$$\left|f - f(x_0)\right| \leq \varepsilon e_0 + \frac{2M}{\delta^r} \rho_{x_0}^r,$$

From the well known property of positive and linear operators, we have

$$\left|L_n(f; x_n) - f(x_0)\right| \leq L_n(\left|f - f(x_0)\right|; x_n) \leq \varepsilon L_n(e_0; x_n) + \frac{2M}{\delta^r} L_n(\rho_{x_0}^r; x_n)$$

for all $n \in \mathbb{N}$. Considering the alpha convergence of the sequences $L_n(e_0)$ and $L_n(\rho^r_{x_0})$, the right-hand side of the above inequality goes to ε as $n \to \infty$. Since, we get

$$\left|\lim_{n\to\infty} L_n(f;x_n) - f(x_0)\right| \leq \varepsilon$$

for arbitrary $\varepsilon > 0$, then we have $\lim\limits_{n\to\infty} L_n(f;x_n) = f(x_0)$. Consequently, we obtain the alpha convergence of the sequence $(L_n(f))$ to f at x_0. Since x_0 is arbitrary, the desired result is obtained. □

Remark 3.2. *If X compact, then the alpha convergence implies the uniform convergence (see prop. 1.3.(4) in [18]). So that, if we take $r = 2$ and a compact interval for the set X equipped with Euclidean metric in Theorem 3.1, we get the classical Korovkin theorem.*

Example 3.3. *For $X = (0,1)$, consider the operators L_n on $C_b(X)$*

$$L_n(f;x) = \begin{cases} f(1/2) + nf(x), & x \leq 1/n \\ f(x), & x > 1/n. \end{cases}$$

It is clear that the operators L_n are linear and positive. Although the sequence (L_n) does not satisfy the conditions of the classical Korovkin theorem, it satisfies the conditions of Theorem 3.1 for $r = 2$.

In the next theorem, let $X \subset \mathbb{R}$ be any set, bounded or unbounded.

Theorem 3.4. *Let (L_n) be a sequence of positive linear operators on $C(X)$. If $(L_n(e_0))$ is exhaustive and bounded at $x_0 \in X$, then $(L_n(f))$ is exhaustive at x_0 for all $f \in C(X)$.*

Proof. Let $f \in C(X)$, $x_0 \in X$ and $\varepsilon > 0$ be given. By exhaustiveness of $(L_n(e_0))$ at x_0, there exists $\delta_0 > 0$ and $N_0 \in \mathbb{N}$ such that for all $x \in X$ and for all $n \geq N_0$ that satisfying $\rho(x,x_0) < \delta$, we have

$$|L_n(e_0;x) - L_n(e_0;x_0)| < \frac{\varepsilon}{3\,(|f(x_0)| + 1)} := A_1(\varepsilon).$$

By boundedness of the sequence $(L_n(e_0;x_0))$, there exists $M > 0$ such that $L_n(e_0;x_0) \leq M$. By the continuty of f at x_0, there exists $\delta_1 > 0$ such that for all $x \in X$ that satisfies $\rho(x,x_0) < \delta_1$, we get

$$\left|f(x) - f(x_0)\right| < \frac{\varepsilon}{3\,(A_1(\varepsilon) + M)} := A_2(\varepsilon).$$

From properties of positive linear operators, we have

$$L_n(|f - f(x_0)|;x) < A_2(\varepsilon)\,|L_n(e_0;x) - L_n(e_0;x_0)| + A_2(\varepsilon)|L_n(e_0;x_0)|.$$

Now, if we choose $\delta = \min\{\delta_0, \delta_1\}$ and $N = N_0$ then for all $n > N$ and $x \in B(x_0;\delta)$, we have

$$\begin{aligned}
\left|L_n(f;x) - L_n(f;x_0)\right| &\leq \left|L_n(f;x) - L_n(f(x_0);x)\right| + \left|L_n(f(x_0);x) - L_n(f(x_0);x_0)\right| \\
&\quad + \left|L_n(f(x_0);x_0) - L_n(f;x_0)\right| \\
&\leq L_n(|f - f(x_0)|;x) + |f(x_0)|\,|L_n(e_0;x) - L_n(e_0;x_0)| \\
&\quad + L_n(|f - f(x_0)|;x_0) \\
&\leq 2A_2(\varepsilon)(A_1(\varepsilon) + M) + |f(x_0)|A_1(\varepsilon) < \varepsilon.
\end{aligned}$$

Hence $(L_n(f))$ is exhaustive at x_0. □

Theorem 3.5. *Let (L_n) be positive linear operators on $C(X)$. If $(L_n(e_0))$ is exhaustive and pointwise bounded on X then $(L_n(f))$ is exhaustive on X for all $f \in C(X)$.*

Proof. Let $f \in C(X)$, $x_0 \in X$ and $\varepsilon > 0$ be given. By exhaustiveness of $(L_n(e_0))$ at x_0, there exists $\delta_0 > 0$ and $N_0 \in \mathbb{N}$ such that for all $x \in X$ and for all $n \geq N_0$ that satisfy $\rho(x, x_0) < \delta_0$, we have

$$\rho\left(L_n(e_0; x), L_n(e_0; x_0)\right) < \frac{\varepsilon}{3\left(|f(x_0)| + 1\right)} = A_1(\varepsilon)$$

Exhaustiveness and pointwise boundedness of $(L_n(e_0))$ on X implies locally almost uniformly boundedness from Corollary 2.9. Therefore, there exists a real number $M > 0$, $\delta_1 > 0$ and $N_1 \in \mathbb{N}$ such that for all $x \in X$ that satisfy $\rho(x, x_0) < \delta_1$ and for all $n \geq N_1$, we have $|L_n(e_0; x)| \leq M$. By the continuity of f at x_0, there exists $\delta_2 > 0$ such that for all $x \in X$ that satisfies $\rho(x, x_0) < \delta_2$, we have

$$|f(x) - f(x_0)| < \frac{\varepsilon}{3M}$$

From properties of positive linear operators, we have

$$L_n\left(|f - f(x_0)|; x\right) < \frac{\varepsilon}{3M} |L_n(e_0; x)|.$$

Now, if we choose $\delta = \min\{\delta_0, \delta_1, \delta_2\}$ and $N = \max\{N_0, N_1\}$ then for all $n > N$ and $x \in B(x_0, \delta)$, we have

$$\begin{aligned}
\left|L_n(f; x) - L_n(f; x_0)\right| &\leq \left|L_n(f; x) - L_n(f(x_0); x)\right| + \left|L_n(f(x_0); x) - L_n(f(x_0); x_0)\right| \\
&\quad + \left|L_n(f(x_0); x_0) - L_n(f; x_0)\right| \\
&\leq L_n(|f - f(x_0)|; x) + |f(x_0)|\, |L_n(e_0; x) - L_n(e_0; x_0)| \\
&\quad + L_n(|f - f(x_0)|; x_0) \\
&\leq 2\tfrac{\varepsilon}{3M} |L_n(e_0; x)| + |f(x_0)| A_1(\varepsilon) < \varepsilon.
\end{aligned}$$

Hence $(L_n(f))$ is exhaustive at x_0. Thus $(L_n(f))$ is exhaustive on X. □

Example 3.6. *Consider the linear positive operators L_n on $C[0,1]$ defined by*

$$L_n(f; x) = \begin{cases} f(x)/n, & x \leq 1/2 \\ f(x)/2n, & x > 1/2. \end{cases}$$

It is clear that $(L_n(e_0))$ is exhausitive at $x = 1/2$ and bounded on $[0,1]$, so for every $f \in C[0,1]$, $(L_n(f))$ is exhaustive at $x = 1/2$. Indeed, for every $\varepsilon > 0$, we choose $\delta < 1/2$ and an integer $n_0 > \sup_{x\in[0,1]} |f(x)|/2\varepsilon$, then $\left|L_n(f; y) - L_n(f; 1/2)\right| < \varepsilon$ hold for all $n \geq n_0$ and for all $y \in B(1/2, \delta)$.

Example 3.7. *Consider the linear positive operators L_n on $C(0,1)$ defined by $L_n(f; x) = f(x) + nf(x_0)$. Let $x_0 \in (0,1)$ be fixed. For a function $f \in C(0,1)$ with $f(x_0) \neq 0$. the sequence $(L_n(f))$ does not converge uniformly on $(0,1)$, but it is exhaustive on $(0,1)$.*

Remark 3.8. *The condition about boundedness can not removed from Theorem 3.5.*

Example 3.9. *Consider the linear positive operators L_n on $C[0,1]$ defined by $L_n(f; x) = nf(x)$. Its clear that $(L_n(e_0))$ is not bounded. Although $(L_n(e_0))$ is exhaustive, the sequence $(L_n(f))$ is not exhaustive on $[0,1]$ for every f which is not constant.*

Remark 3.10. *Korovkin's theorem is not true for the concept of semi-alpha convergence. However, as we can see in the next theorem, it can be written for densely semi-alpha convergence. An example is given after the next theorem.*

Theorem 3.11. *Let (L_n) be a sequence of positive linear operators on $C_b(X)$ and $x_0 \in X$. If the sequence $(L_n(e_0))$ has densely semi-alpha property with respect to e_0 and the sequence $L_n(\rho^r_{x_0})$ has densely semi-alpha property with respect to 0, for some $r > 0$, at x_0 then $L_n(f)$ has densely semi-alpha property with respect to f at x_0 for all $f \in C_b(X)$.*

Proof. Let $f \in C_b(X)$ and $\varepsilon > 0$ be given. Since the sequence $(L_n(e_0))$ has densely semi-alpha property with respect to e_0 at x_0, then there exists a strictly increasing sequence of positive integers $(n^{(1)}_k)$, with $d(\{n^{(1)}_k\}) = 1$, such that $(L_{n^{(1)}_k}(e_0))$ is alpha convergent to e_0 at x_0. Similarly, since the sequence $(L_n(\rho^r_{x_0}))$ has densely semi-alpha property with respect to 0 at x_0 then there exists a strictly increasing sequence of positive integers $(n^{(2)}_k)$, with $d(\{n^{(2)}_k\}) = 1$, such that $(L_{n^{(2)}_k}(\rho^r_{x_0}))$ is alpha convergent to 0 at x_0. Because of the densely semi-alpha property implies the semi-alpha property, if we take the strictly increasing sequence of positive integers (n_k) in the set $\{n^{(1)}_k\} \cap \{n^{(2)}_k\}$ which has natural density 1, we obtain that $L_{n_k}(e_0) \xrightarrow{\alpha} e_0$ and $L_{n_k}(\rho^r_x)$ alpha converges to 0 at x_0 by using Proposition 2.2. Now, the desired result follows from Theorem 3.1. □

Example 3.12. *Let $x_0 \in [0, 1]$ be fixed and consider the linear positive operators L_n on $C[0, 1]$ defined by*

$$L_n(f; x) = \begin{cases} f(x_0), & x = x_0 \\ \int_0^1 f(t)K_n(t, x)dt, & x \neq x_0 \end{cases}$$

where

$$K_n(t, x) = (m + 1)x^m + \frac{1}{n}|x - x_0|, \quad if \quad n \equiv m(mod\, 3)$$

for $n \in \mathbb{N}$. Its obvious that $L_n(e_i) \xrightarrow{semi-\alpha} e_i$ at x_0 for $i = 0, 1, 2$, but $L_n(f)$ does not semi-alpha converge to f at x_0 for $f(x) = x^3$.

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