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Certain vector fields on f-Kenmotsu manifold with Schouten-van Kampen connection

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Abstract. The present study investigates various characteristics of conformal Ricci solitons with a Schouten-van Kampen connection. Characterizations are obtained when the potential vector field involves a torse-forming vector field. Moreover, applications related to submanifolds are also provided. Lastly, we provided an example of conformal Ricci solitons on a three-dimensional f-Kenmotsu manifold to validate our findings.

1. Introduction

Schouten-van Kampen is one of the most intuitive connections adapted to a pair of complementary distributions on a differentiable manifold with an affline connection. Solov'ev conducted a study in 1978 on hyperdistributions in Riemannian manifolds, utilising the Schouten-van Kampen connection [22]. Bejancu investigated the Schouten-van Kampen connection on Foliated manifolds in 2006 [2]. Olszak [17] researched the Schouten-van Kampen connection in 2013 to adapt it to a nearly contact metric structure . Using the Schouten-van Kampen connection, he characterised several classes of nearly contact metric manitolds. The Schouten-van Kampen connection in Sasakian manifolds, f-Kenmotsu manifolds and Kenmotsu manifolds has been investigated by G. Ghosh [10], Yildiz [24] and Chakraborty [5] in recent research. Y. S. Perktas and A. Yildiz [19] done research on f-Kenmotsu 3-manifolds in relation to the Schouten-van Kampen connection.
In 1982, the notion of Ricci flow was introduced by R. S. Hamilton [11]. The equation for Ricci flow is expressed as follows:

$$\frac{\partial g}{\partial t} = -2\bar{S}g.$$

A Riemannian manifold (M, g) is said to be a Ricci soliton if the metric g satisfies the necessary conditions

$$L_v g + 2\bar{S} + 2\lambda g = 0,$$

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the aforementioned equation involves the Lie derivative operator denoted by L, the Ricci tensor denoted by $\bar{S}$, a vector field on the manifold M denoted by v, and a real constant denoted by λ. It is a widely recognised fact in the field that when λ is a smooth function, the soliton is referred to as an almost Ricci soliton. A Ricci soliton can be classified as expanding, steady, or shrinking based on the value of λ, which is positive, zero, or negative, respectively. A modification to Hamilton's Ricci flow equation was proposed by A. E. Fisher [9], which involved the introduction of a conformal Ricci flow equation

$$\frac{\partial g}{\partial t} + 2(\bar{S} + \frac{g}{2n+1}) = -pg, \bar{r}(g) = -1,$$

the aforementioned equation relates the conformal pressure denoted by p to the scalar curvature of the manifold represented by $\bar{r}(g)$. Extensive research has been conducted on solitons in the context of manifolds and their associated connections [12] [13] [14]. Basu and Bhattacharyya have extended the notion of Ricci soliton by proposing the conformal Ricci soliton, which is defined by an equation [1]

$$L_v g + 2\bar{S} + (p + \frac{2}{2n+1} - 2\lambda)g = 0. \tag{1}$$

where λ is constant and p is conformal pressure.

2. Preliminaries

Let $(\bar{M}^{2n+1}, \varphi, \mathcal{N}, \nu, g)$ be a $(2n+1)$ dimensional almost contact metric manifold where φ is $(1,1)$ tensor field, $\mathcal{N}$ is structure vector field, ν is an 1-form and g is compatible Riemannian metric such that

$$\varphi^2(M_1) = -M_1 + \nu(M_1)\mathcal{N},$$

$$\nu(\mathcal{N}) = 1,$$

$$\varphi\mathcal{N} = 0, \nu\varphi = 0, \tag{2}$$

where M_1 is a vector field on $\bar{M}$.
The fundamental 2-form Φ on the manifold $\bar{M}$ is defined by

$$\Phi(M_1, M_2) = g(M_1, \varphi M_2), \tag{3}$$

for all M_1, M_2 on $\bar{M}$.
An almost contact metric manifold is normal if $[\varphi, \varphi](M_1, M_2) + 2d\nu(M_1, M_2)\mathcal{N} = 0$. An almost contact metric structure $(\varphi, \mathcal{N}, \nu, g)$ on a manifold $\bar{M}$ is designated as f-Kenmotsu manifold if the corresponding condition can be expressed [17]

$$(\bar{\nabla}_{M_1}\varphi)M_2 = f\left\{g(\varphi M_1, M_2)\mathcal{N} - \nu(M_2)\varphi M_1\right\}, \tag{4}$$

where $f \in C^\infty(\bar{M})$ such that $df \wedge \nu = 0$ and $\bar{\nabla}$ is Levi-Civita connection on $\bar{M}$. The manifold is an α-Kenmotsu manifold [15] if $f = \alpha =$ constant $\neq 0$. For $\alpha = 1$, α-Kenmotsu manifold reduces to Kenmotsu manifold [16]. If $f = 0$, then α-Kenmotsu manifold become cosymplectic manifold [15]. The condition for f-Kenmotsu manifold to be regular is $f^2 + f' \neq 0$, where $f' = \mathcal{N}(f)$. The following holds true for an f-Kenmotsu manifold

$$\bar{\nabla}_{M_1}\mathcal{N} = f\{M_1 - \nu(M_1)\mathcal{N}\} \tag{5}$$

It follows from above

$$(\bar{\nabla}_{M_1}\nu)M_2 = f\{g(M_1, M_2) - \nu(M_1)\nu(M_2)\} \tag{6}$$

The condition $df \wedge \nu = 0$ is satisfied if $\dim \bar{M} \geq 5$. This is not true generally if $\dim \bar{M} = 3$. In a 3-dimensional f-Kenmotsu manifold $\bar{M}$, we possess [18]

$$\begin{aligned} \bar{\mathcal{R}}(M_1, M_2)M_3 &= \left(\frac{\bar{r}}{2} + 2f^2 + 2f'\right)\{g(M_2, M_3)M_1 - g(M_1, M_3)M_2\} \\ &- \left(\frac{\bar{r}}{2} + 3f^2 + 3f'\right)\{g(M_2, M_3)\nu(M_1)\mathcal{N} - g(M_1, M_3)\nu(M_2)\mathcal{N} \\ &+ \nu(M_2)\nu(M_3)M_1 - \nu(M_1)\nu(M_3)M_2\}, \end{aligned} \tag{7}$$

$$\bar{\mathcal{S}}(M_1, M_2) = \left(\frac{\bar{r}}{2} + f^2 + f'\right)g(M_1, M_2) - \left(\frac{\bar{r}}{2} + 3f^2 + 3f'\right)\nu(M_1)\nu(M_2), \tag{8}$$

$$\bar{\mathcal{Q}}M_1 = \left(\frac{\bar{r}}{2} + f^2 + f'\right)M_1 - \left(\frac{\bar{r}}{2} + 3f^2 + 3f'\right)\nu(M_1)\mathcal{N}, \tag{9}$$

$$\bar{\mathcal{R}}(M_1, M_2)\mathcal{N} = -\left(f^2 + f'\right)\{\nu(M_2)M_1 - \nu(M_1)M_2\}, \tag{10}$$

$$\bar{\mathcal{R}}(\mathcal{N}, M_1)M_2 = -\left(f^2 + f'\right)\{g(M_1, M_2)\mathcal{N} - \nu(M_2)M_1\}, \tag{11}$$

$$\bar{\mathcal{S}}(M_1, \mathcal{N}) = -2(f^2 + f')\nu(M_1), \tag{12}$$

$$\nu(\bar{\mathcal{R}}(\mathcal{N}, M_1)M_2) = -(f^2 + f')\{g(M_1, M_2) - \nu(M_2)\nu(M_1)\}, \tag{13}$$

where $\bar{\mathcal{R}}$, $\bar{\mathcal{S}}$, $\bar{\mathcal{Q}}$, $\bar{r}$ denotes curvature tensor, Ricci tensor, Ricci operator and scalar curvature respectively.

The relationship between the Schouten-van Kampen connection $\bar{\tilde{\nabla}}$ and the Levi-Civita connection $\bar{\nabla}$ on a manifold $\bar{M}$ is defined as follows [17]:

$$\bar{\tilde{\nabla}}_{M_1}M_2 = \bar{\nabla}_{M_1}M_2 - \nu(M_2)\bar{\nabla}_{M_2}\mathcal{N} + (\bar{\nabla}_{M_1}\nu)(M_2)\mathcal{N}, \tag{14}$$

for all the vector field M_1, M_2 on $\bar{M}$. Using the aid of (5), (6), we have

$$\bar{\tilde{\nabla}}_{M_1}M_2 = \bar{\nabla}_{M_1}M_2 + f\{g(M_1, M_2)\mathcal{N} - \nu(M_2)M_1\}. \tag{15}$$

Let $\bar{\mathcal{R}}$ and $\bar{\tilde{\mathcal{R}}}$ be the curvature tensor with respect to Levi-Civita connection $\bar{\nabla}$ and the Schouten-van Kampen connection $\bar{\tilde{\nabla}}$, as a result, $\bar{\mathcal{R}}$ and $\bar{\tilde{\mathcal{R}}}$ are linked by the following formula

$$\begin{aligned} \bar{\tilde{\mathcal{R}}}(M_1, M_2)M_3 &= \bar{\mathcal{R}}(M_1, M_2)M_3 + f^2\{g(M_2, M_3)M_1 - g(M_1, M_3)M_2\} \\ &+ f'\{g(M_2, M_3)\nu(M_1)\mathcal{N} - g(M_1, M_3)\nu(M_2)\mathcal{N} \\ &+ \nu(M_2)\nu(M_3)M_1 - \nu(M_1)\nu(M_3)M_2\}. \end{aligned} \tag{16}$$

Upon computing the inner product of the aforementioned equation with a vector field M_4 and subsequently contracting it, we obtain the following result

$$\bar{\tilde{\mathcal{S}}}(M_2, M_3) = \bar{\mathcal{S}}(M_2, M_3) + (2f^2 + f')g(M_2, M_3) + f'\nu(M_2)\nu(M_3), \tag{17}$$

where $\bar{\tilde{\mathcal{S}}}$ and $\bar{\mathcal{S}}$ denote the Ricci tensor with respect to connection $\bar{\tilde{\nabla}}$ and $\bar{\nabla}$, respectively. As an outcome of the preceding (17), we have the Ricci operator

$$\bar{\tilde{\mathcal{Q}}}M_2 = \bar{\mathcal{Q}}M_2 + (2f^2 + f')M_2 + f'\nu(M_2)\mathcal{N}. \tag{18}$$

Also putting $M_2 = M_3 = e_i$ in (17), we obtain

$$\bar{\tilde{r}} = \bar{r} + 6f^2 + 4f', \tag{19}$$

where $\bar{\tilde{r}}$ and $\bar{r}$ are scalar curvature tensor with respect to connection $\bar{\tilde{\nabla}}$ and $\bar{\nabla}$ respectively. Putting M_3=$\mathcal{N}$ in (17) and using (12), we have

$$\bar{\tilde{\mathcal{S}}}(M_2, \mathcal{N}) = 0. \tag{20}$$

In the realm of f-Kenmotsu manifolds ($\bar{M}^{2n+1}$, g), a non-flat manifold of this type is referred to as a hyper-generalized quasi-Einstein manifold [21] if its Ricci tensor is not identically zero and satisfies the condition

$$\bar{S} = c_1 g + c_2(T_1 \otimes T_1) + c_3(T_1 \otimes T_2 + T_2 \otimes T_1) + c_4(T_1 \otimes T_3 + T_3 \otimes T_1),$$

where c_1, c_2, c_3 and c_4 are functions on $\bar{M}$ called associated functions and T_1, T_2, T_3 are non-zero 1-forms. If $c_3 = c_4 = 0$, then $\bar{M}$ is called a *quasi − Einstein* manifold [4]. If $c_2 = c_3 = c_4 = 0$, then $\bar{M}$ is an *Einstein − manifold* [3].
A vector field defined on an f-Kenmotsu manifold is deemed to be *torse − forming* [23], if it satisfies

$$\bar{\nabla}_{M_1} v = hM_1 + \psi(M_1)v.$$

where ψ is a 1-form, h is a smooth function and $\bar{\nabla}$ is a Levi-Civita connection of g. Specifically, if $\psi = 0$, v is referred to as a concircular vector field [8] and if $h = 0$, v is referred to as a recurrent vector field [20].

3. Conformal Ricci solitons on f-Kenmotsu manifolds with Schouten-van Kampen Connection

This section examines conformal Ricci solitons on an f-Kenmotsu manifold equipped with Schouten-van Kampen connection. First we state the following proposition which we use further to characterize the conformal Ricci solitons on an f-Kenmotsu manifold. Consider $\mathcal{N}$ to be a parallel unit vector field relative to the Levi-Civita connection $\bar{\nabla}$. Using (15), we get

$$\bar{\tilde{\nabla}}_{M_1}\mathcal{N} = f(\nu(M_1)\mathcal{N} - M_1). \tag{21}$$

So we have:

Proposition 3.1. *Let $(\bar{M}^{2n+1}, g, \phi, \mathcal{N}, \nu)$ is a f-Kenmotsu manifold equipped with a Schouten-van Kampen connection. If $\mathcal{N}$ is a parallel unit vector field in relation to the Levi-Civita connection $\bar{\nabla}$ then, $\mathcal{N}$ is a torse-forming vector field with respect to a Schouten-van Kampen connection of the form*

$$\bar{\tilde{\nabla}}_{M_1}\mathcal{N} = f(\nu(M_1)\mathcal{N} - M_1).$$

Theorem 3.2. *Let $(\bar{M}^{2n+1}, g, \phi, \mathcal{N}, \nu)$ be a f-Kenmotsu manifold bearing almost conformal Ricci soliton with Schouten-van Kampen connection. If $\mathcal{N}$ is parallel vector field with Levi-Civita connection then the metric g is quasi-Einstein with respect to Levi-Civita connection as well as Schouten-van Kampen connection. Moreover in this case the soliton is expanding if $\frac{p}{2} + \frac{1}{2n+1} \geq 0$, shrinking if $\frac{p}{2} + \frac{1}{2n+1} \leq 0$ and steady if $\frac{p}{2} + \frac{1}{2n+1} = 0$.*

Proof. If $(g, \lambda, \mathcal{N})$ is conformal Ricci soliton on $\bar{M}$. Then using equation (1) we have

$$g(\bar{\tilde{\nabla}}_{M_1}\mathcal{N}, M_2) + g(M_1, \bar{\tilde{\nabla}}_{M_2}\mathcal{N}) + 2\bar{\tilde{S}}(M_1, M_2) + (p + \frac{2}{2n+1} - 2\lambda)g(M_1, M_2) = 0. \tag{22}$$

Further, if $\mathcal{N}$ is parallel vector field with respect to Levi-Civita connection then making use of proposition (3.1) in (22) we get

$$\bar{\tilde{S}}(M_1, M_2) = (f - \frac{p}{2} - \frac{1}{2n+1} + \lambda)g(M_1, M_2) - f\nu(M_1)(\nu M_2). \tag{23}$$

By virtue of equation (17) and (23) it is easy to see that $\bar{M}$ is quasi-Einstein with respect to Levi-Civita connection as well as Schouten-van Kampen connection. Next, using $M_2 = \mathcal{N}$ in (23) we obtained

$$\bar{\tilde{S}}(M_1, \mathcal{N}) = (\lambda - \frac{p}{2} - \frac{1}{2n+1})\nu(M_1). \tag{24}$$

Finally, equation (20) and (24) yields

$$\lambda = \frac{p}{2} + \frac{1}{2n+1} \tag{25}$$

which proves our assertion. □

Next we prove,

Theorem 3.3. *Let $(\bar{M}^{2n+1}, g)$ be an f-Kenmotsu manifold that admits a Schouten-van Kampen connection, and let v be a torse-forming potential vector field with regard to Levi-Civita connection on $\bar{M}$. Then $(\bar{M}^{2n+1}, g)$ is a conformal Ricci soliton (v, λ, g) if and only if $\tilde{\bar{S}}$ satisfies*

$$\begin{aligned}\tilde{\bar{S}}(M_1, M_2) &= [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) + f\nu(v) - h]g(M_1, M_2) - \frac{1}{2}f[\omega(M_1)\nu(M_2) \\ &+ \omega(M_2)\nu(M_1)] - \frac{1}{2}[\omega(M_2)\psi(M_1) + \omega(M_1)\psi(M_2)].\end{aligned} \tag{26}$$

Proof. Let $\bar{M}$ denote an f-Kenmotsu manifold equipped with a Schouten-van Kampen connection. Then taking the Lie derivative of torse forming potential vector field v with respect to Schouten-van Kampen connection and making use of equation (15) gives

$$(\tilde{\bar{L}}_v)(M_1, M_2) = g(\bar{\nabla}_{M_1} v, M_2) + g(M_1, \bar{\nabla}_{M_2} v) - 2f\nu(v)g(M_1, M_2) + fg(M_1, v)\nu(M_2) + fg(M_2, v)\nu(M_1). \tag{27}$$

Therefore, using the definition of conformal Ricci soliton, we have

$$\begin{aligned}[2\lambda - (p + \frac{2}{2n+1}) + 2f\nu(v)]g(M_1, M_2) &= g(\bar{\nabla}_{M_1} v, M_2) + g(M_1, \bar{\nabla}_{M_2} v) + fg(M_1, v)\nu(M_2) \\ &+ fg(M_2, v)\nu(M_1) - 2\tilde{\bar{S}}(M_1, M_2).\end{aligned} \tag{28}$$

If v be a torse forming potential vector field in relation to a Levi-Civita connection on $\bar{M}$. then we have

$$\bar{\nabla}_{M_1} v = hM_1 + \psi(M_1)v.$$

where h is a smooth function. In accordance with equation (28), it is possible to express the given statement

$$\begin{aligned}\tilde{\bar{S}}(M_1, M_2) &= [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) + f\nu(v) - h]g(M_1, M_2) - \frac{1}{2}[g(M_2, v)\psi(M_1) + g(M_1, v)\psi(M_2)] \\ &- \frac{1}{2}f[g(M_1, v)\nu(M_2) + g(M_2, v)\nu(M_1)].\end{aligned}$$

which complete the proof. □

If v is a concircular vector field relative to the Schouten-van Kampen connection, then the following corollary holds:

Corollary 3.4. *Let $(\bar{M}^{2n+1}, g)$ be an f-Kenmotsu manifold that admits a Schouten-van Kampen connection, and let v be a* concircular potential vector *field with regard to a Schouten-van Kampen connection on $\bar{M}$. Consider that ν is the g dual of v. Then $(\bar{M}^{2n+1}, g)$ is a conformal Ricci soliton (v, λ, g) if and only if $\bar{M}$ is* quasi-Einstein manifold *with associated functions $\lambda - [\frac{p}{2} + \frac{1}{2n+1}] + f\|v\|^2 - h$, $-f$.*

Theorem 3.5. *Let $(\bar{M}^{2n+1}, g)$ be an f-Kenmotsu manifold that admits a Schouten-van Kampen connection, and let v be a torse-forming potential vector field with regard to a Schouten-van Kampen connection on $\bar{M}$. Consider that ω is the g dual of v where ω is 1-form. Then $(\bar{M}^{2n+1}, g)$ is a conformal Ricci soliton (v, λ, g) if and only if $\bar{M}$ is a hyper-generalised quasi-Einstein manifold with associated functions $\lambda - [\frac{p}{2} + \frac{1}{2n+1}] + f\nu(v) - h$, 0, $\frac{f}{2}$, $-\frac{1}{2}$.*

Proof. Now, we have

$$\begin{aligned}\tilde{\bar{S}}(M_1, M_2) &= [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) + f\nu(v) - h]g(M_1, M_2) - \frac{1}{2}[g(M_2, v)\psi(M_1) + g(M_1, v)\psi(M_2)] \\ &- \frac{1}{2}f[g(M_1, v)\nu(M_2) + g(M_2, v)\nu(M_1)].\end{aligned}$$

Considering that ω is a 1-form is the g-dual of v, then from above metioned equation, we get

$$\begin{aligned}\tilde{\bar{S}}(M_1, M_2) &= [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) + f\nu(v) - h]g(M_1, M_2) - \frac{1}{2}f[\omega(M_1)\nu(M_2) + \omega(M_2)\nu(M_1)] \\ &- \frac{1}{2}[\omega(M_2)\psi(M_1) + \omega(M_1)\psi(M_2)].\end{aligned}$$

which complete the proof. □

From (17), the equation (29) is also possible as

$$\begin{aligned}\bar{S}(M_1, M_2) &= [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) + f\nu(v) - h - (2f^2 + f')]g(M_1, M_2) - f'\nu(M_1)\nu(M_2) \\ &- \frac{1}{2}f[g(M_1, v)\nu(M_2) + g(M_2, v)\nu(M_1)] - \frac{1}{2}[g(M_2, v)\psi(M_1) + g(M_1, v)\psi(M_2)].\end{aligned}$$

Therefore we can state the following corollary:

Corollary 3.6. *Let ($\bar{M}^{2n+1}$, g) be an f-Kenmotsu manifold that admits a Schouten-van Kampen connection, and let v be a torse-forming potential vector field with regard to a Levi-Civita connection on $\bar{M}$. Then ($\bar{M}^{2n+1}$, g) is a conformal Ricci soliton (v, λ, g) if and only if*

$$\begin{aligned}\bar{S}(M_1, M_2) &= [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) + f\nu(v) - h - (2f^2 + f')]g(M_1, M_2) - f'\nu(M_1)\nu(M_2) \\ &- \frac{1}{2}f[g(M_1, v)\nu(M_2) + g(M_2, v)\nu(M_1)] - \frac{1}{2}[g(M_2, v)\psi(M_1) + g(M_1, v)\psi(M_2)].\end{aligned}$$

Moreover, the following theorem holds true:

Theorem 3.7. *Let ($\bar{M}^{2n+1}$, g) be an f-Kenmotsu manifold that admits a Schouten-van Kampen connection, and let v be a torse-forming potential vector field and N a parallel unit vector field with regard to a Schouten-van Kampen connection on $\bar{M}$. Consider that ω is the g dual of v where ω is 1-form. Then ($\bar{M}^{2n+1}$, g) is a conformal Ricci soliton (v, λ, g) if and only if $\bar{M}$ is a hyper-generalised quasi-Einstein manifold with associated functions* $\lambda - [\frac{p}{2} + \frac{1}{2n+1}] + f\nu(v) - h - (2f^2 + f'), -f', -\frac{f}{2}, -\frac{1}{2}$.

4. Submanifolds

Let $(\bar{M}, \bar{g})$ be an (2n+1)-dimensional f-Kenmotsu manifold equipped with Schouten-van Kampen connection $\tilde{\bar{\nabla}}$ and Levi-Civita connection $\bar{\nabla}$. Assume that M be an n-dimensional submanifold of $(\bar{M}, \bar{g})$. On the submanifold M, the associated connection is denoted by $\tilde{\nabla}$ and the associated Levi-Civita connection is denoted by ∇.

The Gauss and Weingarten formulations in terms of $\bar{\nabla}$ and $\tilde{\bar{\nabla}}$ can be expressed as:

$$\bar{\nabla}_{M_1} M_2 = \nabla_{M_1} M_2 + \eta(M_1, M_2),$$

$$\tilde{\bar{\nabla}}_{M_1} M_2 = \tilde{\nabla}_{M_1} M_2 + \tilde{\eta}(M_1, M_2)$$

where $M_1, M_2 \in T\bar{M}$, and

and

$$\bar{\nabla}_{M_1} P = -S_P M_1 + \nabla^{\perp}{}_{M_1} P,$$

$$\tilde{\bar{\nabla}}_{M_1} P = -\tilde{S}_P M_1 + \nabla^{\perp}{}_{M_1} P,$$

where $M_1, M_2 \in T\bar{M}$, S_P is the shape operator of M, P is a unit normal vector field and η is the second fundamental form in $(\bar{M}, \bar{g})$ and $\tilde{S}$ is a (1, 1)-tensor field and $\tilde{\eta}$ is second fundamental form on M [6]. Let us denote the tangential parts of $\mathcal{N}$ by $\mathcal{N}^T$ and normal parts of $\mathcal{N}$ by $\mathcal{N}^\perp$. Then, based on the formula [6], we get

$$\tilde{\eta}(M_1, M_2) = \eta(M_1, M_2) - g(M_1, M_2)\mathcal{N}^\perp \tag{29}$$

and

$$\tilde{S}_P M_1 = S_P M_1 - \nu(P)M_1. \tag{30}$$

Also from [6] we have that the associated connection $\tilde{\nabla}$ on the submanifold of an f-Kenmotsu manifold possessed with Schouten-van Kampen connection is also a Schouten-van Kampen connection.

Suppose now that $(\bar{M}, \bar{g})$ is a f-Kenmotsu manifold possessed with Schouten-van Kampen connection and v is a torse-forming vector field with respect to Schouten-van Kampen connection on $\bar{M}$. Let (M, g) denotes the submanifold of $(\bar{M}, \bar{g})$. Let us denote the tangential parts of v by v^T and normal parts of v by $v^\perp$. Then using (15), we have

$$\begin{aligned}\tilde{\bar{\nabla}}_{M_1} v &= \tilde{\bar{\nabla}}_{M_1}(v^T + v^\perp) = \tilde{\bar{\nabla}}_{M_1} v^T + \tilde{\bar{\nabla}}_{M_1} v^\perp \\ &= \bar{\nabla}_{M_1} v^T + f\{g(M_1, v^T)\mathcal{N} - \nu(v^T)M_1\} + \bar{\nabla}_{M_1} v^\perp + f\{g(M_1, v^\perp)\mathcal{N} - \nu(v^\perp)M_1\} \\ &= hM_1 + \psi(M_1)v^T + \psi(M_1)v^\perp.\end{aligned}$$

Utilising the Gauss and Weingarten formulas, as well as the equality between the tangential and normal portions, we find

$$\nabla_{M_1} v^T = (h - f\nu(v^T))M_1 - fg(M_1, v^T)\mathcal{N} + S_{v^\perp} M_1 + \psi(M_1)v^T \tag{31}$$

and

$$\psi(M_1)v^\perp = \eta(M_1, v^T) + \nabla^\perp_{M_1} v^\perp + f\{g(M_1, v^\perp)\mathcal{N} - \nu(v^\perp)M_1\}.$$

then, based on the (31) equation, we obtain

$$\begin{aligned}(L_{v^T} g)(M_1, M_2) &= g(\nabla_{M_1} v^T, M_2) + g(M_1, \nabla_{M_2} v^T) \\ &= 2(h - f\nu(v^T))g(M_1, M_2) - f[g(M_1, v^T)\nu(M_2) + g(M_2, v^T)\nu(M_1)] \\ &+ g(M_1, v^T)\psi(M_2) + g(M_2, v^T)\psi(M_1) + 2\bar{g}(\eta(M_1, M_2), v^T).\end{aligned}$$

Therefore, equation (1) provides us

$$\begin{aligned}\mathcal{S}(M_1, M_2) &= [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) - (h - f\nu(v^T))]g(M_1, M_2) - \bar{g}(\eta(M_1, M_2), v^T) - \frac{f}{2}[g(M_1, v^T)\nu(M_2) \\ &+ g(M_2, v^T)\nu(M_1)] - \frac{1}{2}[g(M_1, v^T)\psi(M_2) + g(M_2, v^T)\psi(M_1)].\end{aligned}$$

Thus, the following theorem can be stated:

Theorem 4.1. *Let M be an n-dimensional submanifold isometrically submerged into a f-Kenmotsu manifold $(\bar{M}, \bar{g})$ equipped with a Schouten-van Kampen connection and let v be a torse-forming potential vector field with regard to a Schouten-van Kampen connection on $\bar{M}$. Then M is a conformal Ricci soliton if and only if the Ricci tensor field $\mathcal{S}$ of M satisfies the condition:*

$$\begin{aligned}\mathcal{S}(M_1, M_2) &= [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) - (h - f\nu(v^T))]g(M_1, M_2) - \bar{g}(\eta(M_1, M_2), v^T) - \frac{f}{2}[g(M_1, v^T)\nu(M_2) \\ &+ g(M_2, v^T)\nu(M_1)] - \frac{1}{2}[g(M_1, v^T)\psi(M_2) + g(M_2, v^T)\psi(M_1)].\end{aligned} \tag{32}$$

for every $M_1, M_2 \in T\bar{M}$.

In the circumstance in which M is $v^{\perp}$-umbilical, it can be deduced that $S_{v^{\perp}}$ is equivalent to JI, where J represents a function on M and I denotes the identity map [7]. Subsequently, utilising the aforementioned equation (32), it can be concluded that

$$\begin{aligned}\mathcal{S}(M_1, M_2) &= [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) - (h - f\nu(v^T)) - J]g(M_1, M_2) + \frac{f}{2}[g(M_1, v^T)\nu(M_2) + g(M_2, v^T)\nu(M_1)] \\ &- \frac{1}{2}[g(M_1, v^T)\psi(M_2) + g(M_2, v^T)\psi(M_1)].\end{aligned}$$

Thus, the following theorem can be stated:

Theorem 4.2. *Let M be an n-dimensional $v^{\perp}$-umbilical submanifold isometrically submerged into a f-Kenmotsu manifold $(\bar{M}, \bar{g})$ equipped with a Schouten-van Kampen connection and let v be a torse-forming potential vector field with regard to a Schouten-van Kampen connection on $\bar{M}$. Consider that ω is the g dual of v^T where ω is 1-form. Then (M^n, g) is a conformal Ricci soliton (v^T, λ, g) if and only if M is a hyper-generalised quasi-Einstein manifold with associated functions $\lambda - (\frac{p}{2} + \frac{1}{2n+1}) - (h - f\nu(v^T)) - J$, 0, $-\frac{f}{2}$, $-\frac{1}{2}$.*

Due to the fact that the induced connection $\tilde{\nabla}$ on the submanifold of a f-Kenmotsu manifold endowed with a Schouten-van Kampen connection is also a Schouten-van Kampen connection. Then, from (29), (32), we get

$$\begin{aligned}\tilde{\mathcal{S}}(M_1, M_2) &= [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) + f\nu(v^T) - h + (2f^2 + f')]g(M_1, M_2) + f'\nu(M_1)\nu(M_2) \\ &- \bar{g}(\tilde{\eta}(M_1, M_2), v^T) - \frac{1}{2}f[g(M_1, v^T)\nu(M_2) + g(M_2, v^T)\nu(M_1)] - \frac{1}{2}[g(M_2, v^T)\psi(M_1) + g(M_1, v^T)\psi(M_2)],\end{aligned} \tag{33}$$

where $\tilde{\mathcal{S}}$ denotes the Ricci tensor of the induced Schouten-van Kampen connection.
Then the following corollary holds:

Corollary 4.3. *Let M be an n-dimensional submanifold isometrically submerged into a f-Kenmotsu manifold $(\bar{M}, \bar{g})$ equipped with a Schouten-van Kampen connection and let v be a torse-forming potential vector field with regard to a Schouten-van Kampen connection on $\bar{M}$. Then (M^n, g) is a conformal Ricci soliton (v^T, λ, g) if and only if the induced Ricci tensor $\tilde{\mathcal{S}}$ with respect to Schouten-van Kampen connection of M satisfies:*

$$\begin{aligned}\tilde{\mathcal{S}}(M_1, M_2) &= [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) + f\nu(v^T) - h + (2f^2 + f')]g(M_1, M_2) + f'\nu(M_1)\nu(M_2) \\ &- \frac{1}{2}f[g(M_1, v^T)\nu(M_2) + g(M_2, v^T)\nu(M_1)] - \frac{1}{2}[g(M_2, v^T)\psi(M_1) + g(M_1, v^T)\psi(M_2)] - \bar{g}(\tilde{\eta}(M_1, M_2), v^T),\end{aligned}$$

for every $M_1, M_2 \in T\bar{M}$.

If M is $v^{\perp}$-umbilical, then by (30), we get

$$\tilde{S}_{v^T} M_1 = (J - \nu(v^T))M_1,$$

which provides us

$$(J - \nu(v^T))g(M_1, M_2) = g(\tilde{S}_{v^T} M_1, M_2) = \bar{g}(\tilde{\eta}(M_1, M_2), v^T).$$

Therefore from (33), we establish

$$\begin{aligned}\tilde{\mathcal{S}}(M_1, M_2) &= [\lambda - (\frac{p}{2} + \frac{1}{2n+1}) + f\nu(v^T) - h + (2f^2 + f') - J + \nu(v^T)]g(M_1, M_2) + f'\nu(M_1)\nu(M_2) \\ &- \frac{1}{2}f[g(M_1, v^T)\nu(M_2) + g(M_2, v^T)\nu(M_1)] - \frac{1}{2}[g(M_2, v^T)\psi(M_1) + g(M_1, v^T)\psi(M_2)].\end{aligned}$$

Thus, the following theorem can be stated:

Theorem 4.4. *Let M be an n-dimensional $v^{\perp}$-umbilical submanifold isometrically submerged into a f-Kenmotsu manifold $(\bar{M}, \bar{g})$ equipped with a Schouten-van Kampen connection and let v be a torse-forming potential vector field with regard to a Schouten-van Kampen connection on $\bar{M}$. Consider that ω is the g dual of v^T where ω is 1-form and $\mathcal{N}$ is a parallel unit vector with regard to a Levi-Civita connection $\bar{\nabla}$. Then (M^n, g) is a conformal Ricci soliton (v^T, λ, g) if and only if M is a hyper-generalised quasi-Einstein manifold with associated functions $\lambda - (\frac{p}{2} + \frac{1}{2n+1}) - h - (1-f)\nu(v^T) - J + (2f^2 + f'), f', -\frac{f}{2}, -\frac{1}{2}$.*

5. Example

We considered an 3-dimension manifold $\bar{M}^{2n+1} = \{(u, v, w)\} \in \mathbf{R}^3$, where (u, v, w) are the standard coordinates in $\mathbf{R}^3$ [19].

We select vector fields that are linearly independent of one another:

$$e_1 = w^2 \frac{\partial}{\partial u}, \quad e_2 = w^2 \frac{\partial}{\partial v}, \quad e_3 = \frac{\partial}{\partial w}$$

Let g denote the Riemannian metric defined by the expression: $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$ and $g(e_i, e_j) = 0$, for $i \neq j$.

Let ν be 1-form defined by $\nu(M_3) = g(M_3, e_3)$ for any $M_3 \in \bar{M}$, let φ be the (1, 1) tensor field defined by:

$$\varphi(e_1) = -e_2, \quad \varphi(e_2) = -e_1, \quad \varphi(e_3) = 0.$$

Using the linearity of g and φ, we have

$$\nu(e_3) = 1, \quad \varphi^2 M_3 = -M_3 + \nu(M_3)e_3, \quad g(\varphi M_3, \varphi M_4) = g(M_3, M_4) - \nu(M_3)\nu(M_4),$$

For Levi-Civita connection $\bar{\nabla}$ we have the following:

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -\frac{2}{w} e_2, \quad [e_1, e_3] = -\frac{2}{w} e_1.$$

Now using the Koszul formula for metric g, we obtain the following:

$$\begin{aligned}
&\bar{\nabla}_{e_1} e_3 = -\frac{2}{w} e_1, \quad \bar{\nabla}_{e_1} e_2 = 0, \quad \bar{\nabla}_{e_1} e_1 = \frac{2}{w} e_3,\\
&\bar{\nabla}_{e_2} e_3 = -\frac{2}{w} e_2, \quad \bar{\nabla}_{e_2} e_2 = \frac{2}{w} e_3, \quad \bar{\nabla}_{e_2} e_1 = 0,\\
&\bar{\nabla}_{e_3} e_3 = 0, \quad \bar{\nabla}_{e_3} e_2 = 0, \quad \bar{\nabla}_{e_3} e_1 = 0.
\end{aligned}$$

From above we finds that manifold satisfies $\bar{\nabla}_{M_1} \mathcal{N} = f(M_1 - \nu(M_1)\mathcal{N})$ for $\mathcal{N} = e_3$, where $f = -\frac{2}{w}$. Hence the manifold is f-Kenmotsu manifold. Also $f^2 + f' \neq 0$. Hence $\bar{M}$ is a regular f-Kenmotsu manifold.

The components of Riemannian curvature($\bar{\mathcal{R}}$) in terms of the Levi-Civita connection $\bar{\nabla}$ are as follows:

$$\begin{aligned}
&\bar{\mathcal{R}}(e_1, e_2)e_3 = 0, \quad \bar{\mathcal{R}}(e_2, e_3)e_3 = -\frac{6}{w^2} e_2, \quad \bar{\mathcal{R}}(e_1, e_3)e_3 = -\frac{6}{w^2} e_1,\\
&\bar{\mathcal{R}}(e_1, e_2)e_2 = -\frac{4}{w^2} e_1, \quad \bar{\mathcal{R}}(e_2, e_3)e_2 = -\frac{6}{w^2} e_3, \quad \bar{\mathcal{R}}(e_1, e_3)e_2 = 0,\\
&\bar{\mathcal{R}}(e_1, e_2)e_1 = \frac{4}{w^2} e_2, \quad \bar{\mathcal{R}}(e_2, e_3)e_1 = 0, \quad \bar{\mathcal{R}}(e_1, e_3)e_1 = \frac{6}{w^2} e_3.
\end{aligned}$$

In view of equation (17), we have

$$\tilde{\tilde{\nabla}}_{e_1}e_3 = (-\frac{2}{w} - f)e_1, \quad \tilde{\tilde{\nabla}}_{e_1}e_2 = 0, \quad \tilde{\tilde{\nabla}}_{e_1}e_1 = \frac{2}{w}(e_3 - \mathcal{N}),$$
$$\tilde{\tilde{\nabla}}_{e_2}e_3 = (-\frac{2}{w} - f)e_2, \quad \tilde{\tilde{\nabla}}_{e_2}e_2 = \frac{2}{w}(e_3 - \mathcal{N}), \quad \tilde{\tilde{\nabla}}_{e_2}e_1 = 0,$$
$$\tilde{\tilde{\nabla}}_{e_3}e_3 = -f(e_3 - \mathcal{N}), \quad \tilde{\tilde{\nabla}}_{e_3}e_2 = 0, \quad \tilde{\tilde{\nabla}}_{e_3}e_1 = 0.$$

From above we see that $\tilde{\tilde{\nabla}}_{e_i}e_j = 0$, $(0 \leq i, j \leq 3)$ for $\mathcal{N} = e_3$ and f = -$\frac{2}{w}$. Hence the manifold is f-Kenmotsu manifold with respect to Schouten-van Kampen connection.

From the above expression of the curvature tensor we obtain the Ricci tensor as follows:

$$\bar{S}(e_1, e_1) = -\frac{10}{w^2}, \quad \bar{S}(e_2, e_2) = -\frac{10}{w^2}, \quad \bar{S}(e_3, e_3) = -\frac{12}{w^2},$$

Therefore, the scalar curvature $\bar{r} = \sum_{i=1}^{3} \bar{S}(e_i, e_i)$ = -$\frac{32}{w^2}$ and $\tilde{\tilde{r}} = \sum_{i=1}^{3} \tilde{\tilde{S}}(e_i, e_i) = 0$ with respect to Levi-Civita connection and Schouten-van Kampen connection respectively.
Let us define a vector field by $v = \mathcal{N}$. Then we obtain:

$$(\tilde{\tilde{L}}_v)(e_1, e_1) = -\frac{4}{z} - 2f, \quad (\tilde{\tilde{L}}_v)(e_2, e_2) = -\frac{4}{z} - 2f, \quad (\tilde{\tilde{L}}_v)(e_3, e_3) = 0.$$

Contracting (1) and using the value of $\tilde{\tilde{r}}$ we have λ=$\frac{3p+2}{6}$. The value of λ satisfies the relation (25). So, g defines a conformal Ricci solitons on 3-dimension f-Kenmotsu manifold for $\lambda = \frac{3p+2}{6}$. Also the Conformal Ricci soliton is expanding if $\lambda \geq 0$ i.e., $\frac{3p+2}{6} \geq 0$, shrinking if $\lambda \leq 0$ i.e., $\frac{3p+2}{6} \leq 0$ and steady if $\lambda = 0$ i.e., $\frac{3p+2}{6}$ = 0.

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