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Almost $*$-Ricci solitons on contact strongly pseudo-convex integrable $\mathcal{CR}$-manifolds

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Abstract. We prove that if contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold admits a $*$-Ricci soliton where the soliton vector Z is contact, then the Reeb vector field ξ is an eigenvector of the Ricci operator at each point if and only if σ is constant. Then we study contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold such that g is a almost $*$-Ricci soliton with potential vector field Z collinear with ξ. To this end, we prove that if a 3-dimensional contact metric manifold M with $Q\varphi = \varphi Q$ which admits a gradient almost $*$-Ricci soliton, then either M is flat or f is constant.

1. Introduction

On a Riemannian manifold (M, g) if there exists a vector field Z and a constant λ satisfying

$$\mathcal{L}_Z g + 2\mathrm{Ric} = 2\lambda g, \tag{1}$$

then it is said that g defines a Ricci soliton (see Hamilton [10, 12]), where Ric denotes the Ricci tensor and $\mathcal{L}_Z$ denotes the Lie-derivative in the direction of Z. Usually, Z and λ are said to be potential vector field and the soliton constant respectively. Obviously, a trivial Ricci soliton is an Einstein metric with Z zero or Killing. Thus, a Ricci soliton may be considered as an apt generalization of an Einstein metric. We say that the Ricci soliton is shrinking when $\lambda > 0$, steady when $\lambda = 0$, and expanding when $\lambda < 0$. If the vector field Z is the gradient of a smooth function f, then g is called a gradient Ricci soliton and the soliton equation (1) becomes

$$Hess_f + \mathrm{Ric} = \lambda g, \tag{2}$$

where $Hess_f$ denotes the Hessian of f. The function f is known as the potential function. In [20], Pigola et al modified the equations (1) and (2) by allowing the constant λ to be a smooth function, and these are called almost Ricci soliton and gradient almost Ricci soliton on M. For more details about the Ricci flow and Ricci soliton we recommend [5] and references therein. The studying of Ricci solitons on almost contact Riemannian manifolds was introduced by Sharma in [21].

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Corresponding to Ricci tensor, Tachibana [22] introduced the concept of $*$-Ricci tensor. In [11] Hamada apply these ideas to real hypersurfaces in complex spaceforms. The $*$-Ricci tensor Ric^* is defined by

$$\mathrm{Ric}^*(X_1, X_2) = \frac{1}{2} trace\{\varphi \circ R(X_1, \varphi X_2)\},$$

for all vector fields X_1, X_2 on M and where φ is a (1,1)-tensor field. If $*$-Ricci tensor is a constant multiple of g, then M is said to be a $*$-Einstein manifold. Hamada gave a complete classification of $*$-Einstein hypersurfaces, and further Ivey and Ryan [14] updated and refined the work of Hamada [11]. Generalizing $*$-Einstein metric, Kaimakamis and Panagiotidou [15] introduced the so-called $*$-Ricci soliton where they essentialy modified the definition of Ricci soliton by replacing the Ricci tensor Ric in Ricci soliton condition with the $*$-Ricci tensor Ric^*.

Definition 1.1. *A Riemannian metric g on M is called a $*$-Ricci soliton if there exists a constant λ and a vector field Z such that*

$$\mathcal{L}_Z g + 2\mathrm{Ric}^* = 2\lambda g, \tag{3}$$

for all vector fields X_1, X_2 on M.

If the soliton constant λ in the defining condition of (3) is a smooth function, then we say that it is an almost $*$-Ricci soliton. Moreover, if the vector field Z is a gradient of a smooth function f, then we say that it is gradient almost $*$-Ricci soliton and in such a case (3) becomes

$$Hess_f + \mathrm{Ric}^* = \lambda g. \tag{4}$$

Note that a $*$-Ricci soliton is trivial if the vector field Z is Killing, and in this case the manifold becomes $*$-Einstein. In this connection, we mention that within the framework of contact geometry $*$-Ricci solitons were first considered by Ghosh and Patra in [9] and further the idea of this concept are studied by Zenkatesha et al [25, 26], Huchchappa et al [13], Dai et al [6], Mandal and Makhal [18]. Motivated by the above cited works we study the $*$-Ricci solitons and almost $*$-Ricci solitons on contact Riemannian manifolds.

This paper is organized as follows. In section 2, the basic information about contact Riemannian manifolds are given. In section 3, we consider $*$-Ricci solitons on contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold M and prove that if (M, g) represents a $*$-Ricci soliton where the soliton vector field Z is contact, then the Reeb vector field ξ is an eigenvector of the Ricci operator at each point if and only if σ is constant. In section 4, first we study a contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold such that g is a almost $*$-Ricci soliton with potential vector field Z collinear with ξ. Finally, we prove that if a 3-dimensional contact Riemannian manifold M on which $Q\varphi = \varphi Q$ admits a gradient almost $*$-Ricci soliton, then either M is flat or f is constant.

2. Preliminaries

A $(2n + 1)$-dimensional Riemannian manifold M is called contact manifold if it has a global 1-form η such that $\eta \wedge (d\eta)^n$ is non-vanishing everywhere on M. For such a 1-form η, there exists a unique vector field ξ, called Reeb vector field, such that $\eta(\xi) = 1$ and $(d\eta)(X_1, \xi) = 0$. A Riemannian metric g on M is said to be an associated metric if there exist a (1,1)-tensor field φ such that

$$\varphi^2 X_1 = -X_1 + \eta(X_1)\xi, \quad \eta(X_1) = g(X_1, \xi), \quad (d\eta)(X_1, X_2) = g(X_1, \varphi X_2). \tag{5}$$

As a result of above equation we have

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad g(\varphi X_1, \varphi X_2) = g(X_1, X_2) - \eta(X_1)\eta(X_2). \tag{6}$$

The manifold M equipped with contact Riemannian structure (φ, ξ, η, g) is called a contact Riemannian manifold. Let us consider a restriction of φ to the contact subbundle $\mathcal{D}$ (defined by $\eta = 0$), and denote this by J. Then $J^2X_1 = -X_1$ and $G(X_1, X_2) = -(d\eta)(X_1, JX_2)$ defines the almost Hermitian structure on $\mathcal{D}$. Thus (M, η, J) is a strongly pseudo-convex $\mathcal{CR}$-manifold (see[23, 24]). We call (M, η, J) a contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold when the complex distribution $\{X_1 - iJX_1; X_1 \text{ in } \mathcal{D}\}$ is integrable. Tanno [23] gave the aforementioned integrability condition by

$$(\nabla_{X_1}\varphi)X_2 = g(X_1 + hX_1, X_2)\xi - \eta(X_2)(X_1 + hX_1), \tag{7}$$

where ∇ is the Riemannian connection of g, and h is the (1,1)-tensor field defined by $2h = \mathcal{L}_\xi\varphi$. Setting a (1,1)-tensor field $\ell = R(\cdot, \xi)\xi$. Then it is not hard to verify that h and ℓ are self-adjoint and satisfy

$$h\xi = \ell\xi = 0, \quad \mathrm{trace}_g h = \mathrm{trace}_g h\varphi = 0, \quad h\varphi + \varphi h = 0. \tag{8}$$

We also have the following formulas for contact Riemannian manifold [1, 3].

$$\nabla_{X_1}\xi = -\varphi X_1 - \varphi h X_1, \quad \nabla_\xi \varphi = 0, \tag{9}$$

$$\mathrm{trace}_g \ell = g(Q\xi, \xi) = 2n - \mathrm{trace}_g h^2, \tag{10}$$

$$R(X_1, X_2)\xi = -(\nabla_{X_1}\varphi)X_2 + (\nabla_{X_2}\varphi)X_1 - (\nabla_{X_1}\varphi h)X_1 + (\nabla_{X_2}\varphi h)X_1, \tag{11}$$

where R is the curvature tensor and Q is the Ricci operator. A contact Riemannian manifold is K-contact (ξ is Killing) if and only if $h = 0$. A contact Riemannian structure is called normal (Sasakian) when almost complex structure J on $M \times \mathbb{R}$, defined by $J(X_1, u\frac{d}{dt}) = (\varphi X_1 - u\xi, \eta(X_1)\frac{d}{dt})$, u being a smooth function on M, is integrable. A contact Riemannian manifold is Sasakian if and only if

$$R(X_1, X_2)\xi = \eta(X_2)X_1 - \eta(X_1)X_2. \tag{12}$$

A Sasakian manifold is K-contact, but the converse is true only when the dimension of M is 3. We remark that any 3-dimensional contact Riemannian manifold satisfies (7) and hence is a contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold. For more details we refer to [4, 7, 16].

On 3-dimensional contact Riemannian manifold with $Q\varphi = \varphi Q$, the following relations hold (see [2]):

$$\begin{aligned} R(X_1, X_2)X_3 =& (\frac{r}{2} - \mathrm{trace}_g\ell)(g(X_2, X_3)X_1 - g(X_1, X_3)X_2) + \frac{1}{2}(3\mathrm{trace}_g\ell - r)(\eta(X_1)g(X_2, X_3)\xi \\ & - \eta(X_2)g(X_1, X_3)\xi + \eta(X_2)\eta(X_3)X_1 - \eta(X_1)\eta(X_3)X_2), \end{aligned} \tag{13}$$

$$QX_1 = \frac{1}{2}(r - \mathrm{trace}_g\ell)X_1 + \frac{1}{2}(3\mathrm{trace}_g\ell - r)\eta(X_1)\xi. \tag{14}$$

Blair et al [2] obtained the following result:

Lemma 2.1. *Let M be a 3-dimensional contact Riemannian manifold with $Q\varphi = \varphi Q$. Then the function* $\mathrm{trace}_g\ell$ *is constant everywhere on M and $\xi r = 0$. Further, if* $\mathrm{trace}_g\ell = 0$ *then M is flat.*

3. ∗-Ricci solitons and contact strongly pseudo-convex integrable $\mathcal{CR}$-manifolds

First we derive the expression of ∗-Ricci tensor on a contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold.

Lemma 3.1. *The ∗-Ricci tensor on a $(2n + 1)$-dimensional contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold M is given by*

$$\begin{aligned} \mathrm{Ric}^*(X_1, X_2) =& \mathrm{Ric}(X_1, X_2) - g(\ell X_1, X_2) - (2n - 2)g(X_1 + hX_1, X_2) \\ & - \eta(X_2)\mathrm{Ric}(X_1, \xi) + (2n - 2)\eta(X_1)\eta(X_2), \end{aligned} \tag{15}$$

for all vector fields X_1, X_2 on M.

Proof. Koufogiorgos [16] obtained the following formula for a contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold:

$$\begin{aligned} R(X_1, X_2)\varphi X_3 - \varphi R(X_1, X_2)X_3 = &\{g(\varphi R(X_1, X_2)\xi, X_3) + \eta(X_1)g(\varphi X_2 + \varphi h X_2, X_3) \\ &- \eta(X_2)g(\varphi X_1 + \varphi h X_1, X_3)\}\xi - g(X_2 + hX_2, X_3)(\varphi X_1 + \varphi h X_1) \\ &+ g(X_1 + hX_1, X_3)(\varphi X_2 + \varphi h X_2) + g(\varphi X_1 + \varphi h X_1, X_3)(X_2 + hX_2) \\ &- g(\varphi X_2 + \varphi h X_2, X_3)(X_1 + hX_1) - \eta(X_3)\{\varphi R(X_1, X_2)\xi \\ &+ \eta(X_1)(\varphi X_2 + \varphi h X_2) - \eta(X_2)(\varphi X_1 + \varphi h X_1)\}. \end{aligned} \tag{16}$$

From (16), making use of skew-symmetry of φ, (5) and (6), we obtain

$$\begin{aligned} g(R(X_1, X_2)\varphi X_3, \varphi X_4) = &g(R(X_1, X_2)X_3, X_4) - \eta(X_4)g(R(X_1, X_2)X_3, \xi) - g(X_2 + hX_2, X_3) \\ &\{g(X_1 + hX_1, X_4) - \eta(X_1)\eta(X_4)\} + g(X_1 + hX_1, X_3)\{g(X_2 + hX_2, X_4) \\ &- \eta(X_2)\eta(X_4)\} + g(\varphi(X_1 + hX_1), X_3)g(X_2 + hX_2, \varphi X_4) - g(\varphi(X_2 + hX_2), X_3) \\ &g(X_1 + hX_1, \varphi X_4) - \eta(X_3)\{g(R(X_1, X_2)\xi, X_4) + \eta(X_1)(g(X_2 + hX_2, X_4) \\ &- \eta(X_2)\eta(X_4)) - \eta(X_2)(g(X_1 + hX_1, X_4) - \eta(X_1)\eta(X_4))\}. \end{aligned} \tag{17}$$

Let $\{e_i\}_{i=1}^{2n+1}$ be a local orthonormal basis of M. Then setting $X_1 = X_4 = e_i$ in the preceding relation and summing over i yields

$$\begin{aligned} g(R(e_i, X_2)\varphi X_3, \varphi e_i) = &\mathrm{Ric}(X_2, X_3) + g(R(X_3, \xi)X_2, \xi) - (2n-2)g(X_2 + hX_2, X_3) \\ &- \eta(X_3)\mathrm{Ric}(X_2, \xi) + (2n-2)\eta(X_2)\eta(X_3), \end{aligned} \tag{18}$$

where we applied the relation (6). The $*$-Ricci tensor on contact Riemannian manifold is defined by (see[1, 8])

$$\mathrm{Ric}^*(X_1, X_2) = g(R(e_i, X_1)\varphi X_2, \varphi e_i) = -\frac{1}{2}g(R(X_1, \varphi X_2)e_i, \varphi e_i).$$

As a result of above relation, the relation (18) transforms into (15). This completes the proof. □

Now we recall the following definition;

Definition 3.2. *A vector field Z on a contact manifold is said to be a contact vector field (or an infinitesimal contact transformation) if there exists a smooth function σ such that $\mathcal{L}_Z\eta = \sigma\eta$. If $\sigma = 0$, then we say that Z is a strict contact transformation.*

It is known from Blair (see p. 34 in [1]) that a vector field Z is a contact vector field if and only if there is a function f on M such that

$$Z = -\frac{1}{2}\varphi \mathrm{grad} f + f\xi, \tag{19}$$

where grad is the gradient operator of g and $\sigma = \xi f$. By virtue of this, we prove

Lemma 3.3. *Let M be a $(2n+1)$-dimensional contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold. If metric g of M is a $*$-Ricci soliton with Z is a contact vector field, then*

$$\begin{aligned} g((Q\varphi + \varphi Q)X_1, X_2) = &(2\lambda - \sigma - 2(2n-2))g(\varphi X_1, X_2) - \frac{1}{4}\{(X_2\sigma)\eta(X_1) - (X_1\sigma)\eta(X_2)\} \\ &+ g((\ell\varphi + \varphi\ell)X_1, X_2). \end{aligned} \tag{20}$$

Proof. By hypothesis, the soliton vector Z is a contact vector field. We take covariant differentiation of (19) and make use of (9) to deduce

$$\nabla_{X_1} Z = -\frac{1}{2}\{(\nabla_{X_1}\varphi)\mathrm{grad} f + \varphi\nabla_{X_1}\mathrm{grad} f\} + (X_1 f)\xi - f(\varphi X_1 + \varphi h X_1). \tag{21}$$

By virtue of this, we easily compute

$$
\begin{aligned}
(\mathcal{L}_Z g)(X_1, X_2) &= g(\nabla_{X_1} Z, X_2) + g(X_1, \nabla_{X_2} Z)\\
&= \frac{1}{2}\{g((\nabla_{X_1}\varphi)X_2, \mathrm{grad} f) + g((\nabla_{X_2}\varphi)X_1, \mathrm{grad} f) + g(\nabla_{X_1}\mathrm{grad} f, \varphi X_2)\\
&\quad + g(\nabla_{X_2}\mathrm{grad} f, \varphi X_1)\} + (X_1 f)\eta(X_2) + (X_2 f)\eta(X_1) + 2fg(h\varphi X_1, X_2).
\end{aligned}
$$

By virtue of (7), the foregoing equation transforms into

$$
\begin{aligned}
(\mathcal{L}_Z g)(X_1, X_2) =&\sigma g(X_1 + hX_1, X_2) + \frac{1}{2}\{g(\nabla_{X_1}\mathrm{grad} f, \varphi X_2) + g(\nabla_{X_2}\mathrm{grad} f, \varphi X_1)\\
&- \eta(X_2) g(X_1 + hX_1, \mathrm{grad} f) - \eta(X_1) g(X_2 + hX_2, \mathrm{grad} f)\}\\
&+ (X_1 f)\eta(X_2) + (X_2 f)\eta(X_1) + 2fg(h\varphi X_1, X_2).
\end{aligned} \tag{22}
$$

As a result of (22), the soliton equation (3) becomes

$$
\begin{aligned}
&\sigma g(X_1 + hX_1, X_2) + \frac{1}{2}\{g(\nabla_{X_1}\mathrm{grad} f, \varphi X_2) + g(\nabla_{X_2}\mathrm{grad} f, \varphi X_1) - \eta(X_2) g(X_1 + hX_1, \mathrm{grad} f)\\
&- \eta(X_1) g(X_2 + hX_2, \mathrm{grad} f)\} + (X_1 f)\eta(X_2) + (X_2 f)\eta(X_1) + 2fg(h\varphi X_1, X_2) - 2\lambda g(X_1, X_2)\\
&+ 2\mathrm{Ric}(X_1, X_2) - 2g(\ell X_1, X_2) - 2(2n-2) g(X_1 + hX_1, X_2) - 2\eta(X_2)\mathrm{Ric}(X_1, \xi)\\
&+ 2(2n-2)\eta(X_1)\eta(X_2) = 0.
\end{aligned} \tag{23}
$$

Take φX_2 instead of X_2 in (23) to obtain

$$
\begin{aligned}
&\sigma g(X_1 + hX_1, \varphi X_2) + \frac{1}{2}\{-g(\nabla_{X_1}\mathrm{grad} f, X_2) + \eta(X_2) g(\nabla_{X_1}\mathrm{grad} f, \xi) + g(\nabla_{\varphi X_2}\mathrm{grad} f, \varphi X_1)\\
&- \eta(X_1) g(\varphi X_2 + h\varphi X_2, \mathrm{grad} f)\} + ((\varphi X_2) f)\eta(X_1) - 2fg(hX_1, X_2) - 2\lambda g(X_1, \varphi X_2)\\
&+ 2g(X_1, Q\varphi X_2) - 2g(\ell X_1, \varphi X_2) - 2(2n-2) g(X_1 + hX_1, \varphi X_2) = 0.
\end{aligned} \tag{24}
$$

From $\sigma = \xi f = g(\mathrm{grad} f, \xi)$, one can easily find that

$$
g(\nabla_{X_1}\mathrm{grad} f, \xi) = g(\mathrm{grad} f, \varphi X_1 + \varphi h X_1) + X_1\sigma. \tag{25}
$$

Anti-symmetrizing the equation (24), making use of Poincare lemma and (25), we deduce

$$
\begin{aligned}
&2\sigma g(X_1, \varphi X_2) + \frac{1}{2}\{(X_1\sigma)\eta(X_2) - (X_2\sigma)\eta(X_1)\} - 4\lambda g(X_1, \varphi X_2) + 2g((Q\varphi + \varphi Q)X_2, X_1)\\
&- 2g((\ell\varphi + \varphi\ell)X_2, X_1) + 4(2n-2) g(X_1, \varphi X_2) = 0,
\end{aligned} \tag{26}
$$

which is equivalent to (20). This completes the proof. □

Theorem 3.4. *If metric g of a contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold M is a $*$-Ricci soliton whose potential vector field Z is a contact vector field, then the Reeb vector field ξ is an eigenvector of the Ricci operator at each point if and only if σ is constant.*

Proof. Contracting (21) over X_1 with respect to orthonormal frame $\{e_i\}_{i=1}^{2n+1}$ and recalling second term of (8), we obtain

$$
div\, Z = -\frac{1}{2}\sum_{i=1}^{2n+1}\{g((\nabla_{e_i}\varphi)\mathrm{grad} f, e_i) + g(\varphi\nabla_{e_i}\mathrm{grad} f, e_i)\} + (\xi f). \tag{27}
$$

It is known from [19] that, the following relation holds for any contact Riemannian manifold;

$$
\sum_{i=1}^{2n+1} g((\nabla_{e_i}\varphi)X_1, e_i) = -2n\eta(X_1). \tag{28}
$$

Let $\{e_\alpha, \varphi e_\alpha, \xi\}, \alpha = 1, 2, 3 \cdots n$, be a φ-basis of M. From which, we compute

$$\sum_{i=1}^{2n+1} g(\varphi \nabla_{e_i} \mathrm{grad} f, e_i) = -\sum_{i=1}^{2n+1} g(\nabla_{e_i} \mathrm{grad} f, \varphi e_i)$$
$$= -\sum_{\alpha=1}^{n} g(\nabla_{e_\alpha} \mathrm{grad} f, \varphi e_\alpha) + \sum_{\alpha=1}^{n} \{g(\nabla_{\varphi e_\alpha} \mathrm{grad} f, e_\alpha)$$
$$+ g(\nabla_\xi \mathrm{grad} f, \varphi \xi)\} = 0. \tag{29}$$

Thus, the utilization of (29), (28) and $\sigma = \xi f$ in (27) provides $div\ Z = (n+1)\sigma$. Now, switching X_1 by ξ in (20) yields

$$\varphi Q \xi = -\frac{1}{4}\{\mathrm{grad}\sigma - (\xi\sigma)\xi\}, \tag{30}$$

where we used first term of (8). Let us suppose that the Reeb vector field ξ is an eigenvector of the Ricci operator, that is, $Q\xi = (\mathrm{trace}_g \ell)\xi$. Then the equation (30) reduces to $\mathrm{grad}\sigma = (\xi\sigma)\xi$. Differentiating this along X_1 and utilization of first term of (9) provides

$$\nabla_{X_1} \mathrm{grad}\sigma = X_1(\xi\sigma)\xi - (\xi\sigma)(\varphi X_1 + \varphi h X_1).$$

Since $g(\nabla_{X_1} \mathrm{grad}\sigma, X_2) = g(X_1, \nabla_{X_2} \mathrm{grad}\sigma)$, the foregoing equation shows

$$X_1(\xi\sigma)\eta(X_2) - X_2(\xi\sigma)\eta(X_1) + (\xi\sigma)d\eta(X_1, X_2) = 0.$$

Replacing X_1 by φX_1 and X_2 by φX_2 and since $d\eta$ is non-zero for any contact Riemannian structure it follows that $(\xi\sigma) = 0$. Hence $\mathrm{grad}\sigma = 0$, i.e. σ is constant. Conversely, if σ is constant, then it follows from (20) that $\varphi Q\xi = 0$. Action of φ on this together with first term of (5) provides $Q\xi = (\mathrm{trace}_g \ell)\xi$. This completes the proof. □

4. Almost ∗-Ricci soliton and contact strongly pseudo-convex integrable $\mathcal{CR}$-manifolds

We shall discuss about some special type of ∗-Ricci soliton where the potential vector field Z is point wise collinear with the Reeb vector field ξ of the contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold.

Theorem 4.1. *Let M be a contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold such that ξ is an eigenvector of the Ricci operator at each point and $(div\ \ell)\xi = 0$. If g represents an almost ∗-Ricci soliton with non-zero potential vector field Z collinear with the Reeb vector field ξ, then M is Sasakian and η-Einstein. In particular if M is complete, then M is compact positive-Sasakian.*

Proof. Since the potential vector field Z on M is collinear with the Reeb vector field ξ, we have $Z = \rho\xi$, where ρ is a non-zero smooth function on M (as Z is non-zero). Differentiating this along X_1 together with the first term of (9) gives

$$\nabla_{X_1} Z = (X_1\rho)\xi - \rho(\varphi X_1 + \varphi h X_1).$$

By virtue of this, the soliton equation (3) can be written as

$$(X_1\rho)\eta(X_2) + (X_2\rho)\eta(X_1) - 2\rho g(\varphi h X_1, X_2) + 2\mathrm{Ric}(X_1, X_2) - 2g(\ell X_1, X_2)$$
$$- 2(2n-2)g(X_1 + hX_1, X_2) + 2((2n-2) - \mathrm{trace}_g \ell)\eta(X_1)\eta(X_2) = 2\lambda g(X_1, X_2), \tag{31}$$

where we used $Q\xi = (\mathrm{trace}_g \ell)\xi$. Plugging ξ in place of X_2 in (31) gives

$$(X_1\rho) + (\xi\rho)\eta(X_1) = 2\lambda\eta(X_1). \tag{32}$$

At this point, putting $X_1 = X_2 = \xi$ in (31) and recalling first term of (10) we obtain

$$(\xi\rho) = \lambda.$$

The foregoing equation along with (32) gives that $\mathrm{grad}\rho = (\xi\rho)\xi$. Next, taking covariant differentiation of this along X_1 together with first term of (9) yields $\nabla_{X_1}\mathrm{grad}\rho = X_1(\xi\rho)\xi - (\xi\rho)(\varphi X_1 + \varphi h X_1)$. By virtue of $g(\nabla_{X_1}\mathrm{grad}\rho, X_2) = g(X_1, \nabla_{X_2}\mathrm{grad}\rho)$, the foregoing equation provides

$$X_1(\xi\rho)\eta(X_2) - X_2(\xi\rho)\eta(X_1) + 2(\xi\rho)d\eta(X_1, X_2) = 0.$$

Choosing X_1, X_2 orthogonal to ξ and remember that $d\eta \neq 0$, the aforementioned equation provides $\xi\rho = 0$. Hence $\mathrm{grad}\rho = 0$ and consequently ρ is constant. By virtue of this, the equation (32) shows that $\lambda = 0$. Thus, the equation (31) reduces to

$$QX_1 - \ell X_1 - (2n-2)(X_1 + hX_1) + ((2n-2) - \mathrm{trace}_g\ell)\eta(X_1)\xi + \rho(h\varphi)X_1 = 0. \tag{33}$$

Taking trace of (33) we obtain $r = 2\mathrm{trace}_g\ell + 2n(2n-2)$, where we used $\lambda = 0$ and $\mathrm{trace}_g h = \mathrm{trace}_g h\varphi = 0$. Further, covariant derivative of (33) along X_2 gives

$$(\nabla_{X_2}Q)X_1 - (\nabla_{X_2}\ell)X_1 - (2n-2)(\nabla_{X_2}h)X_1 - (X_2(\mathrm{trace}_g\ell))\eta(X_1)\xi$$
$$+ ((2n-2) - \mathrm{trace}_g\ell)\{(\nabla_{X_2}\eta)(X_1)\xi + \eta(X_1)\nabla_{X_2}\xi\} + \rho(\nabla_{X_2}h\varphi)X_1 = 0.$$

Contracting this over X_2 provides

$$\frac{1}{2}(X_1 r) - (div\ \ell)X_1 - (2n-2)(div\ h)X_1 - (\xi(\mathrm{trace}_g\ell))\eta(X_1) + \rho(div(h\varphi))X_1 = 0. \tag{34}$$

On the other hand, from the first term of (8) it follows for a contact Riemannian manifold M that

$$(\nabla_{X_1}h)\xi = (h\varphi - h^2\varphi)X_1.$$

Contracting this over X_1 with respect to an orthonormal basis $\{e_i\}$ and noting that $\mathrm{trace}_g(h\varphi) = \mathrm{trace}_g(h^2\varphi) = 0$, we obtain $(div\, h)\xi = 0$. Recall that for any contact Riemannian manifold $(div(h\varphi)X_1) = g(Q\xi, X_1) - 2n\eta(X_1)$. Since $Q\xi = (\mathrm{trace}_g\ell)\xi$, we have

$$(div(h\varphi))X_1 = (\mathrm{trace}_g\ell - 2n)\eta(X_1). \tag{35}$$

At this point, putting $X_1 = \xi$ in (34) and making use of $r = 2(\mathrm{trace}_g\ell) + 2n(2n-2)$, $(div\ h)\xi = 0$ and (35) provides

$$(div\ \ell)\xi + \rho(\mathrm{trace}_g\ell - 2n) = 0.$$

Suppose that $(div\ \ell)\xi = 0$, then from above relation we have $\sigma(\mathrm{trace}_g\ell - 2n) = 0$. From this we have either $\mathrm{trace}_g\ell = 2n$ or $\mathrm{trace}_g\ell \neq 2n$. Suppose that $\mathrm{trace}_g\ell \neq 2n$, then the last equation shows that $\rho = 0$. This contradicts our assumption that Z is non-zero. Thus, the only possibility is that $\mathrm{trace}_g\ell = 2n$. This together with the first term of (10) shows that $h = 0$. Which shows that M is K-contact (ξ is Killing). It is known that K-contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold is Sasakian. Thus, M is Sasakian and by virtue of (33) we have

$$QX_1 = (2n-1)X_1 + \eta(X_1)\xi.$$

This shows that M is η-Einstein. Moreover, if M is complete, then from above equation we can conclude that M is compact and positive-Sasakian. This completes the proof. □

It is known that any 3-dimensional contact Riemannian manifold is a contact strongly pseudo-convex integrable $\mathcal{CR}$-manifold. Thus, it is interesting to study a gradient almost $*$-Ricci soliton in contact Riemannian 3-manifold and here, we prove the following outcome:

Theorem 4.2. *If a 3-dimensional contact Riemannian manifold M such that $Q\varphi = \varphi Q$ admits a gradient almost $*$-Ricci soliton, then either M is flat or potential function f is constant.*

Proof. Using $Q\varphi = \varphi Q$, (10) and $\varphi\xi = 0$ we have that

$$Q\xi = (\mathrm{trace}_g \ell)\xi. \tag{36}$$

As a result of (36), (13) and (14), we have from (15) that

$$\mathrm{Ric}^*(X_1, X_2) = \left(\frac{r}{2} - \mathrm{trace}_g \ell\right)\{g(X_1, X_2) - \eta(X_1)\eta(X_2)\}.$$

Making use of above expression, the gradient almost $*$-Ricci soliton (4) can be exhibited as

$$\nabla_{X_1}\mathrm{grad} f = \left(\lambda - \frac{r}{2} + \mathrm{trace}_g \ell\right)X_1 + \left(\frac{r}{2} - \mathrm{trace}_g \ell\right)\eta(X_1)\xi. \tag{37}$$

By straightforward computations, using the well-known expression of the curvature tensor:

$$R(X_1, X_2) = \nabla_{X_1}\nabla_{X_2} - \nabla_{X_2}\nabla_{X_1} - \nabla_{[X_1,X_2]},$$

and the repeated use of equation (37) gives

$$\begin{aligned} R(X_1, X_2)\mathrm{grad} f =& \frac{(X_2 r)}{2}(X_1 - \eta(X_1)\xi) - \frac{(X_1 r)}{2}(X_2 - \eta(X_2)\xi) + \left(\frac{r}{2} - \mathrm{trace}_g \ell\right)\{2g(X_1, \varphi X_2)\xi \\ &+ \eta(X_1)(\varphi X_2 + \varphi h X_2) - \eta(X_2)(\varphi X_1 + \varphi h X_1)\} + (X_1\lambda)X_2 - (X_2\lambda)X_1. \end{aligned} \tag{38}$$

Taking scalar product of foregoing equation with ξ and employing (6) yields

$$g(R(X_1, X_2)\mathrm{grad} f, \xi) = 2\left(\frac{r}{2} - \mathrm{trace}_g \ell\right)g(X_1, \varphi X_2) + (X_1\lambda)\eta(X_2) - (X_2\lambda)\eta(X_1).$$

Replacing X_2 by ξ in the above equation and utilization of (6), (13) we obtain

$$X_1\left(\frac{\mathrm{trace}_g \ell}{2} f + \lambda\right) = \xi\left(\frac{\mathrm{trace}_g \ell}{2} f + \lambda\right)\eta(X_1).$$

Writing this as: $d\left(\frac{\mathrm{trace}_g \ell}{2} f + \lambda\right) = \xi\left(\frac{\mathrm{trace}_g \ell}{2} f + \lambda\right)\eta$. Applying d to this condition and using the Poincare lemma: $d^2 = 0$ gives $d\left(\xi(\frac{\mathrm{trace}_g \ell}{2} f + \lambda)\right) \wedge d\eta + \xi\left(\frac{\mathrm{trace}_g \ell}{2} f + \lambda\right)d\eta = 0$. Taking wedge product of this equation with η and remember that $\eta \wedge \eta = 0$ and $d\eta \wedge \eta$ is non-vanishing everywhere on contact Riemannian manifold, we conclude that $\xi\left(\frac{\mathrm{trace}_g \ell}{2} f + \lambda\right) = 0$. Consequently, $d\left(\frac{\mathrm{trace}_g \ell}{2} f + \lambda\right) = 0$ on M, and hence

$$\frac{\mathrm{trace}_g \ell}{2} f + \lambda = k, \tag{39}$$

where k is a constant. Substituting ξ for X_2 in (38) and then taking inner product of the resulting equation with X_2 and employing (6), Lemma 2.1 we find

$$g(R(X_1, \xi)\mathrm{grad} f, X_2) = \left(\mathrm{trace}_g \ell - \frac{r}{2}\right)g(\varphi X_1 + \varphi h X_1, X_2) + (X_1\lambda)\eta(X_2) - (\xi\lambda)g(X_1, X_2).$$

As a result of (13), the above equation provides

$$\begin{aligned} &\left(\mathrm{trace}_g \ell - \frac{r}{2}\right)g(\varphi X_1 + \varphi h X_1, X_2) + (X_1\lambda)\eta(X_2) - (\xi\lambda)g(X_1, X_2) \\ &+ \frac{\mathrm{trace}_g \ell}{2}(X_1 f)\eta(X_2) - \frac{\mathrm{trace}_g \ell}{2}(\xi f)g(X_1, X_2) = 0. \end{aligned}$$

By virtue of (39) the preceding equation reduces to $\left(\mathrm{trace}_g\ell - \frac{r}{2}\right)\varphi X_1 + \varphi h X_1 = 0$. Anti-symmetrizing this equation yields $2\left(\mathrm{trace}_g\ell - \frac{r}{2}\right)\varphi X_1 = 0$. From this we obtain $r = 2\mathrm{trace}_g\ell$, which shows that r is constant. On ther hand, contracting (38) over X_1 we get $Q\mathrm{grad}f = \frac{1}{2}\mathrm{grad}r - 2\mathrm{grad}\lambda$. This together with (14) gives

$$\frac{1}{2}(r - \mathrm{trace}_g\ell)\mathrm{grad}f + \frac{1}{2}(3\mathrm{trace}_g\ell - r)(\xi f)\xi + 2\mathrm{grad}\lambda = 0,$$

where we used r is constant. Utilization of (39) and $r = 2\mathrm{trace}_g\ell$ in the above equation yields

$$(\mathrm{trace}_g\ell)(\mathrm{grad}f - (\xi f)\xi) = 0. \tag{40}$$

Since $\mathrm{trace}_g\ell$ is constant, we have either $\mathrm{trace}_g\ell = 0$ or $\mathrm{trace}_g\ell \neq 0$. At this point, suppose that $\mathrm{trace}_g\ell = 0$, then the Lemma 2.1 shows that M is flat. Next, suppose that $\mathrm{trace}_g\ell \neq 0$, then from (40) we obtain $\mathrm{grad}f = (\xi)\xi$. Taking covariant differentiation of this along X_1 together with first term of (9) yields $\nabla_{X_1}\mathrm{grad}f = X_1(\xi f)\xi - (\xi f)(\varphi X_1 + \varphi h X_1)$. By virtue of $g(\nabla_{X_1}\mathrm{grad}f, X_2) = g(X_1, \nabla_{X_2}\mathrm{grad}f)$, the foregoing equation provides

$$X_1(\xi f)\eta(X_2) - X_2(\xi f)\eta(X_1) + 2(\xi f)d\eta(X_1, X_2) = 0.$$

Choosing X_1, X_2 orthogonal to ξ and remember that $d\eta \neq 0$, the aforementioned equation provides $\xi f = 0$. Hence $\mathrm{grad}f = 0$ and consequently f is constant. This completes the proof. □

Remark 4.3. *Our Theorem 4.2 generalizes the Theorem 1.2 of Majhi et al [17].*

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