Filomat 38:2 (2024), 569–576
https://doi.org/10.2298/FIL2402569H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Remarks on pseudocovering spaces in a digital topological setting: A corrigendum

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Abstract. After investigating some properties of the original version of a pseudo-(k_0, k_1)-covering space in the literature, it appears that a pseudo-(k_0, k_1)-covering space is equivalent to a digital (k_0, k_1)-covering space. Hence, as a corrigendum to [7, 9], the paper first revises one of the three conditions for a pseudo-(k_0, k_1)-covering space, which broadens the original version. After that, we suggest some examples for the revised version of a pseudo-(k_0, k_1)-covering map. Since the revised map is so related to the study of several kinds of path liftings, this new version can facilitate some studies in the field of applied topology including digital topology. We note that a weakly local (k_0, k_1)-isomorphic surjection is equivalent to the new version of a pseudo-(k_0, k_1)-covering map instead of the original version of a pseudo-(k_0, k_1)-covering map. The present paper only deals with k-connected digital images (X, k).

1. Introduction

The notion of a pseudo-(k_0, k_1)-covering space was initially introduced in 2012 [7]. Indeed, it was intended to make a digital (k_0, k_1)-covering space in [2–4, 6] more generalized and broader. Hence it was defined by using three conditions among which two of them, i.e., the conditions (1) and (2) for a pseudo-(k_0, k_1)-covering space, are equal to those for a digital (k_0, k_1)-covering space. Meanwhile, the other condition (3) for a pseudo-(k_0, k_1)-covering space is different from the condition (3) for a digital (k_0, k_1)-covering space (see Definitions 3.3 and 3.6 in the present paper). To be specific, the former was defined by using the notion of a weakly local (*WL*-, for brevity)(k_0, k_1)-isomorphism and the latter was characterized by using the concept of a local (k_0, k_1)-isomorphism. However, when combining the two conditions (1) and (2) with each of the conditions (3) for a pseudo-(k_0, k_1)-covering space and a digital (k_0, k_1)-covering space, a pseudo-(k_0, k_1)-covering space implies a digital (k_0, k_1)-covering space (see Theorem 3.3 of [11]). Probably, in [7], there seems to be a gap between the author's intention for establishing a pseudo-(k_0, k_1)-covering space and the mathematical presentation of it. The present paper only deals with k-connected digital images and often uses the notion := to introduce some terms.

The four papers [7, 9, 11, 12] studied various properties of "pseudocovering spaces". Since the two of them [7, 9] have some errors and the others [11, 12] also have some mistakes related to the map in (1.1)

2020 *Mathematics Subject Classification*. Primary 54C08; Secondary 05C40, 54A05, 68U05
Keywords. (Digital) covering space, pseudocovering space, (wealky) local (k_0, k_1)-isomorphisim, unique pseudolifting property, homotopy lifting theorem, digital topology, digital image
Received: 19 April 2023; Revised: 18 July 2023; Accepted: 19 July 2023
Communicated by Ljubiša D. R. Kočinac
The author was supported by the NRF of Korea(2019R1I1A3A03059103) and "Research Base Construction Fund Program funded by Jeonbuk National University in 2022".
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below, the present paper will correct and improve them, which makes them so clear. More precisely, with the original version of a pseudo-(k_0, k_1)-covering map (see Definition 3.3 in the present paper), the map p in (1.1) below is not a pseudo-$(2, k)$-covering map (see Proposition 3.4 in the present paper).

$$\begin{cases} p : (\mathbb{Z}^+, 2) \to SC_k^{n,l} := (x_i)_{i \in [0,l-1]_{\mathbb{Z}}} \text{ defined by} \\ p(t) = x_{t(mod\, l)}, \text{ where } \mathbb{Z}^+ := [0, \infty)_{\mathbb{Z}} := \{t \in \mathbb{Z} \,|\, t \geq 0\}. \end{cases} \tag{1.1}$$

Thus we note that these two finding with Proposition 3.2 and Theorem 3.3 in [11] are basically valid. However, the proof of Proposition 3.2 of [11] is incorrect (see Remark 3.5 in the present paper) and the proof of Theorem 3.3 of [11] is required to be done with more details (see the proof of Theorem 3.7 in the present paper). Thus the present paper will initially correct the proof of Proposition 3.2 of [11] and further, make the proof of Theorem 3.3 of [11] more improved.

Next, we recently found some mistakes on the identity of (4.2) in Proposition 4.4 and Corollary 4.5 of [9]. Besides, the map f_2 in Example 4.3(1) of [7], the map p in (4.1) of [9], and the map p referred to in Remark 4.8 of [7] are also invalid for pseudocovering maps because they do not satisfy the condition (1) for a pseudocovering space (see Definition 3.3 in the present paper). Hence the present paper will correct them and verify that a WL-(k_0, k_1)-surjection is not equivalent to a pseudo-(k_0, k_1)-covering map (see Remark 3.10(2) in the present paper).

Based on the above review, due to an equivalence between a pseudo-(k_0, k_1)-covering map and a digital (k_0, k_1)-covering map, we now have a dilemma in whether to remain the condition (1) for a pseudo-(k_0, k_1)-covering space (see Definition 3.3) or revise it broadly to make a distinction from the condition (1) for a digital (k_0, k_1)-covering map. To be specific, in the case of that we follow the former, a pseudo-(k_0, k_1)-covering map becomes a digital (k_0, k_1)-covering map so that it supports only the unique path lifting (*upl*, for brevity) property [4], a homotopy lifting theorem for a digital (k_0, k_1)-covering map with a radius 2 local isomorphism in a digital topological setting [3] and so forth. However, the notion of a pseudo-(k_0, k_1)-covering space was intended to generalize them such as the unique pseudo-lifting (*usl*, for brevity) (see [7]) and so forth. Hence it is better to revise the condition (1) for a pseudo-(k_0, k_1)-covering map than to remain it (see Definition 4.1). After that, the revised version of a pseudo-(k_0, k_1)-covering map can indeed be broader than a digital (k_0, k_1)-covering map. Furthermore, many kinds of examples for a pseudo-(k_0, k_1)-covering map including the map p in (1.1) are obtained (see Theorem 4.2 and Example 4.3). Finally, the revised version can be more effective than the original version. Besides, with the revised version of a pseudo-(k_0, k_1)-covering space, we prove that a WL-(k_0, k_1)-isomorphic surjection is equivalent to a new version of a pseudo-(k_0, k_1)-covering map as an improvement of Corollary 4.5 of [9].

To sum up, the paper will do corrections and improvements, as follows:

(1) Corrections of the map f_2 in Example 4.3(1) of [7], the map p in Remark 4.8 of [7] and the map p of (4.1) in the proof of Remark 4.3(2) of [9] (see the improvements in Remark 3.5, and Theorem 4.2 in the present paper).

(2) Corrections of the equality of (4.2) of Proposition 4.4, and Corollary 4.5 of [9] (see the improvement of them in Remark 3.10(2), Corollary 3.11, Theorem 4.2, Proposition 4.4, and Example 4.5 in the present paper).

(3) Revision of the notion of a pseudo-(k_0, k_1)-covering space (see Definition 4.1).

(4) Corrections of the proof of Proposition 3.2 of [11] and related works in [12] (see Remark 3.5 in the present paper).

(5) Improvement of the proof of Theorem 3.3 of [11] (see the proof of Theorem 3.7 in the present paper).

In addition, we confirm that the map p in (1.1) now becomes an example for the revised version of a pseudo-$(2, k)$-covering space. With the revised version, some examples for it are shown in Section 4.

2. Preliminaries

In relation to the study of some properties of a pseudo-(k_0, k_1)-covering space, to make the paper self-contained, we will refer to some notions. Naively, a digital image (X, k) can be considered to be a set $X \subseteq \mathbb{Z}^n$

with one of the k-adjacency of $\mathbb{Z}^n$ from (2.1) below (or a digital k-graph on $\mathbb{Z}^n$ [5]). Indeed, the papers [10, 13] considered $(X, k), X \subseteq \mathbb{Z}^n, n \in \{1, 2, 3\}$, with 2-adjacency on $\mathbb{Z}$, 4, 8-adjacency on $\mathbb{Z}^2$, and 6, 18, 26-adjacency on $\mathbb{Z}^3$. As the generalization of the low dimensional cases, the digital k-adjacency relations (or digital k-connectivity) for $X \subseteq \mathbb{Z}^n, n \in \mathbb{N}$, were initially established in [8] (see also [2, 3]), as follows:

For a natural number t, $1 \leq t \leq n$, the distinct points $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n) \in \mathbb{Z}^n$ are $k(t, n)$-adjacent if at most t of their coordinates differ by ± 1 and the others coincide.

According to this statement, the $k(t, n)$-adjacency relations (or digital k-connectivities) of $\mathbb{Z}^n, n \in \mathbb{N}$, are formulated [8] (see also [2, 4]) as follows:

$$k := k(t, n) = \sum_{i=1}^{t} 2^i C_i^n, \text{where } C_i^n := \frac{n!}{(n-i)!\, i!}. \tag{2.1}$$

Using the k-adjacency relations of $\mathbb{Z}^n$ in (2.1), $n \in \mathbb{N}$, we will call the pair (X, k) a digital image on $\mathbb{Z}^n$, $X \subseteq \mathbb{Z}^n$.

A simple closed k-curve (or simple k-cycle) with l elements in $\mathbb{Z}^n, n \geq 2$, denoted by $SC_k^{n,l}$ [4, 10], $l(\geq 4) \in \mathbb{N}$, is defined to be the set $(x_i)_{i \in [0, l-1]_{\mathbb{Z}}} \subseteq \mathbb{Z}^n$ such that x_i and x_j are k-adjacent if and only if $|\, i - j\,| = \pm 1 (mod\, l)$. Then, the number l of $SC_k^{n,l}$ depends on both the dimension n of $\mathbb{Z}^n$ and the k-adjacency. For a digital image (X, k) and $x \in X$, we follow the notation

$$N_k(x, 1) := \{x' \in X \,|\, x \text{ is } k\text{-adjacent to } x'\} \cup \{x\}, \tag{2.2}$$

which is called a digital k-neighborhood of x in (X, k) [4]. Indeed, this notion will be strongly used in studying both pseudocovering spaces and digital covering spaces. For every point x of a digital image (X, k), an $N_k(x, 1)$ always exists in (X, k), the digital continuity of [13] can be represented by the following form.

Proposition 2.1. ([4, 6]) *Let (X, k_0) and (Y, k_1) be digital images on $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. A function $f : X \to Y$ is (k_0, k_1)-continuous if and only if for every point $x \in X$, $f(N_{k_0}(x, 1))$ is a subset of $N_{k_1}(f(x), 1)$.*

Owing to a digital k-graph theoretical feature of a digital image (X, k), we have often used a (k_0, k_1)-*isomorphism* in [5] instead of a (k_0, k_1)-*homeomorphism* in [1], as follows:

Definition 2.2. ([1]; see also [5]) For two digital images (X, k_0) on $\mathbb{Z}^{n_0}$ and (Y, k_1) on $\mathbb{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-*isomorphism* if h is a (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is (k_1, k_0)-continuous. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0-*isomorphism*.

3. Remarks on pseudo-(k_0, k_1)-covering spaces

Since the notions of a digital (k_0, k_1)-covering map and a pseudo-(k_0, k_1)-covering map are so related to the notion of a (weakly) local (k_0, k_1)-isomorphism, we first need to recall it, as follows:

Definition 3.1. ([2, 4, 9]) For two digital images (X, k_0) on $\mathbb{Z}^{n_0}$ and (Y, k_1) on $\mathbb{Z}^{n_1}$, consider a map $h : (X, k_0) \to (Y, k_1)$. Then the map h is said to be a local (k_0, k_1)-isomorphism, if for every $x \in X$, h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $N_{k_1}(h(x), 1)$, i.e., the restriction map $h|_{N_{k_0}(x,1)} : N_{k_0}(x, 1) \to N_{k_1}(h(x), 1)$ is a (k_0, k_1)-isomorphism. If $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a local k_0-isomorphism.

The paper [7] defined the following notion which is weaker than a local (k_0, k_1)-isomorphism.

Definition 3.2. ([7]) For two digital images (X, k_0) on $\mathbb{Z}^{n_0}$ and (Y, k_1) on $\mathbb{Z}^{n_1}$, a map $h : X \to Y$ is called a weakly local (WL-, for brevity) (k_0, k_1)-isomorphism if for every $x \in X$, h maps $N_{k_0}(x, 1)$ (k_0, k_1)-isomorphically onto $h(N_{k_0}(x, 1)) \subseteq (Y, k_1)$, i.e., the restriction map $h|_{N_{k_0}(x,1)} : N_{k_0}(x, 1) \to h(N_{k_0}(x, 1))$ is a (k_0, k_1)-isomorphism. In particular, if $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a weakly local k_0-isomorphism (or a *WL*-k_0-isomorphism).

Using this notion, the paper [7] defined the notion of a pseudo-(k_0, k_1)-covering space, as follows:

Definition 3.3. ([7]) Let (E, k_0) and (B, k_1) be digital images on $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Let $p : E \to B$ be a surjection such that for any $b \in B$,

(1) for some index set M, $p^{-1}(N_{k_1}(b, 1)) = \bigcup_{i\in M} N_{k_0}(e_i, 1)$ with $e_i \in p^{-1}(b) := p^{-1}(\{b\})$;

(2) if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i, 1) \cap N_{k_0}(e_j, 1)$ is an empty set; and

(3) the restriction of p to $N_{k_0}(e_i, 1)$ from $N_{k_0}(e_i, 1)$ to $N_{k_1}(b, 1)$ is a WL-(k_0, k_1)-isomorphism for all $i \in M$.

Then the map p is called a pseudo-(k_0, k_1)-covering map, (E, p, B) is said to be a pseudo-(k_0, k_1)-covering and (E, k_0) is called a pseudo-(k_0, k_1)-covering space over (B, k_1).

In Definition 3.3, in the case of that $n_0 = n_1$ and $k_0 = k_1$, a pseudo-(k_0, k_1)-covering is simply called a pseudo-k_0-covering. Based on the notion of a pseudo-(k_0, k_1)-covering space, the paper [7] suggested an example for a pseudo-$(2, k)$-covering map with the map $p : (\mathbb{Z}^+, 2) \to SC_k^{n,l}$ as in (1.1). Indeed, the paper [7] made a mistake to take this map as a pseudo-$(2, k)$-covering map (see Proposition 3.4 below). By contary to the condition (1) of Definition 3.3, the map p is not a pseudo-$(2, k)$-covering map, as follows:

Proposition 3.4. (Proposition 3.2 of [11]) *The map* $p : (\mathbb{Z}^+, 2) \to SC_k^{n,l} := (c_i)_{i\in[0,l-1]_{\mathbb{Z}}}, l \geq 4$, *in (1.1) is not a pseudo-*$(2, k)$*-covering map.*

Even though this assertion is correct, the proof of this fact in [11] is incorrect. To make the paper self-contained, let us recall the proof of Proposition 3.2 as in [11], as follows (also the same error in the part with lines 13–15 on the page 51 of [12] is shown):

Consider the map as in (1.1), i.e.,

$$p : (\mathbb{Z}^+, 2) \to SC_k^{n,l} := (c_i)_{i\in[0,l-1]_{\mathbb{Z}}}, l \geq 4 \tag{3.1}$$

defined by $p(i) = c_{i(mod\, l)}$, where $\mathbb{Z}^+ := [0, \infty)_{\mathbb{Z}} := \{t \in \mathbb{Z} \,|\, t \geq 0\}$. Then the paper [11] particularly take the element $c_{i-1} \in SC_k^{n,l}$. After that, the paper [11] wrote in the following way.

" Since $c_0 \in N_k(c_{l-1}, 1)$, we obviously have $0 \in p^{-1}(N_k(c_{l-1}, 1))$. By the condition (1) of Definition 3.3, we obtain

$$p^{-1}(N_k(c_{l-1}, 1)) = \bigcup_{t\in\mathbb{N}} N_2((l-1)t, 1) \tag{3.2}$$

But for every $t \in \mathbb{N}$,

$$0 \notin N_2((l-1)t, 1). \tag{3.3}$$

This contradiction to (3.2) shows that p cannot be a pseudo-$(2, k)$-covering map."

To be specific, owing to (3.3), the paper [11] concludes that the identity of (3.2) does not hold, which proves Proposition 3.4.

However, the proof of Proposition 3.4 in above is incorrect, as follows:

Remark 3.5. (Correction of the proof of Proposition 3.4 in the present paper or Proposition 3.2 of [11])

(1) Based on the above proof in [11, 12], we can find some errors in the proof of [11, 12]. Indeed, the part "$(l-1)t$" in (3.2) and (3.3) should be replaced by "$lt - 1$". In relation to the proof of Proposition 3.4 above, even though the paper [12] particularly considered the case of $l = 6$ of $SC_k^{n,6} := (s_i)_{i\in[0,5]_{\mathbb{Z}}}$ (see $SC_8^{2,6}$ as just an example for an $SC_k^{n,6}$ as in Figure 1), the same mistake also appears in the process of disproving a pseudo-$(2, k)$-covering map of the given map p in (3.1) (see the part with lines 13–15 on the page 51 of [12]).

Anyway, unlike the condition (1) of Definition 3.3, we indeed have

$$\bigcup_{t\in\mathbb{N}} N_2(6t-1, 1) \subsetneq p^{-1}(N_k(s_5, 1)) \tag{3.4}$$

where $6t - 1 \in p^{-1}(\{s_5\}), t \in \mathbb{N}$ (see Figure 1).

Similarly, by contrast to the condition (1) of Definition 3.3, given the map $p : (\mathbb{Z}^+, 2) \to SC_k^{n,l}$ of (3.1), we have

$$\bigcup_{t\in\mathbb{N}} N_2(lt-1,1) \subsetneq p^{-1}(N_k(c_{l-1},1)),$$

which implies that the map $p : (\mathbb{Z}^+, 2) \to SC_k^{n,l}$ in (3.1) is not a pseudo-$(2,k)$-covering map as desired.

(2) Using a similar method used in Remark 3.5(1), it is clear that the map p of (4.1) in the proof of Remark 4.3(2) of [9] is not a pseudo-$(8,8)$-covering map either (corrections of the map p of (4.1) of [9] and the map p in Remark 4.8 of [7]).

Similarly, the map f_2 in Example 4.3(1) of [7] is not a pseudo-$(2,k)$-covering map either.

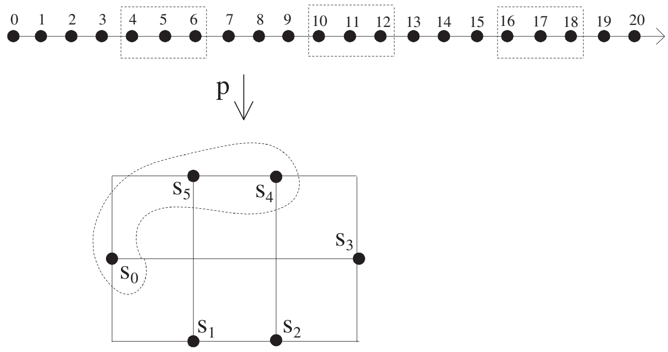

Figure 1: Configurations of a non-pseudo-$(2,8)$-covering map p of (3.1). In particular, for $N_8(s_5,1)$ in $SC_8^{2,6}$, we confirm the formula of (3.4).

To compare between a digital covering space and a pseudocovering space, we need to recall the notion of a digital covering space as follows:

Definition 3.6. ([3, 4, 6]) Let (E, k_0) and (B, k_1) be digital images on $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Let $p : E \to B$ be a surjection such that for any $b \in B$, the conditions (1) and (2) are equal to those of Definition 3.3, and the condition (3) is the following:

The restriction of p to $N_{k_0}(e_i, 1)$ from $N_{k_0}(e_i, 1)$ onto $N_{k_1}(p(e_i), 1)$ is a (k_0, k_1)-isomorphism for all $i \in M$. Then the map p is called a digital (k_0, k_1)-covering map, (E, p, B) is said to be a digital (k_0, k_1)-covering and (E, k_0) is called a digital (k_0, k_1)-covering space over (B, k_1).

In Definition 3.6, in the case of that $n_0 = n_1$ and $k_0 = k_1$, a (k_0, k_1)-covering is simply called a digital k_0-covering. Based on Definitions 3.3 and 3.6, the following is obtained.

Theorem 3.7. (See Theorem 3.3 of [11]) *A pseudo-(k_0, k_1)-covering map is a digital (k_0, k_1)-covering map.*

Indeed, this assertion is valid. However, in view of the proof of [11] we can make it more improved as follows: Since the conditions (1) and (2) of both Definitions 3.3 and 3.6 are equal to each other, we need to focus on only the conditions (3) of them.

Proof. (Improvement of the proof of Theorem 3.3 of [11]) With Definition 3.3, suppose that a pseudo-(k_0, k_1)-covering map $p : (E, k_0) \to (B, k_1)$ is not a digital-(k_0, k_1)-covering map. In view of the conditions (3) of Definitions 3.3 and 3.6, we may assume that there are $b \in B$ and $e \in p^{-1}(\{b\})$ such that the restriction map (or the WL-(k_0, k_1)-isomorphism of p) $p|_{N_{k_0}(e,1)} : N_{k_0}(e, 1) \to N_{k_1}(b, 1) = N_{k_1}(p(e), 1)$ is not surjective.

Thus there is

$$b' \in N_{k_1}(b, 1) \setminus p(N_{k_0}(e, 1)) \neq \emptyset. \qquad (\star 1)$$

Since $b' \in N_{k_1}(b, 1)$ and further, there is a point $e' \in E$ such that $p(e') = b'$ because p is surjective. Due to ($\star$1), $e' \notin N_{k_0}(e, 1)$. Hence we obtain

$$N_{k_0}(e, 1) \cap p^{-1}(\{b'\}) = \emptyset \text{ (equal to (3.1) of [11]).} \qquad (\star 2).$$

Then $e \in p^{-1}(\{b\}) \subseteq p^{-1}(N_{k_1}(b', 1)) = \bigcup_{i \in M} N_{k_0}(e'_i, 1)$, where $e'_i \in p^{-1}(\{b'\})$, which implies that there is $j \in M$ such that $e \in N_{k_0}(e'_j, 1)$, where $e'_j \in N_{k_0}(e, 1) \cap p^{-1}(\{b'\})$, which is a contradiction to ($\star$2).

Based on the surjection of $p|_{N_{k_0}(e,1)} : N_{k_0}(e, 1) \to N_{k_1}(b, 1) = N_{k_1}(p(e), 1)$, since $p|_{N_{k_0}(e'_i,1)} : N_{k_0}(e'_i, 1) \to N_{k_1}(b, 1)$ is a WL-(k_0, k_1)-isomorphism and $p|_{N_{k_0}(e'_i,1)}(N_{k_0}(e'_i, 1)) = N_{k_1}(b, 1)$, the map $p|_{N_{k_0}(e'_i,1)} : N_{k_0}(e'_i, 1) \to N_{k_1}(b, 1)$ is a (k_0, k_1)-isomorphism.

Conversely, since a local (k_0, k_1)-isomorphism is clearly a WL-(k_0, k_1)-isomorphism, it is obvious that a digital (k_0, k_1)-covering map is a pseudo-(k_0, k_1)-covering map [7]. □

In relation to the study of a relation between a WL-(k_0, k_1)-isomorphic surjection and a pseudo-(k_0, k_1)-covering map, there are the following two incorrect statements in [9] (see the identity (3.5) of Proposition 3.8 below) which will be corrected shortly.

Proposition 3.8. (The identity of (4.2) in Proposition 4.4 of [9]) *Let $p : (E, k_0) \to (B, k_1)$ be a WL-(k_0, k_1)-isomorphic surjection. Then, for any $b \in B$ with $e_i \in p^{-1}(\{b\})$, for some index set M we obtain*

$$p^{-1}(N_{k_1}(b, 1)) = \bigcup_{i \in M} N_{k_0}(e_i, 1) \textit{ with } e_i \in p^{-1}(\{b\}). \qquad (3.5)$$

Corollary 3.9. (Corollary 4.5 of [9]) *A WL-local (k_0, k_1)-isomorphic surjection is equivalent to a pseudo-(k_0, k_1)-covering map.*

Based on the notion of a WL-local (k_0, k_1)-isomorphic surjection, we now prove that Proposition 3.4 and Corollary 3.9 above are not valid with the following remark.

Remark 3.10. (1)(Correction of the identity of (4.2) in Proposition 4.4 of [9]) By contrast to the identity in (3.5) above, we have the map p in (3.1) as a counterexample. In detail, while the map p of (3.1) is a WL-$(2, k)$-isomorphic surjection, the map p does not support the identity of (3.5). Indeed, we have

$$\bigcup_{i \in M} N_{k_0}(e_i, 1) \subseteq p^{-1}(N_{k_1}(b, 1)) \text{ with } e_i \in p^{-1}(\{b\}). \qquad (3.6)$$

Hence, the identity of (4.2) in Proposition 4.4 of [9] should be written by (3.6) instead of (3.5).

(2) (Correction of Corollary 4.5 of [9]) Even though Corollary 4.5 of [9] wrote that a WL-(k_0, k_1)-isomorphic surjection is equivalent to a pseudo-(k_0, k_1)-covering map, it is invalid with the map p in (3.1) as a counterexample. More precisely, as shown in (3.6), the WL-$(2, k)$-isomorphic surjection p of (3.1) is not a pseudo-$(2, k)$-covering map (see Remark 3.5), which does not support Corollary 3.9.

In view of Remark 3.10, the following is obtained.

Corollary 3.11. *A pseudo-(k_0, k_1)-covering map implies a WL-(k_0, k_1)-isomorphic surjection, the converse does not hold.*

4. A revision of the condition (1) of Definition 3.3 for a pseudo-(k_0, k_1)-covering space

Based on both Remark 3.5 and Theorem 3.7, since the term "pseudo-(k_0, k_1)-covering" is broader than digital "(k_0, k_1)-covering" in the sense of language, the revision of the original version of a pseudocovering sapce can be admissible. Let us now revise the condition (1) of Definition 3.3 to make a distinction from a digital (k_0, k_1)-covering map.

Definition 4.1. (Revision of Definition 3.3 for a pseudo-(k_0, k_1)-covering space) Let (E, k_0) and (B, k_1) be digital images on $\mathbb{Z}^{n_0}$ and $\mathbb{Z}^{n_1}$, respectively. Let $p : E \to B$ be a surjection such that for any $b \in B$,

(1) for some index set M, $\bigcup_{i\in M} N_{k_0}(e_i, 1)$ is a subset of $p^{-1}(N_{k_1}(b, 1))$ with $e_i \in p^{-1}(\{b\})$ and the other two conditions are equal to the conditions (2) and (3) of Definition 3.3.

Then the map p is called a pseudo-(k_0, k_1)-covering map and (E, k_0) is called a pseudo-(k_0, k_1)-covering space over (B, k_1).

Hereinafter, we will follow Definition 4.1 for a pseudo-(k_0, k_1)-covering space as a revised version of Definition 3.3. Then we have some utilities of it and many good examples for it as follows:

Theorem 4.2. *Let $(E := (e_i)_{i\in[0,\infty)_\mathbb{Z}}, k')$ on $\mathbb{Z}^n$ be a digital image which is $(k', 2)$-isomorphic to $(\mathbb{Z}^+, 2)$. Then (E, k') is a pseudo-(k', k)-covering space over $SC_k^{n,l}$.*

Proof. Based on Definition 4.1, consider the map (see the map q from (c) to (b) in Figure 2 as an example)

$$\left\{\begin{array}{l} q : (E := (e_i)_{i\in[0,\infty)_\mathbb{Z}}, k') \to SC_k^{n,l} := (x_i)_{i\in[0,l-1]_\mathbb{Z}} \text{ defined by} \\ q(e_i) = x_{i(mod\, l)}. \end{array}\right\} \tag{4.1}$$

Then this map q satisfies all of the conditions of Definition 4.1, which implies a pseudo-(k', k)-covering map of q. □

Example 4.3. (1) Consider the map $q : (E := (e_i)_{i\in[0,\infty)_\mathbb{Z}}, 8) \to SC_8^{2,4} := (x_i)_{i\in[0,3]_\mathbb{Z}}$ defined by $q(e_i) = x_{i(mod\, 4)}$. Then it is both a pseudo-$(8, 8)$-covering map and a WL-$(8, 8)$-isomorphic surjection (see the map q from (c) to (b) in Figure 2).

(2) The map p of (3.1) is a pseudo-$(2, k)$-covering map.

(3) As a generalization of the map p in (2) above, let

$$p : ([a, \infty)_\mathbb{Z}, 2) \to SC_k^{n,l} := (s_i)_{i\in[0,l-1]_\mathbb{Z}}, l \geq 4 \tag{4.2}$$

defined by $p(i) = s_{i(mod\, l)}$, where $a = tl, t \in \mathbb{N} \cup \{0\}$. Then this map p is a pseudo-$(2, k)$-covering map.

(4) $p : ([0, m]_\mathbb{Z}, 2) \to SC_k^{n,l} := (s_i)_{i\in[0,l-1]_\mathbb{Z}}, l \geq 4, m \geq l$ defined by $p(i) = s_{i(mod\, l)}$. Then this map p is a pseudo-$(2, k)$-covering map (see Example 4.5 below and Figure 2).

Proposition 4.4. (Improvement of Corollary 4.5 of [9]) *With Definition 4.1, a WL-(k_0, k_1)-isomorphic surjection is equivalent to a pseudo-(k_0, k_1)-covering map.*

Proof. With Definition 4.1, as referred to in (3.6), after replacing the formula of (3.5) by the formula of (3.6), i.e., for some M

$$\bigcup_{i\in M} N_{k_0}(e_i, 1) \subseteq p^{-1}(N_{k_1}(b, 1)), \text{ with } e_i \in p^{-1}(\{b\}),$$

see the proof of Proposition 4.4 of [9] for details. □

Example 4.5. Let

$$\left\{\begin{array}{l} p : ([0, 7]_\mathbb{Z}, 2) \to SC_8^{2,4} = (x_i)_{i\in[0,3]_\mathbb{Z}}, \\ \text{defined by } p(t) = x_{t(mod\, 4)}. \end{array}\right\} \tag{4.3}$$

Then the map p is clearly a WL-$(2, 8)$-isomorphic surjection (see the map p from (a) to (b) of Figure 2). Besides, the map p also satisfies Definition 4.1. In detail, for the element $x_3 \in SC_8^{2,4}$ in (4.3) and Figure 2, unlike the identity in (4.2) of Proposition 4.4 of [9] (or Proposition 3.8 in the present paper), in particular, we obtain

$$\left\{\begin{array}{l} \bigcup_{i\in M:=\{3,7\}} N_2(e_i, 1) = \{2, 3, 4, 6, 7\} \subsetneq p^{-1}(N_8(x_3, 1)) = \{0, 2, 3, 4, 6, 7\} \\ \text{with } e_i \in p^{-1}(\{x_3\}) = \{3, 7\}. \end{array}\right\} \tag{4.4}$$

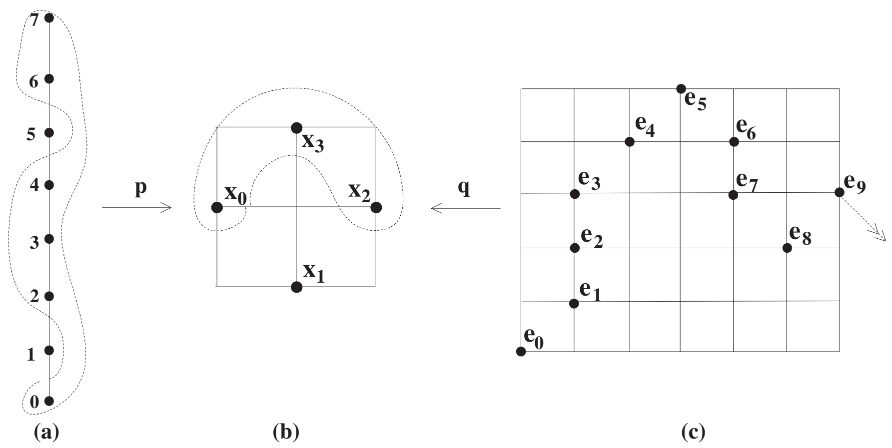

Figure 2: (1) Configuration of a WL-$(2,8)$-isomorphic surjection and a pseudo-$(2,8)$-covering map p in (4.3) followed by Definition 4.1. (2) Configuration of both a WL-8-isomorphic surjection and a pseudo-8-covering space over $SC_8^{2,4}$ based on the map $q : (E,8) \to SC_8^{2,4}$ in (4.1) (see the map q from (c) to (b)), where the set of (c) is just a part of $(E,8)$ in (4.1).

In relation to the unique path lifting property in [4] and the digital homotopy lifting property in [3], we note the following:

Remark 4.6. A pseudo-(k_0,k_1)-covering map of Definition 4.1 need not support the unique path lifting property and the digital homotopy lifting property. But it has the pseudo path lifting property in [7].

5. Summary

The paper revised the condition (1) of the original version of a pseudo-(k_0,k_1)-covering space (see Definition 4.1). Based on this revision, it turns out that while a digital covering space implies a pseudo-covering space (compare Definitions 3.3 and 4.1), the converse does not hold. Besides, it was proved that a WL-(k_0,k_1)-isomorphic surjection is equivalent to a revised version of a pseudo-(k_0,k_1)-covering map.

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