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Relating the annihilation number and the 2-domination number for unicyclic graphs

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Abstract. The *2-domination number* $\gamma_2(G)$ of a graph G is the minimum cardinality of a set $S \subseteq V(G)$ such that every vertex from $V(G) \setminus S$ is adjacent to at least two vertices in S. The *annihilation number* $a(G)$ is the largest integer k such that the sum of the first k terms of the non-decreasing degree sequence of G is at most the number of its edges. It was conjectured that $\gamma_2(G) \leq a(G)+1$ holds for every non-trivial connected graph G. The conjecture was earlier confirmed for graphs of minimum degree 3, trees, block graphs and some bipartite cacti. However, a class of cacti were found as counterexample graphs recently by Yue et al. [9] to the above conjecture. In this paper, we consider the above conjecture from the positive side and prove that this conjecture holds for all unicyclic graphs.

1. Introduction

Given a graph G, we denote by $V(G)$ and $E(G)$ the set of its vertices and edges, respectively. Also, we let $n(G) = |V(G)|$ and $m(G) = |E(G)|$. The *open neighbourhood* of a vertex $v \in V(G)$ is $N_G(v) = \{u|uv \in E(G)\}$. We denote the *degree* of a vertex v by $d_G(v) = |N_G(v)|$. For a pair of vertices $u, v \in V(G)$, the *distance* $d_G(u, v)$ between u and v is the length of a shortest (u, v)-path in G. A path $P = x_1x_2 \dots x_p$ $(p \geq 3)$ in a graph G is said to be a *pendent path* if $d_G(x_1) = 1$, $d_G(x_2) = \cdots = d_G(x_{p-1}) = 2$ and $d_G(x_p) \geq 3$. In particular, when $p = 2$, P is said to be a *pendent edge* and x_1 is said to be a *leaf* or *pendent vertex*. The above x_2 is said to be a *support vertex*. Further, if uvw is a 3-vertex path with $d_G(u) = 1 = d_G(w)$ and $d_G(v) \geq 2$, then v is said to be a *strong support vertex*. A vertex of degree at least 3 is called a *branch vertex*. If $X \subseteq V(G)$, then $G - X$ denotes the graph obtained from G by deleting all vertices in X and all edges incident with them. A connected graph is *unicyclic* if it contains exactly one cycle. A unicyclic graph is a *sun* if each vertex on the cycle is connected to exactly one leaf.

For a graph G of order n and a positive integer $k(\leq n-1)$, a vertex set $D \subseteq V(G)$ is called a *k-dominating set* if each vertex not in D has at least k neighbors in D. The *k-domination number* $\gamma_k(G)$ is the minimum cardinality of such a set D. A k-dominating set of cardinality $\gamma_k(G)$ is called a γ_k-set of G. A 1-dominating set is just the well-studied *dominating set*. The notion of the k-dominating set was introduced by Fink and Jacobson [5], and a survey on k-dominating set can be found in [2].

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For a vertex set $S \subseteq V(G)$, we define $\sum(S,G) = \sum_{v\in S} d_G(v)$. Then S is an *annihilation set* of G if $\sum(S,G) \le m(G)$. Let $v_1, v_2, \dots, v_n$ be an ordering of the vertices of G such that $d_G(v_1) \le d_G(v_2) \le \cdots \le d_G(v_n)$. The *annihilation number* $a(G)$, firstly introduced by Pepper in [8], is the largest integer k such that $\sum_{i=1}^{k} d_G(v_i) \le m(G)$. Further, S is an *optimal annihilation set* if $|S| = a(G)$ and $\max\{d_G(v)|v \in S\} \le \min\{d(u)|u \in V(G)\setminus S\}$.

A conjecture relating the 2-domination number and annihilation number of a graph reads as follows.

Conjecture 1.1 ([3, 4]). *If G is a non-trivial connected graph, then $\gamma_2(G) \le a(G) + 1$.*

From the above definition of annihilation set, every graph satisfies $a(G) \ge \lfloor \frac{n}{2} \rfloor$. Also, it was observed in [1] that $\gamma_2(G) \le \lfloor \frac{n}{2} \rfloor$ for $\delta(G) \ge 3$. Hence, if $\delta(G) \ge 3$, then Conjecture 1.1 holds. It remains for us to study this conjecture for connected graphs with $\delta(G) = 1$ or 2. Inspired by this, Desormeaux et al. [4] studied Conjecture 1.1 for trees, and their result is stated as follows.

Theorem 1.2 ([4]). *If G is a non-trivial tree, then $\gamma_2(G) \le a(G) + 1$.*

It is interesting to note that Theorem 1.2 was re-proven by Lyle [7] by employing a new method in 2017. Later, the result of Theorem 1.2 was extended to the family of block graphs by Jakovac [6]. More recently, Yue et al. [9] disproved Conjecture 1.1 by giving a class of counterexample cacti with leaves. Nevertheless, it still makes sense to consider some special graph family that satisfy Conjecture 1.1. Along this line, Yue et al. [9] proved Conjecture 1.1 holds for a class of bipartite cacti. In this paper, we investigate Conjecture 1.1 for unicyclic graphs, and we obtain the following result.

Theorem 1.3. *Let G be a unicyclic graph. Then $\gamma_2(G) \le a(G) + 1$.*

2. Preliminary results

In this section, we introduce two observations and a critical lemma.

We begin with the following two observations, which can be deduced from the definitions of 2-dominating set and optimal annihilation set immediately.

Observation 1. *Any 2-dominating set of a graph G contains all leaves.*

Observation 2 ([9]). *Any optimal annihilation set of a connected graph G of order $n(\ge 3)$ contains all leaves of G.*

For a unicyclic graph G with $C_\ell = u_1u_2 \dots u_\ell u_1$ being its unique cycle, we denote by T_{u_j} the component containing u_j in $G - \{u_{j-1}, u_{j+1}\}$ (If $j = 1$, we set $u_{j-1} = u_\ell$ and if $j = \ell$, then $u_{j+1} = u_1$). Such a T_{u_j} is also said to be a subtree of G, rooted at u_j.

Definition 2.1. *The* subdivided star *$S_s(K_{1,s+t}, u)(s \ge 1,\ t \ge 0)$ is the graph on $2s+t+1$ vertices which is constructed from the star $K_{1,s+t}$ (with u being the centre) by subdividing any s edges exactly once. In particular, when $s = 1$ and $t = 0$, $S_s(K_{1,s+t}, u) \cong P_3$ with u being one end-vertex. When $s = 1$ and $t = 1$, $S_s(K_{1,s+t}, u) \cong P_4$ with u being a 2-degree vertex. When $s = 2$ and $t = 0$, $S_s(K_{1,s+t}, u) \cong P_5$ with u being the central vertex. When $s + t \ge 3$, u is the maximum degree vertex of $S_s(K_{1,s+t}, u)$.*

Lemma 2.2. *Let G be a unicyclic graph with the unique cycle being C. If C contains a vertex u such that T_u is a subdivided star $S_s(K_{1,s+t}, u)(s \ge 2)$, then $\gamma_2(G) \le a(G) + 1$.*

Proof. For each $i \in [s]$, let $uv_i'v_i$ be a pendent path attached to u and for each $j \in [t]$, let w_j be the leaf adjacent to u if $t \ge 1$, see Figure 1. Let $G' = G - V(S_s(K_{1,s+t}, u))$. Then G' is a non-trivial tree with $m(G') = m(G) - 2s - t - 2$. By Theorem 1.2, we have $\gamma_2(G') \le a(G') + 1$. Let D' be a γ_2-set of G'. From Figure 1 and the definition of 2-dominating set, it can be seen that $D = D' \cup \{u, v_1, v_2, \dots, v_s, w_1, \dots, w_t\}$ is a 2-dominating set of G, yielding that $\gamma_2(G) \le |D| = |D'| + s + t + 1 = \gamma_2(G') + s + t + 1$. Suppose that S' is an optimal annihilation set of G' and let $S = S' \cup \{v_1', v_1, \ldots, v_s, w_1, \ldots, w_t\}$. As $s \ge 2$,

$$\begin{aligned}\sum(S,G) &= \sum(S',G) + d_G(v_1') + d_G(v_1) + \cdots + d_G(v_s) + d_G(w_1) + \cdots + d_G(w_t)\\ &\le \Big(\sum(S',G') + 2\Big) + 2 + s + t\\ &\le m(G') + 4 + s + t\\ &\le m(G) - s + 2\\ &\le m(G).\end{aligned}$$

So $a(G) \ge |S| = |S'| + s + t + 1 = a(G') + s + t + 1$. This gives $\gamma_2(G) \le \gamma_2(G') + s + t + 1 \le a(G') + s + t + 2 \le a(G) + 1$. □

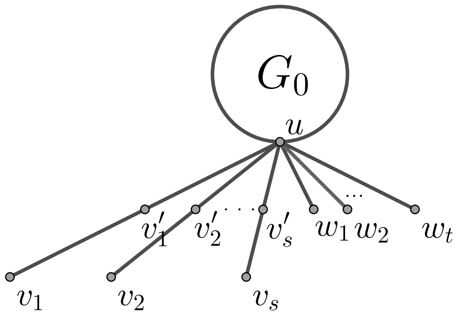

Figure 1: The structure of G in Lemma 2.2, where G_0 is a unicyclic subgraph of G.

3. Proof of Theorem 1.3

In this section we prove Theorem 1.3.

Proof. We proceed by induction on $n = n(G)$. If $n = 3$, then $G \cong C_3$ and $\gamma_2(C_3) = 2 = a(C_3) + 1$. So, we let $n \ge 4$ and assume that for every connected unicyclic graph G' of order $n' < n$, we have $\gamma_2(G') \le a(G') + 1$.

If G is a cycle, it's easy to check that the statement is true. Thus, we may suppose that G contains one cycle as a proper subgraph. Define $\mathcal{L}(G) = \{v \in V(G) | d_G(v) = 1\}$. Since G is a unicyclic graph not isomorphic to a cycle, $\mathcal{L}(G) \ne \emptyset$. Let C_ℓ be the unique cycle in G. For each $u \in V(C_\ell)$ with $d_G(u) \ge 3$, we define $h(u) = \max\{d_G(u,x) | x \in \mathcal{L}(T_u)\}$ and $h(G) = \max\{h(u) | u \in V(C_\ell)$ and $d_G(u) \ge 3\}$.

We first prove the following claim.

Claim 3.1. *Let u be a branch vertex on $V(C_\ell)$ such that T_u has a leaf v_1 with $d_G(v_1, u) = h(G)$. Assume that v_2 is the unique neighbour of v_1 and v_2 is a strong support vertex. Then $\gamma_2(G) \le a(G) + 1$.*

Proof. As v_2 is a strong support vertex, v_2 has at least two leaf-neighbours. Let $v_1, z_1, \ldots, z_t$ ($t \ge 1$) be leaf-neighbours of v_2 and $G' = G - \{v_1, v_2, z_1, \ldots, z_t\}$. Then $m(G') = m(G) - d_G(v_2)$. Obviously, G' has at least two vertices. Let D' be a γ_2-set of G'. Then $D = D' \cup \{v_1, z_1, \ldots, z_t\}$ is a 2-dominating set of G, which implies $\gamma_2(G) \le |D| = |D'| + t + 1 = \gamma_2(G') + t + 1$. Let S' be an optimal annihilation set of G' and $S = S' \cup \{v_1, z_1, \ldots, z_t\}$. Then

$$\begin{aligned}\sum(S,G) &= \sum(S',G) + d_G(v_1) + d_G(z_1) + \cdots + d_G(z_t)\\ &\le \Big(\sum(S',G') + d_G(v_2) - t - 1\Big) + t + 1\\ &\le m(G') + d_G(v_2)\\ &\le m(G),\end{aligned}$$

yielding that $a(G) \geq |S| = |S'| + t + 1 = a(G') + t + 1$. If G' is a non-trivial tree, then by Theorem 1.2, $\gamma_2(G') \leq a(G') + 1$. If G' is a unicyclic graph, then by the induction hypothesis, $\gamma_2(G') \leq a(G') + 1$. Therefore, $\gamma_2(G) \leq \gamma_2(G') + t + 1 \leq a(G') + t + 2 \leq a(G) + 1$. □

We will complete the proof by considering the following cases.

Case 1. $h(G) = 1$.

Since $h(G) = 1$, every vertex outside of C_ℓ is a leaf attached to some vertex of C_ℓ. Clearly, each vertex of C_ℓ is adjacent to at most one leaf. For otherwise, by Claim 3.1, we have $\gamma_2(G) \leq a(G) + 1$, as claimed. So, G is a sun or a unicyclic graph obtained from the sun by removing some leaves.

First, we assume that G is a sun. Let $V(C_\ell) = \{u_1, u_2, \ldots, u_\ell\}$ and w_i is the leaf adjacent to u_i for each $i \in [\ell]$. Clearly, $m(G) = 2\ell$. Take $D = \{u_1, u_4, \ldots, u_{3m+1}, w_1, w_2, \ldots, w_\ell\}$ ($m \in [\frac{\ell-3}{3}, \frac{\ell-1}{3}]$). Then D is a γ_2-set of G, and hence $\gamma_2(G) \leq \lceil \frac{\ell}{3} \rceil + \ell$. Set $S = \{u_1, u_2, \ldots, u_{\lceil \frac{\ell}{3} \rceil - 1}, w_1, w_2, \ldots, w_\ell\}$. Then $\sum(S, G) = (\lceil \frac{\ell}{3} \rceil - 1) \times 3 + \ell \leq 2\ell - 1 < m(G)$, yielding that $a(G) \geq |S| = \lceil \frac{\ell}{3} \rceil + \ell - 1$. So, $\gamma_2(G) \leq a(G) + 1$.

Second, we assume that G is not a sun. There exists a 3-degree vertex, say w, on C_ℓ that has a 2-degree neighbour, say v, on C_ℓ. Denote the pendent vertex adjacent to w with w_1. Set $G' = G - \{w, w_1\}$. Then $m(G') = m(G) - 3$. It follows from Theorem 1.2 that $\gamma_2(G') \leq a(G') + 1$, since G' is a non-trivial tree. Let D' be a γ_2-set of G' and S' be an optimal annihilation set of G'. Since $d_{G'}(v) = 1$, by Observations 1 and 2, we have $v \in D'$ and $v \in S'$. Then $D = D' \cup \{w_1\}$ is a 2-dominating set of G and hence $\gamma_2(G) \leq |D| = |D'| + 1 = \gamma_2(G') + 1$. Also, we have $\sum(S', G) \leq \sum(S', G') + 2$. Take $S = S' \cup \{w_1\}$. Then $\sum(S, G) = \sum(S', G) + d_G(w_1) \leq \left(\sum(S', G') + 2\right) + 1 \leq m(G') + 3 = m(G)$ and hence $a(G) \geq |S| = |S'| + 1 = a(G') + 1$. Thus,

$$\gamma_2(G) \leq \gamma_2(G') + 1 \leq a(G') + 2 \leq a(G) + 1.$$

Case 2. $h(G) = 2$.

Assume that there exists a branch vertex u on C_ℓ such that T_u contains a leaf v_1 satisfying that $d_G(v_1, u) = 2$. Let v_2 be the unique neighbour of v_1. If $d_G(v_2) \geq 3$, then v_2 is a strong support vertex. By Claim 3.1, we have $\gamma_2(G) \leq a(G) + 1$. So, we may assume that $d_G(v_2) = 2$. By the same reason, if $N_G(u) \backslash \left(V(C_\ell) \cup \{v_2\}\right) \neq \emptyset$, then for each $x \in N_G(u) \backslash \left(V(C_\ell) \cup \{v_2\}\right)$, we have $d_G(x) = 1$ or $d_G(x) = 2$. So, $T_u \cong S_s(K_{1,s+t}, u) (s \geq 1, t \geq 0)$. If $s \geq 2$, then we conclude that $\gamma_2(G) \leq a(G) + 1$ by Lemma 2.2. So, we assume that $s = 1$ and $t \geq 0$. Assume that $V(S_1(K_{1,1+t}, u)) = \{u, v_1, v_2, y_1, \ldots, y_t\}$, where u is the vertex defined as in Defintion 2.1, uv_2v_1 is a pendent path and $y_1, \cdots, y_t$ are leaves attached to u if $t \geq 1$.

We consider the following subcases.

Subcase 2.1. *There exists at least a vertex of degree 2, say v, adjacent to u on C_ℓ.*

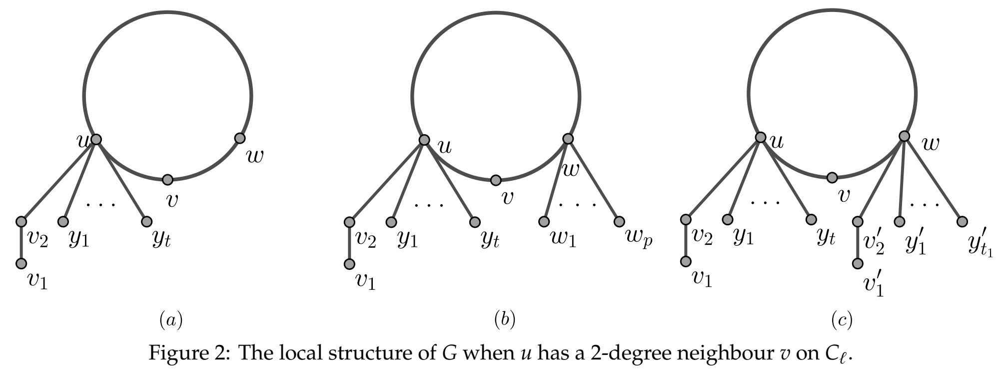

Figure 2: The local structure of G when u has a 2-degree neighbour v on C_ℓ.

Let $N_G(v)\backslash\{u\} = \{w\}$. First, we assume that $d_G(w) = 2$. Then G can be viewed as the graph shown in Figure 2(a). If $\ell \geq 4$, we set $G' = G - \Big(V(S_1(K_{1,1+t}, u)) \cup \{v\}\Big)$. Then G' is a non-trivial tree and $m(G') = m(G) - (t+5)$. According to Theorem 1.2, we have $\gamma_2(G') \leq a(G') + 1$. Let D' be a γ_2-set of G' and S' be an optimal annihilation set of G'. Since $d_{G'}(w) = 1$, we have $w \in D'$ and $w \in S'$ by Observations 1 and 2. Then $D = D' \cup \{u, v_1, y_1, \ldots, y_t\}$ is a 2-dominating set of G. So $\gamma_2(G) \leq |D| = |D'| + t + 2 = \gamma_2(G') + t + 2$. Take $S = S' \cup \{v_1, v_2, y_1, \ldots, y_t\}$. Then $\sum(S,G) = \sum(S',G) + d_G(v_1) + d_G(v_2) + d_G(y_1) + \cdots + d_G(y_t) \leq \Big(\sum(S',G') + 2\Big) + (t+3) \leq m(G') + t + 5 = m(G)$, which implies $a(G) \geq |S| = |S'| + t + 2 = a(G') + t + 2$. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + t + 2 \leq a(G') + t + 3 \leq a(G) + 1.$$

If $\ell = 3$, then $m(G) = t+5$. Take $D = \{u, v, v_1, y_1, \ldots, y_t\}$ and hence D is a minimum 2-dominating set of G. Then $\gamma_2(G) \leq t+3$. Take $S = \{v, v_1, v_2, y_1, \ldots, y_t\}$. Then $\sum(S,G) = t+5 = m(G)$ and we have $a(G) \geq |S| = t+3$. So, $\gamma_2(G) \leq t+3 \leq a(G) < a(G)+1$.

Second, we assume that $d_G(w) \geq 3$ and $h(w) = 1$. Then G can be viewed as the graph shown in Figure 2(b). Let $N_G(w)\backslash V(C_\ell) = \{w_1, \ldots, w_p\}$ $(p \geq 1)$.

Suppose first that $p = 1$. If $\ell \geq 4$, we set $G' = G - \Big(V(S_1(K_{1,1+t}, u)) \cup \{v, w_1\}\Big)$. Obviously, G' is a non-trivial tree and $m(G') = m(G) - (t+6)$. Let D' be a γ_2-set of G' and S' be an optimal annihilation set of G'. Since $d_{G'}(w) = 1$, we obtain $w \in D'$ and $w \in S'$ by Observations 1 and 2. Let $D = D' \cup \{u, v_1, y_1, \ldots, y_t, w_1\}$ and hence D is a 2-dominating set of G. Then $\gamma_2(G) \leq |D| = |D'| + t + 3 = \gamma_2(G') + t + 3$. Set $S = (S'\backslash\{w\}) \cup \{v, v_1, v_2, y_1, \ldots, y_t, w_1\}$. Since $\sum(S'\backslash\{w\}, G) \leq \sum(S'\backslash\{w\}, G') + 1$, we have $\sum(S,G) = \sum(S'\backslash\{w\}, G) + d_G(v) + d_G(v_1) + d_G(v_2) + d_G(y_1) \cdots + d_G(y_t) + d_G(w_1) \leq \Big(\sum(S'\backslash\{w\}, G') + 1\Big) + (t+6) = \sum(S', G') - d_{G'}(w) + t + 7 \leq m(G') + t + 6 = m(G)$ and we have $a(G) \geq |S| = |S'| + t + 3 = a(G') + t + 3$. Since G' is a non-trivial tree, by Theorem 1.2, we have $\gamma_2(G') \leq a(G') + 1$, and hence

$$\gamma_2(G) \leq \gamma_2(G') + t + 3 \leq a(G') + t + 4 \leq a(G) + 1.$$

If $\ell = 3$, then $m(G) = t+6$. Take $D = \{u, v, w_1, v_1, y_1, \ldots, y_t\}$ and hence D is a γ_2-set of G. Then $\gamma_2(G) \leq t+4$. Take $S = \{v, w_1, v_1, v_2, y_1, \ldots, y_t\}$. Then $\sum(S,G) = t+6 = m(G)$ and we have $a(G) \geq |S| = t+4$. Accordingly, $\gamma_2(G) \leq t+4 \leq a(G) < a(G)+1$.

Now, let $p \geq 2$. Set $G' = G - \{w, w_1, w_2, \ldots, w_p\}$. Then G' is a non-trivial tree and $m(G') = m(G) - p - 2$. Let D' be a γ_2-set of G' and S' be an optimal annihilation set of G'. Then $D = D' \cup \{w_1, w_2, \ldots, w_p\}$ is a 2-dominating set of G, yielding that $\gamma_2(G) \leq |D| = |D'| + p = \gamma_2(G') + p$. Take $S = S' \cup \{w_1, w_2, \ldots, w_p\}$. Then $\sum(S,G) = \sum(S',G) + d_G(w_1) + d_G(w_2) + \cdots + d_G(w_p) \leq \Big(\sum(S',G') + 2\Big) + p \leq m(G') + p + 2 = m(G)$. So, $a(G) \geq |S| = |S'| + p = a(G') + p$. As G' is a non-trivial tree, we have $\gamma_2(G') \leq a(G') + 1$ by Theorem 1.2. Therefore,

$$\gamma_2(G) \leq \gamma_2(G') + p \leq a(G') + p + 1 \leq a(G) + 1.$$

Finally, let $d_G(w) \geq 3$ and $h(w) = 2$. By Claim 3.1, it suffices to prove $T_w \cong S_{s_1}(K_{1,s_1+t_1}, w)$ $(s_1 \geq 1, t_1 \geq 0)$. If $s_1 \geq 2$, then by Lemma 2.2 we have $\gamma_2(G) \leq a(G)+1$. So, we assume that $s_1 = 1$. Now, G can be viewed as the graph shown in Figure 2(c). First, we assume that $\ell \geq 4$. Let $G' = G - \Big(V(S_1(K_{1,1+t}, u)) \cup V(S_1(K_{1,1+t_1}, w)) \cup \{v\}\Big)$. Then $m(G') = m(G) - (t + t_1 + 8)$. If $n' = |G'| \geq 2$, then G' is a non-trivial tree. Let D' be a γ_2-set of G'. Then $D = D' \cup \{u, w, v_1, v_1', y_1, \ldots, y_t, y_1', \ldots, y_{t_1}'\}$ is a 2-dominating set of G. Therefore, $\gamma_2(G) \leq |D| = |D'| + t + t_1 + 4 = \gamma_2(G') + t + t_1 + 4$. Let S' be an optimal annihilation set of G', we have $\sum(S',G) \leq \sum(S',G') + 2$. Take $S = S' \cup \{v_1, v_1', v_2, v_2', y_1, \ldots, y_t, y_1', \ldots, y_{t_1}'\}$. Then $\sum(S,G) \leq \Big(\sum(S',G') + 2\Big) + (t + t_1 + 6) \leq m(G') + t + t_1 + 8 = m(G)$ and we have $a(G) \geq |S| = |S'| + t + t_1 + 4 = a(G') + t + t_1 + 4$. Since G' is a non-trivial tree, it follows from Theorem 1.2 that $\gamma_2(G') \leq a(G') + 1$. Hence,

$$\gamma_2(G) \leq \gamma_2(G') + t + t_1 + 4 \leq a(G') + t + t_1 + 5 \leq a(G) + 1.$$

If $n' = |G'| = 1$, then $\ell = 4$ and $m(G) = t + t_1 + 8$. If $\ell = 3$, then $m(G) = t + t_1 + 7$. Upon the case when $n' = 1$ or $\ell = 3$, we take $D = \{u, w, v_1, v_1', y_1, \ldots, y_t, y_1', \ldots, y_{t_1}'\}$ and hence D is a minimum 2-dominating set of G. Then $\gamma_2(G) \leq t + t_1 + 4$. Take $S = \{v_1, v_1', v_2, v_2', y_1, \ldots, y_t, y_1', \ldots, y_{t_1}'\}$. Then $\sum(S,G) = t + t_1 + 6 < m(G)$ and we have $a(G) \geq |S| = t + t_1 + 4$. Therefore, $\gamma_2(G) \leq t + t_1 + 4 \leq a(G) < a(G) + 1$.

Subcase 2.2. *The vertex u has a neighbour v on $V(C_\ell)$ such that $h(v) = 1$ and $N_G(v)\backslash V(C_\ell) = \{z_1, \cdots, z_q\}$ $(q \geq 1)$.*

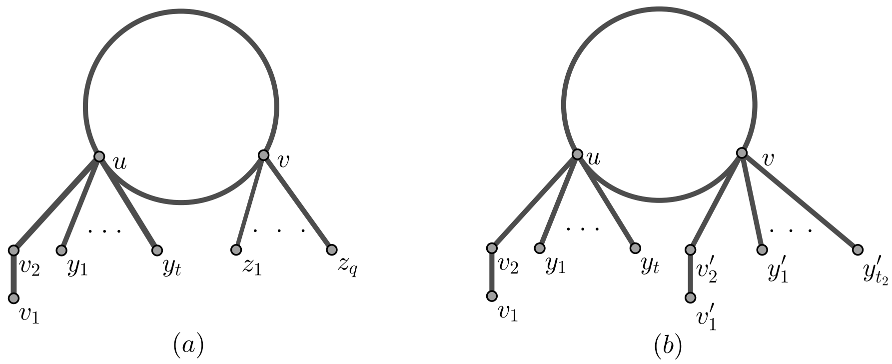

Figure 3: The local structure of G in Subcases 2.2 and 2.3, respectively.

In this subcase, G can be viewed as the graph shown in the Figure 3(a). First, we assume that $q = 1$. Take $G' = G - \left(V(S_1(K_{1,1+t}, u)) \cup \{v, z_1\}\right)$ and $m(G') = m(G) - (t+6)$. If $n' = |G'| = 1$, then G is identical to the graph as shown in Figure 2(b) for the case of $\ell = 3$ and $p = 1$. So, $\gamma_2(G) \leq a(G) + 1$ by our previous proof in Subcase 2.1. Now, we suppose that $n' \geq 2$. Thus, G' is a non-trivial tree. Let D' be a γ_2-set of G'. Then $D = D' \cup \{u, v_1, y_1, \ldots, y_t, z_1\}$ is a 2-dominating set of G, and hence $\gamma_2(G) \leq |D| = |D'| + t + 3 = \gamma_2(G') + t + 3$. Let S' be an optimal annihilation set of G' and $S = S' \cup \{v_1, v_2, y_1, \ldots, y_t, z_1\}$. Then $\sum(S, G) = \sum(S', G) + (t+4) \leq \left(\sum(S', G') + 2\right) + (t+4) \leq m(G') + t + 6 = m(G)$ and we have $a(G) \geq |S| = |S'| + t + 3 = a(G') + t + 3$. Since G' is a non-trivial tree, we obtain $\gamma_2(G') \leq a(G') + 1$ according to Theorem 1.2. Thus,

$$\gamma_2(G) \leq \gamma_2(G') + t + 3 \leq a(G') + t + 4 \leq a(G) + 1.$$

Now, let $q \geq 2$. Set $G' = G - \{v, z_1, z_2, \ldots, z_q\}$. Then G' is a non-trivial tree and $m(G') = m(G) - q - 2$. Let D' be a γ_2-set of G'. Then $D = D' \cup \{z_1, z_2, \ldots, z_q\}$ is a 2-dominating set of G, yielding that $\gamma_2(G) \leq |D| = |D'| + q = \gamma_2(G') + q$. Let S' be an optimal annihilation set of G' and let $S = S' \cup \{z_1, z_2, \ldots, z_q\}$. Then $\sum(S, G) = \sum(S', G) + d_G(z_1) + d_G(z_2) + \cdots + d_G(z_q) \leq \left(\sum(S', G') + 2\right) + q \leq m(G') + q + 2 = m(G)$. So, $a(G) \geq |S| = |S'| + q = a(G') + q$. As G' is a non-trivial tree, we have $\gamma_2(G') \leq a(G') + 1$ by Theorem 1.2. Thus,

$$\gamma_2(G) \leq \gamma_2(G') + q \leq a(G') + q + 1 \leq a(G) + 1.$$

Subcase 2.3. *The vertex u has a neighbour $v \in V(C_\ell)$ such that $h(v) = 2$.*

By Claim 3.1, it suffices to prove that $T_v \cong S_{s_2}(K_{1,s_2+t_2}, v)$ $(s_2 \geq 1, t_2 \geq 0)$. If $s_2 \geq 2$, then by Lemma 2.2, we have $\gamma_2(G) \leq a(G) + 1$. So, we assume that $s_2 = 1$. Now, G can be viewed as the graph shown in Figure 3(b). Let $G' = G - \left(V(S_1(K_{1,1+t}, u)) \cup (V(S_1(K_{1,1+t_2}, v))\backslash\{v\})\right)$. Then G' is a tree with at least two vertices and $m(G') = m(G) - (t + t_2 + 6)$. Let D' be a γ_2-set of G' and S' be an optimal annihilation set of G'. It follows from Observations 1 and 2 that v belongs to any γ_2-set and optimal annihilation set of G' since it is a leaf in G'. Now we let $D = D' \cup \{u, v_1, v_1', y_1, \ldots, y_t, y_1', \ldots, y_{t_2}'\}$ and hence D is a 2-dominating set of G. Then $\gamma_2(G) \leq |D| = |D'| + t + t_2 + 3 = \gamma_2(G') + t + t_2 + 3$. Take $S = (S'\backslash\{v\}) \cup \{v_1, v_1', v_2, v_2', y_1, \ldots, y_t, y_1', \ldots, y_{t_2}'\}$. As $\sum(S'\backslash\{v\}, G') = \sum(S', G') - d_{G'}(v) = \sum(S', G') - 1$, $\sum(S'\backslash\{v\}, G) \leq \sum(S'\backslash\{v\}, G') + 1 = \sum(S', G') \leq m(G')$. So, $\sum(S, G) = \sum(S'\backslash\{v\}, G) + d_G(v_1) + d_G(v_1') + d_G(v_2) + d_G(v_2') + d_G(y_1) + \cdots + d_G(y_t) + d_G(y_1') + \cdots + d_G(y_{t_2}') \leq m(G') + t + t_2 + 6 = m(G)$. Then we have $a(G) \geq |S| = |S'| + t + t_2 + 3 = a(G') + t + t_2 + 3$. By Theorem 1.2, we have $\gamma_2(G') \leq a(G') + 1$ since G' is a non-trivial tree. Accordingly,

$$\gamma_2(G) \leq \gamma_2(G') + t + t_2 + 3 \leq a(G') + t + t_2 + 4 \leq a(G) + 1.$$

Case 3. $h(G) \geq 3$.

Assume that there is a branch vertex $u \in V(C_\ell)$ such that T_u contains a leaf v_1 satisfying that $d_G(u, v_1) = h(G)$. Let $P = v_1v_2v_3 \dots u$ be the shortest path connecting v_1 and u. Since $h(G) \geq 3$, we have $u \neq v_i$ for each $i \in [3]$. If $d_G(v_2) \geq 3$, then v_2 is a strong support vertex. By Claim 3.1, $\gamma_2(G) \leq a(G) + 1$. So, we assume that $d_G(v_2) = 2$. Assume that v_3 have s leaf-neighbors.

If $s \geq 1$, we denote these leaf-neighbors of v_3 with $x_1, x_2, \dots, x_s$. Let $\theta_{v_3} = N_G(v_3) \backslash \{v_4, x_1, x_2, \dots, x_s\}$. Then $|\theta_{v_3}| \geq 1$, as $v_2 \in \theta_{v_3}$. If $|\theta_{v_3}| \geq 2$, let $\theta_{v_3} \backslash \{v_2\} = \{y_1, \dots, y_t\}$. Each vertex in θ_{v_3} must be a support vertex since $d_G(u, v_1) = h(G)$. By Claim 3.1, it suffices to prove that $d(y_i) = 2$ for each $i \in [t]$. Let z_i be the only child of y_i for each $i \in [t]$. It is clear that $d_G(v_3) = s + t + 2$, since the subtree T_u is rooted at u and $u \neq v_3$.

Now, let $G' = G - \{v_1, v_2, x_1, x_2, \dots, x_s, y_1, \dots, y_t, z_1, \cdots, z_t\}$. So $m(G') = m(G) - s - 2t - 2$ and $d_{G'}(v_3) = 1$. By Observations 1 and 2, v_3 belongs to any minimum 2-dominating set and optimal annihilation set of G'. Let D' be a γ_2-set of G' and S' be an optimal annihilation set of G'. Hence, $D = D' \cup \{v_1, x_1, x_2, \dots, x_s, z_1, \dots, z_t\}$ is a 2-dominating set of G, which gives that $\gamma_2(G) \leq |D| = |D'| + s + t + 1 = \gamma_2(G') + s + t + 1$. Let $S = (S' \backslash \{v_3\}) \cup \{v_1, v_2, x_1, x_2, \dots, x_s, z_1, \dots, z_t\}$. Then $\sum(S, G) = \sum(S', G) - d_G(v_3) + d_G(v_1) + d_G(v_2) + d_G(x_1) + \cdots + d_G(x_s) + d_G(z_1) + \cdots + d_G(z_t) \leq (\sum(S', G') + d_G(v_3) - 1) - d_G(v_3) + s + t + 3 \leq m(G') + s + t + 2 = m(G) - t \leq m(G)$. So $a(G) \geq |S| = |S'| + s + t + 1 = a(G') + s + t + 1$. Obviously, G' is a unicyclic graph of order $n' < n$. By the induction hypothesis, $\gamma_2(G') \leq a(G') + 1$. Then

$$\gamma_2(G) \leq \gamma_2(G') + s + t + 1 \leq a(G') + s + t + 2 \leq a(G) + 1.$$

This completes the proof. □

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