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Solving nonlinear matrix and Riesz-Caputo fractional differential equations via fixed point theory in partial metric spaces

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Abstract. A modified implicit relation and ϖ-implicit contractive condition are introduced in the setting of relational partial metric spaces and some related fixed point results are derived. Two suitable examples are provided. As an application, sufficient conditions are derived for the existence of a unique positive definite solution of the non-linear matrix equation $\mathcal{X} = \mathcal{B} + \sum_{i=1}^{k} \mathcal{A}_i^* \mathcal{T}(\mathcal{X}) \mathcal{A}_i$. An example is given, using matrices that are randomly generated, as well as convergence and error analysis and average CPU time analysis. Solving fractional differential equations of Riesz-Caputo type with anti-periodic boundary conditions is also discussed, followed by two illustrations.

1. Introduction

In the last decades, a lot of work has been done on obtaining fixed point results for mappings in partially ordered metric spaces (see, e.g., papers by Matkowski [11, 12], Turinici [20, 21], Ran and Reurings [17], Nieto and Ródríguez-López [13, 14]). Recently, Samet and Turinici [19] obtained a fixed point theorem for nonlinear contractions under symmetric closure of an arbitrary relation, and Ahmadullah et al. [1–3] and Alam and Imdad [4] used an amorphous relation to obtain a relation-theoretic analogue of Banach Contraction Principle (BCP), thus unifying several order-theoretic fixed point theorems.

Several types of generalized distances have been used in order to obtain various fixed point results. Particularly, Matthews [10] introduced the notion of a partial metric space (PMS) and obtained a version of BCP in order to use it in program verification. Later on, fixed point theorems in PMSs, as well as in ordered PMSs were proved by several authors.

Motivated by [16], we introduce in Section 3 a modified implicit relation $\Re$, and define a ϖ-implicit type self-mapping $\mathcal{S}$ on an $\Re$-complete partial metric space. We prove a respective fixed point result, using $\mathcal{S}$-closedness of $\Re$ and $\Re$-continuity of $\mathcal{S}$. In Section 4, we consider some special cases, and provide suitable examples, in order to illustrate the results obtained, and we present in Section 5 a sufficient condition

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ensuring the existence of a unique positive definite solution of a non-linear matrix equation, which is also illustrated by a non-trivial example, including randomly generated matrices. The procedure is visualized by the convergence analysis and a solution graph. Additionally, we use these findings to discuss existence of solutions for fractional differential equations of Riesz-Caputo type with new anti-periodic boundary conditions. The ideas, results, and applications are properly illustrated by examples.

2. Preliminaries

The notations $\mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{R}^+$ will have their usual meanings, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

2.1. Partial metric space

The following definitions can be found, e.g., in [5, 10, 15].

Definition 2.1. *Let Ξ be a nonempty set, and $\varpi : \Xi \times \Xi \to \mathbb{R}^+$ satisfy the following conditions for all $\xi, \eta, \zeta \in \Xi$:*

(p_1) $\xi = \eta$ *if and only if* $\varpi(\xi, \xi) = \varpi(\xi, \eta) = \varpi(\eta, \eta)$,

(p_2) $\varpi(\xi, \xi) \le \varpi(\xi, \eta)$,

(p_3) $\varpi(\xi, \eta) = \varpi(\eta, \xi)$,

(p_4) $\varpi(\xi, \eta) \le \varpi(\xi, \zeta) + \varpi(\zeta, \eta) - \varpi(\zeta, \zeta)$.

Then ϖ is said to be a partial metric on Ξ, and (Ξ, ϖ) is called a partial metric space (PMS).

Clearly, if $\varpi(\xi, \eta) = 0$, then $\xi = \eta$; however, $\varpi(\xi, \xi)$ may not be 0.
If ϖ is a partial metric on Ξ, then by

$$\varpi^s(\xi, \eta) = 2\varpi(\xi, \eta) - \varpi(\xi, \xi) - \varpi(\eta, \eta) \tag{1}$$

a metric ϖ^s on Ξ is defined. The corresponding topology is denoted by τ_ϖ.

Example 2.2. A standard example of a PMS is a pair $(\mathbb{R}^+, \varpi)$, where $\varpi(\xi, \eta) = \max\{\xi, \eta\}$ for all $\xi, \eta \in \mathbb{R}^+$. The corresponding metric is

$$\varpi^s(\xi, \eta) = 2\max\{\xi, \eta\} - \xi - \eta = |\xi - \eta|.$$

Definition 2.3. *Let (Ξ, ϖ) be a PMS. Then:*

1. *It is said that a sequence $\{\xi_n\}$ in (Ξ, ϖ) converges to a point $\xi \in \Xi$ if $\lim_{n\to\infty} \varpi(\xi, \xi_n) = \varpi(\xi, \xi)$.*
2. *It is said that $\{\xi_n\}$ is a Cauchy sequence in (Ξ, ϖ) if $\lim_{n,m\to\infty} \varpi(\xi_n, \xi_m)$ exists and is finite.*
3. *The space (Ξ, ϖ) is called complete if each Cauchy sequence $\{\xi_n\}$ in Ξ converges to a point $\xi \in \Xi$ satisfying $\varpi(\xi, \xi) = \lim_{n,m\to\infty} \varpi(\xi_n, \xi_m)$.*
4. *A mapping $\mathcal{S} : \Xi \to \Xi$ is called continuous at a point $\xi_0 \in \Xi$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\mathcal{S}(B_\varpi(\xi_0, \delta)) \subset B_\varpi(\mathcal{S}\xi_0, \varepsilon)$ (here, $B_\varpi(\xi_0, \delta) = \{\xi \in \Xi : \varpi(\xi, \xi_0) < \delta\}$).*

Remark 2.4. *A limit of a sequence in a PMS need not be unique, and the function $\varpi(\cdot, \cdot)$ need not be continuous.*

Lemma 2.5. *[10, 15] Let (Ξ, ϖ) be a PMS.*
(a) $\{\xi_n\}$ is a Cauchy sequence in (Ξ, ϖ) if and only if it is a Cauchy sequence in (Ξ, ϖ^s).
(b) (Ξ, ϖ) is complete if and only if (Ξ, ϖ^s) is complete. Moreover, $\lim_{n\to\infty} \varpi^s(\xi_n, \xi) = 0$ if and only if

$$\varpi(\xi, \xi) = \lim_{n\to\infty} \varpi(\xi_n, \xi) = \lim_{n,m\to\infty} \varpi(\xi_n, \xi_m).$$

2.2. Relational partial metric spaces

Let Ξ be a nonempty set, $\Re$ be a binary relation on Ξ and let ϖ be a partial metric on Ξ. Then, $(\Xi, \Re)$ will be called a relational set, and $(\Xi, \varpi, \Re)$ will be called an RPMS. We will use the following standard terminology (see, e.g., [4, 7–9, 19]):

1. $\xi \in \Xi$ is $\Re$-related to $\eta \in \Xi$ if $(\xi, \eta) \in \Re$; we will write $[\xi, \eta] \in \Re$ if $(\eta, \xi) \in \Re$ or $(\xi, \eta) \in \Re$.
2. The inverse relation of $\Re$ is $\Re^{-1} = \{(\xi, \eta) \in \Xi \times \Xi : (\eta, \xi) \in \Re\}$; moreover, $\Re^s = \Re \cup \Re^{-1}$.
3. The set $Z \subset \Xi$ is said to be comparable if $[\xi, \eta] \in \Re$ for all $\xi, \eta \in Z$.
4. A sequence $\{\xi_n\}$ in Ξ is called $\Re$-preserving if $(\xi_n, \xi_{n+1}) \in \Re$, for all $n \in \mathbb{N}_0$.
5. $(\Xi, \varpi, \Re)$ is called regular, if for any $\Re$-preserving sequence $\{\xi_n\}$ converging to ξ, $[\xi_n, \xi] \in \Re$ holds for all $n \in \mathbb{N}$.
6. $(\Xi, \varpi, \Re)$ is called $\Re$-complete if for each $\Re$-preserving Cauchy sequence $\{\xi_n\}$ in Ξ, there exists $\xi \in \Xi$ such that
$$\lim_{n,m\to\infty} \varpi(\xi_n, \xi_m) = \varpi(\xi, \xi) = \lim_{n\to\infty} \varpi(\xi_n, \xi).$$
7. $\Re$ is called ϖ-self-closed if for each $\Re$-preserving sequence $\{\xi_n\}$ converging to ξ, there exists its subsequence (ξ_{n_k}), with $[\xi_{n_k}, \xi] \in \Re$, for all $k \in \mathbb{N}_0$.
8. A path of length k in $\Re$ joining $\xi, \eta \in \Xi$ is a finite sequence $\{\zeta_0, \zeta_1, \zeta_2, \dots, \zeta_k\} \subset \Xi$ satisfying:
   (i) $\zeta_0 = \xi$ and $\zeta_k = \eta$,
   (ii) $(\zeta_i, \zeta_{i+1}) \in \Re$ for each i, $0 \le i \le k - 1$.
   (Note that a path of length k involves $k + 1$ elements of Ξ, but they need not be distinct).

Remark 2.6. *A complete PMS is always $\Re$-complete, but the converse need not hold. These notions coincide if the relation $\Re$ is universal.*

Moreover, let $\mathcal{S}$ be a self-mapping on Ξ.

(9) $\Re$ is called $\mathcal{S}$-closed if $(\xi, \eta) \in \Re$ implies $(\mathcal{S}\xi, \mathcal{S}\eta) \in \Re$. It is called weakly $\mathcal{S}$-closed if $(\xi, \eta) \in \Re$ implies $[\mathcal{S}\xi, \mathcal{S}\eta] \in \Re$.

(10) A subset $\mathfrak{Z}$ of Ξ is said to be $\Re$-directed if for all $\xi, \eta \in \mathfrak{Z}$, there is $\zeta \in \Xi$ such that $(\xi, \zeta) \in \Re$ and $(\eta, \zeta) \in \Re$. It is said to be $(\mathcal{S}, \Re)$-directed if for all $\xi, \eta \in \mathfrak{Z}$, there is $\zeta \in \Xi$ such that $(\xi, \mathcal{S}\zeta) \in \Re$ and $(\eta, \mathcal{S}\zeta) \in \Re$.

(11) $\mathcal{S}$ is called $\Re$-continuous at $\xi \in \Xi$ if for every $\Re$-preserving sequence $\{\xi_n\}$ converging to ξ, the sequence $\{\mathcal{S}(\xi_n)\}$ converges to $\mathcal{S}(\xi)$. $\mathcal{S}$ is $\Re$-continuous if it is $\Re$-continuous at each point of Ξ.

Remark 2.7. *A continuous mapping on a PMS is always $\Re$-continuous but the converse need not hold. These notions coincide if the relation $\Re$ is universal.*

In what follows, we will use the the following notations for a relational space $(\Xi, \Re)$ and a self-mapping $\mathcal{S}$ on Ξ.

Fix$(\mathcal{S})$ = the set of all fixed points of $\mathcal{S}$,
$\mathcal{N}(\mathcal{S}, \Re) = \{\xi \in \Xi : (\xi, \mathcal{S}\xi) \in \Re\}$,
$\mathfrak{P}(\xi, \eta, \Re)$ = the set of all paths in $\Re$ joining $\xi, \eta \in \Xi$.

3. ϖ-implicit contractive mappings in partial metric spaces

We are going to introduce a modified version of implicit relation. Some examples are presented in [16]. Let $\mathfrak{G}$ denote the set of all continuous mappings $\mathcal{G} : \mathbb{R}_+^5 \to \mathbb{R}$ that satisfy the conditions:

$(\mathcal{G}_1)$ $\mathcal{G}$ is nonincreasing in the fifth variable;
$(\mathcal{G}_2)$ there exists $h \in [0, 1)$ such that, for all $\tau, \sigma \ge 0$, if $\mathcal{G}(\tau, \sigma, \sigma, \tau, \tau + \sigma) \le 0$ then $\tau \le h\sigma$.

Let $\mathfrak{G}'$ be the set of functions $\mathcal{G}$ from $\mathfrak{G}$ satisfying

($\mathcal{G}_3$) there exists $h \in [0,1)$ such that, for all $\tau,\sigma > 0$, if $\mathcal{G}(\tau,0,0,\sigma,\tau) \le 0$ then $\tau \le h\sigma$.

Let $\mathfrak{G}''$ be the set of functions $\mathcal{G}$ from $\mathfrak{G}'$ satisfying

($\mathcal{G}_4$) $\mathcal{G}(\tau,\tau,0,0,2\tau) > 0$ for each $\tau > 0$.

Example 3.1. *Let* $\mathcal{G}(\tau_1,\dots,\tau_5) = \tau_1 - a\tau_2 - b\tau_3 - c\tau_4 - d\tau_5$, *with* $a > 0$, $b,c,d \ge 0$ *and* $a+b+c+2d < 1$. *Then*

($\mathcal{G}_2$): *If* $\tau \ge 0$, $\sigma > 0$ *and* $\mathcal{G}(\tau,\sigma,\sigma,\tau,\tau+\sigma) = \tau - a\sigma - b\sigma - c\tau - d(\tau+\sigma) \le 0$, *then* $\tau \le h\sigma$, *where* $0 < h = \frac{a+b+d}{1-(c+d)} < 1$.
($\mathcal{G}_3$): *If* $\tau,\sigma \ge 0$ *and* $\mathcal{G}(\tau,0,0,\sigma,\tau) = \tau - c\sigma - d\tau$, *then* $\tau \le (\frac{c}{1-a})\sigma$. *Hence* $\tau \le h\sigma$, *where* $0 < h = \frac{c}{1-a} < 1$.
($\mathcal{G}_4$): *For each* $\tau > 0$, $\mathcal{G}(\tau,\tau,0,0,2\tau) = (1-a-2d)\tau > 0$.

Example 3.2. $\mathcal{G}(\tau_1,\dots,\tau_5) = \tau_1 - k\max\{\tau_2,\tau_3,\tau_4,\tau_5\}$, *with* $k \in (0,\frac{1}{2})$. *Then*

($\mathcal{G}_2$): *If* $\tau \ge 0$, $\sigma > 0$ *and* $\mathcal{G}(\tau,\sigma,\sigma,\tau,\tau+\sigma) = \tau - k(\tau+\sigma) \le 0$, *then* $\tau \le h\sigma$, *where* $0 < h = \frac{k}{1-k} < 1$.
($\mathcal{G}_3$): *If* $\tau > \sigma \ge 0$ *and* $\mathcal{G}(\tau,0,0,\sigma,\tau) = \tau - k\max\{\tau,\sigma\}$, *then* $(1-k)\tau \le 0$, *a contradiction. Thus,* $\tau \le h\sigma$, *where* $0 < h = k < 1$.
($\mathcal{G}_4$): *For each* $\tau > 0$, $\mathcal{G}(\tau,\tau,0,0,2\tau) = (1-2k)\tau > 0$.

Example 3.3. $\mathcal{G}(\tau_1,\dots,\tau_5) = \tau_1^2 - a\tau_2\tau_3 - b\tau_4^2 - c\tau_5^2$, *where* $a > 0$, $b,c \ge 0$ *and* $a+b+4c < 1$. *Then*

($\mathcal{G}_2$): *If* $\tau > \sigma > 0$ *and* $\mathcal{G}(\tau,\sigma,\sigma,\tau,\tau+\sigma) = \tau^2 - a\sigma^2 - b\tau^2 - c(\tau+\sigma)^2 \le 0$, *then* $\tau^2[1-(a+b+4c)] \le 0$, *a contradiction. Hence,* $\tau \le \sigma$ *which implies* $\tau \le h\sigma$, *where* $0 < h = \sqrt{a+b+4c} < 1$.
($\mathcal{G}_3$): *If* τ, $\sigma \ge 0$ *and* $\mathcal{G}(\tau,0,0,\sigma,\tau) = \tau^2 - b\sigma^2 - c\tau^2$, *then* $\tau \le \sqrt{\frac{b}{1-c}}\,\sigma$. *Hence,* $\tau \le h\sigma$, *where* $0 < h = \sqrt{\frac{b}{1-c}} < 1$.
($\mathcal{G}_4$): *For each* $\tau > 0$, $\mathcal{G}(\tau,\tau,0,0,2\tau) = (1-4c)\tau^2 > 0$.

Example 3.4. $\mathcal{G}(\tau_1,\dots,\tau_5) = \tau_1 - a\tau_2 - b\frac{(1+\tau_3)\tau_4}{1+\tau_2} - c\tau_5$, *where* $a > 0$, $b,c \ge 0$ *and* $a+b+2c < 1$. *Then*

($\mathcal{G}_2$): *If* $\tau \ge 0$, $\sigma > 0$ *and* $\mathcal{G}(\tau,\sigma,\sigma,\tau,\tau+\sigma) = \tau - a\sigma - b\tau - c(\tau+\sigma) \le 0$, *then* $\tau \le h\sigma$, *where* $0 < h = \frac{a+c}{1-(b+c)} < 1$.
($\mathcal{G}_3$): *if* τ, $\sigma \ge 0$ *and* $\mathcal{G}(\tau,0,0,\sigma,\tau) = \tau - b\sigma - c\tau$, *then* $\tau \le (\frac{b}{1-c})\sigma$. *Hence,* $\tau \le h\sigma$, *where* $0 < h = \frac{b}{1-c} < 1$.
($\mathcal{G}_4$): *For each* $\tau > 0$, $\mathcal{G}(\tau,\tau,0,0,2\tau) = (1-b-2c)\tau > 0$.

Definition 3.5. *Let* $(\Xi,\varpi,\Re)$ *be an RPMS. A mapping* $\mathcal{S}\colon \Xi \to \Xi$ *is called* ϖ*-implicit contractive if there is* $\mathcal{G} \in \mathfrak{G}$ *such that for all* $\xi,\eta \in \Xi$ *with* $(\xi,\eta) \in \Re$,

$$\mathcal{G}\left(\varpi(\mathcal{S}\xi,\mathcal{S}\eta),\varpi(\xi,\eta),\varpi(\xi,\mathcal{S}\xi),\varpi(\eta,\mathcal{S}\eta),\varpi(\xi,\mathcal{S}\eta)+\varpi(\eta,\mathcal{S}\xi)\right) \le 0. \tag{2}$$

Our first main result is stated as follows:

Theorem 3.6. *Let* $(\Xi,\varpi,\Re)$ *be an RPMS and* $\mathcal{S}\colon \Xi \to \Xi$. *Suppose the following:*

(C_1) $\mathcal{N}(\mathcal{S},\Re) \ne \emptyset$;
(C_2) $\Re$ *is* $\mathcal{S}$*-closed;*
(C_3) Ξ *is* $\Re$*-complete;*
(C_4) $\mathcal{S}$ *is* ϖ*-implicit contractive;*
(C_5) $\mathcal{S}$ *is* $\Re$*-continuous.*

Then $\omega \in \mathrm{Fix}(\mathcal{S})$, *for some* $\omega \in \Xi$.

Proof. Let $\omega_0 \in \mathcal{N}(\mathcal{S}, \Re)$ be arbitrary and let $\omega_{n+1} = \mathcal{S}\omega_n = \mathcal{S}^{n+1}\omega_0$ for all $n \in \mathbb{N}_0$. In the case that $\omega_{n_0+1} = \omega_{n_0}$ for some $n_0 \in \mathbb{N}_0$, then $\omega_{n_0} \in \mathrm{Fix}(\mathcal{S})$. Further, suppose that $\omega_{n+1} \neq \omega_n$, i.e., $\varpi(\mathcal{S}\omega_{n+1}, \mathcal{S}\omega_n) > 0$ for all $n \in \mathbb{N}_0$. Since $(\omega_0,\ \mathcal{S}\omega_0) \in \Re$, it follows by (C_2), that

$$(\mathcal{S}\omega_0, \mathcal{S}^2\omega_0),\ (\mathcal{S}^2\omega_0, \mathcal{S}^3\omega_0),\ \cdots,\ (\mathcal{S}^n\omega_0, \mathcal{S}^{n+1}\omega_0) \in \Re,$$

so that, for each $n \in \mathbb{N}_0$, $(\omega_n, \omega_{n+1}) \in \Re$. Thus, the sequence $\{\omega_n\}$ is $\Re$-preserving. The condition (C_4) with $\xi = \omega_n$, $\eta = \omega_{n+1}$ implies that, for all $n \in \mathbb{N}_0$,

$$\mathcal{G}\begin{pmatrix} \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega_{n+1}), \varpi(\omega_n, \omega_{n+1}), \varpi(\omega_n, \mathcal{S}\omega_n), \\ \varpi(\omega_{n+1}, \mathcal{S}\omega_{n+1}), \varpi(\omega_n, \mathcal{S}\omega_{n+1}) + \varpi(\omega_{n+1}, \mathcal{S}\omega_n) \end{pmatrix} \leq 0,$$

that is,

$$\mathcal{G}\begin{pmatrix} \varpi(\omega_{n+1}, \omega_{n+2}), \varpi(\omega_n, \omega_{n+1}), \varpi(\omega_n, \omega_{n+1}), \\ \varpi(\omega_{n+1}, \omega_{n+2}), \varpi(\omega_n, \omega_{n+2}) + \varpi(\omega_{n+1}, \omega_{n+1}) \end{pmatrix} \leq 0.$$

By the property (p_4),

$$\varpi(\omega_n, \omega_{n+2}) \leq \varpi(\omega_n, \omega_{n+1}) + \varpi(\omega_{n+1}, \omega_{n+2}) - \varpi(\omega_{n+1}, \omega_{n+1}).$$

Then, using $(\mathcal{G}_1)$, we obtain

$$\mathcal{G}\begin{pmatrix} \varpi(\omega_{n+1}, \omega_{n+2}), \varpi(\omega_n, \omega_{n+1}), \varpi(\omega_n, \omega_{n+1}), \\ \varpi(\omega_{n+1}, \omega_{n+2}), \varpi(\omega_n, \omega_{n+1}) + \varpi(\omega_{n+1}, \omega_{n+2}) \end{pmatrix} \leq 0.$$

It follows from $(\mathcal{G}_2)$ that there is $h \in [0, 1)$ such that

$$\varpi(\omega_{n+1}, \omega_{n+2}) \leq h\varpi(\omega_n, \omega_{n+1}). \tag{3}$$

For $m > n$, using (p_4) we obtain

$$\begin{aligned}\varpi(\omega_n, \omega_m) &\leq \varpi(\omega_n, \omega_{n+1}) + \ldots + \varpi(\omega_{n+m-1}, \omega_{n+m}) \\ &\qquad - \varpi(\omega_{n+1}, \omega_{n+1}) - \varpi(\omega_{n+2}, \omega_{n+2}) - \ldots \\ &\leq \varpi(\omega_n, \omega_{n+1}) + \ldots + \varpi(\omega_{n+m-1}, \omega_{n+m}) \\ &\leq (h^n + h^{n+1} + \ldots + h^{m-1})\varpi(\omega_0, \omega_1) \\ &\leq \frac{h^n}{1-h}\varpi(\omega_0, \omega_1).\end{aligned}$$

Thus, $\lim_{n,m\to\infty} \varpi(\omega_n, \omega_m) = 0$ and $\{\omega_n\}$ is a Cauchy sequence (in (Ξ, ϖ), as well as in (Ξ, ϖ^s)). Since the space (Ξ, ϖ), as well as the space (Ξ, ϖ^s), is $\mathcal{R}$-complete, there is $\omega^* \in \Xi$ such that $\lim_{n\to\infty} \omega_n = \omega^*$ in (Ξ, ϖ^s). By the condition (C_5), it follows that

$$\omega^* = \lim_{n\to\infty} \omega_{n+1} = \lim_{n\to\infty} \mathcal{S}\omega_n = \mathcal{S}\omega^*.$$

In other words, $\omega^* \in \mathrm{Fix}\mathcal{S}$. □

Theorem 3.7. *If $\mathcal{G} \in \mathfrak{G}'$ and the condition (C_5) is replaced by:*

(C_5') $(\Xi, \varpi, \Re^s)$ is regular,

then the conclusion of Theorem 3.6 holds.

Proof. As in the proof of Theorem 3.6, we have that $\{\mathcal{S}^n\omega_0\}$ is a Cauchy sequence, and hence there is $\omega^* \in \Xi$, so that

$$\varpi(\omega^*, \omega^*) = \lim_{n\to\infty} \varpi(\omega_n, \omega^*) = \lim_{n,m\to\infty} \varpi(\omega_n, \omega_m) = 0.$$

Since the sequence $\{\omega_n\}$ is $\Re$-preserving and $\omega_n \to^{\tau_\varpi} \omega^*$, then, by (C_5'), $[\omega_n, \omega^*] \in \Re$.

Consider, further, two cases depending on whether $K = \{n \in \mathbb{N} : \mathcal{S}\omega_n = \mathcal{S}\omega^*\}$ is finite or not.

- Let K be finite. Then, for some $n_0 \in \mathbb{N}$, it is $\mathcal{S}\omega_n \neq \mathcal{S}\omega^*$ for all $n \geq n_0$. It follows from (C_4) for $(\omega_n, \omega^*) \in \Re$ that, for all $n \geq n_0$,

$$\mathcal{G}(\varpi(\mathcal{S}\omega_n, \mathcal{S}\omega^*), \varpi(\omega_n, \omega^*), \varpi(\omega_n, \mathcal{S}\omega_n), \varpi(\omega^*, \mathcal{S}\omega^*),$$
$$\varpi(\omega^*, \mathcal{S}\omega_n) + \varpi(\omega_n, \mathcal{S}\omega^*)) \leq 0.$$

By continuity of $\mathcal{G}$, passing to the limit as $n \to \infty$, we obtain that

$$\mathcal{G}(\lim_{n\to\infty} \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega^*), 0, 0, \varpi(\omega^*, \mathcal{S}\omega^*), \lim_{n\to\infty} \varpi(\omega_n, \mathcal{S}\omega^*)) \leq 0.$$

By (p_4), we have

$$\begin{aligned}\varpi(\omega_n, \mathcal{S}\omega^*) &\leq \varpi(\omega_n, \mathcal{S}\omega_n) + \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega^*) - \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega_n)\\ &\leq \varpi(\omega_n, \mathcal{S}\omega_n) + \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega^*),\end{aligned}$$

hence

$$\begin{aligned}\lim_{n\to\infty} \varpi(\omega_n, \mathcal{S}\omega^*) &\leq \lim_{n\to\infty} \varpi(\omega_n, \mathcal{S}\omega_n) + \lim_{n\to\infty} \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega^*)\\ &\leq \lim_{n\to\infty} \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega^*),\end{aligned}$$

and using $(\mathcal{G}_1)$ we obtain

$$\mathcal{G}(\lim_{n\to\infty} \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega^*), 0, 0, \varpi(\omega, \mathcal{S}\omega^*), \lim_{n\to\infty} \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega^*)) \leq 0.$$

By $(\mathcal{G}_2)$, it follows that there exists $h \in [0, 1)$ such that

$$\lim_{n\to\infty} \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega^*) \leq h\varpi(\omega^*, \mathcal{S}\omega^*). \tag{4}$$

But, by (p_4) we have

$$\begin{aligned}\varpi(\omega^*, \mathcal{S}\omega^*)) &\leq \varpi(\omega^*, \mathcal{S}\omega_n) + \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega^*) - \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega_n)\\ &\leq \varpi(\omega_n, \mathcal{S}\omega_n) + \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega^*)\end{aligned}$$

so that

$$\varpi(\omega^*, \mathcal{S}\omega^*) \leq \lim_{n\to\infty} \varpi(\mathcal{S}\omega_n, \mathcal{S}\omega^*). \tag{5}$$

Combining (4) and (5), we get $(1 - h)\varpi(\omega^*, \mathcal{S}\omega^*) \leq 0$, a contradiction. It follows that $\mathcal{S}\omega^* = \omega^*$.

- Let now K be not finite, so that there is a subsequence $\{\omega_{n(k)}\}$ of $\{\omega_n\}$ satisfying

$$\omega_{n(k)+1} = \mathcal{S}\omega_{n(k)} = \mathcal{S}\omega, \quad \text{for all } k \in \mathbb{N}.$$

As $\omega_{n(k)} \to^{\tau_\varpi} \omega^*$, it again follows that $\mathcal{S}\omega^* = \omega^*$.

□

Further, we present a sufficient condition for the uniqueness of fixed point of the mapping $\mathcal{S}$.

Theorem 3.8. *Let the assumptions of Theorem 3.6 hold, and suppose that $\mathcal{G} \in \mathfrak{G}''$ and that $\mathfrak{P}(\omega, \eta; \Re|_\Xi) \neq \emptyset$ for all $\omega, \eta \in \mathrm{Fix}(\mathcal{S})$. Then the fixed point of the mapping $\mathcal{S}$ is unique.*

Proof. Suppose that there exist $\omega^*, \eta^* \in \mathrm{Fix}(\mathcal{S})$ with $\omega^* \neq \eta^*$. By the assumption $\mathfrak{P}(\omega, \eta; \Re|_\Xi) \neq \emptyset$, it follows that there exists a path $\{\zeta_0, \zeta_1, \ldots, \zeta_k\}$ in $\Re$ joining ω^* with η^* (we can assume that $\zeta_i \neq \zeta_{i+1}$ for all $i, 0 \leq i \leq k-1$). This means that

$$\zeta_0 = \omega^*,\ \zeta_k = \eta^*, \text{ and } (\zeta_i, \zeta_{i+1}) \in \Re \text{ for each } i,\ 0 \le i \le k-1.$$

Since $\zeta_i \in \mathrm{Fix}(\mathcal{S})$, it is $\mathcal{S}\zeta_i = \zeta_i$ for all i, $0 \le i \le k-1$, and we have

$$\mathcal{G}\begin{pmatrix} \varpi(\mathcal{S}\zeta_i, \mathcal{S}\zeta_{i+1}), \varpi(\zeta_i, \zeta_{i+1}), \varpi(\zeta_i, \mathcal{S}\zeta_i), \\ \varpi(\zeta_{i+1}, \mathcal{S}\zeta_{i+1}), \varpi(\zeta_i, \mathcal{S}\zeta_{i+1}) + \varpi(\zeta_{i+1}, \mathcal{S}\zeta_i) \end{pmatrix} \le 0,$$

i.e.,

$$\mathcal{G}\begin{pmatrix} \varpi(\zeta_i, \zeta_{i+1}), \varpi(\zeta_i, \zeta_{i+1}), \varpi(\zeta_i, \zeta_i), \\ \varpi(\zeta_{i+1}, \zeta_{i+1}), \varpi(\zeta_i, \zeta_{i+1}) + \varpi(\zeta_{i+1}, \zeta_i) \end{pmatrix} \le 0,$$

hence

$$\mathcal{G}\left(\varpi(\zeta_i, \zeta_{i+1}), \varpi(\zeta_i, \zeta_{i+1}), 0, 0, 2\varpi(\zeta_i, \zeta_{i+1})\right) \le 0,$$

a contradiction with (G_4). Hence, the theorem is proved. □

4. Illustrations

Example 4.1. *Let a partial metric be defined on the set $\Xi = [0, 1]$ by $\varpi(\omega, \eta) = \max\{\omega, \eta\}$, for all $\omega, \eta \in \Xi$, and let a binary relation be given on Ξ by*

$$(\omega, \eta) \in \Re \textit{ if and only if } \omega, \eta \in \{0\} \cup \left\{\frac{1}{8^n} : n \in \mathbb{N}\right\}.$$

Let a self-mapping on Ξ be defined by $\mathcal{S}\omega = \frac{\omega}{8}$ for all $\omega \in \Xi$. Then, the conditions (C_1)–(C_3) and (C_5) of Theorem 3.6 are fulfilled. Let us check condition (C_4). Taking into account Example 3.1, the relation (2) reduces to

$$\varpi(\mathcal{S}\omega, \mathcal{S}\eta) \le a\,\varpi(\omega, \eta) + b\,\varpi(\omega, \mathcal{S}\omega) + c\,\varpi(\eta, \mathcal{S}\eta) + d\,[\varpi(\omega, \mathcal{S}\eta) + \varpi(\eta, \mathcal{S}\omega)]. \tag{6}$$

We have to consider two non-trivial cases for $(\omega, \eta) \in \Re$, i.e., $0 \le \eta, \omega \le \frac{1}{8}$.

1° *Let $\omega = 0$ and $\eta = 1/8^n$, $n \in \mathbb{N}$, (or vice versa). Then (6) reduces to*

$$\frac{1}{8^{n+1}} \le (a + c) \cdot \frac{1}{8^n} + d \cdot \left[\frac{1}{8^{n+1}} + \frac{1}{8^n}\right].$$

2° *If $\eta, \omega \in \{1/8^n \mid n \in \mathbb{N}\}$, $0 < \eta < \omega$, i.e., $\eta \le \omega/8$, then (6) reduces to*

$$\frac{\omega}{8} \le (a + b) \cdot \omega + c \cdot \eta + d \cdot \left[\omega + \frac{\omega}{8}\right].$$

These inequalities are fulfilled for $a = b = c = d = 1/8$ (so that $a + b + c + 2d < 1$).

Hence, the condition (C_4) is also fulfilled. Thus, by Theorem 3.6, there is a (unique) fixed point of $\mathcal{S}$ in Ξ (which is $\omega^* = 0$).

Example 4.2. *Let again the partial metric ϖ be defined on the set $\Xi = [0, 1]$ by $\varpi(\omega, \eta) = \max\{\omega, \eta\}$, for all $\omega, \eta \in \Xi$, and let a binary relation be given on Ξ by*

$$\Re = \left\{(0, 0), (0, 1)\left(\frac{1}{6}, 1\right), \left(\frac{1}{6}, 0\right), \left(0, \frac{1}{6}\right), \left(\frac{1}{6}, \frac{1}{6}\right)\right\}.$$

Let $\mathcal{S} : \Xi \to \Xi$ be defined by

$$\mathcal{S}(\omega) = \begin{cases} 0, & 0 \le \omega \le \frac{1}{6} \\ \frac{1}{6}, & \frac{1}{6} < \omega \le 1. \end{cases}$$

It is again easy to see that the conditions (C_1)–(C_3) *hold true,*

In order to check condition (C_4)*, note that, by Example 3.2, the relation* (2) *reduces to*

$$\varpi(\mathcal{S}\omega, \mathcal{S}\eta) \le k \max\{\varpi(\omega,\eta), \varpi(\omega,\mathcal{S}\omega), \varpi(\eta,\mathcal{S}\eta), \varpi(\omega,\mathcal{S}\eta) + \varpi(\eta,\mathcal{S}\omega)\}. \tag{7}$$

In order to check that this is fulfilled, we have to consider two nontrivial cases:

- *If* $(\omega,\eta) = (0,1)$*, then* $\varpi(\mathcal{S}\omega,\mathcal{S}\eta) = \frac{1}{6}$, $\varpi(\omega,\eta) = 1$, $\varpi(\omega,\mathcal{S}\omega) = 0$, $\varpi(\eta,\mathcal{S}\eta) = 1$, $\varpi(\omega,\mathcal{S}\eta) = \frac{1}{6}$, $\varpi(\eta,\mathcal{S}\omega) = 1$, *and the condition* (7) *reduces to* $1/6 \le 7k/6$.

- *If* $(\omega,\eta) = (\frac{1}{6},1)$*, then* $\varpi(\mathcal{S}\omega,\mathcal{S}\eta) = \frac{1}{6}$, $\varpi(\omega,\eta) = 1$, $\varpi(\omega,\mathcal{S}\omega) = 0$, $\varpi(\eta,\mathcal{S}\eta) = 1$, $\varpi(\omega,\mathcal{S}\eta) = \frac{1}{6}$, $\varpi(\eta,\mathcal{S}\omega) = 1$, *and the condition* (7) *reduces to* $1/6 \le 7k/6$.

Both these inequalities are fulfilled if $k \in (0, 1/2)$*. Thus, the condition* (C_4) *is satisfied.*

Finally, we have to check the condition (C'_5)*. Let* (ω_n) *be an* $\mathfrak{R}$*-preserving sequence in* Ξ *that converges to* ω*. Then,*

$$(\omega_n, \omega_{n+1}) \in \left\{(0,0), \left(\frac{1}{6},0\right), \left(0,\frac{1}{6}\right), \left(\frac{1}{6},\frac{1}{6}\right)\right\}$$

implying that $\omega_n \in \left\{0, \frac{1}{6}\right\}$*. It follows that* $\omega_n \to 0$ *or* $\omega_n \to \frac{1}{6}$ *as* $n \to \infty$ *and we have* $[\omega_n, \omega] \in \mathfrak{R}$ *for all* $n \in \mathbb{N}$*, where* $\omega = 0$ *or* $\omega = \frac{1}{6}$*. Hence* (C'_5) *is satisfied.*

By Theorem 3.7, we conclude that $\mathcal{S}$ *has a fixed point* ($\omega^* = 0$)*.*

5. Application to nonlinear matrix equations

Let $\mathfrak{H}_n$ (resp. $\mathfrak{P}_n$) denote the set of all Hermitian (resp. positive definite) $n \times n$ matrices over $\mathbb{C}$. Denote by $s^+(\mathcal{B}) = \|\mathcal{B}\|_{tr}$ the trace norm (i.e., the sum of all singular values) of a matrix $\mathcal{B} \in \mathfrak{H}_n$. If $\mathcal{C}, \mathcal{D} \in \mathfrak{H}_n$, we will write $\mathcal{C} \succeq \mathcal{D}$ (resp. $\mathcal{C} \succ \mathcal{D}$) if the matrix $\mathcal{C} - \mathcal{D}$ is positive semi-definite (resp. positive definite). Particularly, $\mathcal{X} \in \mathfrak{P}_n$ if $\mathcal{X} \succ O$. The unit $n \times n$ matrix will be denoted by I_n.

Recall the following known facts.

Lemma 5.1. *[17]*

(a) If $\mathcal{A}, \mathcal{B}$ *are positive semi-definite* $n \times n$ *matrices, then*

$$0 \le \operatorname{tr}(\mathcal{A}\mathcal{B}) \le \|\mathcal{A}\| \operatorname{tr}(\mathcal{B}).$$

(b) If $\mathcal{A} \in \mathfrak{H}_n$ *and* $\mathcal{A} \prec I_n$*, then* $\|\mathcal{A}\| < 1$.

We are going to prove the existence and uniqueness of the solution of nonlinear matrix equation

$$\mathcal{X} = \mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{X}) \mathcal{A}_i, \tag{8}$$

where $\mathcal{B} \in \mathfrak{P}_n$, $\mathcal{A}_i^*$ denotes the conjugate transpose of an $n \times n$ matrix $\mathcal{A}_i$ and $\mathcal{T} : \mathfrak{H}_n \to \mathfrak{P}_n$ is an order-preserving and continuous (in the trace norm) mapping satisfying $\mathcal{T}(O) = O$.

Theorem 5.2. *Suppose that the following conditions are fulfilled:*

(H_1) $\sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{B}) \mathcal{A}_i \succ O$;

(H_2) *There exists a positive real number* η *such that* $\sum_{i=1}^{m} \mathcal{A}_i \mathcal{A}_i^* \prec \eta I_n$;

(H_3) *For some* $k \in (0, 1/2)$, *and for all* $\mathcal{X}, \mathcal{Y} \in \mathfrak{P}_n$ *with* $\mathcal{X} \preceq \mathcal{Y}$ *and*

$$\sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{X}) \mathcal{A}_i \neq \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{Y}) \mathcal{A}_i,$$

it is

$$\max \left\{ |s^+(\mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{X}) \mathcal{A}_i)|, |s^+(\mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{Y}) \mathcal{A}_i)| \right\}$$
$$\leq k \cdot \max \left\{ \begin{array}{c} \max\{|s^+(\mathcal{X})|, |s^+(\mathcal{Y})|\}, \\ \max\{|s^+(\mathcal{X})|, |s^+(\mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{X}) \mathcal{A}_i)|\}, \\ \max\{|s^+(\mathcal{Y})|, |s^+(\mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{Y}) \mathcal{A}_i)|\}, \\ \max\{|s^+(\mathcal{X})|, |s^+(\mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{Y}) \mathcal{A}_i)|\}, \\ \max\{|s^+(\mathcal{Y})| + |s^+(\mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{X}) \mathcal{A}_i)|\} \end{array} \right\}.$$

Then the equation (8) *has a unique solution. Moreover, if* $\mathcal{X}_0 \in \mathfrak{P}_n$ *satisfies* $\mathcal{X}_0 \preceq \mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{X}_0) \mathcal{A}_i$, *then the iterations*

$$\mathcal{X}_n = \mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{X}_{n-1}) \mathcal{A}_i \tag{9}$$

converge (in the trace norm) to the mentioned solution of (8).

Proof. Consider the mapping $\mathcal{S} : \mathfrak{P}_n \to \mathfrak{P}_n$ given by

$$\mathcal{S}(\mathcal{X}) = \mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{X}) \mathcal{A}_i, \text{ for } \mathcal{X} \in \mathfrak{P}_n,$$

and the binary relation $\Re = \{(\mathcal{X}, \mathcal{Y}) \in \mathfrak{P}_n \times \mathfrak{P}_n : \mathcal{X} \preceq \mathcal{Y}\}$. It is clear that a fixed point of the mapping $\mathcal{S}$ is a solution of the matrix equation (8). It is easy to see that $\mathcal{S}$ is well defined and $\Re$-continuous, and that $\Re$ is $\mathcal{S}$-closed. The condition (H_1) implies that $(\mathcal{B}, \mathcal{S}(\mathcal{B})) \in \Re$ and hence $\mathfrak{P}_n(\mathcal{S}; \Re) \neq \emptyset$.

If $\varpi \colon \mathfrak{P}_n \times \mathfrak{P}_n \to \mathbb{R}_+$ is defined by

$$\varpi(\mathcal{X}, \mathcal{Y}) = \max\{\|\mathcal{X}\|, \|\mathcal{Y}\|\} \text{ for } \mathcal{X}, \mathcal{Y} \in \mathfrak{P}_n,$$

then $(\mathfrak{P}_n, \varpi, \Re)$ becomes a complete $\Re$-relational PMS.

The following estimate follows from the condition (H_3):

$$\begin{aligned}
\varpi(\mathcal{S}(\mathcal{X}), \mathcal{S}(\mathcal{Y})) &= \max\{\|\mathcal{S}(\mathcal{X})\|, \|\mathcal{S}(\mathcal{Y})\|\} \\
&= \max \left\{ |s^+(\mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{X}) \mathcal{A}_i)|, |s^+(\mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{Y}) \mathcal{A}_i)| \right\} \\
&\leq k \cdot \max \left\{ \begin{array}{c} \max\{|s^+(\mathcal{X})|, |s^+(\mathcal{Y})|\}, \\ \max\{|s^+(\mathcal{X})|, |s^+(\mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{X}) \mathcal{A}_i)|\}, \\ \max\{|s^+(\mathcal{Y})|, |s^+(\mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{Y}) \mathcal{A}_i)|\}, \\ \max\{|s^+(\mathcal{X})|, |s^+(\mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{Y}) \mathcal{A}_i)|\} \\ + \max\{|s^+(\mathcal{Y})|, |s^+(\mathcal{B} + \sum_{i=1}^{m} \mathcal{A}_i^* \mathcal{T}(\mathcal{X}) \mathcal{A}_i)|\} \end{array} \right\} \\
&= k \cdot \max \left\{ \begin{array}{c} \max\{\|\mathcal{X}\|, \|\mathcal{Y}\|\}, \max\{\|\mathcal{X}\|, \|\mathcal{S}(\mathcal{X})\|\}, \\ \max\{\|\mathcal{Y}\|, \|\mathcal{S}(\mathcal{Y})\|\}, \\ \max\{\|\mathcal{X}\|, \|\mathcal{S}(\mathcal{Y})\|\} + \max\{\|\mathcal{Y}\|, \|\mathcal{S}(\mathcal{X})\|\} \end{array} \right\} \\
&= k \cdot \max \left\{ \begin{array}{c} \varpi(\mathcal{X}, \mathcal{Y}), \varpi(\mathcal{X}, \mathcal{S}(\mathcal{X})), \\ \varpi(\mathcal{Y}, \mathcal{S}(\mathcal{Y})), \varpi(\mathcal{X}, \mathcal{S}(\mathcal{Y})) + \varpi(\mathcal{Y}, \mathcal{S}(\mathcal{X})) \end{array} \right\}.
\end{aligned} \tag{10}$$

Consider the function $\mathcal{G} \in \mathfrak{G}$ given as

$$\mathcal{G}(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6) = \tau_1 - k \max\{\tau_2, \tau_3, \tau_4, \tau_5\},$$

where $k \in (0, 1/2)$. Then (10) can be written as

$$\mathcal{G}\left(\begin{array}{c} \varpi(\mathcal{S}(\mathcal{X}), \mathcal{S}(\mathcal{Y})), \varpi(\mathcal{X}, \mathcal{Y}), \varpi(\mathcal{X}, \mathcal{S}(\mathcal{X})), \\ \varpi(\mathcal{Y}, \mathcal{S}(\mathcal{Y})), \varpi(\mathcal{X}, \mathcal{S}(\mathcal{Y})) + \varpi(\mathcal{Y}, \mathcal{S}(\mathcal{X})) \end{array} \right) \leq 0. \tag{11}$$

Hence, the hypotheses of Theorem 3.6 are fulfilled. It follows that there exists $\hat{\mathcal{X}} \in \mathfrak{P}_n$ such that $\mathcal{S}(\hat{\mathcal{X}}) = \hat{\mathcal{X}}$, i.e., the equation (8) has a solution in $\mathfrak{P}_n$. Moreover, since for each pair $\mathcal{X}, \mathcal{Y} \in \mathfrak{P}_n$, there exist its least upper bound and greatest lower bound, we have that $\mathfrak{P}(\mathcal{X}, \mathcal{Y}; \mathfrak{R}|_{\mathfrak{P}_n}) \neq \emptyset$ for all $\mathcal{X}, \mathcal{Y} \in \mathfrak{P}_n$. It follows from Theorem 3.8 that the fixed point of the mapping $\mathcal{S}$ is unique, and hence so is the solution of the equation (8) in $\mathfrak{P}_n$. □

Example 5.3. *Let*

$$\tilde{\mathcal{A}} = \left(\frac{200}{i+j-1}\right)_{ij} \in \mathbb{C}^{n\times n}, \quad \tilde{\mathcal{B}} = \frac{1}{4}\tilde{\mathcal{A}}, \quad \tilde{\mathcal{C}} = \left(\frac{250}{i+j-1}\right)_{ij} \in \mathbb{C}^{n\times n}.$$

and define the matrices

$$\mathcal{D}_1 = \mathcal{I} + \tilde{\mathcal{A}}^*\tilde{\mathcal{A}}, \quad \mathcal{D}_2 = \mathcal{I} + \tilde{\mathcal{B}}^*\tilde{\mathcal{B}}, \quad \mathcal{D}_3 = \mathcal{I} + \tilde{\mathcal{C}}^*\tilde{\mathcal{C}},$$
$$\mathcal{A}_1 = \mathcal{D}_1^{-(1/2)}\tilde{\mathcal{A}}\mathcal{D}_1^{-(1/2)}, \quad \mathcal{A}_2 = \mathcal{D}_2^{-(1/2)}\tilde{\mathcal{B}}\mathcal{D}_2^{-(1/2)}, \quad \mathcal{A}_3 = \mathcal{D}_3^{-(1/2)}\tilde{\mathcal{C}}\mathcal{D}_3^{-(1/2)}.$$

Take $n = 4$, $\eta = 5$,

$$\mathcal{B} = \begin{bmatrix} 0.0010 & 0.0000 & 0.0000 & 0.0000 \\ 0.0000 & 0.0010 & 0.0000 & 0.0000 \\ 0.0000 & 0.0000 & 0.0010 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0010 \end{bmatrix},$$

and $\mathcal{T}(\mathcal{X}) = \mathcal{X}^2$ in order to test our algorithm. As initial values take

$$\mathcal{X}_0 = \begin{bmatrix} 0.0127 & 0.0170 & 0.0212 & 0.0253 \\ 0.0170 & 0.0229 & 0.0288 & 0.0346 \\ 0.0212 & 0.0288 & 0.0364 & 0.0438 \\ 0.0253 & 0.0346 & 0.0438 & 0.0528 \end{bmatrix},$$

$$\mathcal{Y}_0 = \begin{bmatrix} 0.0064 & 0.0085 & 0.0106 & 0.0126 \\ 0.0085 & 0.0115 & 0.0144 & 0.0173 \\ 0.0106 & 0.0144 & 0.0182 & 0.0219 \\ 0.0126 & 0.0173 & 0.0219 & 0.0264 \end{bmatrix},$$

$$\mathcal{Z}_0 = \begin{bmatrix} 0.0032 & 0.0042 & 0.0053 & 0.0063 \\ 0.0042 & 0.0057 & 0.0072 & 0.0086 \\ 0.0053 & 0.0072 & 0.0091 & 0.0109 \\ 0.0063 & 0.0086 & 0.0109 & 0.0132 \end{bmatrix},$$

where $\mathcal{X}_0, \mathcal{Y}_0, \mathcal{Z}_0 \in \mathfrak{P}_4$.

The obtained numerical values are presented in Table 1.

Table 1.

Int. Mat	$\mathcal{T}(\mathcal{X})$	*k*	η	*Dim*	*Iter no.*	*CPU*	*Error*	*Min(Eig)*
$\mathcal{X}_0$	$\mathcal{X}^2$	0.2	5	4	6	0.020962	0.1826	0.0010
$\mathcal{Y}_0$	$\mathcal{Y}^2$	0.2	5	4	6	0.019376	0.0298	0.0010
$\mathcal{Z}_0$	$\mathcal{Z}^2$	0.2	5	4	6	0.019852	0.0118	0.0010

After six iterations, the following positive-definite solution is obtained (in long format):

$$\mathcal{X}^{+} = \begin{bmatrix} 0.00100049335413 & 0.00000065778631 & 0.00000082210697 & 0.00000098614825 \\ 0.00000065778631 & 0.00100088797688 & 0.00000111802585 & 0.00000134774589 \\ 0.00000082210697 & 0.00000111802585 & 0.00100141377430 & 0.00000170914209 \\ 0.00000098614825 & 0.00000134774589 & 0.00000170914209 & 0.00100207011786 \end{bmatrix}$$

The presentation of convergence and solution plot are given in Figures 1 and 2.

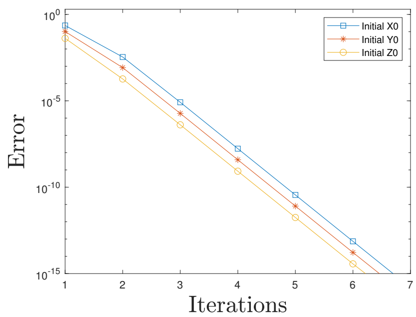

Figure 1: Convergence behavior

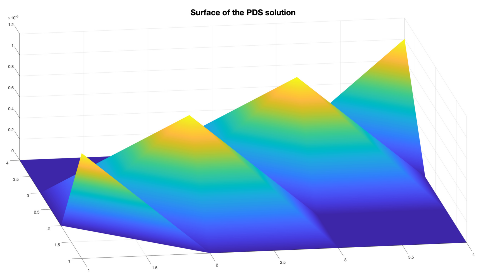

Figure 2: Surface plot

6. Application to Riesz-Caputo fractional differential equations

In this section, we investigate the existence and uniqueness of solution of an anti-periodic boundary value problem (APBVP) for the Riesz-Caputo fractional differential equation of the form

$$ {}^{RC}_{0}\mathcal{D}^{\nu}_{\ell}\varkappa(\wp) = \varphi(\wp, \varkappa(\wp)), \quad \nu \in (2,3], \quad 0 \le \wp \le \ell \tag{12} $$

$$ \varkappa(0) + \varkappa(\ell) = 0, \quad \gamma_1\varkappa'(0) + \delta_1\varkappa'(\ell) = 0, \quad \varkappa''(0) + \varkappa''(\ell) = 0, \tag{13} $$

where ${}^{RC}_{0}\mathcal{D}^{\nu}_{\ell}$ is the Riesz-Caputo derivative and $\varphi : [0, \ell] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

We first introduce some related notions and recall some auxiliary results.

Definition 6.1. *[6] Let $\nu > 0$. The left and right Riemann-Liouville fractional integral of order ν of a function $\varkappa \in C[0, \ell]$ are defined as follows:*

$$ \mathcal{I}^{\nu}_{0}\varkappa(\wp) = \frac{1}{\Gamma(\nu)} \int_0^{\wp} (\wp - \varsigma)^{\nu-1}\varkappa(\varsigma)\, d\varsigma, \quad \wp \in [0, \ell]. $$

$$ {}_{\ell}\mathcal{I}^{\nu}\varkappa(\wp) = \frac{1}{\Gamma(\nu)} \int_{\wp}^{\ell} (\varsigma - \wp)^{\mu-1}\varkappa(\varsigma)\, d\varsigma, \quad \wp \in [0, \ell]. $$

Definition 6.2. *Let $\nu > 0$. The Riesz fractional integral of order ν of a function $\varkappa \in C[0, \ell]$ is defined as*

$$ {}_{0}\mathcal{I}^{\nu}_{\ell}\varkappa(\wp) = \frac{1}{2}\left(\mathcal{I}^{\nu}_{0}\varkappa(\wp) + {}_{\ell}\mathcal{I}^{\nu}\varkappa(\wp)\right). \tag{14} $$

Definition 6.3. *[6] Let $\nu \in (m, m+1]$, $m \in \mathbb{N}$. The left and right Caputo fractional derivative of order ν of a function $\varkappa \in C^{m+1}[0, \ell]$ are defined as:*

$$ \begin{aligned} {}^{C}_{0}\mathcal{D}^{\nu}_{\wp}\varkappa(\wp) &= \frac{1}{\Gamma(m+1-\nu)} \int_0^{\wp} (\wp - s)^{m-\nu}\varkappa^{(m+1)} ds \\ &= \left(\mathcal{I}^{m+1-\nu}_{0}\mathcal{D}^{m+1}\right)\varkappa(\wp) \\ {}^{C}_{\wp}\mathcal{D}^{\nu}_{\ell}\varkappa(\wp) &= \frac{(-1)^{m+1}}{\Gamma(m+1-\nu)} \int_{\wp}^{\ell} (s - \wp)^{m-\nu}\varkappa^{(m+1)} ds \\ &= (-1)^{m+1}\left({}_{\ell}\mathcal{I}^{m+1-\nu}\mathcal{D}^{m+1}\right)\varkappa(\wp) \end{aligned} $$

where $\mathcal{D}$ is the ordinary differential operator.

Definition 6.4. *Let $\nu \in (m, m+1]$, $m \in \mathbb{N}$. The Riesz-Caputo fractional derivative ${}^{RC}_{0}\mathcal{D}^{\nu}\varkappa$ of order ν of a function $\varkappa \in C^{m+1}[0, \ell]$ is defined by*

$$ \begin{aligned} {}^{RC}_{0}\mathcal{D}^{\nu}_{\ell}\varkappa(\wp) &= \frac{1}{\Gamma(m+1-\nu)} \int_0^{\ell} \mid \wp - s \mid^{m-\nu} \varkappa^{(m+1)}(s) ds \\ &= \frac{1}{2}\left({}^{C}_{0}\mathcal{D}^{\nu}_{\wp}\varkappa(\wp) + (-1)^{m+1}{}_{\wp}\mathcal{D}^{\nu}_{\ell}\varkappa(\wp)\right) \\ &= \frac{1}{2}\left(\left(\mathcal{I}^{m+1-\nu}_{0}\mathcal{D}^{m+1}\right)\varkappa(\wp) + (-1)^{m+1}\left({}_{\ell}\mathcal{I}^{m+1-\nu}\mathcal{D}^{m+1}\right)\varkappa(\wp)\right). \end{aligned} $$

Lemma 6.5. *[6] Let $\varkappa \in C^{m}[0, \ell]$ and $\nu \in (m, m+1]$. Then the following hold:*

$$ {}_{0}\mathcal{I}^{\nu}_{\ell}\,{}^{RC}_{0}\mathcal{D}^{\nu}_{\wp}\varkappa(\wp) = \varkappa(\wp) - \sum_{\jmath=0}^{m-1} \frac{\varkappa^{(\jmath)}(\gamma_1)}{\jmath!}(\wp - \gamma_1)^{\jmath}, $$

$$ {}_{\ell}\mathcal{I}^{\nu}_{\wp}\,{}^{RC}\mathcal{D}^{\nu}_{\ell}\varkappa(\wp) = \varkappa(\wp) - \sum_{\jmath=0}^{m-1} \frac{(-1)^{\jmath}\varkappa^{(\jmath)}(\gamma_1)}{\jmath!}(\gamma_1 - \wp)^{\jmath}. $$

When $\nu \in (2,3]$ and $\varkappa(\wp) \in C^3(0,\ell)$, we have

$$
\begin{aligned}
{}_0\mathcal{I}_\ell^{\nu}\,{}_0^{RC}\mathcal{D}_\ell^{\nu}\varkappa(\wp) = &\varkappa(\wp) - \frac{1}{2}[\varkappa(0) + \varkappa(\ell)] - \frac{1}{2}[\varkappa'(0) + \varkappa'(\ell)]\wp + \frac{\ell}{2}\varkappa'(\ell) \\
&- \frac{1}{4}[\varkappa''(0) + \varkappa''(\ell)]\wp^2 - \frac{(\ell^2 - 2\ell\wp)}{4}\varkappa''(\ell).
\end{aligned} \tag{15}
$$

Lemma 6.6. *Suppose that $\chi \in \Lambda := C([0,\ell],\mathbb{R})$, $\varkappa \in C^3([0,\ell])$, and γ_1, δ_1 are nonnegative constants with $\gamma_1 > \delta_1$. Then the fractional APBVP of order $(2,3]$*

$$
{}_0^{RC}\mathcal{D}_\ell^{\nu}\varkappa(\wp) = \chi(\wp), \quad \nu \in (2,3], \quad 0 \le \wp \le \ell \tag{16}
$$

$$
\varkappa(0) + \varkappa(\ell) = 0, \quad \gamma_1\varkappa'(0) + \delta_1\varkappa'(\ell) = 0, \quad \varkappa''(0) + \varkappa''(\ell) = 0, \tag{17}
$$

is equivalent to the integral equation

$$
\begin{aligned}
\varkappa(\wp) = &\frac{-2\wp\ell\gamma_1 + 3\ell^2\gamma_1}{4(\gamma_1 + \delta_1)\Gamma(\nu - 2)} \int_0^\ell (\ell - \varsigma)^{\nu-3}\chi(\varsigma)\, d\varsigma \\
&+ \frac{(\gamma_1 - \delta_1)\wp + \ell\gamma_1}{(\gamma_1 + \delta_1)\Gamma(\nu - 1)} \int_0^\ell (\ell - \varsigma)^{\nu-2}\chi(\varsigma)\, d\varsigma \\
&+ \frac{1}{\Gamma(\nu)} \int_0^\wp (\wp - \varsigma)^{\nu-1}\chi(\varsigma)\, d\varsigma + \frac{1}{\Gamma(\nu)} \int_\wp^\ell (\varsigma - \wp)^{\nu-1}\chi(\varsigma) d\varsigma.
\end{aligned} \tag{18}
$$

Proof. From (16) and (15), we conclude that

$$
\begin{aligned}
\varkappa(\wp) &= \frac{1}{2}[\varkappa(0) + \varkappa(\ell)] + \frac{\wp}{2}(\varkappa'(0) + \varkappa'(\ell)) - \frac{\ell}{2}\varkappa'(\ell) \\
&\quad + \frac{\wp^2}{4}(\varkappa''(0) + \varkappa''(\ell)) + \frac{1}{4}\varkappa''(\ell)(\ell^2 - 2\ell\wp) + {}_0\mathcal{I}_\ell^{\nu}\chi(\wp) \\
&= \frac{\wp}{2}(\varkappa'(0) + \varkappa'(\ell)) - \frac{\ell}{2}\varkappa'(\ell) \\
&\quad + \frac{\wp^2}{4}(\varkappa''(0) + \varkappa''(\ell)) + \frac{1}{4}\varkappa''(\ell)(\ell^2 - 2\ell\wp) \\
&\quad + \frac{1}{\Gamma(\nu)} \int_0^\wp (\wp - \varsigma)^{\nu-1}\chi(\varsigma)\, d\varsigma + \frac{1}{\Gamma(\nu)} \int_\wp^\ell (\varsigma - \wp)^{\nu-1}\chi(\varsigma)\, d\varsigma.
\end{aligned} \tag{19}
$$

Then

$$
\begin{aligned}
\varkappa'(\wp) = &\frac{1}{2}(\varkappa'(0) + \varkappa'(\ell)) + \frac{\wp}{2}(\varkappa''(0) + \varkappa''(\ell)) - \frac{\ell}{2}\varkappa''(\ell) \\
&+ \frac{1}{\Gamma(\nu - 1)} \int_0^\wp (\wp - \varsigma)^{\nu-2}\chi(\varsigma)\, d\varsigma - \frac{1}{\Gamma(\nu - 1)} \int_\wp^\ell (\varsigma - \wp)^{\nu-2}\chi(\varsigma)\, d\varsigma
\end{aligned}
$$

and

$$
\begin{aligned}
\varkappa''(\wp) = &\frac{1}{2}(\varkappa''(0) + \varkappa''(\ell)) + \frac{1}{\Gamma(\nu - 2)} \int_0^\wp (\wp - \varsigma)^{\nu-3}\chi(\varsigma)\, d\varsigma \\
&+ \frac{1}{\Gamma(\nu - 2)} \int_\wp^\ell (\varsigma - \wp)^{\nu-3}\chi(\varsigma)\, d\varsigma.
\end{aligned}
$$

Applying APBVP (17), we have

$$\begin{aligned}\varkappa'(0) &= \frac{2\delta_1\ell}{2(\gamma_1+\delta_1)\Gamma(\nu-2)}\int_0^\ell(\ell-\varsigma)^{\nu-3}\chi(\varsigma)\,d\varsigma\\ &\quad-\frac{2\delta_1}{(\gamma_1+\delta_1)\Gamma(\nu-1)}\int_0^\ell(\ell-\varsigma)^{\nu-2}\chi(\varsigma)\,d\varsigma,\\ \varkappa''(0) &= \frac{-2}{2\Gamma(\nu-2)}\int_0^\ell(\ell-\varsigma)^{\nu-3}\chi(\varsigma)\,d\varsigma\end{aligned}$$

$$\begin{aligned}\varkappa'(\ell) &= \frac{-2\gamma_1\ell}{2(\gamma_1+\delta_1)\Gamma(\nu-2)}\int_0^\ell(\ell-\varsigma)^{\nu-3}\chi(\varsigma)\,d\varsigma\\ &\quad+\frac{2\gamma_1}{(\gamma_1+\delta_1)\Gamma(\nu-1)}\int_0^\ell(\ell-\varsigma)^{\nu-2}\chi(\varsigma)\,d\varsigma,\\ \varkappa''(\ell) &= \frac{2}{2\Gamma(\nu-2)}\int_0^\ell(\ell-\varsigma)^{\nu-3}\chi(\varsigma)\,d\varsigma.\end{aligned}$$

Inserting the quantities that we have obtained from $\varkappa'(0)$ to $\varkappa''(\ell)$ into (19), we get (18). □

We are now ready to introduce the appropriate conditions for our APBVP to have the unique solution. Let $C[0,\ell]$ be the space of continuous functions $\varkappa$ defined on $[0,\ell]$ with norm $\|\varkappa\| = \sup_{\varsigma\in[0,\ell]}|\varkappa(\varsigma)|$.

Theorem 6.7. *Let $\varphi : [0,\ell]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Suppose that there exists a nonnegative real number λ such that for all $(\varsigma,\vartheta),(\varsigma,\varkappa)\in\mathbb{R}^2$, we have*

(A1) $\max\{|\varphi(\varsigma,\vartheta)|,|\varphi(\varsigma,\varkappa)|\}\le\lambda\max\{|\vartheta|,|\varkappa|\},\forall\varsigma\in[0,\ell]$;

(A2) $\left[\frac{\gamma_1\ell^\nu}{4(\gamma_1+\delta_1)\Gamma(\nu-1)}+\frac{-\delta_1\ell^\nu}{(\gamma_1+\delta_1)\Gamma(\nu)}+\frac{2\ell^\nu}{\Gamma(\nu+1)}\right]\lambda<1.$

Then the problem (12) *has a unique solution on* $[0,\ell]$.

Proof. We convert fractional AVBVP (12)-(13) into integral equation using operator $\mathfrak{I}:\Lambda\to\Lambda$ of the form

$$\begin{aligned}\mathfrak{I}\varkappa(\wp) &= \frac{-2\wp\ell\gamma_1+3\ell^2\gamma_1}{4(\gamma_1+\delta_1)\Gamma(\nu-2)}\int_0^\ell(\ell-\varsigma)^{\nu-3}\varphi(\varsigma,\varkappa(\varsigma))\,d\varsigma\\ &\quad+\frac{(\gamma_1-\delta_1)\wp+\ell\gamma_1}{(\gamma_1+\delta_1)\Gamma(\nu-1)}\int_0^\ell(\ell-\varsigma)^{\nu-2}\varphi(\varsigma,\varkappa(\varsigma))\,d\varsigma\\ &\quad+\frac{1}{\Gamma(\nu)}\int_0^\wp(\wp-\varsigma)^{\nu-1}\varphi(\varsigma,\varkappa(\varsigma))\,d\varsigma\\ &\quad+\frac{1}{\Gamma(\nu)}\int_\wp^\ell(\varsigma-\wp)^{\nu-1}\varphi(\varsigma,\varkappa(\varsigma))d\varsigma.\end{aligned}\tag{20}$$

Due to continuity of φ on Λ, $\mathfrak{I}$ is continuous.

For $\varkappa, \widehat{\varkappa} \in \Lambda$ and for each $\wp \in [0, \ell]$, we have

$$\begin{aligned}
&\max\{|\ \Im\varkappa(\wp)|, |\Im\widehat{\varkappa}(\wp)\ |\} \\
&\leq \frac{-2\wp\ell\gamma_1 + 3\ell^2\gamma_1}{4(\gamma_1+\delta_1)\Gamma(\nu-2)} \times \int_0^\ell (\ell-\varsigma)^{\nu-3} \max\{|\ \varphi(\varsigma, \varkappa(\varsigma))|, |\varphi(\varsigma, \widehat{\varkappa}(\varsigma))\ |\} d\varsigma \\
&\quad + \frac{(\gamma_1-\delta_1)\wp + \ell\gamma_1}{(\gamma_1+\delta_1)\Gamma(\nu-1)} \int_0^\ell (\ell-\varsigma)^{\nu-2} \max\{|\ \varphi(\varsigma, \varkappa(\varsigma))|, |\varphi(\varsigma, \widehat{\varkappa}(\varsigma))\ |\} d\varsigma \\
&\quad + \frac{1}{\Gamma(\nu)} \int_0^\wp (\wp-\varsigma)^{\nu-1} \max\{|\ \varphi(\varsigma, \varkappa(\varsigma))|, |\varphi(\varsigma, \widehat{\varkappa}(\varsigma))\ |\} d\varsigma \\
&\quad + \frac{1}{\Gamma(\nu)} \int_\wp^\ell (\varsigma-\wp)^{\nu-1} \max\{|\ \varphi(\varsigma, \varkappa(\varsigma))|, |\varphi(\varsigma, \widehat{\varkappa}(\varsigma))\ |\} d\varsigma \\
&\leq \frac{\gamma_1}{4(\gamma_1+\delta_1)\Gamma(\nu-1)} \ell^\nu \lambda \max\{|\varkappa|, |\widehat{\varkappa}|\} \\
&\quad + \frac{(\gamma_1-\delta_1)\ell + \ell\gamma_1}{(\gamma_1+\delta_1)\Gamma(\nu)} \ell^{\nu-1} \lambda \max\{|\varkappa|, |\widehat{\varkappa}|\} + \frac{2\ell^\nu}{\Gamma(\nu+1)} \lambda \max\{|\varkappa|, |\widehat{\varkappa}|\} \\
&\leq \lambda \left[\frac{\gamma_1 \ell^\nu}{4(\gamma_1+\delta_1)\Gamma(\nu-1)} + \frac{-\delta_1 \ell^\nu}{(\gamma_1+\delta_1)\Gamma(\nu)} + \frac{2\ell^\nu}{\Gamma(\nu+1)} \right] \max\{\|\varkappa\|, \|\widehat{\varkappa}\|\}.
\end{aligned}$$

Set $\xi := \frac{\gamma_1 \ell^\nu}{4(\gamma_1+\delta_1)\Gamma(\nu-1)} + \frac{-\delta_1 \ell^\nu}{(\gamma_1+\delta_1)\Gamma(\nu)} + \frac{2\ell^\nu}{\Gamma(\nu+1)}$. Then

$$|\ \Im\varkappa(\wp) - \Im\widehat{\varkappa}(\wp)\ | \leq \xi\lambda \max\{\|\varkappa\|, \|\widehat{\varkappa}\|\}.$$

Define relational PMS $(\Lambda, \varpi, \leq)$ on Λ^2 as $\varpi(\varkappa, \widehat{\varkappa}) = \max\{\|\varkappa\|, \|\widehat{\varkappa}\|\}$ with usual binary relation. Then

$$\varpi(\Im\varkappa, \Im\widehat{\varkappa}) = \max\{\|\Im\varkappa\|, \|\Im\widehat{\varkappa}\|\} \leq \xi\lambda \max\{\|\varkappa\|, \|\widehat{\varkappa}\|\}.$$

By virtue of Example 3.1, $\Im$ is a ϖ-implicit contractive mapping for $\alpha = \xi\lambda, \beta = 0 = \gamma = \delta$. Hence, following Theorem 3.6, $\Im$ has a unique fixed point, that is, there is a unique solution of the problem (12)–(13). □

Example 6.8. *Consider the following nonlinear FDE with Riesz-Caputo derivative:*

$$\begin{cases} {}^{RC}_{\ 0}\mathcal{D}^{\frac{5}{2}}_1 \varkappa(\wp) = \frac{(\sqrt{\pi}+2)|\varkappa(\wp)|}{\sqrt{\wp^2+121}}, \wp \in [0,1], \\ \varkappa(0) + \varkappa(1) = 0, \quad 3\varkappa'(0) + \frac{1}{2}\varkappa'(1) = 0, \quad \varkappa''(0) + \varkappa''(1) = 0. \end{cases} \tag{21}$$

Here, $\nu = \frac{5}{2}, \gamma_1 = 3, \delta_1 = \frac{1}{2}$ *and* $\varphi(\wp, \varkappa(\wp)) = \frac{(\sqrt{\pi}+2)|\varkappa(\wp)|}{\sqrt{\wp^2+121}}$*. Then, the assumptions* (A_1)*–*(A_2) *are satisfied with* $\lambda = \frac{\sqrt{\pi}+2}{11}$ *with* $\xi\lambda = 0.25249 < 1$*. By using Theorem 6.7, the problem (12)–(13) has a unique solution on* $[0,1]$.

Example 6.9. *Consider the fractional APBVP*

$$\begin{aligned} &{}^{RC}_{\ 0}\mathcal{D}^{\frac{7}{3}}_1 \varkappa(\wp) = \frac{e^{-\wp}}{(3+e^{2\pi})} \sin\left(\frac{|\varkappa(\wp)|}{7+|\varkappa(\wp)|}\right), \quad 0 \leq \wp \leq 1, \\ &\varkappa(0) + \varkappa(1) = 0, \quad \frac{2}{3}\varkappa'(0) + \frac{1}{3}\varkappa'(1) = 0 \quad \varkappa''(0) + \varkappa''(1) = 0. \end{aligned} \tag{22}$$

Here, $\nu = \frac{7}{3}, \gamma_1 = \frac{2}{3}, \delta_1 = \frac{1}{3}$ *and* $\varphi(\wp, \varkappa(\wp)) = \frac{e^{-\wp}}{(3+e^{2\pi})} \sin\left(\frac{|\varkappa(\wp)|}{7+|\varkappa(\wp)|}\right)$*. Then, the assumption* (A_1)*-*(A_2) *is satisfied with* $\lambda = \frac{1}{4}$ *with* $\xi\lambda = 0.1567 < 1$*. By using Theorem 6.7, the problem* (12)-(13) *has a unique solution on* $[0,1]$.

Data Availability

No data were used to support this study.

Competing Interest

The authors declare that there are no competing interests regarding the publication of this manuscript.

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