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Strict fixed point and Ulam-Hyers stability of multivalued asymptotically regular mappings

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Abstract. In this paper, we establish the existence and uniqueness of strict fixed point for an asymptotically regular multivalued mapping in a metric space. We also study the Ulam-Hyers stability, well-posedness and data dependence of the associated strict fixed point problem. We give an example to illustrate our results. Our work extends and complements important results existing in the literature.

1. Introduction

Let (X, d) be a metric space. We denote by $P(X)$, $B(X)$ and $CB(X)$ the family of nonempty subsets of X, the family of bounded subsets of X and the family of closed and bounded subsets of X, respectively. For $\mathcal{B}, \mathcal{G} \subset X$, we adopt the following notations and definitions:

- *The distance from* $m \in X$ *to* $\mathcal{B}$;

$$d(m, \mathcal{B}) := \inf\{d(m, w) : w \in \mathcal{B}\}.$$

- *The diameter of* $\mathcal{B}$ and $\mathcal{G}$;

$$\delta(\mathcal{B}, \mathcal{G}) := \sup\{d(m, w) : m \in \mathcal{B}, w \in \mathcal{G}\}.$$

- *The Hausdorff metric on* $CB(X)$;

$$H(\mathcal{B}, \mathcal{G}) := \max\{\sup_{m\in\mathcal{B}} d(m, \mathcal{G}), \sup_{q\in\mathcal{G}} d(q, \mathcal{B})\}.$$

For a multivalued mapping $F : X \to 2^X$, we say $m \in X$ is (i) a fixed point of F if $m \in Fm$; (ii) a strict fixed point of F if $Fm = \{m\}$. A strict fixed point is also referred to a stationary point [14] or an endpoint [3]. By $Fix(F)$ and $SFix(F)$, we mean the set of fixed points of F and the set of strict fixed points of F, respectively.

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Metric fixed point theory of a multivalued mapping was initiated by Markin [16] and Nadler [17]. Nadler, for example, established the existence of fixed point for a multivalued contraction. The existence of a fixed point does not guarantee the existence of a strict fixed point. Therefore, several authors (see [2], [3], [13], [14]) have studied the existence of strict fixed point for multivalued mappings.

In 1972, Reich proved the following strict fixed point result:

Theorem 1.1 ([23], [13]). *Let (X, d) be a complete metric space and $F : X \to B(X)$ be a multivalued mapping. Suppose there exists $M \geq 0$ and $K \geq 0$ such that $M + 2K < 1$ and for each $m, w \in X$,*

$$\delta(Fm, Fw) \leq Md(m, w) + K[\delta(m, Fm) + \delta(w, Fw)]. \tag{1}$$

Then F has a unique strict fixed point.

Recently, Górnicki[12] generalized the works of Geraghty [11] and Boyd and Wong[6] as follows:

Theorem 1.2. *Let (X, d) be complete metric space and $F : X \to X$ be an asymptotically regular mapping. Suppose there exists $\varphi \in \mathcal{J}$(See Definition 3.1) and $K \in [0, \infty)$ such that for each $m, w \in X$,*

$$d(Fm, Fw) \leq \varphi(d(m, w)) + K[d(m, Fm) + d(w, Fw)]. \tag{2}$$

If F is orbitally continuous or k-continuous, then F has a unique fixed point $z \in X$. Moreover, for each $w \in X$, $F^n w \to z$ as $n \to \infty$.

Motivated by the results of Górnicki [12], Bisht [4], and Reich [23], we study the strict fixed point problem of a multivalued asymptotically regular mapping in a metric space. We also investigate the Ulam-Hyers stability, well-posedness and data dependence for an important consequence of our results.

2. Preliminaries

In this section, we state some needed definitions and lemmas.

Definition 2.1. *Let $F : X \to P(X)$ be a multivalued mapping. For any $w_0 \in X$, $\{w_n\}$ is called orbital sequence of F if $w_{n+1} \in Fw_n$ for all $n = 0, 1, 2, \ldots$.*

Browder and Petryshyn [5] introduced the concept of asymptotic regularity for single-valued mappings. This notion is significant since several contractive mappings are asymptotically regular (see [6], [11]). Abbas et al. [1] studied single-valued asymptotically regular mappings in complex-valued metric spaces. The asymptotic regularity of multivalued mappings has been studied in [10], [20], [24] and [27].

Definition 2.2 ([24]). *A multivalued mapping $F : X \to CB(X)$ is said to be asymptotically regular at w_0 if for each sequence $\{w_n\}$ such that $w_n \in Fw_{n-1}$, we have $\lim_{n\to\infty} d(w_n, w_{n+1}) = 0$.*

F is called asymptotically regular multivalued mapping if it is asymptotically regular at each point of X.

Example 2.3. *Every multivalued contraction $F : X \to CB(X)$ with a strict fixed point is asymptotically regular as follows:*

Let $p \in X$ be a strict fixed point of F. Then for any orbital sequence $\{w_n\}$,

$$\begin{aligned} d(w_n, w_{n+1}) &\leq d(w_n, p) + d(w_{n+1}, p) \\ &= d(w_n, Fp) + d(w_{n+1}, Fp) \\ &\leq H(Fw_{n-1}, Fp) + H(Fw_n, Fp) \\ &\leq Md(w_{n-1}, p) + Md(w_n, p) \\ &\;\;\vdots \\ &\leq M^n d(w_0, p) + M^{n+1} d(w_0, p). \end{aligned}$$

Taking limit as $n \to \infty$, we get $d(w_n, w_{n+1}) \to 0$. Hence F is asymptotically regular.

Following Deimling [9] and Ćirić [7], we have the forms of continuity of a multivalued mapping.

Definition 2.4. *Let (X, d) be a metric space, $F : X \to CB(X)$ a multivalued mapping and $z \in X$. We say*

1. *F is Hausdorff-continuous(or simply H-continuous) if $H(Fw_n, Fz) \to 0$ whenever a sequence $\{w_n\}$ in X converges to z.*
2. *F is orbital H-continuous if $H(Fw_n, Fz) \to 0$ whenever any orbital sequence $\{w_n\}$ in X converges to z. Clearly, H-continuity implies orbital H-continuity.*

Lemma 2.5 ([25]). *Let $\mathcal{B}$ be a nonempty bounded subset of X and $0 < p < 1$ be given. Then for every $x \in X$, there exists $u \in \mathcal{B}$ such that*

$$d(x, u) \geq p\delta(x, \mathcal{B}).$$

Lemma 2.6 ([26]). *Let $F : X \to CB(X)$ a multivalued mapping. Let $m, w \in X$. If $w' \in Fw$, then we have*

$$d(m, w') \leq \delta(m, Gm) + H(Fm, Fw).$$

3. Main Results

Throughout this section, we assume that X is a complete metric space unless stated otherwise.
First, we define some classes of mappings.

Definition 3.1.

1. *Let $\mathcal{S}$ be the family of functions $\alpha : [0, \infty) \to [0, 1)$ satisfying the condition $\alpha(t_n) \to 1$ implies $t_n \to 0$.*
2. *Let $\mathcal{J}$ be the family of functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying the conditions: (i) $\varphi(t) < t$ for all $t > 0$, (ii) φ is upper semi-continuous i.e. $t_n \to t \geq 0$ implies $\limsup_{n\to\infty} \varphi(t_n) \leq \varphi(t)$.*

Theorem 3.2. *Let $F : X \to CB(X)$ be an asymptotically regular mapping. Suppose there exists $\varphi \in \mathcal{J}$ and $K \in [0, \infty)$ such that for each $m, w \in X$,*

$$\delta(Fm, Fw) \leq \varphi(d(m, w)) + K[\delta(m, Fm) + \delta(w, Fw)]. \tag{3}$$

If F is an orbitally H-continuous multivalued mapping, then F has a unique strict fixed point.

Proof. Let $\theta > 1$. Using Lemma 2.4, we can define a single-valued mapping f of X into itself such that $fm \in Fm$ for all $m \in X$, and

$$\delta(m, Fm) \leq \theta d(m, fm) \text{ for all } m \in X.$$

Then, (3.1) implies

$$\begin{aligned} d(fm, fw) &\leq \delta(Fm, Fw) \\ &\leq \varphi(d(m, w)) + K[\delta(m, Fm) + \delta(w, Fw)] \\ &\leq \varphi(d(m, w)) + K\theta[d(m, fm) + d(w, fw)] \end{aligned}$$

for all $m, w \in X$. For any $w_0 \in X$, define $w_{n+1} = fw_n$. Then $w_{n+1} = fw_n \in Fw_n$, and w_{n+1} is an orbital sequence of F. It follows from the asymptotic regularity of F that

$$\lim_{n\to\infty} d(w_n, w_{n+1}) = 0. \tag{4}$$

Next, we show that $\{w_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{w_n\}$ is not Cauchy. Then there exists an $\epsilon > 0$ and sequences of integers $\{m(k)\}$, $\{n(k)\}$ with $m(k) > n(k) \geq k$ such that for $k = 1, 2, \dots$, we have

$$d(w_{m(k)}, w_{n(k)}) \geq \epsilon. \tag{5}$$

By choosing $m(k)$ to be the smallest number exceeding $n(k)$ for which (3.3) holds, we may assume that $d(w_{m(k)-1}, w_{n(k)}) < \epsilon$. Now,

$$\begin{aligned}\epsilon \le d(w_{m(k)}, w_{n(k)}) &\le d(w_{m(k)}, w_{m(k)-1}) + d(w_{m(k)-1}, w_{n(k)}) \\ &< d(w_{m(k)}, w_{m(k)-1}) + \epsilon\end{aligned}$$

Letting $k \to \infty$, it follows by asymptotic regularity of F that

$$\lim_{k\to\infty} d(w_{m(k)}, w_{n(k)}) = \epsilon. \tag{6}$$

Now,

$$\begin{aligned} d(w_n, w_m) &\le d(w_n, w_{n+1}) + d(w_{n+1}, w_{m+1}) + d(w_{m+1}, w_m) \\ &= d(w_n, w_{n+1}) + d(fw_n, fw_m) + d(w_{m+1}, w_m) \\ &\le d(w_n, w_{n+1}) + \varphi(d(w_n, w_m)) + d(w_{m+1}, w_m) \\ &\quad + K\theta[d(w_n, fw_n) + d(w_m, fw_m)] \\ &= \varphi(d(w_n, w_m)) + (K\theta + 1)[d(w_n, w_{n+1}) + d(w_m, w_{m+1})]. \end{aligned}$$

Taking limit as $k \to \infty$, it follows from upper semi-continuity of φ, (3.1) and (3.4) that

$$\epsilon = \lim_{k\to\infty} d(w_{n(k)}, w_{m(k)}) \le \limsup_{k\to\infty} \varphi(d(w_{n(k)}, w_{m(k)})) \le \varphi(\epsilon) < \epsilon.$$

This is a contradiction. Hence $\{w_n\}$ is a Cauchy sequence. Since X is a complete metric space, $\{w_n\}$ converges to $c \in X$.

Using Lemma 2.5, we have

$$\begin{aligned}\delta(c, Fc) &\le d(c, w_n) + \delta(w_n, Fw_n) + H(Fw_n, Fc) \\ &\le d(c, w_n) + \theta d(w_n, w_{n+1}) + H(Fw_n, Fc).\end{aligned} \tag{7}$$

Thus, we get from (3.2), (3.5) and orbital continuity of F that $\delta(c, Fc) = 0$. Hence, c is a strict fixed point of F. Suppose F has a strict fixed point v other than c. Then, we have

$$\begin{aligned} d(v, c) = \delta(Fv, Fc) &\le \varphi(d(v, c)) + K[\delta(v, Fv) + \delta(c, Fc)] \\ &< d(v, c). \end{aligned}$$

This is a contradiction. Hence F has a unique strict fixed point. □

Theorem 3.3. *Let $F : X \to CB(X)$ be an asymptotically regular mapping. Suppose there exists $\alpha \in \mathcal{S}$ and $K \in [0, \infty)$ such that for each $m, w \in X$,*

$$\delta(Fm, Fw) \le \alpha(d(m, w))d(m, w) + K[\delta(m, Fm) + \delta(w, Fw)]. \tag{8}$$

If F is an orbitally H-continuous multivalued mapping, then F has a unique strict fixed point.

Proof. Let $\theta > 1$. Using similar reasoning as in the proof of Theorem 3.2, we can define a single-valued mapping f and sequence $\{w_n\}$ such that

$$d(fm, fw) \le \alpha(d(m, w))d(m, w) + K\theta[d(m, fm) + d(w, fw)]$$

for all $m, w \in X$ and

$$\lim_{n\to\infty} d(w_n, w_{n+1}) = 0. \tag{9}$$

Next, we show that $\{w_n\}$ is a Cauchy sequence. Suppose otherwise. Then, $\limsup_{n,m\to\infty} d(w_n, w_m) > 0$.

Now,

$$\begin{aligned} d(w_n, w_m) &\leq d(w_n, w_{n+1}) + d(w_{n+1}, w_{m+1}) + d(w_{m+1}, w_m) \\ &= d(w_n, w_{n+1}) + d(fw_n, fw_m) + d(w_{m+1}, w_m) \\ &\leq d(w_n, w_{n+1}) + \alpha(d(w_n, w_m))d(w_n, w_m) + d(w_{m+1}, w_m) \\ &\quad +K\theta[d(w_n, fw_n) + d(w_m, fw_m)] \\ &= \alpha(d(w_n, w_m))d(w_n, w_m) \\ &\quad +(K\theta + 1)[d(w_n, w_{n+1}) + d(w_m, w_{m+1})]. \end{aligned}$$

Then,

$$\frac{d(w_n, w_m)}{[d(w_n, w_{n+1}) + d(w_m, w_{m+1})]} \leq \frac{K\theta + 1}{1 - \alpha(d(w_n, w_m))}. \tag{10}$$

Using the assumption that

$$\limsup_{n,m\to\infty} d(w_n, w_m) > 0,$$

(3.7) and (3.8), we have

$$\limsup_{n,m\to\infty} \frac{K\theta + 1}{1 - \alpha(d(w_n, w_m))} = \infty.$$

This implies that

$$\limsup_{n,m\to\infty} \alpha(d(w_n, w_m)) = 1$$

and consequently, since $\alpha \in \mathcal{S}$,

$$\limsup_{n,m\to\infty} d(w_n, w_m) = 0.$$

This is a contradiction. Hence, $\{w_n\}$ is a Cauchy sequence. Completeness of X implies $\{w_n\}$ converges to $c \in X$. Following similar arguments as in the proof of Theorem 3.2, we can show that c is a unique strict fixed point of F. □

As a special case of our Theorems 3.2 and 3.3, we get the following generalization of Theorem 2.1 due to Bisht [4].

Corollary 3.4. *Let $F : X \to CB(X)$ be an asymptotically regular and orbitally H-continuous multivalued mapping. Suppose there exists $M \in [0, 1)$ and $K \in [0, \infty)$ such that for each $m, w \in X$,*

$$\delta(Fm, Fw) \leq Md(m, w) + K[\delta(m, Fm) + \delta(w, Fw)]. \tag{11}$$

Then, F has a unique strict fixed point.

Next, we discuss the well-posedness of the strict fixed point problem.

Definition 3.5 ([21]). *Let (X, d) be a metric space and $F : X \to CB(X)$ a multivalued mapping. The strict fixed point problem*

$$Fm = \{m\}, \ m \in X \tag{12}$$

is well-posed for F if:

(i) $SFix(F) = \{c\}$

(ii) If $\{w_n\}$ is a sequence in X such that $\lim_{n\to\infty} \delta(w_n, Fw_n) = 0$, then $w_n \to c$ as $n \to \infty$.

Theorem 3.6. *Let $F : X \to CB(X)$ be an asymptotically regular and orbitally H-continuous multivalued mapping. Suppose there exists $M \in [0, 1)$ and $K \in [0, \infty)$ such that for each $m, w \in X$,*

$$\delta(Fm, Fw) \leq Md(m, w) + K[\delta(m, Fm) + \delta(w, Fw)]. \tag{13}$$

Then the strict fixed point problem is well-posed for F.

Proof. By Corollary 3.4, it follows that $SFix(F) = \{c\}$. Let $\{w_n\}$ be such that $\lim_{n\to\infty} \delta(w_n, Fw_n) = 0$. Now,

$$\begin{aligned} d(w_n, c) &\leq \delta(w_n, Fw_n) + \delta(Fw_n, Fc) \\ &\leq \delta(w_n, Fw_n) + Md(w_n, c) + K[\delta(w_n, Fw_n) + \delta(c, Fc)] \\ &= Md(w_n, c) + (K + 1)\delta(w_n, Fw_n). \end{aligned}$$

Thus, $(1 - M)d(w_n, c) \leq (K + 1)\delta(w_n, Fw_n)$ and $\lim_{n\to\infty} d(w_n, c) = 0$. □

The Ulam-Hyers stability is an important notion in the theory of differential and integral equations (See [18], [15]). The Ulam-Hyers stability for the strict fixed point problem is defined as follows:

Definition 3.7 ([18]). *Let (X, d) be a metric space and $F : X \to P(X)$ a multivalued mapping. The strict fixed point problem (3.10) is called Ulam-Hyers stable if there exists $\theta > 0$ such that for each $\epsilon > 0$ and for each ϵ-solution $m \in X$ of the strict fixed point problem i.e.*

$$\delta(m, Fm) \leq \epsilon, \tag{14}$$

there exists a solution c of the strict fixed point problem (3.10) such that

$$d(m, c) \leq \theta\epsilon.$$

Theorem 3.8. *Let $F : X \to CB(X)$ be an asymptotically regular and orbitally H-continuous multivalued mapping. Suppose there exists $M \in [0, 1)$ and $K \in [0, \infty)$ such that for each $m, w \in X$,*

$$\delta(Fm, Fw) \leq Md(m, w) + K[\delta(m, Fm) + \delta(w, Fw)]. \tag{15}$$

Then the strict fixed point problem is Ulam-Hyers stable.

Proof. By Corollary 3.4, we have that $SFix(F) = \{c\}$. Let $\epsilon > 0$ and $m \in X$ be such that $\delta(m, Fm) \leq \epsilon$.
Now, we have

$$\begin{aligned} d(m, c) &\leq \delta(m, Fm) + \delta(Fm, Fc) \\ &\leq \delta(y, Fy) + Md(m, w) + K[\delta(m, Fm) + \delta(c, Fc)] \\ &= (K + 1)\delta(m, Fm) + Md(m, w). \end{aligned}$$

Hence,

$$d(m, c) \leq \frac{K + 1}{1 - M}\delta(m, Fm) \leq \frac{K + 1}{1 - M}\epsilon.$$

□

Next, we present a data dependence result for the strict fixed point problem.

Theorem 3.9. *Let $F : X \to CB(X)$ be an asymptotically regular and orbitally H-continuous multivalued mapping. Suppose there exists $M \in [0, 1)$ and $K \in [0, \infty)$ such that for each $m, w \in X$,*

$$\delta(Fm, Fw) \leq Md(m, w) + K[\delta(m, Fm) + \delta(w, Fw)]. \tag{16}$$

Suppose that $R : X \to CB(X)$ is a multivalued mapping with $SFix(R) \neq \emptyset$ and there exists $\xi > 0$ such that $\delta(Fm, Rm) \leq \xi$, for every $m \in X$. Then,

$$\delta(SFix(F), SFix(R)) \leq \frac{K + 1}{1 - M}\xi.$$

Proof. By Corollary 3.4, we have that $SFix(F) = \{c\}$. For any $m \in SFix(R)$, we have

$$\begin{aligned} d(m,c) &= \delta(Rm, Fc) \\ &\leq \delta(Rm, Fm) + \delta(Fm, Fc) \\ &\leq \xi + Md(m,c) + K[\delta(m, Fm) + \delta(c, Fc)] \\ &= \xi + Md(m,c) + K\delta(Gc, Fc) \\ &\leq (K+1)\xi + Md(m,c). \end{aligned}$$

Hence,

$$d(m,c) \leq \frac{K+1}{1-M}\xi$$

and the result follows. □

We illustrate the above results with an example.

Example 3.10. *Let $X = [-1, \frac{1}{2}]$ be endowed with the usual metric. For $m \in X$, define $F : X \to CB(X)$ by*

$$Fm = \begin{cases} \frac{1}{2}, & m \in [-1, 0) \\ [m^3, m^2], & m \in [0, \frac{1}{2}] \end{cases}.$$

We notice that if $M, K \geq 0$, $M + 2K < 1$ and $m = 0$, then there exists $w \in [-1, 0)$ such that

$$\delta(Fm, Fw) \leq Md(m,w) + K[\delta(m, Fm) + \delta(w, Fw)]$$

does not hold. Hence, Reich's Result (Theorem 1.1) is not applicable.

Case 1: *$m \in [-1, 0)$ and $w \in [0, \frac{1}{2}]$. We have*

$$\delta(Fm, Fw) = \frac{1}{2} - w^3, \quad \delta(w, Fw) = w - w^3$$

and $\delta(m, Fm) = \frac{1}{2} - m$. Clearly, $m \leq w$. Thus,

$$m + \frac{1}{2} - w^3 \leq w + \frac{1}{2} - w^3 \quad \text{and} \quad \delta(Fm, Fw) \leq \delta(m, Fm) + \delta(w, Fw).$$

Case 2: *$m \in [0, \frac{1}{2}]$ and $w \in [0, \frac{1}{2}]$. Without loss of generality, let $m \leq w$. Then*

$$\delta(Fm, Fw) = w^2 - m^3, \quad \delta(w, Fw) = w - w^3$$

and $\delta(m, Fm) = m - m^3$. For $w \in [0, \frac{1}{2}]$,

$$y(y^2 + y - 1) \leq 0 \leq m.$$

Hence,

$$\delta(Fm, Fw) \leq \delta(m, Fm) + \delta(w, Fw).$$

We note that F is not H-continuous. Indeed, let $w_n = \frac{-1}{n}$. Then $w_n \to 0$ and $\delta(Fw_n, F0) = H(Fw_n, F0) = \frac{1}{2}$. For $w_0 \in [-1, \frac{1}{2}]$, let $\{w_n\}$ be any orbital sequence of F. Then

$$w_n^3 \leq w_{n+1} \leq w_n^2 \leq w_n \leq w_{n-1}^2 \leq w_{n-1} \leq \ldots \leq w_0 \leq \frac{1}{2}.$$

It follows that $\{w_n\}$ is a nonincreasing sequence and thus converges to $\lambda \geq 0$. If $\lambda > 0$, then

$$w_n^3 \leq w_{n+1} \leq w_n^2 \leq \frac{1}{2},$$

which implies that $\lambda^3 \leq \lambda \leq \lambda^2 \leq \frac{1}{2}$ and $1 \leq \lambda \leq \frac{1}{2}$. This is a contradiction. Hence $\{w_n\}$ converges to 0. Now, we can easily show that F is asymptotically regular and orbitally continuous. By Theorem 3.8, the associated strict fixed point problem is Ulam-Hyers stable.

Remark 3.11.

1. *In view of* (3.5), *orbital H-continuity of F can be replaced by the following condition :* $\delta(Fw_n, Fz) \to 0$ *whenever any orbital sequence* $\{w_n\}$ *in X converges to* $z \in X$ *(See* [8]*).*
2. *Theorem* 3.2 *and Theorem* 3.3 *extend Theorem* 2.2 *and Theorem* 2.1 *in* [12]*, respectively, for multivalued mappings.*
3. *Reich* [22], [23] *and Petrusel and Petrusel* [19] *have extensively used the condition* $M + 2K < 1$*. Our work (Corollary* 3.4*, Theorem* 3.6 *- Theorem* 3.8*) is independent of this condition.*

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