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Different types of approximation operators on $\mathcal{G}_n$-CAS via ideals

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Abstract. A mathematical approach to dealing with the problems of ambiguity and indeterminacy in knowledge is called a rough set theory. It begins by using an equivalence relation to divide the universe into parts. Numerous generalized rough set models have been developed and investigated to increase their adaptability and extend their range of applications. In this context, we introduce new generalized rough set models that are inspired by covering-based rough sets and ideals. In this paper, lower and upper approximations of new types of covering rough sets based on j-neighborhoods, complementary j-neighborhoods, and j-adhesions are defined via ideals. The main features of these approximations are examined. The relationships among them are given by various examples and propositions. Some comparisons between our methods and others' methods such as Abd El-Monsef et al.'s method [2] and Nawar et al.'s method [22] are given. A practical example is given to illustrate one of our methods is more precise.

1. Introduction

The problems of ambiguity and uncertainty in the information system are vital in data analysis. There are many new ways how to manage and perceive knowledge. One of them is the rough set theory. Rough set theory was investigated by Pawlak [24, 25] as a mathematical approach that deals with uncertainty and the vagueness of imprecise data. It has a wide variety of executions in modern-life fields such as biology, physics, engineering, etc. The central idea in this theory is approximation operators which are characterized by equivalence relations. Since these relations restrict the application areas, researchers replaced equivalence relations with binary relations. Many extensions based on binary relations [17, 20, 21, 26, 30, 35] have been made, and thus generalizations of Pawlak's rough set theory have been obtained. Also, many researchers introduced several types of generalization of Pawlak's rough set theory using topological concepts.

Lin [19] and Yao [31] examined rough sets concerning neighborhood systems for the interpretation of granules. Abd El-Monsef et al. [1] defined different neighborhood systems to approximate rough sets and introduced new neighborhood systems which appear as a generalized type of neighborhood spaces. Amer et al. [7] obtained new j-nearly approximations as mathematical instruments modifying and generalizing the j-approximations in the j-neighborhood space. Atef et al. [8] generalized three types of

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rough set models hold on j-neighborhood space. They also introduced the notions of P_j-neighbourhood using j-neighborhoods and investigated their properties. Al-Shami et al. [4] introduced new types of neighborhoods, namely $\mathcal{E}_j$-neighborhoods applying j-neighborhoods and also constructed approximations based on a topology induced by them. They studied their relationships with $\mathcal{N}_j$-neighborhoods and $\mathcal{E}_j$-neighborhoods. Al-Shami [5] defined and investigated C_j-neighborhoods which depend on the inclusion relations between j-neighborhoods and also introduced and studied $\mathcal{M}_j$-neighborhoods [6] defined using union relation Dai et al. [11].

The easing of the partition resulting from an equivalent relationship to a covering is another strategy. Therefore, many extensions based on coverings [9, 27, 32–34, 36] have been made and thus generalizations of Pawlak's rough set theory have been obtained. Zhu [34] defined the covering approximation space and gave a new definition of the neighborhood in this space. Thus, he presented new covering rough sets based on this neighborhood. Abd El-Monsef et al. [2] introduced the generalized covering approximation space as a generalization for covering approximation space defined j-neighborhoods for obtaining new types of approximations here. Then, using j-neighborhoods, Nawar et al. [22] gave the definitions of complementary j-neighborhoods and j-adhesions. Also, they constructed new covering rough sets based on these neighborhoods.

The concept of ideal, which was initially introduced by Kuratowski [18], is one of the major research areas in the field of mathematics. A non-empty collection of sets that is closed by the heredity condition and finite additivity is known as an ideal. In recent years, interest in various ideal rough set models has risen significantly. This approach has the benefit of increasing lower approximations while decreasing upper approximations, which limits a concept's vagueness (uncertainty) to uncertainty areas at their borders. As a result, the accuracy measure is improved and the boundary region is minimized. The ideal can also be thought of as a class of an object in an information system that has specific requirements and can be researched to generate new granulations using information obtained from real-world problems. Firstly, the idea of ideals with "r"-neighborhoods were used by Kandil et al. [16] to expand Pawlak's approximations. They demonstrated that in contrast to Pawlak's method [25], and Yao's method [31] their results reduce the boundary region. Later, many researchers have found the examination of this theory with ideals to be interesting (see [10, 12–15, 23, 28]).

In this paper, the definitions of $\mathcal{I}\varrho_j$-approximations, $\mathcal{I}\mathcal{M}_j$-approximations, and $\mathcal{I}\mathcal{P}_j$-approximations based on j-neighborhoods, complementary j-neighborhoods, and j-adhesions via ideals in generalized covering approximation space are given respectively. Then, the basic properties of these approximations are examined and many counterexamples are given to illustrate counter connections. In addition, the relationships between these approximations are investigated and the best approximations are obtained as $\mathcal{I}\mathcal{P}_j$-approximations. Moreover, $\mathcal{I}\varrho_j$-approximations with j-approximations and $\mathcal{I}\mathcal{M}_j$-approximations with complementary j-approximations are compared and it is seen that our approximations have higher accuracy measures. Finally, a real-life application is given in which $\mathcal{I}\mathcal{P}_j$-approximations and adhesion j-approximations are compared.

2. Preliminaries

Definition 2.1. *[34] Let Σ be a domain of discourse and $\complement$ be a family of subsets of Σ. If none subsets in $\complement$ are empty and $\bigcup \complement = \Sigma$, then $\complement$ is called a covering of Σ. The pair $(\Sigma, \complement)$ is called a covering approximation space.*

Definition 2.2. *[2] Let $\Sigma \neq \emptyset$ be a finite set and ϱ be a binary relation on Σ. Then, the right cover and the left cover of Σ are defined as follows:*

$$\text{right cover: } \complement_r = \{e\varrho : \forall e \in \Sigma \text{ and } \Sigma = \textstyle\bigcup_{e\in\Sigma} e\varrho\},$$
$$\text{left cover: } \complement_l = \{\varrho e : \forall e \in \Sigma \text{ and } \Sigma = \textstyle\bigcup_{e\in\Sigma} \varrho e\}.$$

The triple $(\Sigma, \varrho, \complement_n)$ is called generalized covering approximation space (briefly, $\mathcal{G}_n$-CAS) for $n \in \{r, l\}$.

Definition 2.3. *[2] Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS for $n \in \{r, l\}$. Then, the j-neighborhoods $\mathcal{N}_j(e)$ of $e \in \Sigma$ for each $j \in J = \{r, l, i, u\}$ are defined as follows:*

$$r\text{-neighborhood: } \mathcal{N}_r(e) = \bigcap\{K \in \complement_r : e \in K\},$$
$$l\text{-neighborhood: } \mathcal{N}_l(e) = \bigcap\{K \in \complement_l : e \in K\},$$
$$i\text{-neighborhood: } \mathcal{N}_i(e) = \mathcal{N}_r(e) \cap \mathcal{N}_l(e),$$
$$u\text{-neighborhood: } \mathcal{N}_u(e) = \mathcal{N}_r(e) \cup \mathcal{N}_l(e).$$

Lemma 2.4. *[2] Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS. Then, the following hold:*
(1) $\mathcal{N}_j(e) \neq \emptyset$ for each $e \in \Sigma$ and $j \in J$.
(2) $e \in \mathcal{N}_j(e)$ for each $e \in \Sigma$ and $j \in J$.
(3) If $f \in \mathcal{N}_j(e)$ then $\mathcal{N}_j(f) \subseteq \mathcal{N}_j(e)$ for $e, f \in \Sigma$ and $j \in \{r, l, i\}$.
(4) $\mathcal{N}_j(e)$ represent different coverings of Σ for each $e \in \Sigma$.

Lemma 2.5. *[2] Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS. For each $e \in \Sigma$, the following hold:*
(1) $\mathcal{N}_i(e) \subseteq \mathcal{N}_r(e) \subseteq \mathcal{N}_u(e)$,
(2) $\mathcal{N}_i(e) \subseteq \mathcal{N}_l(e) \subseteq \mathcal{N}_u(e)$.

Definition 2.6. *[2] Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS, $\Delta \subseteq \Sigma$ and $j \in J$. Then, the j-lower approximations, the j-upper approximations, the j-boundary and the j-accuracy measure of Δ are defined respectively as follows:*

$$\underline{\varrho}_j(\Delta) = \{e \in \Delta : \mathcal{N}_j(e) \subseteq \Delta\},\ \overline{\varrho}_j(\Delta) = \{e \in \Sigma : \mathcal{N}_j(e) \cap \Delta \neq \emptyset\},$$
$$\mathcal{B}_j(\Delta) = \overline{\varrho}_j(\Delta) \setminus \underline{\varrho}_j(\Delta) \text{ and } \delta_j(\Delta) = \frac{|\underline{\varrho}_j(\Delta)|}{|\overline{\varrho}_j(\Delta)|}, |\overline{\varrho}_j(\Delta)| \neq \emptyset.$$

Also, Δ is called j-exact set if $\overline{\varrho}_j(\Delta) = \underline{\varrho}_j(\Delta) = \Delta$. Otherwise, Δ is called j-rough set.

Definition 2.7. *[22] Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS for $n \in \{r, l\}$. Then, the complementary j-neighborhoods $\mathcal{M}_j(e)$ of $e \in \Sigma$ for each $j \in J = \{r, l, i, u\}$ are defined as follows:*

$$\mathcal{M}_j(e) = \{f \in \Sigma : e \in \mathcal{N}_j(f)\}.$$

Lemma 2.8. *[22] Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS. Then, the following hold:*
(1) $e \in \mathcal{M}_j(e)$ for each $e \in \Sigma$ and $j \in J$.
(2) If $f \in \mathcal{M}_j(e)$ then $\mathcal{M}_j(f) \subseteq \mathcal{M}_j(e)$ for any $e, f \in \Sigma$ and $j \in \{r, l, i\}$.

Definition 2.9. *[22] Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS, $\Delta \subseteq \Sigma$ and $j \in J$. Then, the complementary j-lower approximations, the complementary j-upper approximations, the complementary j-boundary and the complementary j-accuracy measure of Δ are defined respectively as follows:*

$$\underline{\mathcal{M}}_j(\Delta) = \{e \in \Delta : \mathcal{M}_j(e) \subseteq \Delta\},\ \overline{\mathcal{M}}_j(\Delta) = \{e \in \Sigma : \mathcal{M}_j(e) \cap \Delta \neq \emptyset\},$$
$$\mathcal{BM}_j(\Delta) = \overline{\mathcal{M}}_j(\Delta) \setminus \underline{\mathcal{M}}_j(\Delta) \text{ and } \eta_j(\Delta) = \frac{|\underline{\mathcal{M}}_j(\Delta)|}{|\overline{\mathcal{M}}_j(\Delta)|}, |\overline{\mathcal{M}}_j(\Delta)| \neq \emptyset.$$

Definition 2.10. *[22] Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS for $n \in \{r, l\}$. Then, the j-adhesions $\mathcal{P}_j(e)$ of $e \in \Sigma$ for each $j \in J = \{r, l, i, u\}$ are defined as follows:*

$$\mathcal{P}_j(e) = \{f \in \Sigma : \mathcal{N}_j(e) = \mathcal{N}_j(f)\}.$$

Lemma 2.11. *[22] Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS. Then, for each $e \in \Sigma$ and $j \in J$, the following hold:*
(1) $\mathcal{P}_j(e) \subseteq \mathcal{N}_j(e)$.
(2) $\mathcal{P}_j(e) \subseteq \mathcal{M}_j(e)$.

Definition 2.12. *[22] Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS, $\Delta \subseteq \Sigma$ and $j \in J$. Then, the adhesion j-lower approximations, the adhesion j-upper approximations, the adhesion j-boundary and the adhesion j-accuracy measure of Δ are defined respectively as follows:*

$$\underline{\mathcal{P}}_j(\Delta) = \{e \in \Delta : \mathcal{P}_j(e) \subseteq \Delta\},\ \overline{\mathcal{P}}_j(\Delta) = \{e \in \Sigma : \mathcal{P}_j(e) \cap \Delta \neq \emptyset\},$$
$$\mathcal{BP}_j(\Delta) = \overline{\mathcal{P}}_j(\Delta) \setminus \underline{\mathcal{P}}_j(\Delta) \text{ and } \mu_j(\Delta) = \frac{|\underline{\mathcal{P}}_j(\Delta)|}{|\overline{\mathcal{P}}_j(\Delta)|}, |\overline{\mathcal{P}}_j(\Delta)| \neq \emptyset.$$

3. New Approximation Operators Based on Different Neighborhoods via Ideals

In this section, new rough approximations based on three different neighborhoods and ideals are constructed in $\mathcal{G}_n$-CAS. Their basic properties are obtained and the relationships among them are investigated. Also, the comparisons between these approximations and the previous ones in [2, 22] are discussed.

Definition 3.1. *Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS and $\mathcal{I}$ be an ideal on Σ. Then, the $\mathcal{I}\underline{\varrho}_j$-lower approximations, $\mathcal{I}\overline{\varrho}_j$-upper approximations and $\mathcal{I}\delta_j$-accuracy measures of Δ are defined respectively as follows for $j \in J$ and $\Delta \subseteq \Sigma$.*

$$\mathcal{I}\underline{\varrho}_j(\Delta) = \{e \in \Sigma : \mathcal{N}_j(e) \setminus \Delta \in \mathcal{I}\},$$
$$\mathcal{I}\overline{\varrho}_j(\Delta) = \{e \in \Sigma : \mathcal{N}_j(e) \cap \Delta \notin \mathcal{I}\},$$
$$\mathcal{I}\delta_j(\Delta) = \frac{|\mathcal{I}\underline{\varrho}_j(\Delta) \cap \Delta|}{|\mathcal{I}\overline{\varrho}_j(\Delta) \cup \Delta|}, \ \Delta \neq \emptyset.$$

Remark 3.2. *Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS. If $\mathcal{I} = \{\emptyset\}$, $\mathcal{I}\underline{\varrho}_j$-lower and $\mathcal{I}\overline{\varrho}_j$-upper approximations coincide with j-lower and j-upper approximations.*

Proposition 3.3. *Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS with two ideals $\mathcal{I}, \mathcal{J}$; $j \in J$ and $\Delta_1, \Delta_2 \subseteq \Sigma$.*
(1) $\mathcal{I}\underline{\varrho}_j(\Sigma) = \Sigma$ and $\mathcal{I}\overline{\varrho}_j(\emptyset) = \emptyset$.
(2) $\Delta_1 \subseteq \Delta_2$ implies $\mathcal{I}\underline{\varrho}_j(\Delta_1) \subseteq \mathcal{I}\underline{\varrho}_j(\Delta_2)$ and $\mathcal{I}\overline{\varrho}_j(\Delta_1) \subseteq \mathcal{I}\overline{\varrho}_j(\Delta_2)$.
(3) $\mathcal{I}\underline{\varrho}_j(\Delta_1) \cap \mathcal{I}\underline{\varrho}_j(\Delta_2) = \mathcal{I}\underline{\varrho}_j(\Delta_1 \cap \Delta_2)$ and $\mathcal{I}\overline{\varrho}_j(\Delta_1) \cup \mathcal{I}\overline{\varrho}_j(\Delta_2) = \mathcal{I}\overline{\varrho}_j(\Delta_1 \cup \Delta_2)$.
(4) $\mathcal{I}\underline{\varrho}_j(\Delta_1^c) = (\mathcal{I}\overline{\varrho}_j(\Delta_1))^c$ and $\mathcal{I}\overline{\varrho}_j(\Delta_1^c) = (\mathcal{I}\underline{\varrho}_j(\Delta_1))^c$.
(5) $\mathcal{I}\underline{\varrho}_j(\Delta_1) \cup \mathcal{I}\underline{\varrho}_j(\Delta_2) \subseteq \mathcal{I}\underline{\varrho}_j(\Delta_1 \cup \Delta_2)$ and $\mathcal{I}\overline{\varrho}_j(\Delta_1 \cap \Delta_2) \subseteq \mathcal{I}\overline{\varrho}_j(\Delta_1) \cap \mathcal{I}\overline{\varrho}_j(\Delta_2)$.
(6) If $\Delta_1^c \in \mathcal{I}$, then $\mathcal{I}\underline{\varrho}_j(\Delta_1) = \Sigma$ and if $\Delta_1 \in \mathcal{I}$, then $\mathcal{I}\overline{\varrho}_j(\Delta_1) = \emptyset$.
(7) If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{I}\underline{\varrho}_j(\Delta_1) \subseteq \mathcal{J}\underline{\varrho}_j(\Delta_1)$ and $\mathcal{J}\overline{\varrho}_j(\Delta_1) \subseteq \mathcal{I}\overline{\varrho}_j(\Delta_1)$.

Proof. *(1) For each $e \in \Sigma$, we have $\mathcal{N}_j(e) \setminus \Sigma = \emptyset \in \mathcal{I}$. Thus, $\mathcal{I}\underline{\varrho}_j(\Sigma) = \Sigma$. Also, for each $e \in \Sigma$, we have $\mathcal{N}_j(e) \cap \emptyset = \emptyset \in \mathcal{I}$. Then, $\mathcal{I}\overline{\varrho}_j(\emptyset) = \emptyset$.*
(2) Suppose $\Delta_1 \subseteq \Delta_2$ and $x \in \mathcal{I}\underline{\varrho}_j(\Delta_1)$. Then, we have $\mathcal{N}_j(x) \setminus \Delta_1 \in \mathcal{I}$. By hypothesis, $\mathcal{N}_j(x) \setminus \Delta_2 \in \mathcal{I}$. Thus, $x \in \mathcal{I}\underline{\varrho}_j(\Delta_2)$. The other part is proved similarly.
(3) Since $\Delta_1 \cap \Delta_2 \subseteq \Delta_1$ and $\Delta_1 \cap \Delta_2 \subseteq \Delta_2$, we have $\mathcal{I}\underline{\varrho}_j(\Delta_1 \cap \Delta_2) \subseteq \mathcal{I}\underline{\varrho}_j(\Delta_1)$ and $\mathcal{I}\underline{\varrho}_j(\Delta_1 \cap \Delta_2) \subseteq \mathcal{I}\underline{\varrho}_j(\Delta_2)$ by (2). Thus, $\mathcal{I}\underline{\varrho}_j(\Delta_1 \cap \Delta_2) \subseteq \mathcal{I}\underline{\varrho}_j(\Delta_1) \cap \mathcal{I}\underline{\varrho}_j(\Delta_2)$. Conversely, let $x \in \mathcal{I}\underline{\varrho}_j(\Delta_1) \cap \mathcal{I}\underline{\varrho}_j(\Delta_2)$. Then, $\mathcal{N}_j(x) \setminus \Delta_1 \in \mathcal{I}$ and $\mathcal{N}_j(x) \setminus \Delta_2 \in \mathcal{I}$. By the definition of ideal, we get $(\mathcal{N}_j(x) \setminus \Delta_1) \cup (\mathcal{N}_j(x) \setminus \Delta_2) = \mathcal{N}_j(x) \setminus (\Delta_1 \cap \Delta_2) \in \mathcal{I}$. Thus, $x \in \mathcal{I}\underline{\varrho}_j(\Delta_1 \cap \Delta_2)$. The other part is proved similarly.
(4) Let $x \in \mathcal{I}\underline{\varrho}_j(\Delta_1^c)$. Then, we have $\mathcal{N}_j(x) \setminus \Delta_1^c \in \mathcal{I}$. From here, we get $\mathcal{N}_j(x) \cap \Delta_1 \in \mathcal{I}$, that is $x \notin \mathcal{I}\overline{\varrho}_j(\Delta_1)$. Thus, $x \in (\mathcal{I}\overline{\varrho}_j(\Delta_1))^c$. Conversely, we can prove similarly. The other part is proved similarly.
(5) The proofs are obvious by (2).
(6) Since $\mathcal{N}_j(e) \cap \Delta_1^c \subseteq \Delta_1^c$ for each $e \in \Sigma$, we have $\mathcal{N}_j(e) \cap \Delta_1^c \in \mathcal{I}$ by the hypothesis. Thus, $\mathcal{I}\underline{\varrho}_j(\Delta_1) = \Sigma$. The other proof is proved similarly.
(7) The proofs are clear by hypothesis. □

The following example shows that the converse implications of Proposition 3.3(2) and the converse inclusions of Proposition 3.3(5) are not true in general.

Example 3.4. *Let $\Sigma = \{\mho_1, \mho_2, \mho_3, \mho_4\}$, $\varrho = \{(\mho_1, \mho_1), (\mho_1, \mho_3), (\mho_1, \mho_4), (\mho_2, \mho_1), (\mho_2, \mho_3), (\mho_2, \mho_4), (\mho_3, \mho_1), (\mho_3, \mho_2), (\mho_3, \mho_3), (\mho_4, \mho_1), (\mho_4, \mho_2), (\mho_4, \mho_3)\}$ be a binary relation and $\mathcal{I} = \{\emptyset, \{\mho_3\}, \{\mho_4\}, \{\mho_3, \mho_4\}\}$ be an ideal on Σ. For $\Delta_1 = \{\mho_2\}$ and $\Delta_2 = \{\mho_1\}$, we obtain $\mathcal{I}\underline{\varrho}_r(\Delta_1) = \emptyset$, $\mathcal{I}\underline{\varrho}_r(\Delta_2) = \{\mho_1, \mho_3, \mho_4\}$, $\mathcal{I}\overline{\varrho}_r(\Delta_1) = \{\mho_2\}$ and $\mathcal{I}\overline{\varrho}_r(\Delta_2) = \Sigma$.*

Thus, $\mathcal{I}\underline{\varrho}_r(\Delta_1) \subset \mathcal{I}\underline{\varrho}_r(\Delta_2)$ *and* $\mathcal{I}\overline{\varrho}_r(\Delta_1) \subset \mathcal{I}\overline{\varrho}_r(\Delta_2)$ *but* $\Delta_1 \nsubseteq \Delta_2$*. Also,* $\mathcal{I}\underline{\varrho}_r(\Delta_1) \cup \mathcal{I}\underline{\varrho}_r(\Delta_2) \neq \mathcal{I}\underline{\varrho}_r(\Delta_1 \cup \Delta_2) = \Sigma$ *and* $\mathcal{I}\overline{\varrho}_r(\Delta_1) \cap \mathcal{I}\overline{\varrho}_r(\Delta_2) \neq \mathcal{I}\overline{\varrho}_r(\Delta_1 \cap \Delta_2) = \emptyset$*.*

Note that $\mathcal{I}\underline{\varrho}_j$-lower and $\mathcal{I}\overline{\varrho}_j$-upper approximation operators may not be provide all the properties of j-approximation operators in [2].

Example 3.5. *Consider Example 3.4. Then,*
(1) $\mathcal{I}\overline{\varrho}_l(\Sigma) = \{\mho_1, \mho_2\} \neq \Sigma$ *and* $\mathcal{I}\underline{\varrho}_i(\emptyset) = \{\mho_3, \mho_4\} \neq \emptyset$*.*
(2) $\mathcal{I}\underline{\varrho}_r(\{\mho_1\}) = \{\mho_1, \mho_3, \mho_4\} \nsubseteq \{\mho_1\}$ *and* $\{\mho_2, \mho_3, \mho_4\} \nsubseteq \mathcal{I}\overline{\varrho}_r(\{\mho_2, \mho_3, \mho_4\}) = \{\mho_2\}$*.*
(3) $\mathcal{I}\underline{\varrho}_u(\mathcal{I}\underline{\varrho}_u(\{\mho_1, \mho_4\})) = \emptyset \neq \mathcal{I}\underline{\varrho}_u(\{\mho_1, \mho_4\}) = \{\mho_3, \mho_4\}$ *and* $\mathcal{I}\overline{\varrho}_u(\mathcal{I}\overline{\varrho}_u(\{\mho_2\})) = \Sigma \neq \mathcal{I}\overline{\varrho}_u(\{\mho_2\}) = \{\mho_1, \mho_2\}$*.*

Proposition 3.6. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS with an ideal* $\mathcal{I}$ *and* $\Delta \subseteq \Sigma$*.*
(1) $\mathcal{I}\underline{\varrho}_u(\Delta) \subseteq \mathcal{I}\underline{\varrho}_r(\Delta) \subseteq \mathcal{I}\underline{\varrho}_i(\Delta)$*,*
(2) $\mathcal{I}\underline{\varrho}_u(\Delta) \subseteq \mathcal{I}\underline{\varrho}_l(\Delta) \subseteq \mathcal{I}\underline{\varrho}_i(\Delta)$*,*
(3) $\mathcal{I}\overline{\varrho}_i(\Delta) \subseteq \mathcal{I}\overline{\varrho}_r(\Delta) \subseteq \mathcal{I}\overline{\varrho}_u(\Delta)$*,*
(4) $\mathcal{I}\overline{\varrho}_i(\Delta) \subseteq \mathcal{I}\overline{\varrho}_l(\Delta) \subseteq \mathcal{I}\overline{\varrho}_u(\Delta)$*.*

Proof. (1) and (2) Let $x \in \mathcal{I}\underline{\varrho}_u(\Delta)$*. Then,* $\mathcal{N}_u(x) \setminus \Delta \in \mathcal{I}$*. From Lemma 2.5,* $\mathcal{N}_r(x) \setminus \Delta \subseteq \mathcal{N}_u(x) \setminus \Delta$ *and* $\mathcal{N}_l(x) \setminus \Delta \subseteq \mathcal{N}_u(x) \setminus \Delta$*. By the definition of ideal, we have* $\mathcal{N}_r(x) \setminus \Delta \in \mathcal{I}$ *and* $\mathcal{N}_l(x) \setminus \Delta \in \mathcal{I}$*. Thus,* $x \in \mathcal{I}\underline{\varrho}_r(\Delta)$ *and* $x \in \mathcal{I}\underline{\varrho}_l(\Delta)$*. Also, since* $\mathcal{N}_i(x) \setminus \Delta \subseteq \mathcal{N}_r(x) \setminus \Delta$ *and* $\mathcal{N}_i(x) \setminus \Delta \subseteq \mathcal{N}_l(x) \setminus \Delta$*, we obtain* $x \in \mathcal{I}\underline{\varrho}_i(\Delta)$*.*
(3) and (4) Let $x \in \mathcal{I}\overline{\varrho}_i(\Delta)$*. Then, we have* $\mathcal{N}_i(x) \cap \Delta \notin \mathcal{I}$*. From Lemma 2.5, we get* $\mathcal{N}_i(x) \cap \Delta \subseteq \mathcal{N}_r(x) \cap \Delta$ *and* $\mathcal{N}_i(x) \cap \Delta \subseteq \mathcal{N}_l(x) \cap \Delta$*. By the definition of ideal,* $\mathcal{N}_r(x) \cap \Delta \notin \mathcal{I}$ *and* $\mathcal{N}_l(x) \cap \Delta \notin \mathcal{I}$*. Hence,* $x \in \mathcal{I}\overline{\varrho}_r(\Delta)$ *and* $x \in \mathcal{I}\overline{\varrho}_l(\Delta)$*. Moreover, we have* $\mathcal{N}_u(x) \cap \Delta \notin \mathcal{I}$ *since* $\mathcal{N}_r(x) \subseteq \mathcal{N}_u(x)$ *and* $\mathcal{N}_l(x) \subseteq \mathcal{N}_u(x)$*. Thus,* $x \in \mathcal{I}\overline{\varrho}_u(\Delta)$*.* □

Corollary 3.7. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS with an ideal* $\mathcal{I}$ *and* $\Delta \subseteq \Sigma$*.*
(1) $\mathcal{I}\delta_u(\Delta) \leq \mathcal{I}\delta_r(\Delta) \leq \mathcal{I}\delta_i(\Delta)$*,*
(2) $\mathcal{I}\delta_u(\Delta) \leq \mathcal{I}\delta_l(\Delta) \leq \mathcal{I}\delta_i(\Delta)$*.*

Remark 3.8. *In Table 1, we calculate* $\mathcal{I}\underline{\varrho}_j$*-lower approximations,* $\mathcal{I}\overline{\varrho}_j$*-upper approximations and* $\mathcal{I}\delta_j$*-accuracy measures of all subsets of* Σ *for* $j \in J$ *according to Example 3.4. In this way, we see that the highest* $\mathcal{I}\delta_j$*-accuracy measures can be obtained for* $j = i$*.*

Theorem 3.9. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS with an ideal* $\mathcal{I}$ *and* $\Delta \subseteq \Sigma$*. For* $j \in J$*, the following hold:*
(1) $\underline{\varrho}_j(\Delta) \subseteq \mathcal{I}\underline{\varrho}_j(\Delta)$*,*
(2) $\mathcal{I}\overline{\varrho}_j(\Delta) \subseteq \overline{\varrho}_j(\Delta)$*.*

Proof. (1) Let $x \in \underline{\varrho}_j(\Delta)$*. Then, we have* $\mathcal{N}_j(x) \subseteq \Delta$ *for* $x \in \Delta \subseteq \Sigma$*. Thus,* $\mathcal{N}_j(x) \setminus \Delta = \emptyset \in \mathcal{I}$*, that is,* $x \in \mathcal{I}\underline{\varrho}_j(\Delta)$*.*
(2) Let $x \in \mathcal{I}\overline{\varrho}_j(\Delta)$*. Then, we get* $\mathcal{N}_j(x) \cap \Delta \notin \mathcal{I}$ *for* $x \in \Sigma$*. Hence,* $\mathcal{N}_j(x) \cap \Delta \neq \emptyset$*, that is,* $x \in \overline{\varrho}_j(\Delta)$*.* □

Corollary 3.10. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS with an ideal* $\mathcal{I}$ *and* $\Delta \subseteq \Sigma$*. For* $j \in J$*,* $\delta_j(\Delta) \leq \mathcal{I}\delta_j(\Delta)$*.*

Remark 3.11. *Let* $\Sigma = \{\mho_1, \mho_2, \mho_3\}$*,* $\varrho = \{(\mho_1, \mho_1), (\mho_1, \mho_2), (\mho_2, \mho_3), (\mho_3, \mho_1)\}$ *be a binary relation and* $\mathcal{I} = \{\emptyset, \{\mho_1\}\}$ *be an ideal on* Σ*. When lower approximations, upper approximations, and accuracy measures of all subsets of* Σ *are calculated for* $j = r$ *by using Abd El-Monsef et al.'s approach [2] and our approach,* $\mathcal{I}\delta_r$*-accuracy measures are higher than* δ_r*-accuracy measures as seen in Table 2.*

Definition 3.12. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS and* $\mathcal{I}$ *be an ideal on* Σ*. Then, the* $\mathcal{I}\underline{\mathcal{M}}_j$*-lower approximations,* $\mathcal{I}\overline{\mathcal{M}}_j$*-upper approximations and* $\mathcal{I}\eta_j$*-accuracy measures of* Δ *are defined respectively as follows for* $j \in J$ *and* $\Delta \subseteq \Sigma$*.*

Table 1: $\mathcal{I}\underline{\varrho}_j$-lower approximations, $\mathcal{I}\overline{\varrho}_j$-upper approximations and $\mathcal{I}\delta_j$-accuracy measures of all subsets of Σ for $j \in J$

Δ	$j=r$			$j=l$			$j=u$			$j=i$		
	$\mathcal{I}\underline{\varrho}_r(\Delta)$	$\mathcal{I}\overline{\varrho}_r(\Delta)$	$\mathcal{I}\delta_r(\Delta)$	$\mathcal{I}\underline{\varrho}_l(\Delta)$	$\mathcal{I}\overline{\varrho}_l(\Delta)$	$\mathcal{I}\delta_l(\Delta)$	$\mathcal{I}\underline{\varrho}_u(\Delta)$	$\mathcal{I}\overline{\varrho}_u(\Delta)$	$\mathcal{I}\delta_u(\Delta)$	$\mathcal{I}\underline{\varrho}_i(\Delta)$	$\mathcal{I}\overline{\varrho}_i(\Delta)$	$\mathcal{I}\delta_i(\Delta)$
$\{\mho_1\}$	$\{\mho_1,\mho_3,\mho_4\}$	Σ	$\frac{1}{4}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	0	$\{\mho_3,\mho_4\}$	Σ	0	$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{2}$
$\{\mho_2\}$	$\emptyset$	$\{\mho_2\}$	0	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	0	$\emptyset$	$\{\mho_1,\mho_2\}$	0	$\{\mho_3,\mho_4\}$	$\{\mho_2\}$	0
$\{\mho_3\}$	$\emptyset$	$\emptyset$	0	$\{\mho_3,\mho_4\}$	$\emptyset$	1	$\emptyset$	$\emptyset$	0	$\{\mho_3,\mho_4\}$	$\emptyset$	1
$\{\mho_4\}$	$\emptyset$	$\emptyset$	0	$\{\mho_3,\mho_4\}$	$\emptyset$	1	$\emptyset$	$\emptyset$	0	$\{\mho_3,\mho_4\}$	$\emptyset$	1
$\{\mho_1,\mho_2\}$	Σ	Σ	$\frac{1}{2}$	Σ	$\{\mho_1,\mho_2\}$	1	Σ	Σ	$\frac{1}{2}$	Σ	$\{\mho_1,\mho_2\}$	1
$\{\mho_1,\mho_3\}$	$\{\mho_1,\mho_3,\mho_4\}$	Σ	$\frac{1}{2}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{3}$	$\{\mho_3,\mho_4\}$	Σ	$\frac{1}{4}$	$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{2}{3}$
$\{\mho_1,\mho_4\}$	$\{\mho_1,\mho_3,\mho_4\}$	Σ	$\frac{1}{2}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{3}$	$\{\mho_3,\mho_4\}$	Σ	$\frac{1}{4}$	$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{2}{3}$
$\{\mho_2,\mho_3\}$	$\emptyset$	$\{\mho_2\}$	0	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{3}$	$\emptyset$	$\{\mho_1,\mho_2\}$	0	$\{\mho_3,\mho_4\}$	$\{\mho_2\}$	$\frac{1}{2}$
$\{\mho_2,\mho_4\}$	$\emptyset$	$\{\mho_2\}$	0	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{3}$	$\emptyset$	$\{\mho_1,\mho_2\}$	0	$\{\mho_3,\mho_4\}$	$\{\mho_2\}$	$\frac{1}{2}$
$\{\mho_3,\mho_4\}$	$\emptyset$	$\emptyset$	0	$\{\mho_3,\mho_4\}$	$\emptyset$	1	$\emptyset$	$\emptyset$	0	$\{\mho_3,\mho_4\}$	$\emptyset$	1
$\{\mho_1,\mho_2,\mho_3\}$	Σ	Σ	$\frac{3}{4}$	Σ	$\{\mho_1,\mho_2\}$	1	Σ	Σ	$\frac{3}{4}$	Σ	$\{\mho_1,\mho_2\}$	1
$\{\mho_1,\mho_2,\mho_4\}$	Σ	Σ	$\frac{3}{4}$	Σ	$\{\mho_1,\mho_2\}$	1	Σ	Σ	$\frac{3}{4}$	Σ	$\{\mho_1,\mho_2\}$	1
$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1,\mho_3,\mho_4\}$	Σ	$\frac{3}{4}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{2}$	$\{\mho_3,\mho_4\}$	Σ	$\frac{1}{2}$	$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{3}{4}$
$\{\mho_2,\mho_3,\mho_4\}$	$\emptyset$	$\{\mho_2\}$	0	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{2}$	$\emptyset$	$\{\mho_1,\mho_2\}$	0	$\{\mho_3,\mho_4\}$	$\{\mho_2\}$	$\frac{2}{3}$
Σ	Σ	Σ	1	Σ	$\{\mho_1,\mho_2\}$	1	Σ	Σ	1	Σ	$\{\mho_1,\mho_2\}$	1

Table 2: Comparison between Abd El-Monsef et al.'s approach [2] and our approach for $j = r$.

Δ	Abd El-Monsef et al.'s approach			our approach		
	$\underline{\varrho}_r(\Delta)$	$\overline{\varrho}_r(\Delta)$	$\delta_r(\Delta)$	$\mathcal{I}\underline{\varrho}_r(\Delta)$	$\mathcal{I}\overline{\varrho}_r(\Delta)$	$\mathcal{I}\delta_r(\Delta)$
$\{\mho_1\}$	$\{\mho_1\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{2}$	$\{\mho_1\}$	$\emptyset$	1
$\{\mho_2\}$	$\emptyset$	$\{\mho_2\}$	0	$\{\mho_1,\mho_2\}$	$\{\mho_2\}$	1
$\{\mho_3\}$	$\{\mho_3\}$	$\{\mho_3\}$	1	$\{\mho_1,\mho_3\}$	$\{\mho_3\}$	1
$\{\mho_1,\mho_2\}$	$\{\mho_1,\mho_2\}$	$\{\mho_1,\mho_2\}$	1	$\{\mho_1,\mho_2\}$	$\{\mho_2\}$	1
$\{\mho_1,\mho_3\}$	$\{\mho_1,\mho_3\}$	Σ	$\frac{2}{3}$	$\{\mho_1,\mho_3\}$	$\{\mho_3\}$	1
$\{\mho_2,\mho_3\}$	$\{\mho_3\}$	$\{\mho_2,\mho_3\}$	$\frac{1}{2}$	Σ	$\{\mho_2,\mho_3\}$	1
Σ	Σ	Σ	1	Σ	$\{\mho_2,\mho_3\}$	1

$$\mathcal{I}\underline{\mathcal{M}}_j(\Delta) = \{e \in \Sigma : \mathcal{M}_j(e) \setminus \Delta \in \mathcal{I}\},$$
$$\mathcal{I}\overline{\mathcal{M}}_j(\Delta) = \{e \in \Sigma : \mathcal{M}_j(e) \cap \Delta \notin \mathcal{I}\},$$
$$\mathcal{I}\eta_j(\Delta) = \frac{|\mathcal{I}\underline{\mathcal{M}}_j(\Delta) \cap \Delta|}{|\mathcal{I}\overline{\mathcal{M}}_j(\Delta) \cup \Delta|}, \ \Delta \neq \emptyset.$$

Remark 3.13. *Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS. If $\mathcal{I} = \{\emptyset\}$, $\mathcal{I}\underline{\mathcal{M}}_j$-lower and $\mathcal{I}\overline{\mathcal{M}}_j$-upper approximations coincide with complementary j-lower and complementary j-upper approximations.*

Proposition 3.14. *Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS with two ideals $\mathcal{I}, \mathcal{J}$; $j \in J$ and $\Delta_1, \Delta_2 \subseteq \Sigma$.*
(1) $\mathcal{I}\underline{\mathcal{M}}_j(\Sigma) = \Sigma$ and $\mathcal{I}\overline{\mathcal{M}}_j(\emptyset) = \emptyset$.
(2) $\Delta_1 \subseteq \Delta_2$ implies $\mathcal{I}\underline{\mathcal{M}}_j(\Delta_1) \subseteq \mathcal{I}\underline{\mathcal{M}}_j(\Delta_2)$ and $\mathcal{I}\overline{\mathcal{M}}_j(\Delta_1) \subseteq \mathcal{I}\overline{\mathcal{M}}_j(\Delta_2)$.
(3) $\mathcal{I}\underline{\mathcal{M}}_j(\Delta_1) \cap \mathcal{I}\underline{\mathcal{M}}_j(\Delta_2) = \mathcal{I}\underline{\mathcal{M}}_j(\Delta_1 \cap \Delta_2)$ and $\mathcal{I}\overline{\mathcal{M}}_j(\Delta_1) \cup \mathcal{I}\overline{\mathcal{M}}_j(\Delta_2) = \mathcal{I}\overline{\mathcal{M}}_j(\Delta_1 \cup \Delta_2)$.
(4) $\mathcal{I}\underline{\mathcal{M}}_j(\Delta_1^c) = (\mathcal{I}\overline{\mathcal{M}}_j(\Delta_1))^c$ and $\mathcal{I}\overline{\mathcal{M}}_j(\Delta_1^c) = (\mathcal{I}\underline{\mathcal{M}}_j(\Delta_1))^c$.
(5) $\mathcal{I}\underline{\mathcal{M}}_j(\Delta_1) \cup \mathcal{I}\underline{\mathcal{M}}_j(\Delta_2) \subseteq \mathcal{I}\underline{\mathcal{M}}_j(\Delta_1 \cup \Delta_2)$ and $\mathcal{I}\overline{\mathcal{M}}_j(\Delta_1 \cap \Delta_2) \subseteq \mathcal{I}\overline{\mathcal{M}}_j(\Delta_1) \cap \mathcal{I}\overline{\mathcal{M}}_j(\Delta_2)$.
(6) If $\Delta_1^c \in \mathcal{I}$, then $\mathcal{I}\underline{\mathcal{M}}_j(\Delta_1) = \Sigma$ and if $\Delta_1 \in \mathcal{I}$, then $\mathcal{I}\overline{\mathcal{M}}_j(\Delta_1) = \emptyset$.
(7) If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{I}\underline{\mathcal{M}}_j(\Delta_1) \subseteq \mathcal{J}\underline{\mathcal{M}}_j(\Delta_1)$ and $\mathcal{J}\overline{\mathcal{M}}_j(\Delta_1) \subseteq \mathcal{I}\overline{\mathcal{M}}_j(\Delta_1)$.

Proof. *Proofs are done similarly to the proofs of Proposition 3.3.* □

The following example shows that the converse implications of Proposition 3.14 (2) and the converse inclusions of Proposition 3.14(5) are not true in general.

Example 3.15. *Consider Example 3.4. For $\Delta_1 = \{\mho_1\}$ and $\Delta_2 = \{\mho_2\}$, we obtain $\mathcal{I}\underline{\mathcal{M}}_r(\Delta_1) = \{\mho_4\}$, $\mathcal{I}\underline{\mathcal{M}}_r(\Delta_2) = \{\mho_2, \mho_4\}$, $\mathcal{I}\overline{\mathcal{M}}_r(\Delta_1) = \{\mho_1, \mho_3\}$ and $\mathcal{I}\overline{\mathcal{M}}_r(\Delta_2) = \{\mho_1, \mho_2, \mho_3\}$. Thus, $\mathcal{I}\underline{\mathcal{M}}_r(\Delta_1) \subset \mathcal{I}\underline{\mathcal{M}}_r(\Delta_2)$ and $\mathcal{I}\overline{\mathcal{M}}_r(\Delta_1) \subset \mathcal{I}\overline{\mathcal{M}}_r(\Delta_2)$ but $\Delta_1 \nsubseteq \Delta_2$. Also, $\mathcal{I}\underline{\mathcal{M}}_r(\Delta_1) \cup \mathcal{I}\underline{\mathcal{M}}_r(\Delta_2) \neq \mathcal{I}\underline{\mathcal{M}}_r(\Delta_1 \cup \Delta_2) = \Sigma$ and $\mathcal{I}\overline{\mathcal{M}}_r(\Delta_1) \cap \mathcal{I}\overline{\mathcal{M}}_r(\Delta_2) \neq \mathcal{I}\overline{\mathcal{M}}_r(\Delta_1 \cap \Delta_2) = \emptyset$.*

Note that $\mathcal{I}\underline{\mathcal{M}}_j$-lower and $\mathcal{I}\overline{\mathcal{M}}_j$-upper approximation operators may not provide all the properties of complementary j-approximation operators in [22].

Example 3.16. *Consider Example 3.4. Then,*
(1) $\mathcal{I}\overline{\mathcal{M}}_r(\Sigma) = \{\mho_1, \mho_2, \mho_3\} \neq \Sigma$ and $\mathcal{I}\underline{\mathcal{M}}_i(\emptyset) = \{\mho_3, \mho_4\} \neq \emptyset$.
(2) $\mathcal{I}\underline{\mathcal{M}}_r(\{\mho_1\}) = \{\mho_4\} \nsubseteq \{\mho_1\}$ and $\{\mho_1, \mho_4\} \nsubseteq \mathcal{I}\overline{\mathcal{M}}_r(\{\mho_1, \mho_4\}) = \{\mho_1, \mho_3\}$.
(3) If we change the ideal as $\mathcal{I} = \{\emptyset, \{\mho_2\}\}$ in Example 3.4, then $\mathcal{I}\underline{\mathcal{M}}_u(\mathcal{I}\underline{\mathcal{M}}_u(\{\mho_1\})) = \emptyset \neq \mathcal{I}\underline{\mathcal{M}}_u(\{\mho_1\}) = \{\mho_2\}$ and $\mathcal{I}\overline{\mathcal{M}}_u(\mathcal{I}\overline{\mathcal{M}}_u(\{\mho_1\})) = \Sigma \neq \mathcal{I}\overline{\mathcal{M}}_u(\{\mho_1\}) = \{\mho_1, \mho_2, \mho_3\}$.

Proposition 3.17. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS with an ideal* $\mathcal{I}$ *and* $\Delta \subseteq \Sigma$.
(1) $\mathcal{I}\underline{\mathcal{M}}_u(\Delta) \subseteq \mathcal{I}\underline{\mathcal{M}}_r(\Delta) \subseteq \mathcal{I}\underline{\mathcal{M}}_i(\Delta)$,
(2) $\mathcal{I}\underline{\mathcal{M}}_u(\Delta) \subseteq \mathcal{I}\underline{\mathcal{M}}_l(\Delta) \subseteq \mathcal{I}\underline{\mathcal{M}}_i(\Delta)$,
(3) $\mathcal{I}\overline{\mathcal{M}}_i(\Delta) \subseteq \mathcal{I}\overline{\mathcal{M}}_r(\Delta) \subseteq \mathcal{I}\overline{\mathcal{M}}_u(\Delta)$,
(4) $\mathcal{I}\overline{\mathcal{M}}_i(\Delta) \subseteq \mathcal{I}\overline{\mathcal{M}}_l(\Delta) \subseteq \mathcal{I}\overline{\mathcal{M}}_u(\Delta)$.

Proof. Proofs are done similarly to the proofs of Proposition 3.6. □

Corollary 3.18. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS with an ideal* $\mathcal{I}$ *and* $\Delta \subseteq \Sigma$.
(1) $\mathcal{I}\eta_u(\Delta) \leq \mathcal{I}\eta_r(\Delta) \leq \mathcal{I}\eta_i(\Delta)$,
(2) $\mathcal{I}\eta_u(\Delta) \leq \mathcal{I}\eta_l(\Delta) \leq \mathcal{I}\eta_i(\Delta)$.

Remark 3.19. *In Table 3, we calculate* $\mathcal{I}\underline{\mathcal{M}}_j$*-lower approximations,* $\mathcal{I}\overline{\mathcal{M}}_j$*-upper approximations and* $\mathcal{I}\eta_j$*-accuracy measures of all subsets of* Σ *for* $j \in J$ *according to Example 3.4. In this way, we see that the highest* $\mathcal{I}\eta_j$*-accuracy measures can be obtained for* $j = i$.

Remark 3.20. *Comparing Table 1 and Table 3 shows that* $\mathcal{I}\underline{\varrho}_j$*-lower approximations and* $\mathcal{I}\underline{\mathcal{M}}_j$*-lower approximations,* $\mathcal{I}\overline{\varrho}_j$*-upper approximations and* $\mathcal{I}\overline{\mathcal{M}}_j$*-upper approximations,* $\mathcal{I}\delta_j$*-accuracy measures and* $\mathcal{I}\eta_j$*-accuracy measures are not comparable for* $j \in J$.

Theorem 3.21. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS with an ideal* $\mathcal{I}$ *and* $\Delta \subseteq \Sigma$. *For* $j \in J$, *the following hold:*
(1) $\underline{\mathcal{M}}_j(\Delta) \subseteq \mathcal{I}\underline{\mathcal{M}}_j(\Delta)$,
(2) $\mathcal{I}\overline{\mathcal{M}}_j(\Delta) \subseteq \overline{\mathcal{M}}_j(\Delta)$.

Proof. Proofs are done similarly to the proofs of Theorem 3.9. □

Corollary 3.22. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS with an ideal* $\mathcal{I}$ *and* $\Delta \subseteq \Sigma$. *For* $j \in J$, $\eta_j(\Delta) \leq \mathcal{I}\eta_j(\Delta)$.

Remark 3.23. *Consider the example in the Remark 3.11 with the ideal* $\mathcal{I} = \{\emptyset, \{\mho_2\}\}$. *When lower approximations, upper approximations, and accuracy measures of all subsets of* Σ *are calculated for* $j = r$ *by using Nawar et al.'s approach [22] and our approach,* $\mathcal{I}\eta_r$*-accuracy measures are higher than* η_r*-accuracy measures as seen in Table 4.*

Definition 3.24. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS and* $\mathcal{I}$ *be an ideal on* Σ. *Then, the* $\mathcal{I}\underline{\mathcal{P}}_j$*-lower approximations,* $\mathcal{I}\overline{\mathcal{P}}_j$*-upper approximations and* $\mathcal{I}\mu_j$*-accuracy measures of* Δ *are defined respectively as follows for* $j \in J$ *and* $\Delta \subseteq \Sigma$.

$$\mathcal{I}\underline{\mathcal{P}}_j(\Delta) = \{e \in \Sigma : \mathcal{P}_j(e) \setminus \Delta \in \mathcal{I}\},$$
$$\mathcal{I}\overline{\mathcal{P}}_j(\Delta) = \{e \in \Sigma : \mathcal{P}_j(e) \cap \Delta \notin \mathcal{I}\},$$
$$\mathcal{I}\mu_j(\Delta) = \frac{|\mathcal{I}\underline{\mathcal{P}}_j(\Delta) \cap \Delta|}{|\mathcal{I}\overline{\mathcal{P}}_j(\Delta) \cup \Delta|}, \ \Delta \neq \emptyset.$$

Remark 3.25. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS. If* $\mathcal{I} = \{\emptyset\}$, $\mathcal{I}\underline{\mathcal{P}}_j$*-lower and* $\mathcal{I}\overline{\mathcal{P}}_j$*-upper approximations coincide with adhesion j-lower and adhesion j-upper approximations.*

Proposition 3.26. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS with two ideals* $\mathcal{I}, \mathcal{J}$; $j \in J$ *and* $\Delta_1, \Delta_2 \subseteq \Sigma$.
(1) $\mathcal{I}\underline{\mathcal{P}}_j(\Sigma) = \Sigma$ and $\mathcal{I}\overline{\mathcal{P}}_j(\emptyset) = \emptyset$.
(2) $\Delta_1 \subseteq \Delta_2$ implies $\mathcal{I}\underline{\mathcal{P}}_j(\Delta_1) \subseteq \mathcal{I}\underline{\mathcal{P}}_j(\Delta_2)$ and $\mathcal{I}\overline{\mathcal{P}}_j(\Delta_1) \subseteq \mathcal{I}\overline{\mathcal{P}}_j(\Delta_2)$.
(3) $\mathcal{I}\underline{\mathcal{P}}_j(\Delta_1) \cap \mathcal{I}\underline{\mathcal{P}}_j(\Delta_2) = \mathcal{I}\underline{\mathcal{P}}_j(\Delta_1 \cap \Delta_2)$ and $\mathcal{I}\overline{\mathcal{P}}_j(\Delta_1) \cup \mathcal{I}\overline{\mathcal{P}}_j(\Delta_2) = \mathcal{I}\overline{\mathcal{P}}_j(\Delta_1 \cup \Delta_2)$.
(4) $\mathcal{I}\underline{\mathcal{P}}_j(\Delta_1^c) = (\mathcal{I}\overline{\mathcal{P}}_j(\Delta_1))^c$ and $\mathcal{I}\overline{\mathcal{P}}_j(\Delta_1^c) = (\mathcal{I}\underline{\varrho}_j(\Delta_1))^c$.
(5) $\mathcal{I}\underline{\mathcal{P}}_j(\Delta_1) \cup \mathcal{I}\underline{\mathcal{P}}_j(\Delta_2) \subseteq \mathcal{I}\underline{\mathcal{P}}_j(\Delta_1 \cup \Delta_2)$ and $\mathcal{I}\overline{\mathcal{P}}_j(\Delta_1 \cap \Delta_2) \subseteq \mathcal{I}\overline{\mathcal{P}}_j(\Delta_1) \cap \mathcal{I}\overline{\mathcal{P}}_j(\Delta_2)$.
(6) If $\Delta^c \in \mathcal{I}$, then $\mathcal{I}\underline{\mathcal{P}}_j(\Delta_1) = \Sigma$ and if $\Delta_1 \in \mathcal{I}$, then $\mathcal{I}\overline{\mathcal{P}}_j(\Delta_1) = \emptyset$.
(7) If $\mathcal{I} \subseteq \mathcal{J}$, then $\mathcal{I}\underline{\mathcal{P}}_j(\Delta_1) \subseteq \mathcal{J}\underline{\mathcal{P}}_j(\Delta_1)$ and $\mathcal{J}\overline{\mathcal{P}}_j(\Delta_1) \subseteq \mathcal{I}\overline{\mathcal{P}}_j(\Delta_1)$.

Table 3: $\mathcal{I}\underline{\mathcal{M}}_j$-lower approximations, $\mathcal{I}\overline{\mathcal{M}}_j$-upper approximations and $\mathcal{I}\eta_j$-accuracy measures of all subsets of Σ for $j \in J$

Δ	$j = r$			$j = l$			$j = u$			$j = i$		
	$\mathcal{I}\underline{\mathcal{M}}_r(\Delta)$	$\mathcal{I}\overline{\mathcal{M}}_r(\Delta)$	$\mathcal{I}\eta_r(\Delta)$	$\mathcal{I}\underline{\mathcal{M}}_l(\Delta)$	$\mathcal{I}\overline{\mathcal{M}}_l(\Delta)$	$\mathcal{I}\eta_l(\Delta)$	$\mathcal{I}\underline{\mathcal{M}}_u(\Delta)$	$\mathcal{I}\overline{\mathcal{M}}_u(\Delta)$	$\mathcal{I}\eta_u(\Delta)$	$\mathcal{I}\underline{\mathcal{M}}_i(\Delta)$	$\mathcal{I}\overline{\mathcal{M}}_i(\Delta)$	$\mathcal{I}\eta_i(\Delta)$
$\{\mho_1\}$	$\{\mho_4\}$	$\{\mho_1, \mho_3\}$	0	$\{\mho_3, \mho_4\}$	$\{\mho_1, \mho_2\}$	0	$\{\mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	0	$\{\mho_3, \mho_4\}$	$\{\mho_1\}$	0
$\{\mho_2\}$	$\{\mho_2, \mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	$\frac{1}{3}$	$\{\mho_3, \mho_4\}$	$\{\mho_1, \mho_2\}$	0	$\{\mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	0	$\{\mho_2, \mho_3, \mho_4\}$	$\{\mho_1, \mho_2\}$	$\frac{1}{2}$
$\{\mho_3\}$	$\{\mho_4\}$	$\emptyset$	0	$\{\mho_3, \mho_4\}$	$\emptyset$	1	$\{\mho_4\}$	$\emptyset$	0	$\{\mho_3, \mho_4\}$	$\emptyset$	1
$\{\mho_4\}$	$\{\mho_4\}$	$\emptyset$	1	$\{\mho_3, \mho_4\}$	$\emptyset$	1	$\{\mho_4\}$	$\emptyset$	1	$\{\mho_3, \mho_4\}$	$\emptyset$	1
$\{\mho_1, \mho_2\}$	Σ	$\{\mho_1, \mho_2, \mho_3\}$	$\frac{2}{3}$	Σ	$\{\mho_1, \mho_2\}$	1	Σ	$\{\mho_1, \mho_2, \mho_3\}$	$\frac{2}{3}$	Σ	$\{\mho_1, \mho_2\}$	1
$\{\mho_1, \mho_3\}$	$\{\mho_4\}$	$\{\mho_1, \mho_3\}$	0	$\{\mho_3, \mho_4\}$	$\{\mho_1, \mho_2\}$	$\frac{1}{3}$	$\{\mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	0	$\{\mho_3, \mho_4\}$	$\{\mho_1\}$	$\frac{1}{2}$
$\{\mho_1, \mho_4\}$	$\{\mho_4\}$	$\{\mho_1, \mho_3\}$	$\frac{1}{3}$	$\{\mho_3, \mho_4\}$	$\{\mho_1, \mho_2\}$	$\frac{1}{3}$	$\{\mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	$\frac{1}{4}$	$\{\mho_3, \mho_4\}$	$\{\mho_1\}$	$\frac{1}{2}$
$\{\mho_2, \mho_3\}$	$\{\mho_2, \mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	$\frac{1}{3}$	$\{\mho_3, \mho_4\}$	$\{\mho_1, \mho_2\}$	$\frac{1}{3}$	$\{\mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	0	$\{\mho_2, \mho_3, \mho_4\}$	$\{\mho_1, \mho_2\}$	$\frac{2}{3}$
$\{\mho_2, \mho_4\}$	$\{\mho_2, \mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	$\frac{1}{2}$	$\{\mho_3, \mho_4\}$	$\{\mho_1, \mho_2\}$	$\frac{1}{3}$	$\{\mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	$\frac{1}{4}$	$\{\mho_2, \mho_3, \mho_4\}$	$\{\mho_1, \mho_2\}$	$\frac{2}{3}$
$\{\mho_3, \mho_4\}$	$\{\mho_4\}$	$\emptyset$	$\frac{1}{2}$	$\{\mho_3, \mho_4\}$	$\emptyset$	1	$\{\mho_4\}$	$\emptyset$	$\frac{1}{2}$	$\{\mho_3, \mho_4\}$	$\emptyset$	1
$\{\mho_1, \mho_2, \mho_3\}$	Σ	$\{\mho_1, \mho_2, \mho_3\}$	1	Σ	$\{\mho_1, \mho_2\}$	1	Σ	$\{\mho_1, \mho_2, \mho_3\}$	1	Σ	$\{\mho_1, \mho_2\}$	1
$\{\mho_1, \mho_2, \mho_4\}$	Σ	$\{\mho_1, \mho_2, \mho_3\}$	$\frac{3}{4}$	Σ	$\{\mho_1, \mho_2\}$	1	Σ	$\{\mho_1, \mho_2, \mho_3\}$	$\frac{3}{4}$	Σ	$\{\mho_1, \mho_2\}$	1
$\{\mho_1, \mho_3, \mho_4\}$	$\{\mho_4\}$	$\{\mho_1, \mho_3\}$	$\frac{1}{3}$	$\{\mho_3, \mho_4\}$	$\{\mho_1, \mho_2\}$	$\frac{1}{2}$	$\{\mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	$\frac{1}{4}$	$\{\mho_3, \mho_4\}$	$\{\mho_1\}$	$\frac{2}{3}$
$\{\mho_2, \mho_3, \mho_4\}$	$\{\mho_2, \mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	$\frac{1}{2}$	$\{\mho_3, \mho_4\}$	$\{\mho_1, \mho_2\}$	$\frac{1}{2}$	$\{\mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	$\frac{1}{4}$	$\{\mho_2, \mho_3, \mho_4\}$	$\{\mho_1, \mho_2\}$	$\frac{3}{4}$
Σ	Σ	$\{\mho_1, \mho_2, \mho_3\}$	1	Σ	$\{\mho_1, \mho_2\}$	1	Σ	$\{\mho_1, \mho_2, \mho_3\}$	1	Σ	$\{\mho_1, \mho_2\}$	1

Table 4: Comparison between Nawar et al.'s approach [22] and our approach for $j = r$.

Δ	Nawar et al.'s approach			our approach		
	$\underline{\mathcal{M}}_r(\Delta)$	$\overline{\mathcal{M}}_r(\Delta)$	$\eta_r(\Delta)$	$\mathcal{I}\underline{\mathcal{M}}_r(\Delta)$	$\mathcal{I}\overline{\mathcal{M}}_r(\Delta)$	$\mathcal{I}\eta_r(\Delta)$
$\{\mho_1\}$	$\emptyset$	$\{\mho_1\}$	0	$\{\mho_1, \mho_2\}$	$\{\mho_1\}$	1
$\{\mho_2\}$	$\{\mho_2\}$	$\{\mho_1, \mho_2\}$	$\frac{1}{2}$	$\{\mho_2\}$	$\emptyset$	1
$\{\mho_3\}$	$\{\mho_3\}$	$\{\mho_3\}$	1	$\{\mho_2, \mho_3\}$	$\{\mho_3\}$	1
$\{\mho_1, \mho_2\}$	$\{\mho_1, \mho_2\}$	$\{\mho_1, \mho_2\}$	1	$\{\mho_1, \mho_2\}$	$\{\mho_1\}$	1
$\{\mho_1, \mho_3\}$	$\{\mho_3\}$	$\{\mho_1, \mho_3\}$	$\frac{1}{2}$	Σ	$\{\mho_1, \mho_3\}$	1
$\{\mho_2, \mho_3\}$	$\{\mho_2, \mho_3\}$	Σ	$\frac{2}{3}$	$\{\mho_2, \mho_3\}$	$\{\mho_3\}$	1
Σ	Σ	Σ	1	Σ	$\{\mho_1, \mho_3\}$	1

Proof. *Proofs are done similarly to the proofs of Proposition 3.3.* □

The following example shows that the converse implications of Proposition 3.26 (2) and the converse inclusions of Proposition 3.26(5) are not true in general.

Example 3.27. *Consider the Example 3.4.*
(1) For $\Delta_1 = \{\mho_2\}$ *and* $\Delta_2 = \{\mho_3\}$*, we obtain* $\mathcal{I}\underline{\mathcal{P}}_r(\Delta_1) = \{\mho_2, \mho_4\}$, $\mathcal{I}\underline{\mathcal{P}}_r(\Delta_2) = \{\mho_4\}$, $\mathcal{I}\overline{\mathcal{P}}_r(\Delta_1) = \{\mho_2\}$ *and* $\mathcal{I}\overline{\mathcal{P}}_r(\Delta_2) = \emptyset$*. Thus,* $\mathcal{I}\underline{\mathcal{P}}_r(\Delta_2) \subset \mathcal{I}\underline{\mathcal{P}}_r(\Delta_1)$ *and* $\mathcal{I}\overline{\mathcal{P}}_r(\Delta_2) \subset \mathcal{I}\overline{\mathcal{P}}_r(\Delta_1)$ *but* $\Delta_2 \not\subseteq \Delta_1$.
(2) For $\Delta_1 = \{\mho_1\}$ *and* $\Delta_2 = \{\mho_2\}$*, we get* $\mathcal{I}\underline{\mathcal{P}}_l(\Delta_1) = \mathcal{I}\underline{\mathcal{P}}_l(\Delta_2) = \{\mho_3, \mho_4\}$*. Hence,* $\mathcal{I}\underline{\mathcal{P}}_l(\Delta_1) \cup \mathcal{I}\underline{\mathcal{P}}_l(\Delta_2) \neq \mathcal{I}\underline{\mathcal{P}}_l(\Delta_1 \cup \Delta_2) = \Sigma$.
(3) For $\Delta_1 = \{\mho_1, \mho_3, \mho_4\}$ *and* $\Delta_2 = \{\mho_2, \mho_3, \mho_4\}$*, we have* $\mathcal{I}\overline{\mathcal{P}}_l(\Delta_1) = \mathcal{I}\overline{\mathcal{P}}_l(\Delta_2) = \{\mho_1, \mho_2\}$*. Then,* $\mathcal{I}\overline{\mathcal{P}}_l(\Delta_1) \cap \mathcal{I}\overline{\mathcal{P}}_l(\Delta_2) \neq \mathcal{I}\overline{\mathcal{P}}_l(\Delta_1 \cap \Delta_2) = \emptyset$.

Note that $\mathcal{I}\underline{\mathcal{P}}_j$-lower and $\mathcal{I}\overline{\mathcal{P}}_j$-upper approximation operators may not provide all the properties of adhesion j-approximation operators in [22].

Example 3.28. *Consider Example 3.4. Then,*
(1) $\mathcal{I}\overline{\mathcal{P}}_l(\Sigma) = \{\mho_1, \mho_2\} \neq \Sigma$ *and* $\mathcal{I}\underline{\mathcal{P}}_r(\emptyset) = \{\mho_4\} \neq \emptyset$.
(2) $\mathcal{I}\underline{\mathcal{P}}_r(\{\mho_1\}) = \{\mho_1, \mho_3, \mho_4\} \not\subseteq \{\mho_1\}$ *and* $\{\mho_2, \mho_3, \mho_4\} \not\subseteq \mathcal{I}\overline{\mathcal{P}}_r(\{\mho_2, \mho_3, \mho_4\}) = \{\mho_2\}$.

Proposition 3.29. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS with an ideal* $\mathcal{I}$ *and* $\Delta \subseteq \Sigma$.
(1) $\mathcal{I}\underline{\mathcal{P}}_u(\Delta) \subseteq \mathcal{I}\underline{\mathcal{P}}_r(\Delta) \subseteq \mathcal{I}\underline{\mathcal{P}}_i(\Delta)$,
(2) $\mathcal{I}\underline{\mathcal{P}}_u(\Delta) \subseteq \mathcal{I}\underline{\mathcal{P}}_l(\Delta) \subseteq \mathcal{I}\underline{\mathcal{P}}_i(\Delta)$,
(3) $\mathcal{I}\overline{\mathcal{P}}_i(\Delta) \subseteq \mathcal{I}\overline{\mathcal{P}}_r(\Delta) \subseteq \mathcal{I}\overline{\mathcal{P}}_u(\Delta)$,
(4) $\mathcal{I}\overline{\mathcal{P}}_i(\Delta) \subseteq \mathcal{I}\overline{\mathcal{P}}_l(\Delta) \subseteq \mathcal{I}\overline{\mathcal{P}}_u(\Delta)$.

Proof. *Proofs are done similarly to the proofs of Proposition 3.6.* □

Corollary 3.30. *Let* $(\Sigma, \varrho, \complement_n)$ *be a* $\mathcal{G}_n$*-CAS with an ideal* $\mathcal{I}$ *and* $\Delta \subseteq \Sigma$.
(1) $\mathcal{I}\mu_u(\Delta) \leq \mathcal{I}\mu_r(\Delta) \leq \mathcal{I}\mu_i(\Delta)$,
(2) $\mathcal{I}\mu_u(\Delta) \leq \mathcal{I}\mu_l(\Delta) \leq \mathcal{I}\mu_i(\Delta)$.

Remark 3.31. *In Table 5, we calculate* $\mathcal{I}\underline{\mathcal{P}}_j$*-lower approximations,* $\mathcal{I}\overline{\mathcal{P}}_j$*-upper approximations and* $\mathcal{I}\mu_j$*-accuracy measures of all subsets of* Σ *for* $j \in J$ *according to Example 3.4. In this way, we see that the highest* $\mathcal{I}\mu_j$*-accuracy measures can be obtained for* $j = i$.

Table 5: $\mathcal{I}\underline{\mathcal{P}}_j$-lower approximations, $\mathcal{I}\overline{\mathcal{P}}_j$-upper approximations and $\mathcal{I}\mu_j$-accuracy measures of all subsets of Σ for $j \in J$

Δ	$j = r$			$j = l$			$j = u$			$j = i$		
	$\mathcal{I}\underline{\mathcal{P}}_r(\Delta)$	$\mathcal{I}\overline{\mathcal{P}}_r(\Delta)$	$\mathcal{I}\mu_r(\Delta)$	$\mathcal{I}\underline{\mathcal{P}}_l(\Delta)$	$\mathcal{I}\overline{\mathcal{P}}_l(\Delta)$	$\mathcal{I}\mu_l(\Delta)$	$\mathcal{I}\underline{\mathcal{P}}_u(\Delta)$	$\mathcal{I}\overline{\mathcal{P}}_u(\Delta)$	$\mathcal{I}\mu_u(\Delta)$	$\mathcal{I}\underline{\mathcal{P}}_i(\Delta)$	$\mathcal{I}\overline{\mathcal{P}}_i(\Delta)$	$\mathcal{I}\mu_i(\Delta)$
$\{\mho_1\}$	$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1,\mho_3\}$	$\frac{1}{2}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	0	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	0	$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1\}$	1
$\{\mho_2\}$	$\{\mho_2,\mho_4\}$	$\{\mho_2\}$	1	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	0	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	0	$\{\mho_2,\mho_3,\mho_4\}$	$\{\mho_2\}$	1
$\{\mho_3\}$	$\{\mho_4\}$	$\emptyset$	0	$\{\mho_3,\mho_4\}$	$\emptyset$	1	$\{\mho_3,\mho_4\}$	$\emptyset$	1	$\{\mho_3,\mho_4\}$	$\emptyset$	1
$\{\mho_4\}$	$\{\mho_4\}$	$\emptyset$	1	$\{\mho_3,\mho_4\}$	$\emptyset$	1	$\{\mho_3,\mho_4\}$	$\emptyset$	1	$\{\mho_3,\mho_4\}$	$\emptyset$	1
$\{\mho_1,\mho_2\}$	Σ	$\{\mho_1,\mho_2,\mho_3\}$	$\frac{2}{3}$	Σ	$\{\mho_1,\mho_2\}$	1	Σ	$\{\mho_1,\mho_2\}$	1	Σ	$\{\mho_1,\mho_2\}$	1
$\{\mho_1,\mho_3\}$	$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1,\mho_3\}$	1	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{3}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{3}$	$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1\}$	1
$\{\mho_1,\mho_4\}$	$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1,\mho_3\}$	$\frac{2}{3}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{3}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{3}$	$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1\}$	1
$\{\mho_2,\mho_3\}$	$\{\mho_2,\mho_4\}$	$\{\mho_2\}$	$\frac{1}{2}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{3}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{3}$	$\{\mho_2,\mho_3,\mho_4\}$	$\{\mho_2\}$	1
$\{\mho_2,\mho_4\}$	$\{\mho_2,\mho_4\}$	$\{\mho_2\}$	1	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{3}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{3}$	$\{\mho_2,\mho_3,\mho_4\}$	$\{\mho_2\}$	1
$\{\mho_3,\mho_4\}$	$\{\mho_4\}$	$\emptyset$	$\frac{1}{2}$	$\{\mho_3,\mho_4\}$	$\emptyset$	1	$\{\mho_3,\mho_4\}$	$\emptyset$	1	$\{\mho_3,\mho_4\}$	$\emptyset$	1
$\{\mho_1,\mho_2,\mho_3\}$	Σ	$\{\mho_1,\mho_2,\mho_3\}$	1	Σ	$\{\mho_1,\mho_2\}$	1	Σ	$\{\mho_1,\mho_2\}$	1	Σ	$\{\mho_1,\mho_2\}$	1
$\{\mho_1,\mho_2,\mho_4\}$	Σ	$\{\mho_1,\mho_2,\mho_3\}$	$\frac{3}{4}$	Σ	$\{\mho_1,\mho_2\}$	1	Σ	$\{\mho_1,\mho_2\}$	1	Σ	$\{\mho_1,\mho_2\}$	1
$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1,\mho_3\}$	1	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{2}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{2}$	$\{\mho_1,\mho_3,\mho_4\}$	$\{\mho_1\}$	1
$\{\mho_2,\mho_3,\mho_4\}$	$\{\mho_2,\mho_4\}$	$\{\mho_2\}$	$\frac{2}{3}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{2}$	$\{\mho_3,\mho_4\}$	$\{\mho_1,\mho_2\}$	$\frac{1}{2}$	$\{\mho_2,\mho_3,\mho_4\}$	$\{\mho_2\}$	1
Σ	Σ	$\{\mho_1,\mho_2,\mho_3\}$	1	Σ	$\{\mho_1,\mho_2\}$	1	Σ	$\{\mho_1,\mho_2\}$	1	Σ	$\{\mho_1,\mho_2\}$	1

Theorem 3.32. *Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS with an ideal $\mathcal{I}$ and $\Delta \subseteq \Sigma$ For $j \in J$, the following hold:*
(1) $\underline{\mathcal{P}}_j(\Delta) \subseteq \mathcal{I}\underline{\mathcal{P}}_j(\Delta)$,
(2) $\mathcal{I}\overline{\mathcal{P}}_j(\Delta) \subseteq \overline{\mathcal{P}}_j(\Delta)$.

Proof. *Proofs are done similarly to the proofs of Theorem 3.9.* □

Example 3.33. *Let $\Sigma = \{\mho_1, \mho_2, \mho_3, \mho_4\}$, $\varrho = \{(\mho_1, \mho_1), (\mho_1, \mho_3), (\mho_1, \mho_4), (\mho_2, \mho_1), (\mho_2, \mho_3), (\mho_2, \mho_4), (\mho_3, \mho_1), (\mho_3, \mho_2), (\mho_3, \mho_4), (\mho_4, \mho_1), (\mho_4, \mho_4)\}$ be a binary relation and $\mathcal{I} = \{\emptyset, \{\mho_1\}, \{\mho_4\}, \{\mho_1, \mho_4\}\}$ be an ideal on Σ. For $\Delta = \{\mho_1\}$, we obtain $\mathcal{I}\underline{\mathcal{P}}_r(\Delta) = \{\mho_1, \mho_4\}$, $\underline{\mathcal{P}}_r(\Delta) = \emptyset$, $\overline{\mathcal{P}}_r(\Delta) = \{\mho_1, \mho_4\}$, $\mathcal{I}\overline{\mathcal{P}}_r(\Delta) = \emptyset$. Thus $\underline{\mathcal{P}}_r(\Delta) \neq \mathcal{I}\underline{\mathcal{P}}_r(\Delta)$ and $\mathcal{I}\overline{\mathcal{P}}_r(\Delta) \neq \overline{\mathcal{P}}_r(\Delta)$.*

Corollary 3.34. *Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS with an ideal $\mathcal{I}$ and $\Delta \subseteq \Sigma$ For $j \in J$, $\mu_j(\Delta) \leq \mathcal{I}\mu_j(\Delta)$.*

Theorem 3.35. *Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS with an ideal $\mathcal{I}$ and $\Delta \subseteq \Sigma$ For $j \in J$, the following hold:*
(1) $\mathcal{I}\underline{\varrho}_j(\Delta) \subseteq \mathcal{I}\underline{\mathcal{P}}_j(\Delta)$,
(2) $\mathcal{I}\underline{\mathcal{M}}_j(\Delta) \subseteq \mathcal{I}\underline{\mathcal{P}}_j(\Delta)$,
(3) $\mathcal{I}\overline{\mathcal{P}}_j(\Delta) \subseteq \mathcal{I}\overline{\varrho}_j(\Delta)$,
(4) $\mathcal{I}\overline{\mathcal{P}}_j(\Delta) \subseteq \mathcal{I}\overline{\mathcal{M}}_j(\Delta)$.

Proof. *(1) Let $e \in \mathcal{I}\underline{\varrho}_j(\Delta)$. Then $\mathcal{N}_j(e) \setminus \Delta \in \mathcal{I}$. By the Lemma 2.11, $\mathcal{P}_j(e) \setminus \Delta \in \mathcal{I}$. Hence $e \in \mathcal{I}\underline{\mathcal{P}}_j(\Delta)$. So we obtain $\mathcal{I}\underline{\varrho}_j(\Delta) \subseteq \mathcal{I}\underline{\mathcal{P}}_j(\Delta)$.*
(2) The proof is similar to that of (1).
(3) Let $e \in \mathcal{I}\overline{\mathcal{P}}_j(\Delta)$. Then $\mathcal{P}_j(e) \cap \Delta \notin \mathcal{I}$. By the Lemma 2.11, $\mathcal{N}_j(e) \cap \Delta \notin \mathcal{I}$. Hence $e \in \mathcal{I}\overline{\varrho}_j(\Delta)$. So we obtain $\mathcal{I}\overline{\mathcal{P}}_j(\Delta) \subseteq \mathcal{I}\overline{\varrho}_j(\Delta)$.
(4) The proof is similar to that of (3). □

The following example shows that we may not replace by equality relation in Theorem 3.35.

Example 3.36. *Consider the Example 3.4.*
(1) For $\Delta = \{\mho_2, \mho_4\}$, we obtain $\mathcal{I}\underline{\mathcal{P}}_r(\Delta) = \{\mho_2, \mho_4\}$ and $\mathcal{I}\underline{\varrho}_r(\Delta) = \emptyset$. Thus, $\mathcal{I}\underline{\mathcal{P}}_r(\Delta) \neq \mathcal{I}\underline{\varrho}_r(\Delta)$.
(2) For $\Delta = \{\mho_1\}$, we have $\mathcal{I}\overline{\mathcal{P}}_r(\Delta) = \{\mho_1, \mho_3\}$ and $\mathcal{I}\overline{\varrho}_r(\Delta) = \Sigma$. Hence, $\mathcal{I}\overline{\mathcal{P}}_r(\Delta) \neq \mathcal{I}\overline{\varrho}_r(\Delta)$.
(3) For $\Delta = \{\mho_1, \mho_3, \mho_4\}$, we get $\mathcal{I}\underline{\mathcal{P}}_r(\Delta) = \{\mho_1, \mho_3, \mho_4\}$ and $\mathcal{I}\underline{\mathcal{M}}_r(\Delta) = \{\mho_4\}$. From here, $\mathcal{I}\underline{\mathcal{P}}_r(\Delta) \neq \mathcal{I}\underline{\mathcal{M}}_r(\Delta)$.
(4) For $\Delta = \{\mho_2\}$, we obtain $\mathcal{I}\overline{\mathcal{P}}_r(\Delta) = \{\mho_2\}$ and $\mathcal{I}\overline{\mathcal{M}}_r(\Delta) = \{\mho_1, \mho_2, \mho_3\}$. Then, $\mathcal{I}\overline{\mathcal{P}}_r(\Delta) \neq \mathcal{I}\overline{\mathcal{M}}_r(\Delta)$.

Corollary 3.37. *Let $(\Sigma, \varrho, \complement_n)$ be a $\mathcal{G}_n$-CAS with an ideal $\mathcal{I}$ and $\Delta \subseteq \Sigma$ For $j \in J$, $\mathcal{I}\delta_j(\Delta) \leq \mathcal{I}\mu_j(\Delta)$ and $\mathcal{I}\eta_j(\Delta) \leq \mathcal{I}\mu_j(\Delta)$.*

Remark 3.38. *Comparing Table 1,Table 3 and Table 5 shows that $\mathcal{I}\mu_j$-accuracy measures are higher than $\mathcal{I}\delta_j$-accuracy measures and $\mathcal{I}\eta_j$-accuracy measures.*

4. An Application

The main purpose of this section is to present a simple practice example to compare Nawar et al's approach and one of our approaches. We use Walczak's example in Chemistry. Let $\Sigma = \{\mho_1, \mho_2, \mho_3, \mho_4, \mho_5, \mho_6, \mho_7\}$ be seven amino acids represented in terms of five attributes: a_1=PIE, a_2=PIF, a_3=SAC=surface area, a_4=MR=molecular refractivity, a_5=Vol=molecular volume [29].

Table 6: Quantitative attributes of seven amino acids

	a_1	a_2	a_3	a_4	a_5
$\mho_1$	1.85	2.25	-2.70	401.80	5.7550
$\mho_2$	0	0	0	224.9	1.6620
$\mho_3$	0.15	0.13	-0.25	337.20	3.8560
$\mho_4$	0.23	0.31	-0.55	254.20	2.1260
$\mho_5$	0.71	1.22	-1.60	295.10	3.0540
$\mho_6$	0.89	0.96	-1.70	377.80	4.9710
$\mho_7$	0.17	0.26	-0.58	282.9	2.7480

Example 4.1. *Consider Table 6 containing data about 7 amino acids.*

We have five relations described as $\varrho_m = \{(\mho_i, \mho_j) : |\mho_i(a_m) - \mho_j(a_m)| \leq \frac{\sigma_m}{2}, i, j = 1, ...7, m = 1, ..., 5\}$ *where* σ_m *represents the standard of the quantitative attributes* a_m*, m=1...5. and we compute* $\mho_i\varrho = \bigcap \mho_j\varrho_m$*,* $m = 1, .., 5$*. Thus, we can get the r-neighborhoods of all elements using Definition 2.3, as follows:* $\mathcal{N}_r(\mho_1) = \{\mho_1\}$*,* $\mathcal{N}_r(\mho_2) = \{\mho_2\}$*,* $\mathcal{N}_r(\mho_3) = \{\mho_3\}$*,* $\mathcal{N}_r(\mho_4) = \{\mho_4, \mho_7\}$*,* $\mathcal{N}_r(\mho_5) = \{\mho_5\}$*,* $\mathcal{N}_r(\mho_6) = \{\mho_6\}$*,* $\mathcal{N}_r(\mho_7) = \{\mho_4, \mho_7\}$*. Then* $\mathcal{P}_r(\mho_4) = \mathcal{P}_r(\mho_7) = \{\mho_4, \mho_7\}$*,* $\mathcal{P}_r(\mho_1) = \{\mho_1\}$*,* $\mathcal{P}_r(\mho_2) = \{\mho_2\}$*,* $\mathcal{P}_r(\mho_3) = \{\mho_3\}$*,* $\mathcal{P}_r(\mho_5) = \{\mho_5\}$ *and* $\mathcal{P}_r(\mho_6) = \{\mho_6\}$ *. Then taking the ideal* $\mathcal{I} = \{\emptyset, \{\mho_4\}, \{\mho_7\}, \{\mho_4, \mho_7\}\}$*. We get Table 7.*

Table 7: Comparisons between Nawar et al's approach [22] approach and our approach

Δ	Nawar et al's approach [22]			our approach		
	lower app.	upper app	accuracy	lower app.	upper app	accuracy
$\{\mho_4\}$	$\emptyset$	$\{\mho_4, \mho_7\}$	0	$\{\mho_4, \mho_7\}$	$\emptyset$	1
$\{\mho_1, \mho_4\}$	$\{\mho_1\}$	$\{\mho_1, \mho_4, \mho_7\}$	$\frac{1}{3}$	$\{\mho_4, \mho_7\}$	$\{\mho_1\}$	1
$\{\mho_2, \mho_6\}$	$\{\mho_2, \mho_6\}$	$\{\mho_2, \mho_6\}$	1	$\{\mho_2, \mho_6, \mho_4, \mho_7\}$	$\{\mho_4, \mho_7\}$	1
$\{\mho_6, \mho_7\}$	$\{\mho_6\}$	$\{\mho_4, \mho_6, \mho_7\}$	$\frac{1}{3}$	$\{\mho_6, \mho_7\}$	$\{\mho_6\}$	1
$\{\mho_3, \mho_5, \mho_7\}$	$\{\mho_3, \mho_5\}$	$\{\mho_3, \mho_4, \mho_5, \mho_7\}$	$\frac{1}{2}$	$\{\mho_3, \mho_4, \mho_5, \mho_6, \mho_7\}$	$\{\mho_3, \mho_5\}$	1
$\{\mho_1, \mho_2, \mho_3, \mho_4\}$	$\{\mho_1, \mho_2, \mho_3\}$	$\{\mho_1, \mho_2, \mho_3, \mho_4, \mho_7\}$	$\frac{3}{5}$	$\{\mho_1, \mho_2, \mho_3, \mho_4, \mho_7\}$	$\{\mho_1, \mho_2, \mho_3\}$	1
$\{\mho_1, \mho_3, \mho_4, \mho_5, \mho_6\}$	$\{\mho_1, \mho_3, \mho_5, \mho_6\}$	$\{\mho_1, \mho_3, \mho_4, \mho_5, \mho_6, \mho_7\}$	$\frac{2}{3}$	Σ	$\{\mho_1, \mho_3, \mho_5, \mho_6\}$	1

From this practical example, the accuracy measures of our approach are higher than Nawar's approach [22]. Hence, these approaches are very useful in rough set theory.

5. Conclusion

The problem of managing and perceiving knowledge is a crucial issue in the area of information systems. There are many new ways how to manage and perceive knowledge. One of them is the rough set theory. Rough set theory was investigated by Pawlak as a mathematical approach that deals with uncertainty and the vagueness of imprecise data. It has a wide variety of executions in modern life fields such as biology, chemistry, engineering, etc. The central idea in this theory is approximation operators, which are characterized by equivalence classes. However, the equivalence relations are limited to theoretical and practical viewpoints. Therefore, many researchers introduced several types of generalization of Pawlak's rough set theory using topological concepts and they are constructed based on the concept of coverings and the ideals. In this work, we defined new types of lower and upper approximations using j-neighborhoods, complementary j-neighborhoods, and j-adhesions via ideals in generalized covering approximation space. Then, we gave some of their basic characteristics and showed that the best accuracy measures are obtained in case $j = i$. Besides, we compared these approximations both with each other and with the two previously proposed approximations. We supported all the results we obtained with various examples and tables. In

future studies,
(1) Search how the suggested methods can be used in other application areas.
(2) Study new approaches can be obtained by using different types of neighborhoods in generalized covering approximation spaces.
(3) Investigate these concepts and results presented here to soft covering rough sets and fuzzy covering rough sets by using soft set, fuzzy set, and ideals.

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