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On density topology using ideals in the space of reals

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Abstract. In this paper we have introduced the notion of $\mathcal{I}$-density topology in the space of reals introducing the notions of upper $\mathcal{I}$-density and lower $\mathcal{I}$-density where $\mathcal{I}$ is an ideal of subsets of the set of natural numbers. We have further studied certain separation axioms of this topology.

1. Introduction and Preliminaries

The idea of convergence of real sequences was generalized to the notion of statistical convergence in [9, 27]. For $K \subset \mathbb{N}$, the set of natural numbers and $n \in \mathbb{N}$ let $K_n = \{k \in K : k \leq n\}$. The natural density of the set K is defined by $d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$, provided the limit exists, where $|K_n|$ stands for the cardinality of the set K_n. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to x_0 if for each $\epsilon > 0$ the set $K(\epsilon) = \{k \in \mathbb{N} : |x_k - x_0| \geq \epsilon\}$ has natural density zero.

After this pioneering work, in the year 2000 the theory of statistical convergence of real sequences were generalized to the idea of $\mathcal{I}$-convergence of real sequences by P. Kostyrko et al. [18] using the notion of ideal $\mathcal{I}$ of subsets of $\mathbb{N}$, the set of natural numbers. A subcollection $\mathcal{I} \subset 2^{\mathbb{N}}$ is called an ideal if $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and $A \in \mathcal{I}, B \subset A$ imply $B \in \mathcal{I}$. $\mathcal{I}$ is called nontrivial ideal if $\mathcal{I} \neq \{\phi\}$ and $\mathbb{N} \notin \mathcal{I}$. $\mathcal{I}$ is called admissible if it contains all the singletons. It is easy to verify that the family $\mathcal{I}_d = \{A \subset \mathbb{N} : d(A) = 0\}$ forms a nontrivial admissible ideal of subsets of $\mathbb{N}$. If $\mathcal{I}$ is a nontrivial ideal then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} : \mathbb{N} \setminus M \in \mathcal{I}\}$ is a filter on $\mathbb{N}$ and it is called the filter associated with the ideal $\mathcal{I}$ of $\mathbb{N}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-convergent [18] to x_0 if the set $K(\epsilon) = \{k \in \mathbb{N} : |x_k - x_0| \geq \epsilon\}$ belongs to $\mathcal{I}$ for each $\epsilon > 0$. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\mathcal{I}$-bounded if there is a real number $M > 0$ such that $\{k \in \mathbb{N} : |x_k| > M\} \in \mathcal{I}$. Further many works were carried out in this direction by many authors [2, 3, 22].

Demirci [8] introduced the notion of $\mathcal{I}$-limit superior and inferior of real sequence and proved several basic properties.

Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$ and $x = \{x_n\}_{n\in\mathbb{N}}$ be a real sequence. Let, $B_x = \{b \in \mathbb{R} : \{k : x_k > b\} \notin \mathcal{I}\}$ and $A_x = \{a \in \mathbb{R} : \{k : x_k < a\} \notin \mathcal{I}\}$. Then the $\mathcal{I}$-limit superior of x is given by,

$$\mathcal{I} - \limsup x = \begin{cases} \sup B_x & \text{if } B_x \neq \phi \\ -\infty & \text{if } B_x = \phi \end{cases}$$

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and the $\mathcal{I}$-limit inferior of x is given by,

$$\mathcal{I} - \liminf x = \begin{cases} \inf A_x & \text{if } A_x \neq \phi \\ \infty & \text{if } A_x = \phi \end{cases}$$

Further Lahiri and Das [21] carried out more works in this direction. Throughout the paper the ideal $\mathcal{I}$ will always stand for a nontrivial admissible ideal of subsets of $\mathbb{N}$.

We shall use the notation $m^\star$ for the outer Lebesgue measure, $m_\star$ for the inner Lebesgue measure, $\mathcal{L}$ for the σ-algebra of Lebesgue measurable sets and m for the Lebesgue measure. Throughout $\mathbb{R}$ stands for the set of all real numbers. The symbol $\mathfrak{T}_U$ stands for the natural topology on $\mathbb{R}$. Wherever we write $\mathbb{R}$ it means that $\mathbb{R}$ is equipped with natural topology unless otherwise stated. By 'Euclidean F_σ and Euclidean G_δ set'we mean F_σ and G_δ set in $\mathbb{R}$ equipped with natural topology. The symmetric difference of two sets A and B is $(A \setminus B) \cup (B \setminus A)$ and it is denoted by $A \triangle B$. For $x \in \mathbb{R}$ and $A \subset \mathbb{R}$ we define $dist(x, A) = \inf\{|x - a| : a \in A\}$. By 'a sequence of closed intervals $\{J_n\}_{n\in\mathbb{N}}$ about a point p'we mean $p \in \bigcap_{n\in\mathbb{N}} J_n$.

The idea of density functions and the corresponding density topology [4, 13, 17, 24, 25, 32] were studied in several spaces like the space of real numbers [26], Euclidean n-space [29], metric spaces [20], abstract measure spaces [23] etc. Goffman et al. [11, 12] and H.E. White [30] studied further on some properties of density topology on the space of real numbers.

For, $E \in \mathcal{L}$ and $x \in \mathbb{R}$ the upper density of E at the point x denoted by $d^-(x, E)$ and the lower density of E at the point x denoted by $d_-(x, E)$ are defined in [30] as follows:

$$d^-(x, E) = \lim_{n\to\infty} \left(\sup \left\{ \frac{m(E \cap I)}{m(I)} : I \text{ is a closed interval}, x \in I, 0 < m(I) < \frac{1}{n} \right\} \right)$$

$$d_-(x, E) = \lim_{n\to\infty} \left(\inf \left\{ \frac{m(E \cap I)}{m(I)} : I \text{ is a closed interval}, x \in I, 0 < m(I) < \frac{1}{n} \right\} \right)$$

If $d_-(x, E) = d^-(x, E) = \gamma$ we say E has density γ at the point x and denote γ by $d(x, E)$. Moreover $x \in \mathbb{R}$ is a density point of E if and only if $d(x, E) = 1$. Let us take the family

$$\mathfrak{T}_d = \{E \in \mathcal{L} : d(x, E) = 1 \text{ for all } x \in E\}$$

Then $\mathfrak{T}_d$ is ordinary density topology on $\mathbb{R}$ [12] and it is finer than the usual topology $\mathfrak{T}_U$. Any member of $\mathfrak{T}_d$ is called a d-open set.

The idea of metric density was studied by Martin [23] in a totally finite measure space as follows. Let $(X, \mathcal{S}, m)$ be a totally finite measure space in which $m(X) = 1$ and m is complete. For a subset E of X the outer measure $m^\star(E)$ of E is defined to be $m^\star(E) = \inf\{m(F) : E \subset F \in \mathcal{S}\}$. Let $\mathscr{K}$ be a collection of sequences $\{K_n\}$ of sets from $\mathcal{S}$ such that for each $p \in X$ there exists at least one sequence $\{K_n\} \in \mathscr{K}$ satisfying (i) $p \in K_n$ for each n and (ii) $m(K_n) \to 0$ as $n \to \infty$. Any sequence $\{K_n\} \in \mathscr{K}$ satisfying condition (i) and (ii) is said to be convergent to p. Let $\mathscr{K}(p)$ denote the collection of sequences in $\mathscr{K}$ which converge to p. Then, for $E \subset X$ and for any point p in X the upper outer density of E at p denoted by $D^{-\star}(E, p)$ and the lower outer density of E at p denoted by $D_-^\star(E, p)$ are defined by equations

$$D^{-\star}(E, p) = \sup\{\limsup \frac{m^\star(E \cap K_n)}{m(K_n)} : \{K_n\}_{n\in\mathbb{N}} \in \mathscr{K}(p)\}$$

and

$$D_-^\star(E, p) = \inf\{\liminf \frac{m^\star(E \cap K_n)}{m(K_n)} : \{K_n\}_{n\in\mathbb{N}} \in \mathscr{K}(p)\}.$$

When $D^{-\star}(E, p) = D_{-\star}(E, p)$, we say that the outer density of E exists at p and it is denoted by $D^\star(E, p)$. If E is measurable, we omit the word 'outer'and call it respectively the upper and lower density of E at p and we denote these by $D^-(E, p)$ and $D_-(E, p)$. If $D^-(E, p) = D_-(E, p)$ we say that the density of E exists at p and denote the common value by $D(E, p)$.

In the recent years the notion of classical Lebesgue density point were generalised by weakening the assumptions on the sequences of intervals and consequently several notions like $\langle s\rangle$-density point by M. Filipczak and J. Hejduk [10], $\mathcal{J}$-density point by J. Hejduk and R. Wiertelak [14], $\mathcal{S}$-density point by F. Strobin and R. Wiertelak [28] were obtained. Significant works on density topology are also seen in [5, 6, 31, 33–35].

In this paper we have tried to generalize the classical Lebesgue density point using the notion of ideal $\mathcal{I}$ of subsets of naturals. We have given the notion of $\mathcal{I}$-density in the space of reals introducing the notions of upper $\mathcal{I}$-density and lower $\mathcal{I}$-density. In Section 3 we have proved Lebesgue $\mathcal{I}$-density Theorem and in Section 4 we have given $\mathcal{I}$-density topology on the real line. We have shown that $\mathcal{I}$-density topology is finer than the density topology on the real line. We have also studied the idea of $\mathcal{I}$-approximate continuity and it is proved that $\mathcal{I}$-approximately continuous functions are indeed continuous if the real number space is endowed with $\mathcal{I}$-density topology. The existence of bounded $\mathcal{I}$-approximately continuous functions has been given using Lusin-Menchoff condition for $\mathcal{I}$-density. In the last section we have proved that $\mathcal{I}$-density topology is completely regular.

2. $\mathcal{I}$-density

Definition 2.1. *For $E \in \mathcal{L}$, $p \in \mathbb{R}$ and $n \in \mathbb{N}$ the upper $\mathcal{I}$-density of E at the point p denoted by $\mathcal{I} - d^{-}(p, E)$ and the lower $\mathcal{I}$-density of E at the point p denoted by $\mathcal{I} - d_{-}(p, E)$ are defined as follows: Suppose $\{J_n\}_{n\in\mathbb{N}}$ be a sequence of closed intervals about p such that*

$$\mathscr{S}(J_n) = \{n \in \mathbb{N} : 0 < m(J_n) < \tfrac{1}{n}\} \in \mathcal{F}(\mathcal{I})$$

For any such $\{J_n\}_{n\in\mathbb{N}}$ we take

$$x_n = \frac{m(J_n \cap E)}{m(J_n)} \text{ for all } n \in \mathbb{N}.$$

Then $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of non-negative real numbers. Now, if

$$B_{x_k} = \{b \in \mathbb{R} : \{k : x_k > b\} \notin \mathcal{I}\}$$

and

$$A_{x_k} = \{a \in \mathbb{R} : \{k : x_k < a\} \notin \mathcal{I}\}$$

we define,

$$\begin{aligned}\mathcal{I} - d^{-}(p, E) &= \sup\{\sup B_{x_n} : \{J_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I})\}\\ &= \sup\{\mathcal{I} - \limsup\ x_n : \{J_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I})\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{I} - d_{-}(p, E) &= \inf\{\inf A_{x_n} : \{J_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I})\}\\ &= \inf\{\mathcal{I} - \liminf\ x_n : \{J_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I})\}.\end{aligned}$$

In the above two expressions supremum and infimum are taken over the class of sequences $\{J_n\}_{n\in\mathbb{N}}$ satisfying the condition that $\mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I})$ and it is to be understood that $\{J_n\}_{n\in\mathbb{N}}$'s are closed intervals about the point p. Now, if $\mathcal{I} - d_{-}(p, E) = \mathcal{I} - d^{-}(p, E)$ then we denote the common value by $\mathcal{I} - d(p, E)$ which we call as $\mathcal{I}$-density of E at the point p.

A point $p_0 \in \mathbb{R}$ is called an *$\mathcal{I}$-density point* of $E \in \mathcal{L}$ if $\mathcal{I} - d(p_0, E) = 1$.

If a point $p_0 \in \mathbb{R}$ is an $\mathcal{I}$-density point of the set $\mathbb{R} \setminus E$, then p_0 is called an *$\mathcal{I}$-dispersion point* of E.

Remark 2.2. *The notion of $\mathcal{I}$-density point is more general than the notion of density point as the collection of intervals about the point p considered in case of $\mathcal{I}$-density is larger than that considered in case of classical density which is illustrated in the following example.*

Example 2.3. *Let us consider the ideal $\mathcal{I}_d$ of subsets of $\mathbb{N}$ where $\mathcal{I}_d$ is the ideal containing all those subsets of $\mathbb{N}$ whose natural density is zero and $\mathcal{I}_{fin}$, the ideal containing all finite subsets of $\mathbb{N}$. Now, for any point $x \in \mathbb{R}$ consider the following collections of sequences of intervals:*

$$\mathcal{J}_x = \{\{J_n\}_{n\in\mathbb{N}} : \{J_n\} \text{is a sequence of closed intervals about } x \text{ such that } \mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I}_{fin})\} \text{ and}$$

$$\mathcal{G}_x = \{\{J_n\}_{n\in\mathbb{N}} : \{J_n\} \text{is a sequence of closed intervals about } x \text{ such that } \mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I}_d)\}$$

We claim that $\mathcal{J}_x \subsetneq \mathcal{G}_x$. Since any finite subset of $\mathbb{N}$ has natural density zero so $\mathcal{I}_{fin} \subset \mathcal{I}_d$. Clearly, $\{J_n\}_{n\in\mathbb{N}} \in \mathcal{J}_x$ implies $\mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I}_{fin})$. So, $\mathbb{N} \setminus \mathscr{S}(J_n) \in \mathcal{I}_{fin}$ which implies that $\mathbb{N} \setminus \mathscr{S}(J_n) \in \mathcal{I}_d$. Thus, $\mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I}_d)$. Hence, $\{J_n\}_{n\in\mathbb{N}} \in \mathcal{G}_x$. So, $\mathcal{J}_x \subseteq \mathcal{G}_x$.
Now in particular let us take the following sequence $\{K_n\}_{n\in\mathbb{N}}$ of closed intervals about a point x.

$$K_n = \begin{cases} \left[x - \frac{1}{2n+1}, x + \frac{1}{2n+1}\right] & \text{for } n \neq m^2 \text{ where } m \in \mathbb{N} \\ [x-n, x+n] & \text{for } n = m^2 \text{ where } m \in \mathbb{N} \end{cases}$$

We observe that for $n \neq m^2$, $m(K_n) = \frac{2}{2n+1} < \frac{1}{n}$ and for $n = m^2$, $m(K_n) = 2n \nless \frac{1}{n}$. Therefore, $\mathscr{S}(K_n) = \{n \in \mathbb{N} : 0 < m(K_n) < \frac{1}{n}\} = \{n : n \neq m^2, \text{for some } m \in \mathbb{N}\} \in \mathcal{F}(\mathcal{I}_d)$. But since $\mathbb{N} \setminus \mathscr{S}(K_n) = \{n : n = m^2, \text{where } m \in \mathbb{N}\}$ is not a finite set so it does not belong to $\mathcal{I}_{fin}$. Therefore, $\mathcal{J}_x \subsetneq \mathcal{G}_x$.

Let us take the set E to be the open interval $(-1, 1)$ and the point x to be 0. Let $\{K_n\}_{n\in\mathbb{N}} \in \mathcal{G}_0 \setminus \mathcal{J}_0$ be taken as above. Now if $x_n = \frac{m(K_n\cap E)}{m(K_n)}$ then

$$x_n = \begin{cases} 1 & \text{if } n \neq m^2 \text{ where } m \in \mathbb{N} \\ \frac{1}{m^2} & \text{if } n = m^2 \text{ where } m \in \mathbb{N} \end{cases}$$

Now let us calculate lim sup *and* lim inf *of the sequence $\{x_n\}$.*

$$\limsup x_n = \inf_n \sup_{k\geq n} x_k = 1 \text{ and } \liminf x_n = \sup_n \inf_{k\geq n} x_k = 0.$$

Consequently, $\lim_n x_n$ does not exist. Next we will show that 0 is $\mathcal{I}_d$-density point of the set E.
Given any sequence of closed intervals $\{J_n\}_{n\in\mathbb{N}}$ about the point 0 such that $\mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I}_d)$ we have $\{n \in \mathbb{N} : J_n \subset E\} \in \mathcal{F}(\mathcal{I}_d)$. For if $\mathscr{S}(J_n) = \{k_1, k_2, \cdots, k_n, \cdots\}$ (say). Then there exists $n_0 \in \mathbb{N}$ such that for $k_n > k_{n_0}$, $J_{k_n} \subset E$. Thus, $\{n : J_n \subset E\} \supset \mathscr{S}(J_n) \setminus \{k_1, k_2, \cdots, k_{n_0}\}$. Since $\mathbb{N} \setminus \{k_1, k_2, \cdots, k_{n_0}\} \in \mathcal{F}(\mathcal{I}_d)$ so

$$\mathscr{S}(J_n) \setminus \{k_1, k_2, \cdots, k_{n_0}\} = \mathscr{S}(J_n) \cap (\mathbb{N} \setminus \{k_1, k_2, \cdots, k_{n_0}\}) \in \mathcal{F}(\mathcal{I}_d).$$

Now if, $J_n \subset E$ then $r_n = \frac{m(J_n\cap E)}{m(J_n)} = \frac{m(J_n)}{m(J_n)} = 1$. Thus, $\{n : r_n = 1\} \supset \{n : J_n \subset E\}$. Therefore, $\{n : r_n = 1\} \in \mathcal{F}(\mathcal{I}_d)$. Therefore, $B_{r_n} = (-\infty, 1)$ and $A_{r_n} = (1, \infty)$ and so, $\mathcal{I}_d - \limsup r_n = \sup B_{r_n} = 1$ and $\mathcal{I}_d - \liminf r_n = \inf A_{r_n} = 1$. This is true for all $\{J_n\}_{n\in\mathbb{N}} \in \mathcal{G}_0$. Hence,

$$\mathcal{I}_d - d^-(0, E) = \sup\{\sup B_{r_n} : \{J_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I}_d)\} = 1$$

and

$$\mathcal{I}_d - d_-(0, E) = \inf\{\inf A_{r_n} : \{J_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I}_d)\} = 1.$$

Hence $\mathcal{I}_d - d(0, E)$ exists and equals to 1. So, 0 is an $\mathcal{I}_d$-density point of the set E.

Note 2.4. *It is evident that for any sequence of intervals* $\{J_n\}_{n\in\mathbb{N}}$ *such that* $\mathscr{S}(J_n) = \{n \in \mathbb{N} : 0 < m(J_n) < \frac{1}{n}\} \in \mathcal{F}(\mathcal{I}_{fin})$ *we have* $m(J_n) \to 0$ *as* $n \to \infty$. *For, let* $\mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I}_{fin})$. *This implies* $\mathbb{N} \setminus \mathscr{S}(J_n)$ *is a finite set say* $\{n_1, n_2, \cdots, n_k\}$. *Take* $N_1 = max\{n_1, n_2, \cdots, n_k\}$. *Then* $n \in \mathscr{S}(J_n)$ *for every* $n > N_1$. *Let* $\epsilon > 0$ *be arbitrary. Then there exists* $N_2 \in \mathbb{N}$ *such that* $\frac{1}{N_2} < \epsilon$. *If we choose* $N_3 = max\{N_1, N_2\}$ *then for every* $n > N_3$ *we have* $n \in \mathscr{S}(J_n)$ *and* $\frac{1}{n} < \frac{1}{N_2}$. *So,* $m(J_n) < \frac{1}{n} < \frac{1}{N_3} \le \frac{1}{N_2} < \epsilon$ *which implies* $m(J_n) \to 0$ *as* $n \to \infty$. *Also note that if ideal* $\mathcal{I} = \mathcal{I}_{fin}$ *then for any bounded real sequence* $\{x_n\}$,

$$\mathcal{I}_{fin} - \limsup x_n = \limsup x_n \quad \textit{and} \quad \mathcal{I}_{fin} - \liminf x_n = \liminf x_n$$

So when $\mathcal{I} = \mathcal{I}_{fin}$ *the definition of upper and lower density points take the forms*

$$\begin{aligned}\mathcal{I}_{fin} - d^-(p,E) &= \sup\{\mathcal{I}_{fin} - \limsup\ x_n : \{J_n\}_{n\in\mathbb{N}} \textit{ such that } \mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I}_{fin})\}\\ &= \sup\{\limsup\ x_n : \{J_n\}_{n\in\mathbb{N}} \textit{ such that } \mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I}_{fin})\}\\ &= \sup\{\limsup\ x_n : \{J_n\}_{n\in\mathbb{N}} \in \mathscr{K}(p)\}\\ &= D^-(E,p), \quad \textit{the upper metric density}\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{I}_{fin} - d_-(p,E) &= \inf\{\mathcal{I}_{fin} - \liminf\ x_n : \{J_n\}_{n\in\mathbb{N}} \textit{ such that } \mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I}_{fin})\}\\ &= \inf\{\liminf\ x_n : \{J_n\}_{n\in\mathbb{N}} \textit{ such that } \mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I}_{fin})\}\\ &= \inf\{\liminf\ x_n : \{J_n\}_{n\in\mathbb{N}} \in \mathscr{K}(p)\}\\ &= D_-(E,p), \quad \textit{the lower metric density}\end{aligned}$$

Thus in particular if $\mathcal{I} = \mathcal{I}_{fin}$ *our definition of* $\mathcal{I}$*-density coincides with definition of metric density as introduced by [23] and* $\mathcal{I}_{fin} - d(p,E) = D(E,p)$. *Also it was mentioned in [23] that for the family of all regular sequences of intervals converging to x we get ordinary density.*

The following theorem was given by K. Demirci [8].

Theorem 2.5. *For any real sequence* x, $\mathcal{I} - \liminf x \le \mathcal{I} - \limsup x$.

Here we are proving some important results which will be needed later in our discussion.

Theorem 2.6. *For any Lebesgue measurable set* $A \subset \mathbb{R}$ *and any point* $p \in \mathbb{R}$,

$$\mathcal{I} - d_-(p,A) \le \mathcal{I} - d^-(p,A).$$

Proof. Let $\{I_k\}_{k\in\mathbb{N}}$ be any sequence of closed intervals about the point p such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$. Let us take the real sequence $x_n = \frac{m(A\cap I_n)}{m(I_n)}$. Then clearly, $\mathcal{I} - \liminf x_n \le \mathcal{I} - \limsup x_n$. So,

$$\begin{aligned}\mathcal{I} - d_-(p,A) &= \inf\{\mathcal{I} - \liminf x_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\}\\ &\le \inf\{\mathcal{I} - \limsup x_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\}\\ &\le \sup\{\mathcal{I} - \limsup x_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\}\\ &= \mathcal{I} - d^-(p,A).\end{aligned}$$

□

The following theorem is useful to prove our next results.

Theorem 2.7 ([21]). *If* $x = \{x_n\}_{n\in\mathbb{N}}$ *and* $y = \{y_n\}_{n\in\mathbb{N}}$ *are two* $\mathcal{I}$*-bounded real number sequences, then*

(i) $\mathcal{I} - \limsup(x + y) \le \mathcal{I} - \limsup x + \mathcal{I} - \limsup y$

(ii) $\mathcal{I}-\liminf(x+y) \geq \mathcal{I}-\liminf x+\mathcal{I}-\liminf y$

Proposition 2.8. *Given an $\mathcal{I}$-bounded real sequence $\{x_n\}_{n\in\mathbb{N}}$ and a real number c,*

(i) $\mathcal{I}-\liminf(c+x_n)=c+\mathcal{I}-\liminf x_n$

(ii) $\mathcal{I}-\limsup(c+x_n)=c+\mathcal{I}-\limsup x_n$

Proof. (i) It is obvious that $\mathcal{I}-\liminf(c+x_n) \geq c+\mathcal{I}-\liminf x_n$. Now we are to show that $\mathcal{I}-\liminf(c+x_n) \leq c+\mathcal{I}-\liminf x_n$. Let $y_n=c+x_n$. Then $\mathcal{I}-\liminf x_n=\mathcal{I}-\liminf(y_n-c) \geq \mathcal{I}-\liminf y_n-c$. Therefore, $\mathcal{I}-\liminf y_n \leq c+\mathcal{I}-\liminf x_n$. So, we can conclude that $\mathcal{I}-\liminf(c+x_n)=c+\mathcal{I}-\liminf x_n$. The proof of (ii) is analogous. □

Proposition 2.9. *For any real sequence $x=\{x_n\}_{n\in\mathbb{N}}$,*

(i) $\mathcal{I}-\limsup(-x)=-(\mathcal{I}-\liminf x)$

(ii) $\mathcal{I}-\liminf(-x)=-(\mathcal{I}-\limsup x)$

Proof. (i) Let us take $B_x=\{b\in\mathbb{R}:\{k: x_k>b\}\notin\mathcal{I}\}$ and $A_x=\{a\in\mathbb{R}:\{k: x_k<a\}\notin\mathcal{I}\}$. Then clearly, $B_{(-x)}=-A_x$.

Therefore, $\mathcal{I}-\limsup(-x)=\sup B_{(-x)}=\sup(-A_x)=-\inf A_x=-\mathcal{I}-\liminf(x)$. In a similar manner we can prove (ii). □

Lemma 2.10. *For any two disjoint Lebesgue measurable subsets A and B of $\mathbb{R}$ and any point $p\in\mathbb{R}$ if $\mathcal{I}-d(p,A)$ and $\mathcal{I}-d(p,B)$ exist, then $\mathcal{I}-d(p,A\cup B)$ exists and $\mathcal{I}-d(p,A\cup B)=\mathcal{I}-d(p,A)+\mathcal{I}-d(p,B)$.*

Proof. Let $\{I_k\}_{k\in\mathbb{N}}$ be any sequence of closed intervals about the point p such that $\mathscr{S}(I_k)\in\mathcal{F}(\mathcal{I})$. Now let us take the real sequences $\{x_n\}_{n\in\mathbb{N}},\{y_n\}_{n\in\mathbb{N}},\{z_n\}_{n\in\mathbb{N}}$ defined as $x_n=\frac{m(A\cap I_n)}{m(I_n)}$, $y_n=\frac{m(B\cap I_n)}{m(I_n)}$ and $z_n=\frac{m((A\cup B)\cap I_n)}{m(I_n)}$. Then each of $\{x_n\}_{n\in\mathbb{N}},\{y_n\}_{n\in\mathbb{N}},\{z_n\}_{n\in\mathbb{N}}$ is bounded and hence $\mathcal{I}$-bounded. Since A and B are disjoint sets, we have for any $n\in\mathscr{S}(I_k)$, $m((A\cup B)\cap I_n)=m(A\cap I_n)+m(B\cap I_n)$. So, $z_n=x_n+y_n$ for $n\in\mathscr{S}(I_k)$. Hence,

$$\begin{aligned}
\mathcal{I}-d^-(p,A\cup B) &= \sup\{\mathcal{I}-\limsup z_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n)\in\mathcal{F}(\mathcal{I})\}\\
&= \sup\{\mathcal{I}-\limsup(x_n+y_n) : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n)\in\mathcal{F}(\mathcal{I})\}\\
&\leq \sup\{\mathcal{I}-\limsup x_n+\mathcal{I}-\limsup y_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n)\in\mathcal{F}(\mathcal{I})\}\\
&\leq \sup\{\mathcal{I}-\limsup x_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n)\in\mathcal{F}(\mathcal{I})\}\\
&\quad\quad + \sup\{\mathcal{I}-\limsup y_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n)\in\mathcal{F}(\mathcal{I})\}\\
&= \mathcal{I}-d^-(p,A)+\mathcal{I}-d^-(p,B)\\
&= \mathcal{I}-d_-(p,A)+\mathcal{I}-d_-(p,B) \qquad (1)\\
&= \inf\{\mathcal{I}-\liminf x_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n)\in\mathcal{F}(\mathcal{I})\}\\
&\quad\quad + \inf\{\mathcal{I}-\liminf y_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n)\in\mathcal{F}(\mathcal{I})\}\\
&\leq \inf\{\mathcal{I}-\liminf x_n+\mathcal{I}-\liminf y_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n)\in\mathcal{F}(\mathcal{I})\}\\
&\leq \inf\{\mathcal{I}-\liminf(x_n+y_n) : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n)\in\mathcal{F}(\mathcal{I})\}\\
&= \inf\{\mathcal{I}-\liminf z_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n)\in\mathcal{F}(\mathcal{I})\}\\
&= \mathcal{I}-d_-(p,A\cup B).
\end{aligned}$$

Also, by Theorem 2.6, $\mathcal{I}-d^-(p,A\cup B)\geq\mathcal{I}-d_-(p,A\cup B)$. Therefore, $\mathcal{I}-d(p,A\cup B)$ exists and $\mathcal{I}-d^-(p,A\cup B)=\mathcal{I}-d_-(p,A\cup B)=\mathcal{I}-d(p,A\cup B)$. From (1) it is clear that $\mathcal{I}-d(p,A\cup B)\leq\mathcal{I}-d(p,A)+\mathcal{I}-d(p,B)\leq\mathcal{I}-d(p,A\cup B)$. Hence, $\mathcal{I}-d(p,A\cup B)=\mathcal{I}-d(p,A)+\mathcal{I}-d(p,B)$. □

Lemma 2.11. *For any two Lebesgue measurable subsets A and B of $\mathbb{R}$ and any point $p\in\mathbb{R}$ if $\mathcal{I}-d(p,A)$ and $\mathcal{I}-d(p,B)$ exist and $A\subset B$, then $\mathcal{I}-d(p,B\setminus A)$ exists and $\mathcal{I}-d(p,B\setminus A)=\mathcal{I}-d(p,B)-\mathcal{I}-d(p,A)$.*

Proof. Since A and B are measurable sets, for any sequence of closed intervals $\{I_k\}_{k\in\mathbb{N}}$ about the point p such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$ we have for $n \in \mathscr{S}(I_k)$, $m((B \setminus A) \cap I_n) = m(B \cap I_n) - m(A \cap I_n)$. Consider x_n and y_n as in previous lemma. Take $p_n = \frac{m((B\setminus A)\cap I_n)}{m(I_n)}$. So, $p_n = y_n - x_n$. It is easy to see that $\{p_n\}_{n\in\mathbb{N}}$ is bounded and hence an $\mathcal{I}$-bounded sequence. So,

$$\begin{aligned}
\mathcal{I} - d_-(p, B \setminus A) &= \inf\{\mathcal{I} - \liminf p_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= \inf\{\mathcal{I} - \liminf(y_n - x_n) : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&\geq \inf\{\mathcal{I} - \liminf y_n - \mathcal{I} - \limsup x_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&\geq \inf\{\mathcal{I} - \liminf y_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&\quad - \sup\{\mathcal{I} - \limsup x_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= \mathcal{I} - d_-(p, B) - \mathcal{I} - d^-(p, A) \\
&= \mathcal{I} - d^-(p, B) - \mathcal{I} - d_-(p, A) \\
&= \sup\{\mathcal{I} - \limsup y_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&\quad - \inf\{\mathcal{I} - \liminf x_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&\geq \sup\{\mathcal{I} - \limsup y_n - \mathcal{I} - \liminf x_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&\geq \sup\{\mathcal{I} - \limsup(y_n - x_n) : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= \sup\{\mathcal{I} - \limsup(p_n) : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= \mathcal{I} - d^-(p, B \setminus A).
\end{aligned}$$

Therefore, $\mathcal{I} - d(p, B \setminus A)$ exists and $\mathcal{I} - d^-(p, B \setminus A) = \mathcal{I} - d_-(p, B \setminus A) = \mathcal{I} - d(p, B \setminus A)$. So, $\mathcal{I} - d(p, B \setminus A) \geq \mathcal{I} - d(p, B) - \mathcal{I} - d(p, A) \geq \mathcal{I} - d(p, B \setminus A)$. Hence, $\mathcal{I} - d(p, B \setminus A) = \mathcal{I} - d(p, B) - \mathcal{I} - d(p, A)$. □

Theorem 2.12. *For any measurable set H, $\mathcal{I}$-density of H at a point $p \in \mathbb{R}$ exists if and only if $\mathcal{I} - d^-(p, H) + \mathcal{I} - d^-(p, H^c) = 1$.*

Proof. Let $\{I_k\}_{k\in\mathbb{N}}$ be any sequence of closed intervals about the point p such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$ and let H be a measurable subset. Let $x_n = \frac{m(I_n\cap H)}{m(I_n)}$ and $y_n = \frac{m(I_n\cap H^c)}{m(I_n)}$. Then $x_n + y_n = 1\ \forall n \in \mathscr{S}(I_k)$. Both $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are $\mathcal{I}$-bounded sequences.
Necessary part: Let $\mathcal{I}$-density of a measurable set H at the point p exists. Now

$$\begin{aligned}
\mathcal{I} - d^-(p, H) &= \mathcal{I} - d_-(p, H) \\
&= \inf\{\mathcal{I} - \liminf x_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= \inf\{\mathcal{I} - \liminf(1 - y_n) : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= \inf\{1 - \mathcal{I} - \limsup y_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= 1 - \sup\{\mathcal{I} - \limsup y_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= 1 - \mathcal{I} - d^-(p, H^c).
\end{aligned}$$

Sufficient part: Let $\mathcal{I} - d^-(p, H) + \mathcal{I} - d^-(p, H^c) = 1$. Then,

$$\begin{aligned}
\mathcal{I} - d^-(p, H) &= 1 - \mathcal{I} - d^-(p, H^c) \\
&= 1 - \sup\{\mathcal{I} - \limsup y_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= \inf\{1 - \mathcal{I} - \limsup y_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= \inf\{1 + \mathcal{I} - \liminf(-y_n) : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= \inf\{\mathcal{I} - \liminf(1 - y_n) : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= \inf\{\mathcal{I} - \liminf x_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\
&= \mathcal{I} - d_-(p, H).
\end{aligned}$$

Hence, $\mathcal{I}$-density of H at p exists. □

3. Lebesgue $\mathcal{I}-$density theorem

Let $H \subset \mathbb{R}$ be a measurable set. Let us denote the set of points of $\mathbb{R}$ at which H has $\mathcal{I}$-density 1 by $\Theta_{\mathcal{I}}(H)$.

Theorem 3.1. *For any measurable set $H \subset \mathbb{R}$, $\Theta_{\mathcal{I}}(H) \setminus H \subset H^c \setminus \Theta_{\mathcal{I}}(H^c)$.*

Proof. It is obvious that $\Theta_{\mathcal{I}}(H) \setminus H \subset H^c$. Now we show if $x \in \Theta_{\mathcal{I}}(H)$, then $x \notin \Theta_{\mathcal{I}}(H^c)$. Suppose if possible, $x \in \Theta_{\mathcal{I}}(H) \cap \Theta_{\mathcal{I}}(H^c)$. Then $\mathcal{I} - d(x, H) = 1$ and $\mathcal{I} - d(x, H^c) = 1$. But this leads to a contradiction to Theorem 2.12. Therefore, $\Theta_{\mathcal{I}}(H) \cap \Theta_{\mathcal{I}}(H^c)$ is an empty set. Thus, $\Theta_{\mathcal{I}}(H) \setminus H \subset H^c \setminus \Theta_{\mathcal{I}}(H^c)$. □

Here we prove an analogue of classical Lebesgue density theorem by the idea presented in [25] (Theorem 3.20).

Theorem 3.2. *For any measurable set $H \subset \mathbb{R}$, $m(H \triangle \Theta_{\mathcal{I}}(H)) = 0$ where $H \triangle \Theta_{\mathcal{I}}(H)$ stands for the symmetric difference of H and $\Theta_{\mathcal{I}}(H)$.*

Proof. It is sufficient to show that for a measurable subset H of $\mathbb{R}$, $H \setminus \Theta_{\mathcal{I}}(H)$ is a null set, since $\Theta_{\mathcal{I}}(H) \setminus H \subset H^c \setminus \Theta_{\mathcal{I}}(H^c)$ and H^c is measurable. Let us assume that, without any loss of generality, H is bounded because if H is unbounded, it can be written as $\bigcup_{n=1}^{\infty} H_n$ where each H_n is bounded.

For $\mu > 0$ let us take

$$C_\mu = \{x \in H : \mathcal{I} - d_-(x, H) < 1 - \mu\}. \tag{2}$$

Then, for $\mu_1 < \mu_2$ we have $C_{\mu_2} \subset C_{\mu_1}$ and $H \setminus \Theta_{\mathcal{I}}(H) = \bigcup_{\mu > 0} C_\mu$. We are to show that $m^\star(C_\mu) = 0$. Let, if possible $m^\star(C_\mu) > 0$ for some $\mu > 0$. Since $C_\mu \subset H$ and H is bounded, so C_μ is bounded. Then there exists a bounded open set $G \supset C_\mu$ such that $(1 - \mu)m(G) < m^\star(C_\mu)$. Let $\mathcal{F}$ be the family of all closed intervals I such that $I \subset G$ and $m(H \cap I) \leq (1 - \mu)m(I)$. Then for each $x \in C_\mu$ $\exists J \in \mathcal{F}$ such that $x \in J$ and $m(J) < \epsilon$ for arbitrary small $\epsilon > 0$. So, C_μ is covered by $\mathcal{F}$ in the sense of Vitali. For any disjoint sequence $\{I_k\}_{k \in \mathbb{N}}$ of elements of $\mathcal{F}$,

$$\begin{aligned} m^\star(C_\mu \cap (\bigcup_{k \in \mathbb{N}} I_k)) = m^\star(\bigcup_{k \in \mathbb{N}} (C_\mu \cap I_k)) &\leq \sum_{k \in \mathbb{N}} m^\star(C_\mu \cap I_k) \leq \sum_{k \in \mathbb{N}} m(H \cap I_k) \\ &\leq (1 - \mu) \sum_{k \in \mathbb{N}} m(I_k) < (1 - \mu)m(G) < m^\star(C_\mu). \end{aligned}$$

Therefore,

$$m^\star(C_\mu \setminus \bigcup_{k \in \mathbb{N}} I_k) > 0. \tag{3}$$

We construct a disjoint sequence $\{J_k\}_{k \in \mathbb{N}}$ of elements in $\mathcal{F}$ as follows. Let $\alpha_0 = \sup_{J \in \mathcal{F}} m(J)$. Choose $J_1 \in \mathcal{F}$ such that $m(J_1) > \frac{\alpha_0}{2}$. Take $\mathcal{F}_1 = \{J \in \mathcal{F} : J \cap J_1 = \phi\}$. Then $\mathcal{F}_1$ is nonempty, since $m^\star(C_\mu \setminus J_1) > 0$, by (3). Let $\alpha_1 = \sup_{J \in \mathcal{F}_1} m(J)$. Choose $J_2 \in \mathcal{F}_1$ such that $m(J_2) > \frac{\alpha_1}{2}$. Take $\mathcal{F}_2 = \{J \in \mathcal{F}_1 : J \cap J_2 = \phi\}$. Then $\mathcal{F}_2$ is nonempty, by (3). Likewise we choose $J_1, J_2, \ldots, J_n$. By induction, let us take $\mathcal{F}_n = \{J \in \mathcal{F}_{n-1} : J \cap J_n = \phi\}$. Then $\mathcal{F}_n$ is nonempty, by (3). Let $\alpha_n = \sup\{m(J) : J \in \mathcal{F}_n\}$. Choose $J_{n+1} \in \mathcal{F}_n$ such that $m(J_{n+1}) > \frac{\alpha_n}{2}$. Take $B = C_\mu \setminus \bigcup_{k \in \mathbb{N}} J_k$. Then, by (3), $m^\star(B) > 0$. Since $J_k \subset G$ $\forall k \in \mathbb{N}$, it follows that $\bigcup_{k \in \mathbb{N}} J_k \subset G$. Thus $\sum_{k=1}^{\infty} m(J_k) \leq m(G) < \infty$. Therefore, $\exists$ $n_0 \in \mathbb{N}$ such that $\sum_{k=n_0+1}^{\infty} m(J_k) < \frac{m^\star(B)}{4}$. For $k > n_0$ let Q_k denote the interval concentric with J_k such that $m(Q_k) = 4m(J_k)$. Now, $\sum_{k=n_0+1}^{\infty} m(Q_k) = 4 \sum_{k=n_0+1}^{\infty} m(J_k) < m^\star(B)$. So, the family of intervals $\{Q_k\}_{k > n_0}$ does not cover B.

Let us take $b \in B \setminus \bigcup_{k=n_0+1}^{\infty} Q_k$. Then, $b \in C_\mu \setminus \bigcup_{k=1}^{n_0} J_k$. Since, $\mathcal{F}$ is a Vitali cover of C_μ, $\exists$ an interval $J \in \mathcal{F}_{n_0}$ such that $b \in J$ and b is the center of J. Clearly for some $k > n_0$, $J \cap J_k \neq \phi$. Because if $J \cap J_k = \phi$ $\forall k > n_0$, then since $J \in \mathcal{F}_{n_0}$, $J \cap J_k = \phi$ for $k = 1, 2, \ldots, n_0$. Hence, $J \cap J_k = \phi$ $\forall k \in \mathbb{N}$. Thus, $J \in \mathcal{F}_n$ $\forall n \in \mathbb{N}$ which implies that $m(J) \leq \alpha_n < 2m(J_{n+1}) \forall n \in \mathbb{N}$. Again, since $\sum_{k=1}^{\infty} m(J_k) \leq m(G) < \infty$, so for given any $\epsilon > 0$ $\exists$ $k_0 \in \mathbb{N}$ such that $\sum_{k=k_0}^{\infty} m(J_k) < \epsilon$. But, $\sum_{k=k_0}^{\infty} m(J_k) > \sum_{k=k_0}^{\infty} (\frac{\alpha_{k-1}}{2})$. So we get a contradiction.

So, let k_0 be the least positive integer for which $J \cap J_{k_0} \neq \phi$. Then, $k_0 > n_0$ and $J \in \mathcal{F}_{k_0-1}$. Therefore, $m(J) \leq \alpha_{k_0-1} < 2m(J_{k_0}) = \frac{m(Q_{k_0})}{2}$. Now for $b \in J$ and $J \cap J_{k_0} \neq \phi$ we have the following two cases.

1. If $b \in J_{k_0}$, then $b \in Q_{k_0}$.
2. If $b \notin J_{k_0}$, then also we claim $b \in Q_{k_0}$.
Since b is the center of J, let us take $J = [b - \frac{m(J)}{2}, b + \frac{m(J)}{2}]$. Let x_{k_0} be the center of J_{k_0}. Then take $J_{k_0} = [x_{k_0} - \frac{m(J_{k_0})}{2}, x_{k_0} + \frac{m(J_{k_0})}{2}]$.

Consequently, $Q_{k_0} = [x_{k_0} - 2m(J_{k_0}), x_{k_0} + 2m(J_{k_0})]$.
Let $x \in J \cap J_{k_0}$.Then, $|b - x| \leq \frac{m(J)}{2}$ and $|x - x_{k_0}| \leq \frac{m(J_{k_0})}{2}$. Hence,

$$|b - x_{k_0}| \leq |b - x| + |x - x_{k_0}| \leq \frac{m(J)}{2} + \frac{m(J_{k_0})}{2} < m(J_{k_0}) + \frac{m(J_{k_0})}{2} = \frac{3}{2}m(J_{k_0}) < 2m(J_{k_0})$$

Hence, $b \in Q_{k_0}$ which implies that $b \in \bigcup_{k=n_0+1}^{\infty} Q_k$. This leads to a contradiction to our choice of b in $B \setminus \bigcup_{k=n_0+1}^{\infty} Q_k$. So, $m^\star(C_\mu) = 0$ for each $\mu > 0$. Therefore, $m(H \setminus \Theta_{\mathcal{I}}(H)) = 0$. □

The statement of this theorem may also be stated as follows: 'Almost all points of an arbitrary measurable set H are the $\mathcal{I}$-density points of H'.

4. $\mathcal{I}$-density topology

Definition 4.1. *A measurable set $E \subset \mathbb{R}$ is $\mathcal{I} - d$ open iff $\mathcal{I} - d_-(x, E) = 1$ $\forall x \in E$.*

Let us take the collection $\mathfrak{T}_{\mathcal{I}} = \{A \subset \mathbb{R} : A$ is $\mathcal{I} - d$ open $\}$.

Theorem 4.2. *The collection $\mathfrak{T}_{\mathcal{I}}$ is a topology on $\mathbb{R}$.*

Proof. By voidness, $\phi \in \mathfrak{T}_{\mathcal{I}}$. Since $\mathbb{R} \in \mathcal{L}$, so for $E = \mathbb{R}$ and any $r \in \mathbb{R}$ let $\{I_k\}_{k\in\mathbb{N}}$ be any sequence of closed intervals about the point r such that $\mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})$. It is clear that $\mathbb{R} \cap I_k = I_k$ for all k. Therefore $x_k = \frac{m(\mathbb{R}\cap I_k)}{m(I_k)} = 1$ for all $k \in \mathbb{N}$. Then

$$A_{x_k} = \{a \in \mathbb{R} : \{k : x_k < a\} \notin \mathcal{I}\} = (1, \infty).$$

Thus, $\mathcal{I} - d_-(r, \mathbb{R}) = \inf\{\inf A_{x_n} : \{I_n\}_{n\in\mathbb{N}}$ such that $\mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} = 1$ $\forall r \in \mathbb{R}$. Therefore, $\mathbb{R} \in \mathfrak{T}_{\mathcal{I}}$.

Next, let Λ be an arbitrary indexing set and $\{A_\alpha\}_{\alpha\in\Lambda}$ be a collection of sets in $\mathfrak{T}_{\mathcal{I}}$. We are to show, $\bigcup_{\alpha\in\Lambda} A_\alpha \in \mathfrak{T}_{\mathcal{I}}$. Clearly A_α is measurable and $\mathcal{I} - d$ open for each $\alpha \in \Lambda$. First we have to show, $\bigcup_{\alpha\in\Lambda} A_\alpha$ is measurable. Let us take $A = \bigcup_{\alpha\in\Lambda} A_\alpha$.

Let us take a point $p \in A$. So $p \in A_\alpha$ for some $\alpha \in \Lambda$. Since A_α is $\mathcal{I} - d$ open so $\mathcal{I} - d_-(p, A_\alpha) = 1$. Therefore, there exists a sequence $\{I_n^p\}_{n\in\mathbb{N}}$ of closed intervals about p such that $\mathscr{S}(I_n^p) \in \mathcal{F}(\mathcal{I})$ and $\mathcal{I} - \liminf_n \frac{m(A_\alpha \cap I_n^p)}{m(I_n^p)} = 1$. Thus, $\mathcal{I} - \liminf_n \frac{m(A_\alpha\cap I_n^p)}{m(I_n^p)} \leq \mathcal{I} - \limsup_n \frac{m(A_\alpha\cap I_n^p)}{m(I_n^p)} \leq 1$ implies $\mathcal{I} - \lim_n \frac{m(A_\alpha\cap I_n^p)}{m(I_n^p)} = 1$. This means that for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\forall$ $n \in \mathscr{S}(I_n^p)$ and $n > n_0$ we have

$$1 - \epsilon < \frac{m(A_\alpha \cap I_n^p)}{m(I_n^p)} < 1 + \epsilon.$$

So for some suitable k we have $\frac{m(A_\alpha\cap I_k^p)}{m(I_k^p)} > 1 - \epsilon$. Since A_α is measurable so $A_\alpha \cap I_k^p$ is measurable subset of A. If A is bounded, by Vitali Covering Theorem for $\mathbb{R}$, A contains a measurable set G such that $m^\star(A \setminus G) < \epsilon\, m(G)$. Therefore, A is measurable. If A is unbounded, then A can be written as $A = \bigcup_{n=1}^{\infty} A_n$ where each A_n is bounded and measurable. Therefore, A is measurable.

Now we will show that for all $p \in A$, $\mathcal{I} - d_-(p, A) = 1$. If $p \in A$, then $p \in A_\alpha$ for some α. So, $\mathcal{I} - d_-(p, A_\alpha) = 1$. Since, $\mathcal{I} - d_-(p, A) \geq \mathcal{I} - d_-(p, A_\alpha) = 1$. Therefore, $\mathcal{I} - d_-(p, A) = 1$ $\forall p \in A$. Hence, $A = \bigcup_{\alpha\in\Lambda} A_\alpha \in \mathfrak{T}_{\mathcal{I}}$.

Finally, for any two set $A, B \in \mathfrak{T}_{\mathcal{I}}$ we are to show $A \cap B \in \mathfrak{T}_{\mathcal{I}}$. Since both A and B are measurable, $A \cap B$ is measurable. Now, for any $p \in A \cap B$ we are to show that $\mathcal{I} - d_{-}(p, A \cap B) = 1$. It is sufficient to show that $\mathcal{I} - d_{-}(p, A \cap B) \geq 1\ \forall p \in A \cap B$. Let $\{I_k\}_{k\in\mathbb{N}}$ be any sequence of closed intervals about a point p such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$. Let us define $a_n = \frac{m(A\cap I_n)}{m(I_n)}$, $b_n = \frac{m(B\cap I_n)}{m(I_n)}$ and $p_n = \frac{m(A\cap B\cap I_n)}{m(I_n)}$. Then for all $n \in \mathscr{S}(I_k)$, $m(A \cap I_n) + m(B \cap I_n) - m(A \cap B \cap I_n) \leq m(I_n)$

So,

$$\frac{m(A \cap I_n)}{m(I_n)} + \frac{m(B \cap I_n)}{m(I_n)} \leq 1 + \frac{m(A \cap B \cap I_n)}{m(I_n)}.$$

Hence, $a_n + b_n \leq 1 + p_n$. Taking $\mathcal{I} - \liminf$ on both sides we have

$$\mathcal{I} - \liminf\{a_n + b_n\} \leq \mathcal{I} - \liminf\{1 + p_n\} = 1 + \mathcal{I} - \liminf p_n.$$

Thus,

$$\begin{aligned} &\inf\{\mathcal{I} - \liminf\{a_n + b_n\} : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\ &\qquad\qquad \leq 1 + \inf\{\mathcal{I} - \liminf p_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\}. \end{aligned}$$

Since,

$$\mathcal{I} - \liminf a_n + \mathcal{I} - \liminf b_n \leq \mathcal{I} - \liminf\{a_n + b_n\}.$$

So,

$$\begin{aligned} &\inf\{\mathcal{I} - \liminf a_n + \mathcal{I} - \liminf b_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\ &\leq \inf\{\mathcal{I} - \liminf\{a_n + b_n\} : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\ &\qquad\qquad \leq 1 + \inf\{\mathcal{I} - \liminf p_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\}. \end{aligned}$$

Hence,

$$\begin{aligned} &\inf\{\mathcal{I} - \liminf a_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\ &+ \inf\{\mathcal{I} - \liminf b_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\ &\leq \inf\{\mathcal{I} - \liminf a_n + \mathcal{I} - \liminf b_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} \\ &\qquad\qquad \leq 1 + \inf\{\mathcal{I} - \liminf p_n : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\}. \end{aligned}$$

Therefore,

$$\mathcal{I} - d_{-}(p, A) + \mathcal{I} - d_{-}(p, B) \leq 1 + \mathcal{I} - d_{-}(p, A \cap B).$$

Now since $A, B \in \mathfrak{T}_{\mathcal{I}}$ we have $\mathcal{I} - d_{-}(p, A \cap B) \geq 1$. So, $\mathfrak{T}_{\mathcal{I}}$ is a topology on $\mathbb{R}$. □

The topology $\mathfrak{T}_{\mathcal{I}}$ is called the $\mathcal{I}$-density topology on $\mathbb{R}$ and the pair $(\mathbb{R}, \mathfrak{T}_{\mathcal{I}})$ is the corresponding topological space.

Theorem 4.3. *The family $\mathfrak{T}_{\mathcal{I}}$ is a topology on the real line finer than the natural topology $\mathfrak{T}_U$.*

Proof. Let us take an open set U in $\mathfrak{T}_U$. Since any $\mathfrak{T}_U$-open set in $\mathbb{R}$ can be written as countable union of disjoint open intervals, so without any loss of generality, let U be an open interval (a, b) where $a, b \in \mathbb{R}$ and $a < b$. We are to prove that U is $\mathcal{I} - d$ open. Clearly U is Lebesgue measurable. Now given any point p in U suppose $\{J_n\}_{n\in\mathbb{N}}$ be any sequence of closed interval about p such that $\mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I})$. Then there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ and $n \in \mathscr{S}(J_n)$ we have $J_n \subset U$. So for $n > n_0$ and $n \in \mathscr{S}(J_n)$, $x_n = \frac{m(J_n\cap U)}{m(J_n)} = 1$. Therefore, $\{k \in \mathbb{N} : x_k = 1\} \supset \mathscr{S}(J_n) \cap (\mathbb{N} \setminus \{1, 2, \cdots, n_o\})$. Thus, $\{k \in \mathbb{N} : x_k = 1\} \in \mathcal{F}(\mathcal{I})$. So, $A_{x_k} = (1, \infty)$ and $\mathcal{I} - d_{-}(p, U) = \inf\{\inf A_{x_k} : \{J_k\}_{k\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} = 1$. Hence, U is $\mathcal{I} - d$ open. Thus, any set that is open in natural topology $\mathfrak{T}_U$ on $\mathbb{R}$ is also $\mathcal{I} - d$ open. So the topology $\mathfrak{T}_{\mathcal{I}}$ is finer than the topology $\mathfrak{T}_U$. □

Definition 4.4. *A set $F \subset \mathbb{R}$ is said to be $\mathcal{I}-d$ closed if F^c is $\mathcal{I}-d$ open.*

Definition 4.5. *A point $x \in \mathbb{R}$ is called an $\mathcal{I}-d$ limit point of a set $E \subset \mathbb{R}$ (not necessarily measurable) if and only if $\mathcal{I}-d^-(x,E) > 0$ where instead of taking measure m outer measure m^* is taken.*

Theorem 4.6. *In the space $(\mathbb{R}, \mathfrak{T}_\mathcal{I})$ given any Lebesgue measurable set $E \subset \mathbb{R}$, $m(E) = 0$ if and only if E is $\mathcal{I}-d$ closed and discrete.*

Proof. Necessary part: Let $m(E) = 0$. Then for any point $p \in \mathbb{R}$ and any sequence $\{I_n\}_{n\in\mathbb{N}}$ of closed intervals about p such that $\mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})$ take $x_n = \frac{m(I_n\cap E)}{m(I_n)}$. Then $x_n = 0\ \forall n \in \mathbb{N}$. So, $B_{x_n} = \{b \in \mathbb{R} : \{k : x_k > b\} \notin \mathcal{I}\} = (-\infty, 0)$. Thus, $\mathcal{I}-d^-(p,E) = \sup\{\sup B_{x_k} : \{I_k\}_{k\in\mathbb{N}}$ such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})\} = 0$. Hence, p is not an $\mathcal{I}-d$ limit point of E. So, E has no $\mathcal{I}-d$ limit points. Therefore, E is $\mathcal{I}-d$ closed and discrete.

Sufficient part: Let E be $\mathcal{I}-d$ closed and discrete. Then E has no $\mathcal{I}-d$ limit points and so $\mathcal{I}-d^-(p,E) = 0$ $\forall p \in \mathbb{R}$. Thus $\mathcal{I}-d(p,E) = 0\ \forall p \in \mathbb{R}$. But, by Lebesgue $\mathcal{I}$-density theorem, $\mathcal{I}-d(p,E) = 1$ for almost all $p \in E$. Therefore, $m(E) = 0$. □

Remark 4.7. *Though $\mathbb{Q}$ is neither open nor closed in $(\mathbb{R}, \mathfrak{T}_U)$ and since $m(\mathbb{Q}) = 0$, by Theorem 4.6, it is $\mathcal{I}-d$ closed in $(\mathbb{R}, \mathfrak{T}_\mathcal{I})$. So a natural question arises whether a subset of $\mathbb{R}$ exists which is neither $\mathcal{I}-d$ open nor $\mathcal{I}-d$ closed. In the following example we have shown that such sets do exist in $(\mathbb{R}, \mathfrak{T}_\mathcal{I})$.*

Example 4.8. *There exists a subset of $\mathbb{R}$ which is neither $\mathcal{I}-d$ open nor $\mathcal{I}-d$ closed. Here we are giving a construction of a collection of such sets in $\mathbb{R}$. Let us take an open interval $I = (x_1, x_2)$ where $x_1, x_2 \in \mathbb{Q}$ and $x_1 < x_2$. Since I is open in $(\mathbb{R}, \mathfrak{T}_\mathcal{I})$ it is $\mathcal{I}-d$ open. Now, let $b = \frac{(x_1+x_2)}{2}$. Then b is the center of I and $b \in \mathbb{Q}$. Take $J = \left[b - \frac{|x_2-x_1|}{4}, b + \frac{|x_2-x_1|}{4}\right]$. Then $J \subset I$. Let $I' = I \setminus (J \cap \mathbb{Q}^c)$. We claim that I' is neither $\mathcal{I}-d$ open nor $\mathcal{I}-d$ closed. Let $\{I_k\}_{k\in\mathbb{N}}$ be a sequence of closed intervals about $b \in I'$ such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$. Take, $x_k = \frac{m(I_k\cap I')}{m(I_k)}$. For large $k \in \mathscr{S}(I_k)$, $(I_k \cap I') \subset \mathbb{Q}$. Thus $m(I_k \cap I') = 0$. Thus, $B_{x_k} = \{b \in \mathbb{R} : \{k : x_k > b\} \notin \mathcal{I}\} = (-\infty, 0)$. Therefore, $\mathcal{I}-d^-(b,I') = \sup\{\sup B_{x_k} : \{I_k\}_{k\in\mathbb{N}}$ such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})\} = 0$. Thus, $\mathcal{I}-d_-(b,I') = 0$ and so b is not an $\mathcal{I}$-density point of I'. Hence, I' is not $\mathcal{I}-d$ open.*

Now, to show I' is not $\mathcal{I}-d$ closed we are to show $(I')^c$ is not $\mathcal{I}-d$ open. We see, $(I')^c = (-\infty, x_1] \cup (J\cap\mathbb{Q}^c) \cup [x_2, \infty)$. Let $\{J_k\}_{k\in\mathbb{N}}$ be any sequence of closed intervals about the point x_1 such that $\mathscr{S}(J_n) \in \mathcal{F}(\mathcal{I})$ where in particular we choose $J_k = \left[x_1 - \frac{1}{2^{k+1}}, x_1\right]\ \forall k \in \mathbb{N}$. Take, $z_k = \frac{m(J_k\cap(I')^c)}{m(J_k)}$ where $0 < m(J_k) = \frac{1}{2^{k+1}} < \frac{1}{k}\ \forall k$. So $\mathscr{S}(J_k) = \mathbb{N} \in \mathcal{F}(\mathcal{I})$. Then, $m(J_k \cap (I')^c) = 0\ \forall k$ implies $z_k = 0\ \forall k$. So, $\inf\{\mathcal{I}-\liminf z_k : \{J_k\}_{k\in\mathbb{N}}$ such that $\mathscr{S}(J_k) \in \mathcal{F}(\mathcal{I})\} = 0$ which implies $\mathcal{I}-d_-(x_1,(I')^c) = 0$. Therefore, x_1 is not an $\mathcal{I}$-density point of $(I')^c$. So, $(I')^c$ is not $\mathcal{I}-d$ open.

5. $\mathcal{I}$-approximate continuity

The notion of approximate continuity introduced by A. Denjoy is connected with the notion of Lebesgue density point. Since the idea of classical Lebesgue density point has been generalized to $\mathcal{I}$-density point, subsequently in this section, we should obtain the notion of $\mathcal{I}$-approximate continuity.

Definition 5.1 (cf.[4]). *A function $f : \mathbb{R} \to \mathbb{R}$ is called $\mathcal{I}$-approximately continuous at $x_0 \in \mathbb{R}$ if there exists a set $E_{x_0} \in \mathcal{L}$ such that $\mathcal{I}-d(x_0, E_{x_0}) = 1$ and $f|_{E_{x_0}}$ is continuous at x_0.*

If the function f is $\mathcal{I}$-approximately continuous at every point of $\mathbb{R}$ then we simply say f is $\mathcal{I}$-approximately continuous. We use the notation $\mathcal{I}-\mathbb{AC}$ to denote $\mathcal{I}$-approximate continuity of f. If 'f is $\mathcal{I}$-approximately continuous at x'we simply write (in short) 'f is $\mathcal{I}-\mathbb{AC}$ at x'.

Now we prove the following results with suitable modification of classical proofs.

Theorem 5.2. *If $f, g : \mathbb{R} \to \mathbb{R}$ is $\mathcal{I}-\mathbb{AC}$ at x_0, then the functions $f+g$, $f\cdot g$ and $a\cdot g$ for any $a \in \mathbb{R}$ are $\mathcal{I}-\mathbb{AC}$ at x_0. If $g(x) \neq 0$ for any $x \in (x_0-\delta, x_0+\delta)$ where $\delta > 0$ then $\frac{1}{g}$ is $\mathcal{I}-\mathbb{AC}$ at x_0.*

Proof. At first we show for any two Lebesgue measurable subsets A and B of $\mathbb{R}$ and a point x_0 in $\mathbb{R}$ if $\mathcal{I} - d(x_0, A) = 1$ and $\mathcal{I} - d(x_0, B) = 1$ then $\mathcal{I} - d(x_0, A \cap B) = 1$. It is sufficient to show $\mathcal{I} - d_-(x_0, A \cap B) \geq 1$. Let $\{I_k\}_{k\in\mathbb{N}}$ be any sequence of closed intervals about x_0 such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$. Then for $k \in \mathscr{S}(I_k)$ we have

$$\frac{m(A \cap I_k)}{m(I_k)} + \frac{m(B \cap I_k)}{m(I_k)} \leq 1 + \frac{m((A \cap B) \cap I_k)}{m(I_k)}.$$

Let us take $x_k = \frac{m(A\cap I_k)}{m(I_k)}, y_k = \frac{m(B\cap I_k)}{m(I_k)}, z_k = \frac{m((A\cap B)\cap I_k)}{m(I_k)}$. So, $z_k \geq x_k + y_k - 1$. Thus,

$$\begin{aligned}\mathcal{I} - \liminf z_n &\geq \mathcal{I} - \liminf(x_n + y_n - 1)\\ &\geq \mathcal{I} - \liminf(x_n + y_n) - 1\\ &\geq \mathcal{I} - \liminf x_n + \mathcal{I} - \liminf y_n - 1.\end{aligned}$$

Hence,

$$\begin{aligned}&\inf\{\mathcal{I} - \liminf z_n : \{I_n\} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\}\\ &\geq \inf\{\mathcal{I} - \liminf x_n + \mathcal{I} - \liminf y_n - 1 : \{I_n\} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\}\\ &\geq \inf\{\mathcal{I} - \liminf x_n : \{I_n\} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\}\\ &\qquad + \inf\{\mathcal{I} - \liminf y_n : \{I_n\} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\} - 1.\end{aligned}$$

So,

$$\begin{aligned}\mathcal{I} - d_-(x_0, A \cap B) &= \inf\{\mathcal{I} - \liminf z_n : \{I_n\} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\}\\ &\geq \mathcal{I} - d_-(x_0, A) + \mathcal{I} - d_-(x_0, B) - 1\\ &= 1 + 1 - 1 = 1.\end{aligned}$$

Now since f and g are $\mathcal{I} - \mathbb{AC}$ at x_0, so there exists two sets E_f and E_g in $\mathbb{R}$ such that x_0 is an $\mathcal{I}$-density point of both E_f and E_g and hence $\mathcal{I} - d(x_0, E_f \cap E_g) = 1$. Also $f|_{E_f}$ and $g|_{E_g}$ are continuous at x_0. So,

$$(f + g)|_{E_f\cap E_g} = f|_{E_f\cap E_g} + g|_{E_f\cap E_g}.$$

Hence, $(f + g)$ is $\mathcal{I} - \mathbb{AC}$ at x_0. Again,

$$(f \cdot g)|_{E_f\cap E_g} = f|_{E_f\cap E_g} \cdot g|_{E_f\cap E_g}.$$

Hence, $(f \cdot g)$ is $\mathcal{I} - \mathbb{AC}$ at x_0. Similarly for any $a \in \mathbb{R}$, $(a \cdot f)$ is $\mathcal{I} - \mathbb{AC}$ at x_0.

Moreover, since $g(x) \neq 0$ for any $x \in (x_0 - \delta, x_0 + \delta)$ where $\delta > 0$, so $g|_{E_g\cap(x_0-\delta,x_0+\delta)} \neq 0$ and continuous at x_0. Then $(\frac{1}{g})|_{E_g\cap(x_0-\delta,x_0+\delta)}$ is continuous at x_0 and x_0 is an $\mathcal{I}$-density point of $E_g \cap (x_0 - \delta, x_0 + \delta)$. Hence $\frac{1}{g}$ is $\mathcal{I} - \mathbb{AC}$ at x_0. □

Theorem 5.3. *If f is $\mathcal{I} - \mathbb{AC}$ at x_0 and g is continuous at $f(x_0)$ then $(g \circ f)$ is $\mathcal{I} - \mathbb{AC}$ at x_0.*

Proof. By hypothesis, there exists a subset E_f of $\mathbb{R}$ such that $\mathcal{I} - d(x_0, E_f) = 1$ and $f|_{E_f}$ is continuous at x_0. Now $(g \circ f)|_{E_f} = g \circ f|_{E_f}$. Since composition of two continuous functions is continuous so $(g \circ f)|_{E_f}$ is continuous at x_0. Thus, $(g \circ f)$ is $\mathcal{I} - \mathbb{AC}$ at x_0. □

We state here the Lusin's Theorem for our future purpose.

Theorem 5.4 ([25]). *A real valued function f on $\mathbb{R}$ is measurable if and only if for each $\epsilon > 0$ there exists a set E with $m(E) < \epsilon$ such that the restriction of f to $\mathbb{R} \setminus E$ is continuous.*

Theorem 5.5. *A function $g : \mathbb{R} \to \mathbb{R}$ is measurable if and only if it is $\mathcal{I} - \mathbb{AC}$ almost everywhere.*

Proof. Necessary part: Let g be measurable. For $\epsilon > 0$, by Lusin's Theorem, there exists a continuous function ψ such that $m(\{x : g(x) \neq \psi(x)\}) < \epsilon$. Let $E = \{x : g(x) \neq \psi(x)\}$. Since E is measurable so E^c is measurable. By Theorem 3.2, almost every point of E^c is a point of $\mathcal{I}-$density of E^c and $g|_{E^c} = \psi$ is continuous. So g is $\mathcal{I} - \mathbb{AC}$ at almost every point of E^c. Thus g is $\mathcal{I} - \mathbb{AC}$ except on E where outer measure of E is less than ϵ. So, g is $\mathcal{I} - \mathbb{AC}$ almost everywhere, since $\epsilon > 0$ is arbitrary.

Sufficient part: Suppose g is $\mathcal{I} - \mathbb{AC}$ almost everywhere. We show g is measurable. For $r \in \mathbb{R}$ let $E_r = \{x : g(x) < r\}$. It is sufficient to show that E_r is measurable. Without any loss of generality let E_r be uncountable. Let $B = \{x \in \mathbb{R} : g \text{ is } \mathcal{I} - \mathbb{AC} \text{ at } x\}$. Then

$$E_r = (E_r \cap B) \cup (E_r \setminus B).$$

From hypothesis $m(\mathbb{R} \setminus B) = 0$. Since m is a complete measure so $E_r \setminus B \in \mathcal{L}$. It is enough to show $E_r \cap B \in \mathcal{L}$. Let $t \in E_r \cap B$. Since $t \in B$ so there exists a set $D_t \in \mathcal{L}$ such that $\mathcal{I} - d(t, D_t) = 1$ and $f|_{D_t}$ is continuous at t. Without any loss of generality D_t can be chosen inside $E_r \cap B$. Therefore

$$E_r \cap B = \bigcup_{t \in E_r \cap B} D_t.$$

If possible, let $E_r \cap B$ be not measurable. Then there exists an Euclidean F_σ set P and Euclidean G_δ set H such that $P \subset E_r \cap B \subset H$ and

$$m(P) = m_\star(E_r \cap B) < m^\star(E_r \cap B) = m(H).$$

Thus $m(H \setminus P) > 0$. By Theorem 3.2, almost every point of $H \setminus P$ is a point of $\mathcal{I}$-density of $H \setminus P$. Since $m(H \setminus P) = m^\star((E_r \cap B) \setminus P)$, so $m^\star((E_r \cap B) \setminus P) > 0$. There exists $t_0 \in (E_r \cap B) \setminus P \subset H \setminus P$ such that $\mathcal{I} - d(t_0, H \setminus P) = 1$. Now $t_0 \in (E_r \cap B)$. So there exists set $D_{t_0} \subset E_r \cap B$ such that $\mathcal{I} - d(t_0, D_{t_0}) = 1$. We claim that $m(D_{t_0} \setminus P) > 0$. For, if possible, let

$$m(D_{t_0} \setminus P) = 0. \tag{4}$$

Then $\mathcal{I} - d(t_0, D_{t_0} \setminus P) = 0$. Now $H = D_{t_0} \cup (H \setminus D_{t_0})$. So, by Theorem 2.13,

$$\mathcal{I} - d(t_0, H \setminus D_{t_0}) = 0, \text{ since } \mathcal{I} - d(t_0, D_{t_0}) = 1. \tag{5}$$

Here

$$H \setminus P = (D_{t_0} \setminus P) \cup ((H \setminus P) \setminus D_{t_0}).$$

Now from (4) and (5) we have

$$\mathcal{I} - d(t_0, H \setminus P) = \mathcal{I} - d(t_0, D_{t_0} \setminus P) + \mathcal{I} - d(t_0, (H \setminus P) \setminus D_{t_0}) = 0.$$

This is a contradiction. Now $m(D_{t_0} \setminus P) > 0$ implies $m_\star(D_{t_0} \setminus P) > 0$. Then $m_\star((E_r \cap B) \setminus P) > 0$. This contradicts to the fact that $m(P) = m_\star(E_r \cap B)$. Thus $E_r \cap B \in \mathcal{L}$. □

Definition 5.6 ([24]). *The set of all continuous functions defined on interval I is called as the null Baire class of functions. If the function $g(x)$ defined on I is not in the null class but is representable in the form*

$$g(x) = \lim_{n \to \infty} g_n(x) \tag{6}$$

where all the functions $g_n(x)$ are continuous then $g(x)$ is said to be a function of the first Baire class. In general the functions of Baire class $m \in \mathbb{N}$ are functions which are not in any of the preceeding classes but can be represented as the limit of sequence of functions of Baire class $(m-1)$ as in (6).

In this way all the classes of functions with finite indices are defined. We denote these classes by $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_m, \ldots$

Theorem 5.7 ([24]). *Let I be a fixed interval and $g : I \to \mathbb{R}$ be a function of class not greater than m_1 and let ψ be a function of class not greater than m_2 whose values lie in I. Then $(g \circ \psi)$ is a function of class $\leq m_1 + m_2$.*

Theorem 5.8 ([24]). *Let I be a fixed interval. Then $g : I \to \mathbb{R}$ is a function of Baire class not greater than the first if and only if for arbitrary $\alpha \in \mathbb{R}$ the sets $C^\alpha = \{x : g(x) < \alpha\}$ and $C_\alpha = \{x : g(x) > \alpha\}$ are of type Euclidean F_σ.*

Theorem 5.9. *Given any fixed interval I if $g : I \to \mathbb{R}$ is $\mathcal{I} - \mathbb{AC}$ function, then g belongs to first Baire class.*

Proof. Since g is $\mathcal{I} - \mathbb{AC}$, so by Theorem 5.5, g is measurable. First let us take g to be bounded. Then there exists a positive number M such that $|g(x)| < M$ for $x \in I$. Now for $a \in I$ define

$$G(x) = \int_a^x g(t)dt.$$

Then $G : I \to \mathbb{R}$ is a continuous function. We claim for each $r \in I$,

$$\lim_{k\to 0} \frac{G(r+k) - G(r)}{k} = g(r). \tag{7}$$

i.e., given any $\epsilon > 0$ there exists $\delta > 0$ such that $|\frac{1}{k}\int_r^{r+k} g(t)dt - g(r)| < \epsilon$ whenever $k < \delta$. Since g is $\mathcal{I} - \mathbb{AC}$ on I, so for $r \in I$ there exists $B_r \subset I$ such that $\mathcal{I} - d(r, B_r) = 1$ and $g|_{B_r}$ is continuous at r. So for each $k > 0$

$$\begin{aligned} \left|\frac{1}{k}\int_r^{r+k} g(t)dt - g(r)\right| &= \left|\frac{1}{k}\int_r^{r+k} (g(t) - g(r))dt\right| \\ &\leq \frac{1}{k}\int_r^{r+k} |g(t) - g(r)|dt \\ &= \frac{1}{k}\int_{[r,r+k]\cap B_r} |g(t) - g(r)|dt + \frac{1}{k}\int_{[r,r+k]\setminus B_r} |g(t) - g(r)|dt. \end{aligned} \tag{8}$$

Now for given any $\epsilon > 0$ we choose $\delta > 0$ such that the following hold:

1. Since $g|_{B_r}$ is continuous at r, so for $t \in B_r \cap (r - \delta, r + \delta)$ we have $|g(t) - g(r)| < \frac{\epsilon}{2}$.
2. Since $\mathcal{I} - d(r, B_r) = 1$, so $\mathcal{I} - d(r, B_r^c) = 0$ and so for some $k < \delta$ we have $\frac{m([r,r+k]\setminus B_r)}{k} < \frac{\epsilon}{4M}$.

For $k < \delta$ from (8) we obtain

$$\begin{aligned} \left|\frac{1}{k}\int_r^{r+k} g(t)dt - g(r)\right| &\leq \frac{1}{k}\cdot\frac{\epsilon}{2}\cdot m([r, r+k]) + \frac{1}{k}\cdot 2M \cdot m([r, r+k]\setminus B_r) \\ &< \frac{1}{k}\cdot\frac{\epsilon}{2}\cdot k + \frac{1}{k}\cdot 2M\cdot\frac{\epsilon k}{4M} \\ &= \epsilon. \end{aligned} \tag{9}$$

Similarly calculating for $k < 0$ we obtain (7). Thus for each $r \in I$ we have

$$g(r) = \lim_{k\to 0}\frac{G(r+k) - G(r)}{k} = \lim_{n\to\infty}\frac{G(r+\frac{1}{n}) - G(r)}{\frac{1}{n}} = \lim_{n\to\infty} n\left\{G\left(r + \frac{1}{n}\right) - G(r)\right\}.$$

Now let $G_n(r) = n\{G(r + \frac{1}{n}) - G(r)\}$. Then G_n is continuous, since G is continuous. Therefore g is in first Baire class.

Now if $g : I \to \mathbb{R}$ is unbounded then let $h : \mathbb{R} \to (0, 1)$ be a homeomorphism. So, h and h^{-1} are continuous. Also by Theorem 5.3, $h \circ g : I \to (0, 1)$ is $\mathcal{I} - \mathbb{AC}$ and $(h \circ g)$ is bounded. So by the first part $(h \circ g)$ is in first Baire class. Now $g = h^{-1} \circ (h \circ g)$. Hence by Theorem 5.7, g is in first Baire class. □

The next lemma is based on the idea presented in [32](Theorem 3.1) and the condition presented in this lemma will be called the condition (J_2) of J. M. Jedrzejewski.

Lemma 5.10. *Let $\{G_n\}_{n\in\mathbb{N}}$ be any decreasing sequence of Lebesgue measurable sets such that for some $x_0 \in \mathbb{R}$, $\mathcal{I} - d(x_0, G_n) = 1 \ \ \forall n \in \mathbb{N}$. Then there exists a decreasing sequence $\{s_n\}_{n\in\mathbb{N}}$ of positive real numbers converging to zero such that*

$$A_{x_0} = \bigcup_{n=1}^{\infty}(G_n \setminus (x_0 - s_n, x_0 + s_n)) \text{ and } \mathcal{I} - d(x_0, A_{x_0}) = 1.$$

Proof. Let $\{\delta_n\}_{n\in\mathbb{N}}$ be a strictly decreasing sequence such that $0 < \delta_n < 1 \ \forall n \in \mathbb{N}$ and $\delta_n \to 0$ as $n \to \infty$. Now since $\mathcal{I} - d(x_0, G_n) = 1 \ \forall n \in \mathbb{N}$, so $\mathcal{I} - d_-(x_0, G_n) = 1$ and $\mathcal{I} - d^-(x_0, G_n) = 1$. Clearly,

$$\inf\left\{\mathcal{I} - \liminf \frac{m(G_n \cap I_k)}{m(I_k)} : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\right\} = 1$$

and

$$\sup\left\{\mathcal{I} - \limsup \frac{m(G_n \cap I_k)}{m(I_k)} : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\right\} = 1.$$

So for any sequence of closed intervals $\{I_k\}_{k\in\mathbb{N}}$ about x_0 such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$ we have

$$1 \leq \mathcal{I} - \liminf \frac{m(G_n \cap I_k)}{m(I_k)} \leq \mathcal{I} - \limsup \frac{m(G_n \cap I_k)}{m(I_k)} \leq 1.$$

Therefore, $\mathcal{I} - \liminf_k \frac{m(G_n \cap I_k)}{m(I_k)} = \mathcal{I} - \limsup_k \frac{m(G_n \cap I_k)}{m(I_k)} = \mathcal{I} - \lim_k \frac{m(G_n \cap I_k)}{m(I_k)} = 1$. So for given any $\epsilon > 0$ and for each $n \in \mathbb{N}$,

$$C_\epsilon^{(n)} = \left\{k : \frac{m(G_n \cap I_k)}{m(I_k)} > 1 - \epsilon\right\} \in \mathcal{F}(\mathcal{I}).$$

Now for $\epsilon = \delta_n$ there exists $k_n \in \mathbb{N}$ such that, for $k \geq k_n$ and $k \in C_{\delta_n}^{(n)}$,

$$\frac{m(G_n \cap I_k)}{m(I_k)} > 1 - \delta_n.$$

We choose k_n's so that $\{k_n\}_{n\in\mathbb{N}}$ is increasing and the sequence $\{m(I_{k_n})\}_{n\in\mathbb{N}}$ is decreasing. Thus consider a subsequence $\{I_{k_n}\}_{k_n \in C_{\delta_n}^{(n)}}$ of the sequence $\{I_k\}_{k\in C_{\delta_n}^{(n)}}$ and put

$$s_n = \delta_n m(I_{k_{n+1}}) \text{ for } n \in \mathbb{N} \text{ and } k_{n+1} \in C_{\delta_n}^{(n)}.$$

Since $\delta_n \to 0$ and $m(I_{k_{n+1}}) < \frac{1}{k_{n+1}}$, so $s_n \to 0$ as $n \to \infty$. Since δ_n is decreasing and $m(I_{k_n})$ is decreasing, s_n is decreasing. Without any loss of generality we can assume that $m(I_k)$ is decreasing for $k \in C_{\delta_n}^{(n)}$. For $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $3\delta_n < \delta$ for $n > n_0$. Moreover there exists $l_1 \in \mathbb{N}$ such that $m(I_k) < m(I_{k_{n_0+1}})$ for $k > l_1$ and $k \in C_{\delta_n}^{(n)}$. Now fix $k > l_1$ and $k \in C_{\delta_n}^{(n)}$. So there exists $n_1 > n_0$ such that

$$m(I_{k_{n_1+1}}) \leq m(I_k) < m(I_{k_{n_1}}).$$

Since $\{m(I_k)\}_{k\in C^{(n)}_{\delta_n}}$ is decreasing sequence, so $k > k_{n_1}$. Thus for fixed $n = n_1$ we have

$$\begin{aligned}\frac{m\left((G_{n_1} \setminus (x_0 - s_{n_1}, x_0 + s_{n_1})) \cap I_k\right)}{m(I_k)} &= \frac{m\left((G_{n_1} \cap I_k) \setminus (x_0 - s_{n_1}, x_0 + s_{n_1})\right)}{m(I_k)}\\ &\geq \frac{m(G_{n_1} \cap I_k) - 2s_{n_1}}{m(I_k)}\\ &= \frac{m(G_{n_1} \cap I_k)}{m(I_k)} - \frac{2s_{n_1}}{m(I_k)}\\ &> 1 - \delta_{n_1} - \frac{2\delta_{n_1} m(I_{k_{n_1+1}})}{m(I_k)}\\ &> 1 - \delta_{n_1} - 2\delta_{n_1}\\ &= 1 - 3\delta_{n_1}\\ &> 1 - \delta.\end{aligned} \tag{10}$$

So, since for all $k \in \mathbb{N}$

$$\frac{m(A_{x_0} \cap I_k)}{m(I_k)} \geq \frac{m\left((G_{n_1} \setminus (x_0 - s_{n_1}, x_0 + s_{n_1})) \cap I_k\right)}{m(I_k)},$$

we have

$$\left\{k : \frac{m((G_{n_1} \setminus (x_0 - s_{n_1}, x_0 + s_{n_1})) \cap I_k)}{m(I_k)} > 1 - \delta\right\} \subset \left\{k : \frac{m(A_{x_0} \cap I_k)}{m(I_k)} > 1 - \delta\right\}.$$

Moreover, since $\mathcal{I}$ is an admissible ideal,

$$\left\{k : \frac{m((G_{n_1} \setminus (x_0 - s_{n_1}, x_0 + s_{n_1})) \cap I_k)}{m(I_k)} > 1 - \delta\right\} \supset C^{(n)}_{\delta_n} \cap (\mathbb{N} \setminus \{1, 2, \cdots, l_1\}) \in \mathcal{F}(\mathcal{I}).$$

Hence, $\left\{k : \frac{m(A_{x_0} \cap I_k)}{m(I_k)} > 1 - \delta\right\} \in \mathcal{F}(\mathcal{I})$. Therefore, $\left\{k : 1 - \delta < \frac{m(A_{x_0} \cap I_k)}{m(I_k)} < 1 + \delta\right\} \in \mathcal{F}(\mathcal{I})$ and so $\mathcal{I}-\lim_k \frac{m(A_{x_0} \cap I_k)}{m(I_k)} = 1$. Clearly for any sequence of closed intervals $\{I_k\}_{k\in\mathbb{N}}$ about x_0 such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$ we have

$$\mathcal{I} - \liminf_k \frac{m(A_{x_0} \cap I_k)}{m(I_k)} = \mathcal{I} - \limsup_k \frac{m(A_{x_0} \cap I_k)}{m(I_k)} = \mathcal{I} - \lim_k \frac{m(A_{x_0} \cap I_k)}{m(I_k)} = 1.$$

Thus $\mathcal{I} - d_-(x_0, A_{x_0}) = \mathcal{I} - d^-(x_0, A_{x_0}) = 1$. So, $\mathcal{I} - d(x_0, A_{x_0}) = 1$. This completes the proof. □

Theorem 5.11. *Given any fixed interval I, $g : I \to \mathbb{R}$ is $\mathcal{I} - \mathbb{AC}$ function if and only if for each $\mu \in \mathbb{R}$ both the sets $C^\mu = \{x : g(x) < \mu\}$ and $C_\mu = \{x : g(x) > \mu\}$ belongs to the topology $\mathfrak{T}_{\mathcal{I}}$.*

Proof. Necessary part: Let the function g be $\mathcal{I} - \mathbb{AC}$. Then by Theorem 5.9, g is in the first Baire class. So by Theorem 5.8, for each $\mu \in \mathbb{R}$, C^μ and C_μ are of type Euclidean F_σ. So, both C^μ and C_μ belongs to $\mathcal{L}$. Now we are to show that for each $x \in C^\mu$, $\mathcal{I} - d(x, C^\mu) = 1$.

Let us fix $\mu \in \mathbb{R}$ and let us take $x_0 \in C^\mu$. Then $g(x_0) < \mu$. So, $\mu - g(x_0) > 0$. Since g is $\mathcal{I} - \mathbb{AC}$ at x_0, so there exists $E \in \mathcal{L}$ such that $\mathcal{I} - d(x_0, E) = 1$ and $g|_E$ is continuous at x_0. Hence, for given any $\epsilon > 0$ there exists $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta) \cap E$ implies $g(x_0) - \epsilon < g(x) < g(x_0) + \epsilon$. In particular if we choose $\epsilon_0 = \frac{\mu - g(x_0)}{M}$ for some $M \in \mathbb{N}$ and $M > 1$, then $g(x_0) = \mu - M\epsilon_0$. So for suitably chosen $\delta_0 > 0$ and for $x \in (x_0 - \delta_0, x_0 + \delta_0) \cap E$ we have

$$g(x) < g(x_0) + \epsilon_0 = \mu - M\epsilon_0 + \epsilon_0 < \mu.$$

Thus, $(x_0 - \delta_0, x_0 + \delta_0) \cap E \subset C^\mu$. Since x_0 is an $\mathcal{I}$-density point of $(x_0 - \delta_0, x_0 + \delta_0)$ and E, so it is $\mathcal{I}$-density point of $(x_0 - \delta_0, x_0 + \delta_0) \cap E$. Therefore, $\mathcal{I} - d(x_0, C^\mu) = 1$.

Sufficient part: Let $x_0 \in I$. Without any loss of generality, we choose x_0 in I without being the end points of I. Let $\{\epsilon_n\}_{n\in\mathbb{N}}$ be a decreasing sequence of positive real numbers converging to zero. For each $n \in \mathbb{N}$ let $A_n = \{x : g(x) < g(x_0) + \epsilon_n\}$ and $B_n = \{x : g(x) > g(x_0) - \epsilon_n\}$. By hypothesis, $A_n, B_n \in \mathfrak{T}_{\mathcal{I}}$. Let $C_n = A_n \cap B_n$ $\forall n \in \mathbb{N}$. Then $C_n = \{x : |g(x) - g(x_0)| < \epsilon_n\}$. We observe $C_n \in \mathfrak{T}_{\mathcal{I}}$. Since $x_0 \in C_n$, so $\mathcal{I} - d(x_0, C_n) = 1$ $\forall n \in \mathbb{N}$. Since $\{C_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of measurable sets, so by lemma 5.10, there exists a strictly decreasing sequence $\{s_n\}_{n\in\mathbb{N}}$ of positive real numbers converging to zero such that

$$A_{x_0} = \bigcup_{n=1}^{\infty}(C_n \setminus (x_0 - s_n, x_0 + s_n)) \text{ and } \mathcal{I} - d(x_0, A_{x_0}) = 1.$$

Then $A_{x_0} \in \mathcal{L}$. Now we are to show that $g|_{A_{x_0}}$ is continuous at x_0. For fixed $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\epsilon_n < \epsilon$ $\forall n > n_0$. Now if $x \in A_{x_0} \cap (x_0 - s_{n_0}, x_0 + s_{n_0})$, then $x \in \bigcup_{n=n_0+1}^{\infty}(C_n \setminus (x_0 - s_n, x_0 + s_n))$. So there exists $n_1 > n_0$ such that $x \in C_{n_1}$. Let us choose $\delta = s_{n_0}$. Then for $x \in A_{x_0} \cap (x_0 - \delta, x_0 + \delta)$ we have $x \in C_{n_1}$ i.e., $|g(x) - g(x_0)| < \epsilon_{n_1} < \epsilon$. Therefore $g|_{A_{x_0}}$ is continuous at x_0. Hence g is $\mathcal{I} - \mathbb{AC}$ at x_0.
□

Definition 5.12 (cf.[16]). *A function $g : \mathbb{R} \to \mathbb{R}$ is called $\mathcal{I}$-approximately upper semi-continuous at a point $x_0 \in \mathbb{R}$ if for every $\alpha > g(x_0)$ there exists a set $E_{x_0} \in \mathcal{L}$ such that $\mathcal{I} - d(x_0, E_{x_0}) = 1$ and $g(x) < \alpha$ for every $x \in E_{x_0}$.*

Moreover, g is called $\mathcal{I}$-approximately upper semi-continuous if it is $\mathcal{I}$-approximately upper semi-continuous at every point $x \in \mathbb{R}$. Similarly we define $\mathcal{I}$-approximately lower semi-continuity.

Theorem 5.13. *A function $g : \mathbb{R} \to \mathbb{R}$ is $\mathcal{I} - \mathbb{AC}$ if and only if it is $\mathcal{I}$-approximately upper and $\mathcal{I}$-approximately lower semi-continuous.*

Proof. Necessary part: Let g be $\mathcal{I} - \mathbb{AC}$ at $x_0 \in \mathbb{R}$. So there exists $E \in \mathcal{L}$ such that $\mathcal{I} - d(x_0, E) = 1$ and $g|_E$ is continuous at x_0. So given any $\epsilon > 0$ there exists $\delta > 0$ such that, whenever $x \in (x_0 - \delta, x_0 + \delta) \cap E$, $g(x_0) - \epsilon < g(x) < g(x_0) + \epsilon$. Now for every $c \in \mathbb{R}$ and $c > g(x_0)$ choose $\epsilon > 0$ such that $g(x_0) + \epsilon < c$. For this $\epsilon > 0$ we choose $\delta > 0$ such that for every $x \in (x_0 - \delta, x_0 + \delta) \cap E$ we have $g(x) < g(x_0) + \epsilon < c$. Moreover x_0 is an $\mathcal{I}$-density point of $(x_0 - \delta, x_0 + \delta) \cap E$. Thus g is $\mathcal{I}$-approximately upper semi-continuous at x_0. Since choice of $x_0 \in \mathbb{R}$ is arbitrary, g is $\mathcal{I}$-approximately upper semi-continuous at every $x \in \mathbb{R}$. Similarly it can be shown that g is $\mathcal{I}$-approximately lower semi-continuous at every $x \in \mathbb{R}$.

Sufficient part: Let g be $\mathcal{I}$-approximately upper and $\mathcal{I}$-approximately lower semi-continuous. For any $\alpha \in \mathbb{R}$ let $C^\alpha = \{x \in \mathbb{R} : g(x) < \alpha\}$. Now take $x_0 \in C^\alpha$. Then $g(x_0) < \alpha$. Since g is $\mathcal{I}$-approximately upper semi-continuous at x_0 so there exists $E_{x_0} \in \mathcal{L}$ such that $\mathcal{I} - d(x_0, E_{x_0}) = 1$ and, $\forall\, x \in E_{x_0}$, $g(x) < \alpha$. Let us take $\widehat{E}_{x_0} = \{x_0\} \cup E_{x_0}$. Then $\widehat{E}_{x_0} \in \mathcal{L}$. Now define

$$V_{x_0} = \{y \in \widehat{E}_{x_0} : \mathcal{I} - d(y, \widehat{E}_{x_0}) = 1\}.$$

Then V_{x_0} is $\mathcal{I} - d$ open and $V_{x_0} \in \mathfrak{T}_{\mathcal{I}}$. Moreover,

$$y \in V_{x_0} \implies y \in \widehat{E}_{x_0} \implies g(y) < \alpha \implies y \in C^\alpha.$$

Thus $V_{x_0} \subset C^\alpha$. Since choice of x_0 is arbitrary, so $V_x \subset C^\alpha$ for all $x \in C^\alpha$. Therefore, $C^\alpha = \bigcup_{x\in C^\alpha} V_x$ where $V_x \in \mathfrak{T}_{\mathcal{I}}$. Consequently, $C^\alpha \in \mathfrak{T}_{\mathcal{I}}$.

In a similar approach we can show for any $\beta \in \mathbb{R}$, $C_\beta = \{x \in \mathbb{R} : g(x) > \beta\} \in \mathfrak{T}_{\mathcal{I}}$. Thus by Theorem 5.11, it can be concluded g is $\mathcal{I} - \mathbb{AC}$. □

We now proceed to prove the main result of this section.

Theorem 5.14. *A function $g : (\mathbb{R}, \mathfrak{T}_{\mathcal{I}}) \to (\mathbb{R}, \mathfrak{T}_U)$ is continuous if and only if g is $\mathcal{I} - \mathbb{AC}$ at every $x \in \mathbb{R}$*

Proof. Necessary part: Let $g : (\mathbb{R}, \mathfrak{T}_{\mathcal{I}}) \to (\mathbb{R}, \mathfrak{T}_U)$ be continuous at x_0. So given any $\mathfrak{T}_U$-open set V containing $g(x_0)$ there exists $\mathcal{I} - d$ open set U containing x_0 such that $x_0 \in U \subset g^{-1}(V)$. Since U is $\mathcal{I} - d$ open set and

$x_0 \in U$, $\mathcal{I} - d(x_0, U) = 1$ and so $g|_U$ is continuous at x_0. Hence g is $\mathcal{I} - \mathbb{AC}$ at x_0. Since choice of x_0 is arbitrary, so g is $\mathcal{I} - \mathbb{AC}$ at every x.

Sufficient part: Let g be $\mathcal{I} - \mathbb{AC}$. Then by Theorem 5.11, for any $\mu \in \mathbb{R}$ we have $C^\mu = \{x : g(x) < \mu\}$ and $C_\mu = \{x : g(x) > \mu\}$ where both C^μ and C_μ are in $\mathfrak{T}_\mathcal{I}$. Then let g be $\mathcal{I} - \mathbb{AC}$ at x_0 for some $x_0 \in \mathbb{R}$. Let V be an open set in $(\mathbb{R}, \mathfrak{T}_U)$ containing $g(x_0)$. Without any loss of generality let $V = (g(x_0) - \epsilon', g(x_0) + \epsilon)$ for some $\epsilon, \epsilon' > 0$. We are to show that there exists a set $U \in \mathfrak{T}_\mathcal{I}$ containing x_0 such that $g(U) \subset V$. Let $C^\star = \{x : g(x) < g(x_0) + \epsilon\}$ and $C_\star = \{x : g(x) > g(x_0) - \epsilon'\}$. Then

$$C^\star \cap C_\star = \{x : g(x_0) - \epsilon' < g(x) < g(x_0) + \epsilon\}.$$

Let $C^\star \cap C_\star = U$. Then $U \in \mathfrak{T}_\mathcal{I}$. Observe that $x_0 \in U$. Now any $x \in U$ implies $g(x) \in (g(x_0) - \epsilon', g(x_0) + \epsilon)$. Therefore $g(U) \subset V$. Hence g is continuous at x_0. This completes the proof. □

6. Lusin-Menchoff Theorem

The Lusin-Menchoff theorem plays a vital role in proving complete regularity of density topology [36]. In this paper, since we attempt to prove complete regularity of $\mathcal{I}$-density topology, we try to prove analogue of Lusin-Menchoff theorem for $\mathcal{I}$-density.

Definition 6.1 ([17]). *A topological space is called Polish if it is separable and completely metrizable.*

Example 6.2. $(\mathbb{R}, \mathfrak{T}_U)$ *is a Polish space.*

Definition 6.3 ([17]). *A topological space X is called perfect if all of its points are limit points or equivalently it contains no isolated points.*

If P is a subset of a topological space X then P is called perfect in X if P is closed and perfect in its relative topology. The following theorem is known as Cantor-Bendixon theorem.

Theorem 6.4 ([17]). *Let X be a Polish space. Then X can be written uniquely as $X = P \cup C$, where P is a perfect subset of X and C is countable open.*

The above result holds good if we take any closed set instead of X. Now we state the Perfect set Theorem for Borel sets.

Theorem 6.5 ([17]). *Let X be a Polish space and $A \subset X$ be Borel. Then either A is countable or else it contains a Cantor set.*

Now we will prove some lemmas which will be needed later in this section.

Lemma 6.6. *Let B be a Borel set. Then for $x \in B$ such that $\mathcal{I} - d(x, B) = 1$ there exists a $\mathfrak{T}_U$ perfect set P such that $x \in P \subset B$.*

Proof. For $x \in B$, $\mathcal{I} - d(x, B) = 1$ implies $\mathcal{I} - d^-(x, B) = \mathcal{I} - d_-(x, B) = 1$. For $\{I_n\}_{n\in\mathbb{N}}$ being any sequence of closed intervals about x such that $\mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})$ we have

$$\inf\left\{\mathcal{I} - \liminf \frac{m(B \cap I_k)}{m(I_k)} : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\right\} = 1$$

and

$$\sup\left\{\mathcal{I} - \limsup \frac{m(B \cap I_k)}{m(I_k)} : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\right\} = 1.$$

So for any sequence of closed intervals $\{I_k\}_{k\in\mathbb{N}}$ about x such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$, we have

$$1 \leq \mathcal{I} - \liminf \frac{m(B \cap I_k)}{m(I_k)} \leq \mathcal{I} - \limsup \frac{m(B \cap I_k)}{m(I_k)} \leq 1.$$

So, $\mathcal{I} - \limsup \frac{m(B\cap I_k)}{m(I_k)} = 1$. For given $\epsilon > 0$ let $A_\epsilon = \left\{n : \frac{m(B\cap I_n)}{m(I_n)} > 1-\epsilon\right\}$. Then $A_\epsilon \in \mathcal{F}(\mathcal{I})$. For $n \in A_\epsilon$,

$$\frac{m(B\cap I_n)}{m(I_n)} > 1-\epsilon \qquad \Longrightarrow m(I_n\cap B) > (1-\epsilon)m(I_n) \qquad \Longrightarrow m(I_n\cap B) > 0. \tag{11}$$

Let us take a sequence $\{J_n\}_{n\in\mathbb{N}} = \{[c_n, d_n]\}_{n\in\mathbb{N}}$ of pairwise disjoint intervals such that $dist(x, J_n) \to 0$ as $n \to \infty$ and without any loss of generality assume $m(J_n \cap B) > 0\ \forall n \in A_\epsilon$. So, $J_n \cap B$ is not countable $\forall n \in A_\epsilon$. Since both J_n and B are Borel sets, so, $J_n \cap B$ is Borel. Now since $(\mathbb{R}, \mathfrak{T}_U)$ is a Polish space, by Theorem 6.5, $\forall n \in A_\epsilon$, there exists a $\mathfrak{T}_U$-perfect set P_n such that $P_n \subset J_n \cap B$. Since J_n's are pairwise disjoint, so $\{P_n\}_{n\in A_\epsilon}$ is a collection of pairwise disjoint $\mathfrak{T}_U$-perfect set.

Now let $P = \{x\} \cup (\bigcup_{n\in A_\epsilon} P_n)$. Then $x \in P \subset B$.
We claim that P is $\mathfrak{T}_U$-perfect set.

First we show P has no isolated points. Now since for $i \in A_\epsilon$ each P_i is $\mathfrak{T}_U$-perfect, so P_i has no isolated point. Hence $\bigcup_{i\in A_\epsilon} P_i$ has no isolated point. Now we show x is not an isolated point of P. Let $N(x)$ be any open neighbourhood about x. Then for some $n_0 \in A_\epsilon$, $J_{n_0} \cap (N(x) \setminus \{x\}) \neq \phi$. Then for $n_0^{'} > n_0$ and $n_0^{'} \in A_\epsilon$ there exists a $\mathfrak{T}_U$-perfect set $P_{n_0^{'}}$ such that $P_{n_0^{'}} \cap (N(x) \setminus \{x\})$ is nonempty. Hence $P \cap (N(x) \setminus \{x\})$ is nonempty. So, x is not an isolated point of P. Therefore, P has no isolated points.

Next we show P is $\mathfrak{T}_U$-closed. Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence in P such that $s_n \to s$. We are to show $s \in P$. We have the following two cases:
Case(i) Let there be finitely many s_n in each P_i for $i \in A_\epsilon$. Then without any loss of generality we may assume that $s_i \in P_i$ for each $i \in A_\epsilon$. We claim, $s = x$. For large $i \in A_\epsilon$,

$$|s-x| \le |s-s_i| + |s_i - x| \le |s-s_i| + dist(x, P_{i'}). \tag{12}$$

Here i' in the subscript of $P_{i'}$ is the immediate predecessor of i in A_ϵ. Also since $dist(x, J_n) \to 0$ as $n \to \infty$, so $dist(x, P_i) \to 0$ as $i \to \infty$. So, as $i \to \infty$, from (12) we can conclude $s = x$. Hence $s \in P$.
Case(ii) If at least one of P_n say P_i contains infinitely many of s_n, then suppose that there exists a subsequence $\{s_{n_k}\}_{k\in\mathbb{N}}$ of $\{s_n\}_{n\in\mathbb{N}}$ such that $\{s_{n_k}\} \subset P_i$. Since $s_{n_k} \to x$ and P_i is $\mathfrak{T}_U$-perfect so, $s \in P_i$. Therefore, $s \in P$.
Hence, P is $\mathfrak{T}_U$-closed. Consequently, P is $\mathfrak{T}_U$-perfect. □

Lemma 6.7. *Let B be a Borel set. Then for every countable set C such that $cl(C) \subset B$ and $\mathcal{I} - d(x, B) = 1 \forall x \in C$ there exists a $\mathfrak{T}_U$ perfect set P such that $C \subset P \subset B$. Here $cl(C)$ stands for $\mathfrak{T}_U$-closure of C.*

Proof. Let us take $C = \{x_i : i \in \mathbb{N}\} \subset B$. Now put $B_i = B \cap [x_i - \frac{1}{2^i}, x_i + \frac{1}{2^i}]$ for $i \in \mathbb{N}$. Then B_i is a Borel set containing x_i for each i. We claim that $\mathcal{I} - d(x_i, B_i) = 1$. Now since $\mathcal{I} - d(x_i, B) = 1$ so $\mathcal{I} - d_-(x_i, B) = 1$ and $\mathcal{I} - d^-(x_i, B) = 1$. For $\{I_n\}_{n\in\mathbb{N}}$ being any sequence of closed intervals about x_i such that $\mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})$ we have

$$\inf\left\{\mathcal{I} - \liminf \frac{m(B\cap I_k)}{m(I_k)} : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\right\} = 1$$

and

$$\sup\left\{\mathcal{I} - \limsup \frac{m(B\cap I_k)}{m(I_k)} : \{I_n\}_{n\in\mathbb{N}} \text{ such that } \mathscr{S}(I_n) \in \mathcal{F}(\mathcal{I})\right\} = 1.$$

So for any sequence of closed intervals $\{I_k\}_{k\in\mathbb{N}}$ about x such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$ we have

$$1 \le \mathcal{I} - \liminf \frac{m(B\cap I_k)}{m(I_k)} \le \mathcal{I} - \limsup \frac{m(B\cap I_k)}{m(I_k)} \le 1.$$

So, $\mathcal{I} - \limsup \frac{m(B\cap I_k)}{m(I_k)} = 1$. For given $\epsilon > 0$ let $A_\epsilon = \left\{n : \frac{m(B\cap I_n)}{m(I_n)} > 1-\epsilon\right\}$. Then $A_\epsilon \in \mathcal{F}(\mathcal{I})$.

Since $B_i \subset B$, $\exists n_0 \in \mathbb{N}$ such that $\forall n > n_0$ and $n \in A_\epsilon$ we have $m((B \setminus B_i) \cap I_n) = 0$. Therefore, $\forall n > n_0$ and $n \in A_\epsilon$, $\frac{m(B\cap I_n)}{m(I_n)} = \frac{m(B_i\cap I_n)}{m(I_n)}$. So, $\left\{n : \frac{m(B_i\cap I_n)}{m(I_n)} > 1-\epsilon\right\} = A_\epsilon \setminus \{1, 2, \cdots, n_0\} \in \mathcal{F}(\mathcal{I})$. So, $\left\{n : 1-\epsilon < \frac{m(B_i\cap I_n)}{m(I_n)} < 1+\epsilon\right\} \in$

$\mathcal{F}(\mathcal{I})$ and $\mathcal{I}-\lim_k \frac{m(B_i\cap I_k)}{m(I_k)} = 1$. Clearly for any sequence of closed intervals $\{I_k\}_{k\in\mathbb{N}}$ about x_i such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$ we have

$$\mathcal{I}-\liminf_k \frac{m(B_i\cap I_k)}{m(I_k)} = \mathcal{I}-\limsup_k \frac{m(B_i\cap I_k)}{m(I_k)} = \mathcal{I}-\lim_k \frac{m(B_i\cap I_k)}{m(I_k)} = 1.$$

Thus $\mathcal{I}-d_-(x_i,B_i) = \mathcal{I}-d^-(x_i,B_i) = 1$. So, $\mathcal{I}-d(x_i,B_i) = 1$ for each $i \in \mathbb{N}$. By Lemma 6.6 there exists a $\mathfrak{T}_U$-perfect set P_i such that $x_i \in P_i \subset B_i$ for each $i \in \mathbb{N}$.

Now let $P = cl(C) \cup (\bigcup_{i\in\mathbb{N}} P_i)$. Then clearly $C \subset P \subset B$.

We claim, P is $\mathfrak{T}_U$-perfect.

It is clear that $\bigcup_{i\in\mathbb{N}} P_i$ has no isolated points. Now if $x \in C$. Then $x = x_i$ for some i and $x_i \in P_i$ where P_i is $\mathfrak{T}_U$-perfect. So, x is not an isolated point of P. Again if $x \in cl(C)\setminus C$ then there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \in C$ such that $z_n \to x$. Then any open neighbourhood $N(x)$ about x contains some $z_i \neq x$. Consequently, x is not an isolated point of P.

So, P has no isolated points.

Next we are to show P is $\mathfrak{T}_U$-closed. Let $\{s_n\}_{n\in\mathbb{N}}$ be a sequence in P such that $s_n \to s$. To show $s \in P$. We have the following three cases:

Case(i): If $cl(C)$ contains infinitely many of s_n, then suppose $\{s_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{s_n\}_{n\in\mathbb{N}}$ such that $\{s_{n_k}\}_{k\in\mathbb{N}} \subset cl(C)$. Since $s_{n_k} \to x$ and $cl(C)$ is $\mathfrak{T}_U$-closed so $s \in cl(C)$. Hence, $s \in P$.

Case(ii): If atleast one of P_i contains infinitely many of s_n then suppose $\{s_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence of $\{s_n\}_{n\in\mathbb{N}}$ such that $\{s_{n_k}\}_{k\in\mathbb{N}} \subset P_i$. Since $s_{n_k} \to x$ and P_i is $\mathfrak{T}_U$-perfect so, $s \in P_i$. Hence $s \in P$.

Case(iii): Let there be finitely many s_n in each P_i for $i \in \mathbb{N}$. Then without any loss of generality we may assume that $s_i \in P_i$ for each $i \in \mathbb{N}$. Since $m(P_i) \le m(B_i) < \frac{1}{2^{i-1}}$ for each $i \in \mathbb{N}$, therefore, $|x_i - s_i| < \frac{1}{2^{i-1}}$ for each $i \in \mathbb{N}$. Now for each $k \in \mathbb{N}$,

$$|x_k - s| \le |x_k - s_k| + |s_k - s| < \frac{1}{2^{k-1}} + |s_k - s|.$$

From the above inequality it can be concluded that $x_k \to s$ as $k \to \infty$. Therefore, $s \in cl(C)$, since $x_i \in C\ \forall i$. Thus, $s \in P$. □

Lemma 6.8. *Let H be a Lebesgue measurable set. Then for every $\mathfrak{T}_U$ closed subset Z of H such that $\mathcal{I}-d(x,H) = 1$ $\forall x \in Z$ there exists a $\mathfrak{T}_U$-perfect set P such that $Z \subset P \subset H$.*

Proof. Since H is a measurable subset of $\mathbb{R}$, so there exists an Euclidean F_σ set $A \subset H$ such that $m(H\setminus A) = 0$. For $x \in \mathbb{R}$, let $\{I_k\}_{k\in\mathbb{N}}$ be any sequence of closed intervals about x such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$. Let $h_k = \frac{m(H\cap I_k)}{m(I_k)}$ and $a_k = \frac{m(A\cap I_k)}{m(I_k)}$. Then,

$$\begin{aligned} h_k = \frac{m(H\cap I_k)}{m(I_k)} = \frac{m((A\cup(H\setminus A))\cap I_k)}{m(I_k)} &= \frac{m(A\cap I_k)}{m(I_k)} + \frac{m((H\setminus A)\cap I_k)}{m(I_k)} \\ &= \frac{m(A\cap I_k)}{m(I_k)} = a_k. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{I}-d_-(x,A) &= \inf\{\mathcal{I}-\liminf a_k : \{I_k\}_{k\in\mathbb{N}} \text{ such that } \mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})\} \\ &= \inf\{\mathcal{I}-\liminf h_k : \{I_k\}_{k\in\mathbb{N}} \text{ such that } \mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})\} \\ &= \mathcal{I}-d_-(x,H) \\ &= 1. \end{aligned}$$

Similarly, $\mathcal{I}-d^-(x,A) = 1$. Therefore, $\mathcal{I}-d(x,A) = 1\ \forall x \in Z$. Since both A and Z are Borel sets, so $B = A\cup Z$ is Borel and $Z \subset B \subset H$. Also $\mathcal{I}-d(x,B) = 1\ \forall x \in Z$, since $A \subset B$. Since $(\mathbb{R},\mathfrak{T}_U)$ is a Polish space, so by

Theorem 6.4, $Z = P_1 \cup C$ where P_1 is $\mathfrak{T}_U$-perfect and C is countable set. Now since Z is $\mathfrak{T}_U$-closed we have $cl(C) \subset Z \subset B$ and $\mathcal{I} - d(x, B) = 1\ \forall x \in C$. By Lemma 6.7, there exists a $\mathfrak{T}_U$-perfect set P_2 such that $C \subset P_2 \subset B$. Therefore, $P_1 \cup C \subset P_1 \cup P_2 \subset B$. Take $P = P_1 \cup P_2$. Then P is $\mathfrak{T}_U$-perfect and $Z \subset P \subset B \subset H$. □

Now we prove an analogue of Lusin-Menchoff Theorem for $\mathcal{I}$-density.

Theorem 6.9. *Let H be a measurable set. Then for every $\mathfrak{T}_U$ closed set Z such that $Z \subset H$ and $\mathcal{I} - d(x, H) = 1$ $\forall x \in Z$ there exists a $\mathfrak{T}_U$ perfect set P such that $Z \subset P \subset H$ and $\mathcal{I} - d(x, P) = 1\ \forall x \in Z$.*

Proof. By hypothesis and Lemma 6.8, there exists $\mathfrak{T}_U$-perfect set K such that $Z \subset K \subset H$. Now define,

$$H_n = \{z \in H : \tfrac{1}{n+1} < dist(z, Z) \le \tfrac{1}{n}\} \text{ for } n \in \mathbb{N}$$

and let

$$H_0 = \{z \in H : dist(z, Z) > 1\}.$$

Then, $H = Z \cup (\bigcup_{n=0}^{\infty} H_n)$. Without any loss of generality let us assume that each H_n is nonempty. Since *dist* function is continuous, so H_n's are measurable for each $n \in \mathbb{N} \cup \{0\}$. So for every $n \in \mathbb{N} \cup \{0\}$ we can find a closed set $F_n \subset H_n$ such that $m(H_n \setminus F_n) < \frac{1}{2^{n+1}}$. By Cantor Bendixon theorem, since every closed set can be expressed as a union of a perfect set and a countable set, for each n there exists $\mathfrak{T}_U$-perfect set $P_n \subset F_n \subset H_n$ such that $m(H_n \setminus P_n) < \frac{1}{2^{n+1}}$. Put,

$$P = K \cup (\bigcup_{n=1}^{\infty} P_n).$$

Then P is nonempty $\mathfrak{T}_U$-perfect set such that $Z \subset P \subset H$.

Now we are to show that $\mathcal{I} - d(x, P) = 1\ \forall x \in Z$.
For $x \in Z$, by hypothesis $\mathcal{I} - d(x, H) = 1$. Since $H = Z \cup (\bigcup_{n=0}^{\infty} H_n)$. So

$$\begin{aligned} H \setminus P &= (Z \setminus P) \cup \left(\left\{ \bigcup_{n=0}^{\infty} H_n \right\} \setminus P \right) = \left\{ \bigcup_{n=0}^{\infty} H_n \right\} \setminus P \\ &= \bigcup_{n=0}^{\infty} (H_n \setminus P) = \bigcup_{n=0}^{\infty} H_n \setminus (K \cup \bigcup_{m=1}^{\infty} P_m) \\ &= \bigcup_{n=0}^{\infty} \left((H_n \setminus K) \cap (H_n \setminus \bigcup_{m=1}^{\infty} P_m) \right) \\ &= \bigcup_{n=0}^{\infty} \left((H_n \setminus K) \cap \left(\bigcap_{m=1}^{\infty} (H_n \setminus P_m) \right) \right). \end{aligned}$$

Let $\{I_k\}_{k \in \mathbb{N}}$ be any sequence of closed intervals about x such that $\mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})$. From here k will be chosen from $\mathscr{S}(I_k)$. So, we have

$$I_k \cap (H \setminus P) = \bigcup_{n=0}^{\infty} \left(I_k \cap (H_n \setminus K) \cap \left(\bigcap_{m=1}^{\infty} (H_n \setminus P_m) \right) \right). \tag{13}$$

Now for a fixed k there are two possibilities:

1. $\exists\, n_k \in \mathbb{N}$ such that $I_k \cap H_n = \phi$ for $n < n_k$ but $I_k \cap H_{n_k} \neq \phi$
2. $I_k \cap H_n = \phi\ \forall\, n$. In this case we put $n_k = \infty$.

For case (2) the R.H.S. in (13) is empty set. So $m(I_k \cap (H \setminus P)) = 0$. Therefore, $\frac{m(I_k \cap H)}{m(I_k)} = \frac{m(I_k \cap P)}{m(I_k)}$. Hence,

$$\begin{aligned}\mathcal{I} - d_-(x, P) &= \inf\left\{\mathcal{I} - \liminf \frac{m(I_k \cap P)}{m(I_k)} : \{I_k\}_{k \in \mathbb{N}} \text{ such that } \mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})\right\} \\ &= \inf\left\{\mathcal{I} - \liminf \frac{m(I_k \cap H)}{m(I_k)} : \{I_k\}_{k \in \mathbb{N}} \text{ such that } \mathscr{S}(I_k) \in \mathcal{F}(\mathcal{I})\right\} \\ &= \mathcal{I} - d_-(x, H) \\ &= 1\end{aligned}$$

Similarly, $\mathcal{I} - d^-(x, P) = 1$. Hence, $\mathcal{I} - d(x, P) = 1$.

For case (1) from (13) we have

$$I_k \cap (H \setminus P) = \bigcup_{n=n_k}^{\infty}\left(I_k \cap (H_n \setminus K) \cap \left(\bigcap_{m=1}^{\infty}(H_n \setminus P_m)\right)\right). \tag{14}$$

Thus,

$$\begin{aligned}m(I_k \cap (H \setminus P)) &\leq \sum_{n=n_k}^{\infty} m\left(I_k \cap (H_n \setminus K) \cap \left(\bigcap_{m=1}^{\infty}(H_n \setminus P_m)\right)\right) \\ &\leq \sum_{n=n_k}^{\infty} m(H_n \setminus P_n) \\ &< \sum_{n=n_k}^{\infty} \frac{1}{2^{n+1}} = \frac{1}{2^{n_k}}.\end{aligned} \tag{15}$$

Now we will consider the following two subcases:
Subcase (i): Let us assume for each k, $n_k < \infty$. We claim that as $k \to \infty$ then $n_k \to \infty$. To show given $N \in \mathbb{N}$ there exists $k_0 \in \mathbb{N}$ such that if $k > k_0$ then $n_k > N$.

For given any large $N \in \mathbb{N}$ let $k_0 = N + 1$. If $k > k_0$, then $m(I_k) < \frac{1}{k} < \frac{1}{k_0}$. Also $I_k \cap H_{n_k} \neq \phi$. Let $y \in I_k \cap H_{n_k}$. Since $x, y \in I_k$, so

$$|x - y| < m(I_k) < \frac{1}{k_0}. \tag{16}$$

Moreover since $x \in Z$ and $y \in H_{n_k}$,

$$|x - y| \geq dist(H_{n_k}, Z) > \frac{1}{n_k + 1}. \tag{17}$$

From equation (16) and (17) we have $\frac{1}{n_k+1} < \frac{1}{k_0} = \frac{1}{N+1}$, which implies that $n_k > N$. Also note that $m(I_k) > \frac{1}{n_k+1}$, since $\frac{1}{n_k+1} < |x - y| < m(I_k)$ by (16).
Now for $k > k_0$,

$$\begin{aligned}\frac{m(I_k \cap H)}{m(I_k)} &= \frac{m(I_k \cap P)}{m(I_k)} + \frac{m(I_k \cap (H \setminus P))}{m(I_k)} \\ &< \frac{m(I_k \cap P)}{m(I_k)} + \frac{n_k + 1}{2^{n_k}}.\end{aligned} \tag{18}$$

Therefore, $\mathcal{I}-\liminf \frac{m(I_k\cap H)}{m(I_k)} \le \mathcal{I}-\liminf \frac{m(I_k\cap P)}{m(I_k)}$, by (18). So,

$$\begin{aligned}\mathcal{I}-d_-(x,P) &= \inf\left\{\mathcal{I}-\liminf \frac{m(I_k\cap P)}{m(I_k)} : \{I_k\}_{k\in\mathbb{N}} \text{ such that } \mathscr{S}(I_k)\in\mathcal{F}(\mathcal{I})\right\}\\ &\ge \inf\left\{\mathcal{I}-\liminf \frac{m(I_k\cap H)}{m(I_k)} : \{I_k\}_{k\in\mathbb{N}} \text{ such that } \mathscr{S}(I_k)\in\mathcal{F}(\mathcal{I})\right\}\\ &= \mathcal{I}-d_-(x,H)\\ &= 1.\end{aligned}$$

So, $1 \le \mathcal{I}-d_-(x,P) \le \mathcal{I}-d^-(x,P) \le 1$. Hence $\mathcal{I}-d(x,P)=1$.

Subcase (*ii*): Let $\{k\in\mathscr{S}(I_k) : n_k=\infty\}$ be an infinite subset of $\mathscr{S}(I_k)$. Say, $\{k\in\mathscr{S}(I_k) : n_k=\infty\} = \{k_1 < k_2 < \cdots < k_l < \cdots\}$. So, there exists a subsequence $\{k_l\}$ of $\{k\}$ such that $n_{k_l}=\infty$ and $k_l\to\infty$ as $l\to\infty$. So, $I_{k_l}\cap H_n=\phi\ \forall n$. Hence $m(I_{k_l}\cap(H\setminus P))=0$. So, $\frac{m(I_{k_l}\cap H)}{m(I_{k_l})} = \frac{m(I_{k_l}\cap P)}{m(I_{k_l})}$. Thus, by subcase (*i*) we can write $\mathcal{I}-\liminf \frac{m(I_k\cap H)}{m(I_k)} \le \mathcal{I}-\liminf \frac{m(I_k\cap P)}{m(I_k)}$. Therefore,

$$\begin{aligned}\mathcal{I}-d_-(x,P) &= \inf\left\{\mathcal{I}-\liminf \frac{m(I_k\cap P)}{m(I_k)} : \{I_k\}_{k\in\mathbb{N}} \text{ such that } \mathscr{S}(I_k)\in\mathcal{F}(\mathcal{I})\right\}\\ &\ge \inf\left\{\mathcal{I}-\liminf \frac{m(I_k\cap H)}{m(I_k)} : \{I_k\}_{k\in\mathbb{N}} \text{ such that } \mathscr{S}(I_k)\in\mathcal{F}(\mathcal{I})\right\}\\ &= \mathcal{I}-d_-(x,H)\\ &= 1\end{aligned}$$

So, $1 \le \mathcal{I}-d_-(x,P) \le \mathcal{I}-d^-(x,P) \le 1$. Hence $\mathcal{I}-d(x,P)=1$. This completes the proof. □

7. Some separation axioms

The purpose of this section is to provide some information about separation axioms for the space $(\mathbb{R},\mathfrak{T}_\mathcal{I})$. Since by Theorem 4.3, $\mathfrak{T}_U\subset\mathfrak{T}_\mathcal{I}$ we obtain immediately the following result.

Proposition 7.1. *The space $(\mathbb{R},\mathfrak{T}_\mathcal{I})$ is a Hausdorff space.*

In the next theorem we obtain a bounded $\mathcal{I}-\mathbb{AC}$ function. Given any two sets A and B we use the notation $A\subset\bullet B$ to mean $A\subset B$ and $\mathcal{I}-d(x,B)=1\ \forall x\in A$ (cf. [4]).

Theorem 7.2. *Let H be a subset of $\mathbb{R}$ of type Euclidean F_σ such that $\mathcal{I}-d(x,H)=1\ \forall x\in H$. Then there exists an $\mathcal{I}-\mathbb{AC}$ function $g:\mathbb{R}\to\mathbb{R}$ such that*

$$\begin{aligned}&(1)\ 0<g(x)\le 1 \text{ for } x\in H\\ &(2)\ g(x)=0 \quad \text{ for } x\notin H.\end{aligned}$$

Proof. If $H=\phi$, then $g(x)=0\ \forall x\in\mathbb{R}$ and so g is $\mathcal{I}-\mathbb{AC}$. Let H be a nonempty Euclidean F_σ set. So, $H=\bigcup_{n=1}^{\infty}K_n$ where each K_n is nonempty $\mathfrak{T}_U-$closed set. Now we construct a family of $\mathfrak{T}_U$ closed sets $\{Q_\beta : \beta\in\mathbb{R} \text{ and } \beta\ge 1\}$ such that $Q_{\beta_1}\subset\bullet Q_{\beta_2}$ if $\beta_1<\beta_2$ and $H=\bigcup_{\beta\ge1}Q_\beta$.

Let $Q_1=K_1$. Since $Q_1\subset H$ where H is measurable and Q_1 is $\mathfrak{T}_U$-closed set and $\mathcal{I}-d(x,H)=1\ \forall x\in Q_1$, so by Theorem 6.9, $\exists\ \mathfrak{T}_U$ closed set B_2 such that $Q_1\subset B_2\subset H$ and $\mathcal{I}-d(x,B_2)=1\ \forall x\in Q_1$. So $Q_1\subset\bullet B_2\subset\bullet H$. Now take $Q_2=K_2\cup B_2$. Then $Q_1\subset\bullet Q_2\subset\bullet H$. We proceed inductively. Suppose $\exists\ \mathfrak{T}_U$ closed set Q_n satisfying $Q_{n-1}\subset\bullet Q_n\subset\bullet H$ and $K_n\subset Q_n$. Then by Theorem 6.9, $\exists\ \mathfrak{T}_U$ closed set B_{n+1} such that $Q_n\subset\bullet B_{n+1}\subset\bullet H$. Let $Q_{n+1}=K_{n+1}\cup B_{n+1}$. Then $Q_n\subset\bullet Q_{n+1}\subset\bullet H$ and $K_{n+1}\subset Q_{n+1}$. By induction we obtain the collection $\{Q_n\}_{n\in\mathbb{N}}$ such that $K_n\subset Q_n\ \forall n\in\mathbb{N}$ and $Q_n\subset H\ \forall n\in\mathbb{N}$. Therefore,

$$H=\bigcup_{n\in\mathbb{N}}Q_n. \tag{19}$$

.

Now by Theorem 6.9, for each $l \in \mathbb{N} \cup \{0\}$ and $n \geq 2^l$ we define a $\mathfrak{T}_U-$closed set $Q_{\frac{n}{2^l}}$ such that

$$Q_{\frac{n}{2^l}} \subset \bullet\, Q_{\frac{(n+1)}{2^l}}. \tag{20}$$

. So we have the following cases:
For $l = 0$ we get $Q_1 \subset \bullet\, Q_2 \subset \bullet\, Q_3 \subset \bullet \cdots$
For $l = 1$ we get $Q_1 \subset \bullet\, Q_{\frac{3}{2}} \subset \bullet\, Q_2 \subset \bullet\, Q_{\frac{5}{2}} \subset \bullet\, Q_3 \subset \bullet \cdots$
For $l = 2$ we get $Q_1 \subset \bullet\, Q_{\frac{5}{4}} \subset \bullet\, Q_{\frac{3}{2}} \subset \bullet\, Q_{\frac{7}{4}} \subset \bullet\, Q_2 \subset \bullet\, Q_{\frac{9}{4}} \subset \bullet\, Q_{\frac{5}{2}} \subset \bullet \cdots$
and so on.

Suppose for fixed l_0 we choose $Q_{\frac{n}{2^{l_0}}}$ $\forall n \geq 2^{l_0}$ such that $Q_{\frac{n}{2^{l_0}}} \subset \bullet\, Q_{\frac{(n+1)}{2^{l_0}}}$. Since $Q_{\frac{n}{2^{l_0}}} = Q_{\frac{2n}{2^{l_0+1}}}$. So by (20) and Theorem 6.9, we have $Q_{\frac{2n}{2^{l_0+1}}} \subset \bullet\, Q_{\frac{2n+1}{2^{l_0+1}}}$ and $Q_{\frac{2n+1}{2^{l_0+1}}} \subset \bullet\, Q_{\frac{2n+2}{2^{l_0+1}}}$.

Therefore, $Q_{\frac{n}{2^{l_0}}} \subset \bullet\, Q_{\frac{2n+1}{2^{l_0+1}}} \subset \bullet\, Q_{\frac{(n+1)}{2^{l_0}}}$. In particular we get

$$Q_1 \subset \bullet \cdots \subset \bullet\, Q_{\frac{9}{8}} \subset \bullet \cdots \subset \bullet\, Q_{\frac{5}{4}} \subset \bullet \cdots \subset \bullet\, Q_{\frac{3}{2}} \subset \bullet \cdots \subset \bullet\, Q_{\frac{7}{4}} \subset \bullet \cdots \subset \bullet\, Q_{\frac{15}{8}} \cdots \subset \bullet\, Q_2 \cdots$$

For each real number $\beta \geq 1$ we define

$$Q_\beta = \bigcap_{\frac{n}{2^l} \geq \beta} Q_{\frac{n}{2^l}}.$$

.

Moreover, since each $Q_{\frac{n}{2^l}}$ is $\mathfrak{T}_U-$closed, so Q_β is $\mathfrak{T}_U-$closed. Now if $\beta_1 < \beta_2$ we can choose sufficiently large l_0 so that for some $n_0 \in \mathbb{N}$ we have $2^{l_0}\beta_1 < n_0 < (n_0 + 1) < 2^{l_0}\beta_2$. Observe that $Q_{\frac{(n_0+1)}{2^{l_0}}} \subset Q_{\frac{n}{2^l}}$ $\forall \frac{n}{2^l} \geq \beta_2$. Hence $Q_{\frac{(n_0+1)}{2^{l_0}}} \subset \bigcap_{\frac{n}{2^l} \geq \beta_2} Q_{\frac{n}{2^l}} = Q_{\beta_2}$. So, $Q_{\beta_1} \subset Q_{\frac{n_0}{2^{l_0}}} \subset \bullet\, Q_{\frac{(n_0+1)}{2^{l_0}}} \subset Q_{\beta_2}$. Consequently, $Q_{\beta_1} \subset \bullet\, Q_{\beta_2}$. Thus

$$H = \bigcup_{\beta \geq 1} Q_\beta.$$

We define $g : \mathbb{R} \to \mathbb{R}$ where

$$g(x) = \begin{cases} \frac{1}{\inf\{\beta : x \in Q_\beta\}} & \text{if} \quad x \in H \\ 0 & \text{if} \quad x \notin H \end{cases} \tag{21}$$

Since $\beta \geq 1$ so $g(x) \leq 1$ $\forall x \in \mathbb{R}$. Now take $x \in H$. From (19), we see $\exists$ some $n_0 \in \mathbb{N}$ such that $x \in Q_{n_0}$. So, $\inf\{\beta : x \in Q_\beta\} \leq n_0$ which means $g(x) \geq \frac{1}{n_0} > 0$. So, $0 < g(x) \leq 1$ $\forall x \in H$.

Next we are to prove g is $\mathcal{I}-\mathbb{AC}$. We first show that g is continuous on H^c. Let $x_0 \in H^c$. So by (19), $x_0 \in Q_n^c$ $\forall n$. Take in particular $n = N$ and then $x_0 \in Q_N^c$. Since Q_N is $\mathfrak{T}_U$ closed $\exists\, \delta > 0$ such that $Q_N \cap (x_0 - \delta, x_0 + \delta) = \phi$. Now since $Q_{\beta_1} \subset Q_{\beta_2}$ for $\beta_1 < \beta_2$, therefore for $\beta \leq N$ we get $Q_\beta \cap (x_0 - \delta, x_0 + \delta) = \phi$. Thus if $\beta \leq N$ and $x \in (x_0 - \delta, x_0 + \delta)$, then $x \in Q_\beta^c$. Thus $\inf\{\beta : x \in Q_\beta\} \geq N$ and so $g(x) \leq \frac{1}{N}$ for $x \in (x_0 - \delta, x_0 + \delta)$. Since choice of N is arbitrary, so $g(x_0) = 0$ $\forall x_0 \in H^c$. So, g is continuous on H^c.

Now we prove g is upper semi-continuous at any $x_0 \in H$. Let $g(x_0) = \frac{1}{\lambda'}$. Then for $\lambda < \lambda'$ we observe that $x_0 \notin Q_\lambda$. Since Q_λ is $\mathfrak{T}_U$ closed, so for sufficiently small $\delta > 0$ we have $(x_0 - \delta, x_0 + \delta) \subset Q_\lambda^c$. Thus for any $x \in (x_0 - \delta, x_0 + \delta)$, since $\inf\{\beta : x \in Q_\beta\} > \lambda$, we have $g(x) - g(x_0) < \frac{1}{\lambda} - \frac{1}{\lambda'}$. So we are done.

Now we show g is $\mathcal{I}$-approximately lower semi-continuous at points $x \in H$. Let $x_0 \in H$ and suppose $g(x_0) = \frac{1}{\lambda}$. For any $\alpha < g(x_0)$ let $C_\alpha = \{x : g(x) > \alpha\}$. It is enough to show $\mathcal{I} - d(x_0, C_\alpha) = 1$. Since $\alpha < \frac{1}{\lambda}$, there exists $\delta > 0$ such that $\alpha < \frac{1}{\lambda+2\delta} < \frac{1}{\lambda}$. Now we observe $\lambda = \inf\{\beta : x_0 \in Q_\beta\}$. So clearly $x_0 \in Q_{\lambda+\delta}$. From the properties of the family $\{Q_\beta : \beta \geq 1\}$ we have $Q_{\lambda+\delta} \subset \bullet\, Q_{\lambda+2\delta}$. Therefore $\mathcal{I} - d(x_0, Q_{\lambda+2\delta}) = 1$. We claim that $Q_{\lambda+2\delta} \subset C_\alpha$. For any $x \in Q_{\lambda+2\delta}$ we have $\inf\{\beta : x \in Q_\beta\} \leq \lambda + 2\delta$. That means $g(x) \geq \frac{1}{\lambda+2\delta}$. Since $\frac{1}{\lambda+2\delta} > \alpha$, so $g(x) > \alpha$. Consequently, $x \in C_\alpha$. Hence $Q_{\lambda+2\delta} \subset C_\alpha$. So, $\mathcal{I} - d(x_0, C_\alpha) = 1$. Hence, g is $\mathcal{I}$-approximately lower semi-continuous.

Thus g is $\mathcal{I} - \mathbb{AC}$ function. □

We now show $(\mathbb{R}, \mathfrak{T}_{\mathcal{I}})$ is completely regular. To prove this theorem we need the following lemma.

Lemma 7.3. *Let P_1, P_2, G be pairwise disjoint subsets of $\mathbb{R}$ such that*

(i) $P_1 \cup P_2 \cup G = \mathbb{R}$

(ii) $P_1 \cup G$ *and* $P_2 \cup G$ *are* $\mathcal{I} - d$ *open and of type Euclidean* F_σ.

Then there exists an $\mathcal{I} - \mathbb{AC}$ function g such that

(i) $g(x) = 0$ *for* $x \in P_1$

(ii) $0 < g(x) < 1$ *for* $x \in G$

(iii) $g(x) = 1$ *for* $x \in P_2$.

Proof. Since $P_1 \cup G$ and $P_2 \cup G$ both are Euclidean F_σ and also $(P_1 \cup G)^c = P_2$ and $(P_2 \cup G)^c = P_1$, so by Theorem 7.2, there exists two $\mathcal{I} - \mathbb{AC}$ functions g_1 and g_2 such that

$$0 < g_1(x) \leq 1 \text{ for } x \in P_2 \cup G \text{ and } g_1(x) = 0 \text{ for } x \in P_1$$
$$0 < g_2(x) \leq 1 \text{ for } x \in P_1 \cup G \text{ and } g_2(x) = 0 \text{ for } x \in P_2.$$

Now take, $\psi : (\mathbb{R} \times \mathbb{R}) \setminus \{(0,0)\} \to [0,1]$ where $\psi(x_1, x_2) = \frac{|x_1|}{|x_1|+|x_2|}$. Then,

$$\psi(0, x_2) = 0 \text{ for } x_2 \neq 0$$
$$\psi(x_1, 0) = 1 \text{ for } x_1 \neq 0$$
$$0 < \psi(x_1, x_2) < 1 \text{ for } x_1 \neq 0, x_2 \neq 0.$$

Then ψ is continuous except at $\{(0,0)\}$. We consider, $g(x) = \psi(g_1(x), g_2(x))$. Since modulus function is continuous, so by Theorem 5.3, $|g_1(x)|$ and $|g_2(x)|$ are $\mathcal{I} - \mathbb{AC}$. Moreover $|g_1(x)| + |g_2(x)| \neq 0$ for all x. Hence, by Theorem 5.2, g is $\mathcal{I} - \mathbb{AC}$.

Then for $x \in P_1$, $g(x) = \psi(0, g_2(x)) = 0$, since $g_2(x) \neq 0$ and for $x \in P_2$, $g(x) = \psi(g_1(x), 0) = 1$, since $g_1(x) \neq 0$. Finally for $x \in G$, $g_1(x) \neq 0$ and $g_2(x) \neq 0$. So, $g(x) = \frac{|g_1(x)|}{|g_1(x)|+|g_2(x)|}$. Thus $0 < g(x) < 1$ for $x \in G$.
□

Theorem 7.4. *The space $(\mathbb{R}, \mathfrak{T}_{\mathcal{I}})$ is completely regular.*

Proof. Let F be $\mathcal{I} - d$ closed set in $\mathbb{R}$ and $p_0 \notin F$. Since every $\mathcal{I} - d$ open set is measurable, F is measurable. Let H be an Euclidean G_δ-set such that $F \subset H$, $m(H \setminus F) = 0$ and $p_0 \notin H$. Let us put $P_1 = H, P_2 = \{p_0\}$ and $G = \mathbb{R} \setminus (P_1 \cup P_2)$. Then, $P_1 \cup G = \mathbb{R} \setminus \{p_0\} = (-\infty, p_0) \cup (p_0, \infty)$. Since each of $(-\infty, p_0)$ and (p_0, ∞) are Euclidean F_σ-set so their union is Euclidean F_σ-set. Moreover, $(-\infty, p_0)$ and (p_0, ∞) are $\mathfrak{T}_U$ open so $\mathcal{I} - d$ open. Again, $P_2 \cup G = \mathbb{R} \setminus H$ is Euclidean F_σ-set, H being an Euclidean G_δ-set. We observe $\mathbb{R} \setminus H = (\mathbb{R} \setminus F) \setminus (H \setminus F)$. Since $\mathbb{R} \setminus F$ is $\mathcal{I} - d$ open and $m(H \setminus F) = 0$ so $\mathbb{R} \setminus H$ is $\mathcal{I} - d$ open. By Lemma 7.3, there exists an $\mathcal{I} - \mathbb{AC}$ function $g : \mathbb{R} \to \mathbb{R}$ such that

1. $g(x) = 0$ for $x \in H$ and $H \supset F$
2. $0 < g(x) < 1$ for $x \in G$
3. $g(x) = 1$ for $x = p_0$

Therefore, $g(x) = 0$ on F and $g(p_0) = 1$. So, by Theorem 5.14, g is a continuous function on $(\mathbb{R}, \mathfrak{T}_{\mathcal{I}})$. Hence, $(\mathbb{R}, \mathfrak{T}_{\mathcal{I}})$ is completely regular. □

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