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On ϕ-S-1-absorbing δ-primary ideals of commutative rings

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Abstract. Let R be a commutative ring with unity $(1 \neq 0)$ and let $\mathfrak{I}(R)$ be the set of all ideals of R. Let $\phi : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ be a reduction function of ideals of R and let $\delta : \mathfrak{I}(R) \to \mathfrak{I}(R)$ be an expansion function of ideals of R. We recall that a proper ideal I of R is called a ϕ-1-absorbing δ-primary ideal of R, if whenever $abc \in I - \phi(I)$ for some nonunit elements $a, b, c \in R$, then $ab \in I$ or $c \in \delta(I)$. In this paper, we introduce a new class of ideals that is a generalization to the class of ϕ-1-absorbing δ-primary ideals. Let S be a multiplicative subset of R such that $1 \in S$ and let I be a proper ideal of R with $S \cap I = \emptyset$, then I is called a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$, if whenever $abc \in I - \phi(I)$ for some nonunit elements $a, b, c \in R$, then $sab \in I$ or $sc \in \delta(I)$. In this paper, we have presented a range of different examples, properties, characterizations of this new class of ideals.

1. Introduction

Throughout this paper all rings are commutative with unity $(1 \neq 0)$. Let $\mathfrak{I}(R)$ be the set of all ideals of R. In [17], Yassine et al. introduced the concept of 1-absorbing prime ideals as a generalization of prime ideals. A proper ideal I of R is called a 1-absorbing prime ideal if whenever $xyz \in I$ for some nonunit elements x, $y, z \in R$ then either $xy \in I$ or $z \in I$. After that in [13], Koc et al. defined weakly 1-absorbing prime ideals as a generalization of 1-absorbing prime ideals. Then in [2], Badawi and Celikel defined 1-absorbing primary ideals and in [3] they defined weakly 1-absorbing primary ideals. In a recent study, D. Zhao [20] introduced the concept of expansion function of ideals of R. Let δ be an expansion function of ideals of R, recall from [20] that a proper ideal I of R is said to be a δ-primary ideal of R, if $a, b \in R$ with $ab \in I$, then $a \in I$ or $b \in \delta(I)$. This concept of δ-primary ideals is a generalization of the concepts of prime ideals and primary ideals. Let δ be an expansion function of ideals of R and ϕ a reduction function of ideals of R. In a very recent studies, Yildiz et al. [19] defined ϕ-1-absorbing prime ideals as a generalization of 1-absorbing prime ideals and El Khalfi et al. [5] defined 1-absorbing δ-primary ideals as a generalization of 1-absorbing prime ideals. Let S be a multiplicative subset of R such that $1 \in S$. In [10], A. Hamed and A. Malek introduced the concept of S-prime ideal as a generalization of prime ideals. Recall from [10] that a proper ideal I of R with $I \cap S = \emptyset$ is said to be an S-prime if there exists $s \in S$ such that for all $a, b \in R$ with $ab \in I$ implies that $sa \in I$ or $sb \in I$. In [12] the author introduced the concept of ϕ-δ-primary ideals and this concept is a generalization of the concept of δ-primary ideals in [20], after that in [11] the author introduced the concept of ϕ-δ-S-primary ideals which is a generalization of the concept of ϕ-δ-primary ideals. In the most recent research, Mahdou et al. [14] defined an S-1-absorbing prime ideals and weakly S-1-absorbing prime ideals as generalizations

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of 1-absorbing prime ideals and weakly 1-absorbing prime ideals.

Let ϕ and δ be a reduction and an expansion functions of ideals of R, respectively. Motivated and inspired by the previous works, the purpose of this article is to extend the concepts of ϕ-S-prime ideals of R and ϕ-δ-S-primary ideals of R to the concept of ϕ-S-1-absorbing δ-primary ideals of R, where S is a multiplicative subset of R such that $1 \in S$. This means that the concept of ϕ-S-1-absorbing δ-primary ideals is a generalization of the concepts of ϕ-S-prime ideals of R and ϕ-δ-S-primary ideals of R. In Example 2.6(i) and Example 2.9, we show that the next right arrows of ideals are irreversible:

S-prime $\Rightarrow$ ϕ-S-prime ideals $\Rightarrow$ ϕ-S-1-absorbing prime ideals $\Rightarrow$ ϕ-S-1-absorbing δ-primary ideals.
ϕ-δ-primary ideals $\Rightarrow$ ϕ-1-absorbing δ-primary ideals $\Rightarrow$ ϕ-S-1-absorbing δ-primary ideals.

The main goal of our article is to study the reversibility of the above right arrows of ideals in a commutative ring with unity ($1 \neq 0$) and to present a range of different examples, properties, and characterizations of the concept of ϕ-S-1-absorbing δ-primary ideals.

Let ϕ, δ be a reduction function and an expansion function of ideals of R, respectively, and let S be a multiplicative subset of R such that $1 \in S$. In this paper, we call a proper ideal I of R, with $I \cap S = \emptyset$, a ϕ-S-1-absorbing δ-primary ideal of R associated to some $s \in S$ if whenever a, b, c are nonunit elements in R such that $abc \in I - \phi(I)$, then $sab \in I$ or $sc \in \delta(I)$. Among many results in the article, it is shown (Proposition 2.22) that if I is a ϕ-S-1-absorbing δ-primary ideal of R associated to some $s \in S$ such that it is not an S-1-absorbing δ-primary where (x, y, z) is a ϕ-S-1-δ-triple zero of I with $sxz, syz \notin I$, then $I^3 \subseteq \phi(I)$. Theorem 2.25 proves that a proper ideal I of R is a ϕ-S-1-absorbing δ-primary ideal of R associated to some $s \in S$ if and only if for each a, b nonunit elements in R such that $ab \notin (I : s)$ we have either $(I : ab) \subseteq (\delta(I) : s)$ or $(I : ab) = (\phi(I) : ab)$. Also, in the same theorem we prove that a proper ideal I of R is a ϕ-S-1-absorbing δ-primary ideal of R associated to some $s \in S$ if and only if for each proper ideals J, K, L of R such that $JKL \subseteq I$ but $JKL \nsubseteq \phi(I)$, either $sJK \subseteq I$ or $sL \subseteq \delta(I)$. Moreover, in the case when $\phi(I : a) = (\phi(I) : a)$, $\delta(I : a) = (\delta(I) : a)$ for each $a \in R$ and S satisfies the conditions $(I : t) \subseteq (I : s)$, $\phi(I) = (\phi(I) : t)$ for each $t \in S$, where s is a nonunit element in S, it is proved (Theorem 2.35) that I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ if and only if $(I : s)$ is a ϕ-1-absorbing δ-primary ideal of R. In section 3, let $f : X \to Y$ be a nonzero $(\delta, \phi) - (\gamma, \psi)$-surjective homomorphism. In Theorem 3.3, we prove that f induces one-to-one correspondence between ϕ-S-1-absorbing δ-primary ideals of X associated to some $s \in S$ consisting $\ker(f)$ and ψ-$f(S)$-1-absorbing γ-primary ideals of Y associated to $f(s) \in f(S)$. Also, in Lemma 3.6, we prove that if a,b,c are nonunit elements in X, then (a, b, c) is a ϕ-S-1-δ-triple zero of I, where I is a ϕ-S-1-absorbing δ-primary ideals of X associated to some $s \in S$ consisting $\ker(f)$, if and only if $(f(a), f(b), f(c))$ is a ψ-$f(S)$-1-γ-triple zero of $f(I)$. In the last section, we determine ϕ-S-1-absorbing δ-primary ideals in direct product of rings and we prove some results concerning ϕ-S-1-absorbing δ-primary ideals in direct product of rings. (See, Theorem 4.1, Corollary 4.2 and Theorem 4.4).

2. Properties of ϕ-S-1-absorbing δ-Primary ideals

Our aim in this section is to present a range of different properties, characterizations, and examples of ϕ-S-1-absorbing δ-primary ideals of R, where R is a commutative ring with unity ($1 \neq 0$). First, we start with the following basic definition.

Definition 2.1. *Let R be a commutative ring with unity ($1 \neq 0$), and let $\Im(R)$ be the set of all ideals of R.*
(1) Recall from [20] that a function $\delta : \Im(R) \to \Im(R)$ is called an expansion function of ideals of R if whenever I, J, K are ideals of R with $J \subseteq I$, then $\delta(J) \subseteq \delta(I)$ and $K \subseteq \delta(K)$.
(2) Recall from [12] that a function $\phi : \Im(R) \to \Im(R)$ is called a reduction function of ideals of R if $\phi(I) \subseteq I$ for all ideals I of R and if whenever $P \subseteq Q$, where P and Q are ideals of R, then $\phi(P) \subseteq \phi(Q)$.

Next, we define the concepts of S-1-absorbing δ-primary and ϕ-S-1-absorbing δ-primary ideals of R.

Definition 2.2. *Let R be a commutative ring with unity ($1 \neq 0$), and S a multiplicative subset of R. Suppose δ, ϕ are expansion and reduction functions of ideals of R, respectively.*
(1) A proper ideal I of R satisfying $I \cap S = \emptyset$ is said to be an S-1-absorbing δ-primary ideal of R associated to $s \in S$, if whenever $abc \in I$ for some nonunit elements $a, b, c \in R$, then $sab \in I$ or $sc \in \delta(I)$.
(2) A proper ideal I of R satisfying $I \cap S = \emptyset$ is said to be a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$, if whenever $abc \in I - \phi(I)$ for some nonunit elements $a, b, c \in R$, then $sab \in I$ or $sc \in \delta(I)$.

In the following example, we recall from [4] some examples of expansion functions of ideals of a given ring R.

Example 2.3.
(1) The identity function δ_0, where $\delta_0(I) = I$ for any $I \in \mathfrak{I}(R)$, is an expansion function of ideals in R.
(2) For each ideal I of R define $\delta_1(I) = \sqrt{I}$. Then δ_1 is an expansion function of ideals in R.
(3) Let J be a proper ideal of R. If $\delta(I) = I + J$ for every ideal I in $\mathfrak{I}(R)$, then δ is an expansion function of ideals in R.
(4) Let J be a proper ideal of R. If $\delta(I) = (I : J)$ for every ideal I in $\mathfrak{I}(R)$, then δ is an expansion function of ideals in R.
(5) Assume that δ_1, δ_2 are expansion functions of ideals of R. Let $\delta : \mathfrak{I}(R) \to \mathfrak{I}(R)$ such that $\delta(I) = \delta_1(I) + \delta_2(I)$. Then δ is an expansion function of ideals of R.
(6) Assume that δ_1, δ_2 are expansion functions of ideals of R. Let $\delta : \mathfrak{I}(R) \to \mathfrak{I}(R)$ such that $\delta(I) = \delta_1(I) \cap \delta_2(I)$. Then δ is an expansion function of ideals of R.
(7) Assume that $\delta_1,...,\delta_n$ are expansion functions of ideals of R. Let $\delta : \mathfrak{I}(R) \to \mathfrak{I}(R)$ such that $\delta(I) = \cap_{i=1}^{n} \delta_i(I)$ then δ is also an expansion function of ideals of R.
(8) Assume that δ_1, δ_2 are expansion functions of ideals of R. Let $\delta : \mathfrak{I}(R) \to \mathfrak{I}(R)$ such that $\delta(I) = \delta_1(\delta_2(I))$. Then δ is an expansion function of ideals of R.

Recall that if $\psi_1, \psi_2 : \mathfrak{I}(R) \to \mathfrak{I}(R) \cup \{\emptyset\}$ are expansion (reduction) functions of ideals of R, we define $\psi_1 \leq \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$ for each $I \in \mathfrak{I}(R)$.
In the following example, we recall from [1] some examples of reduction functions of ideals of a given ring R.

Example 2.4.
(1) The function $\phi_\emptyset$, where $\phi_\emptyset(I) = \emptyset$ for any $I \in \mathfrak{I}(R)$ is an ideal reduction.
(2) The function ϕ_0, where $\phi_0(I) = \{0\}$ for any $I \in \mathfrak{I}(R)$ is an ideal reduction.
(3) The function ϕ_2, where $\phi_2(I) = I^2$ for any $I \in \mathfrak{I}(R)$ is an ideal reduction.
(4) The function ϕ_n, where $\phi_n(I) = I^n$ for any $I \in \mathfrak{I}(R)$ is an ideal reduction.
(5) The function ϕ_ω, where $\phi_\omega(I) = \cap_{n=1}^{\infty} I^n$ for any $I \in \mathfrak{I}(R)$ is an ideal reduction.
(6) The function ϕ_1, where $\phi_1(I) = I$ for any $I \in \mathfrak{I}(R)$ is an ideal reduction.
Observe that $\phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_{n+1} \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1$.

Remark 2.5.
(1) If $\delta \leq \gamma$. Then every ϕ-S-1-absorbing δ-primary ideal of R is a ϕ-S-1-absorbing γ-primary ideal. In particular, every ϕ-S-1-absorbing prime ideal of R is a ϕ-S-1-absorbing δ-primary ideal. However, the converse is not true in general.
(2) If $\phi \leq \psi$. Then every ϕ-S-1-absorbing δ-primary ideal of R is a ψ-S-1-absorbing δ-primary ideal. In particular, every S-1-absorbing δ-primary ideal of R is a ϕ-S-1-absorbing δ-primary ideal. However, the converse is not true in general.

Example 2.6.
(i) Set $R = \mathbb{Z}_{24}$, $I = 8\mathbb{Z}_{24}$. Then $\delta_1(I) = \sqrt{I} = 2\mathbb{Z}_{24}$. Take $S = \{1\}$, $\phi = \phi_\emptyset$. Then it is easy to check that I is an S-1-absorbing δ_1-primary ideal of R, since if a, b, c are nonunit elements in R such that $abc \in I$, then $2/abc$. If $2/c$ then $c \in \delta_1(I)$. If not, then $8/ab$ implies that $ab \in I$. Moreover, I is not an S-1-absorbing prime ideal, since $(2)(2)(2) = 8 \in I$ but neither $4 \in I$ nor $2 \in I$.

(ii) Set $R = \mathbb{Z}_{24}$, $S = \{1, 5\}$. Then S is a multiplicative subset of R. Let $I = \{0\}$. Then $\delta_1(I) = 6\mathbb{Z}_{24}$, $\phi_2(I) = I^2 = (0)$. So, I is an almost-S-1-absorbing δ_1-primary ideal of R associated to $s = 5$. Moreover, $(3)(2)(4) = 0 \in I$ but neither $(5)(3)(2) \in I$ nor $(5)(4) \in \delta_1(I)$. Thus, I is not an S-1-absorbing δ_1-primary ideal of R associated to $s = 5$.

Proposition 2.7. Let $\{J_i : i \in \triangle\}$ be a directed set of ϕ-S-1-absorbing δ-primary ideals of R associated to $s \in S$. Then the ideal $J = \cup_{i\in\triangle} J_i$ is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$.

Proof.
Let $abc \in J - \phi(J)$, where a, b, c are nonunit elements in R. Suppose $sab \notin J$. We want to show that $sc \in \delta(J)$. Since $abc \notin \phi(J)$, we have $abc \notin \phi(J_i)$ for all $i \in\triangle$. Let $t \in\triangle$ such that $abc \in J_t - \phi(J_t)$, then $sab \in J_t$ or $sc \in \delta(J_t)$, since J_t is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$. Since $sab \notin J$, we have $sab \notin J_t$ which implies that $sc \in \delta(J_t) \subseteq \delta(J)$. Hence J is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$. ■

Proposition 2.8. Let $\{Q_i : i \in\triangle\}$ be a directed set of ϕ-S-1-absorbing δ-primary ideals of R associated to $s \in S$. Suppose $\phi(Q_i) = \phi(Q_j)$ and $\delta(Q_i) = \delta(Q_j)$ for every $i, j \in\triangle$. If ϕ, δ have the intersection property, then the ideal $J = \cap_{i\in\triangle} Q_i$ is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$.

Proof.
Let $t \in\triangle$, since $\phi(Q_i) = \phi(Q_t)$ and $\delta(Q_i) = \delta(Q_t)$ for every $i \in\triangle$, and since ϕ, δ have the intersection property, then $\phi(J) = \phi(Q_t)$ and $\delta(J) = \delta(Q_t)$. Let $abc \in J - \phi(J)$, where a, b, c are nonunit elements in R such that $sc \notin \delta(J)$. Then $abc \in Q_t - \phi(Q_t)$. Since Q_t is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$, we conclude that $sab \in Q_t$ or $sc \in \delta(Q_t)$. Since $sc \notin \delta(J)$, we get $sc \notin \delta(Q_t) = \delta(J)$. Hence we conclude that $sab \in Q_t$ for each $t \in\triangle$ which implies that $sab \in J$. Thus, J is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$.■

Obviously, every ϕ-1-absorbing δ-primary ideal of R is a ϕ-S-1-absorbing δ-primary ideal associated to $s \in S$. In particular, every weakly 1-absorbing δ_1-primary ideal of R is a weakly S-1-absorbing δ_1-primary ideal associated to $s \in S$. However, the next example shows that the converse is not true in general.

Example 2.9. Let $R = \mathbb{Z}_{24}[x]$, $P = 12\mathbb{Z}_{24}[x]$. Let $\phi = \phi_0$ and $\delta = \delta_1$, Let $S = \{4^k : k \geq 0\} = \{1, 4, 16\}$. Then S is a multiplicative subset of R such that $P \cap S = \emptyset$, and $\phi_0(P) = \{0\}$, $\delta_1(P) = 6\mathbb{Z}_{24}[x]$. We show that P is a weakly S-1-absorbing δ_1-primary ideal of R associated to $s = 4 \in S$. Let $f(x)$, $g(x)$, $h(x)$ be nonunit polynomials in R such that $0 \neq fgh \in P$. If $3/h(x)$ then $12/4h(x)$ implies that $4h(x) \in \delta_1(P)$. If not, then $3/f(x)g(x)$ implies that $12/4f(x)g(x)$ and hence, $4f(x)g(x) \in P$. Thus, we conclude that P is a weakly S-1-absorbing δ_1-primary ideal of R associated to $s = 4$. Since $0 \neq (3x)(3)(4x) = 12x^2 \in P$ and neither $9x \in P$ nor $4x \in \delta_1(P)$, we get that P is not a weakly 1-absorbing δ_1-primary ideal of R. Moreover, P is also not a weakly δ_1-primary ideal of R.

Following to [16], we give the following definition about quasi-local rings.

Definition 2.10. *A commutative ring R is said to be a quasi-local ring if it has a unique maximal ideal. Otherwise, we say R is a non-quasi-local ring.*

In the next result, we show that if a ring R admits an S-1-absorbing δ-primary ideal that is not a δ-S-primary, then R is a quasi-local ring.

Theorem 2.11. If I is an S-1-absorbing δ-primary ideal of R associated to $s \in S$ such that I is not a δ-S-primary, then R is a quasi-local ring.

Proof.
Since I is not a δ-S-primary ideal of R associated to s, there exist a, b in R such that $ab \in I$ with $sa \notin I$ and $sb \notin \delta(I)$. Then it is easy to see that a and b are nonunit elements in R. Now, let d be a nonunit element in R, then $adb \in I$. Thus, $sad \in I$, since $sb \notin \delta(I)$. Suppose for a unit element c in R, $d + c$ is a nonunit in R. Then $a(d + c)b \in I$ and $sb \notin \delta(I)$ implies that $sa(d + c) \in I$. So, $sac \in I$ implies that $sa \in I$, since $sad \in I$, a contradiction. Hence, $d + c$ is a unit in R. Thus the result follows from [2, Lemma 1]. ■

Theorem 2.12. Let R be a non-quasi-local ring and I proper ideal of R. Suppose that $(\phi(I) : x)$ is not a maximal ideal in R for each $x \in I$. Then the following statements are equivalent
(1) I is a ϕ-δ-S-primary ideal of R associated to $s \in S$.
(2) I is a ϕ-S-1-absorbing δ-primary ideal of R associated to s.

Proof.
($1 \to 2$): Clear.
($2 \to 1$): Let $x, y \in R$ such that $xy \in I - \phi(I)$. If x or y is a unit in R, then $sx \in I$ or $sy \in I \subseteq \delta(I)$. Therefore, we may assume that x, y are nonunit elements in R. Since $xy \notin \phi(I)$, we get that $(\phi(I) : xy)$ is a proper ideal of R. Let M be a maximal ideal of R such that $(\phi(I) : xy)$ is properly contained in M. Choose a maximal ideal N of R such that $N \neq M$, since R is a non-quasi-local ring. Let $z \in N - M$. Then z is a nonunit in R with $z \notin (\phi(I) : xy)$, since $(\phi(I) : xy) \subseteq M$. So, $zxy \in I - \phi(I)$ implies that $szx \in I$ or $sy \in \delta(I)$. If $sy \in \delta(I)$, then we are done. If not, then $szx \in I$. Let $a \in R$ such that $1 + az \in M$, since $z \notin M$. Then $1 + az$ is a nonunit in R. If $1 + az \notin (\phi(I) : xy)$, then $(1 + az)xy \in I - \phi(I)$ which implies that $s(1 + az)x \in I$, since $sy \notin \delta(I)$. Hence, $sx \in I$, since $szxa \in I$. Assume that $1 + az \in (\phi(I) : xy)$, then $(1 + az)xy \in \phi(I)$. Since $(\phi(I) : xy)$ is properly contained in M, choose $b \in M - (\phi(I) : xy)$, then b is a nonunit in R with $(1 + b + az)xy \in I - \phi(I)$. Moreover, $(1 + b + az)$ is not a unit in R since $(1 + b + az) \in M$, and $(1 + b + az)xy = (1 + az)xy + bxy \in I - \phi(I)$ implies that $bxy \in I - \phi(I)$. So, $sbx \in I$ since $sy \notin \delta(I)$. Also, $(1 + b + az)xy \in I - \phi(I)$ and $sy \notin \delta(I)$ implies that $s(1 + b + az)x = sx + sbx + szxa \in I$. Hence, $sx \in I$. Accordingly, I is a ϕ-δ-S-primary ideal of R associated to s. ■

Lemma 2.13. Let R be any ring. If I is an S-1-absorbing prime ideal of R associated to $s \in S$ that is not S-prime, then $(R, \mathbf{m})$ is a quasi-local ring and I is a 1-absorbing prime ideal of R that is not a prime such that $\mathbf{m}^2 \subseteq I \subsetneq \mathbf{m}$.

Proof.
Suppose that I is an S-1-absorbing prime ideal of R associated to $s \in S$ that is not S-prime. By Theorem 2.11, $(R, \mathbf{m})$ is a quasi-local ring with a unique maximal ideal $\mathbf{m}$. Moreover, since I is not an S-prime, there exist $x, y \in R$ such that $xy \in I$ and $sx \notin I$, $sy \notin I$. If x is a unit in R, then $y \in I$ implies that $sy \in I$, a contradiction. Similarly, if y is a unit, then we get again a contradiction. Therefore, we may assume that x, y are nonunit elements in R. Let $a, b \in \mathbf{m}$, then $abxy \in I$ implies that $sabx \in I$, since $sy \notin I$. So, $sabx \in I$ implies that $s^2ab = abs^2 \in I$, since $sx \notin I$. Thus, $abs^2 \in I$ implies that $sab \in I$ or $s^3 \in I$. Since $I \cap S = \emptyset$ and $s^3 \in S$, we get that $sab \in I$ implies that $s\mathbf{m}^2 \subseteq I$. If s is not a unit in R, then $s \in \mathbf{m}$ implies that $s^3 \in s\mathbf{m}^2 \subseteq I$, a contradiction. Therefore, s is a unit in R. Thus, I is a 1-absorbing prime ideal of R that is not a prime. Hence, by [6, Lemma 1], $\mathbf{m}^2 \subseteq I \subsetneq \mathbf{m}$. ■

Proposition 2.14. Let I be a proper ideal of R such that I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ such that $\sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$ and $\sqrt{\phi(I)} \subseteq \phi(\sqrt{I})$, then $\sqrt{I}$ is a ϕ-S-δ-primary ideal of R associated to s.

Proof.
Let $a, b \in R$ such that $ab \in \sqrt{I} - \phi(\sqrt{I})$. Then $ab \in \sqrt{I}$ which implies that $a^n b^n \in I$ for some $n \geq 1$. If $a^n b^n \in \phi(I)$, then $ab \in \sqrt{\phi(I)} \subseteq \phi(\sqrt{I})$, a contradiction. Thus, $a^n b^n \in I - \phi(I)$. So $a^k a^k b^n \in I - \phi(I)$ for some positive integer k. Since a, b are nonunit elements in R and I is a ϕ-S-1-absorbing δ-primary ideal of R associated to s, we conclude that $sa^{2k} \in I$ or $sb^n \in \delta(I)$. Thus, $sa \in \sqrt{I}$ or $sb \in \sqrt{\delta(I)} \subseteq \delta(\sqrt{I})$. Hence, $\sqrt{I}$ is a ϕ-δ-S-primary ideal of R associated to s. ■

Corollary 2.15. Let I be a proper ideal of R such that I is a ϕ-S-1-absorbing primary ideal of R associated to $s \in S$. Suppose that $\sqrt{\phi(I)} \subseteq \phi(\sqrt{I})$. Then $\sqrt{I}$ is a ϕ-S-prime ideal of R associated to s.

Proof.
Let $\delta(J) = \sqrt{J}$ for every ideal J in R. Then, by the above proposition, if I is a ϕ-S-1-absorbing primary ideal of R associated to s then $\sqrt{I}$ is a ϕ-S-prime ideal of R associated to s. ■

Proposition 2.16. Let I be a proper ideal of R such that I is a ϕ-S-1-absorbing primary ideal of R associated to $s \in S$. Suppose that $\sqrt{\phi(I)} \subseteq \phi(\sqrt{I})$ and $(\phi(\sqrt{I}) : x) \subseteq (\phi(\sqrt{I}) : s)$ for each $x \in S$. If $a \in R - (\sqrt{I} : s)$, then $S \cap (\sqrt{I} : a) = \emptyset$.

Proof.
It is easy to see that $\sqrt{I} \cap S = \emptyset$, since $I \cap S = \emptyset$. Also, by the above corollary, $\sqrt{I}$ is a ϕ-S-prime ideal of R associated to s. We show that $S \cap (\sqrt{I} : a) = \emptyset$. Let $t \in S$ such that $ta \in \sqrt{I}$. If $ta \in \phi(\sqrt{I})$, then $a \in (\phi(\sqrt{I}) : t) \subseteq (\phi(\sqrt{I}) : s)$ which implies that $sa \in \phi(\sqrt{I}) \subseteq \sqrt{I}$, a contradiction. Thus, $ta \in \sqrt{I} - \phi(\sqrt{I})$ implies that $sa \in \sqrt{I}$ or $st \in \sqrt{I}$, which is a contradiction again, since $a \notin (\sqrt{I} : s)$ and $S \cap \sqrt{I} = \emptyset$. Thus, $S \cap (\sqrt{I} : a) = \emptyset$. ■

Corollary 2.17. Let I be a proper ideal of R such that I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ with $\delta(I) \subseteq \sqrt{I}$. Suppose $(\phi(\sqrt{I}) : x) \subseteq (\phi(\sqrt{I}) : s)$ for each $x \in S$ and $(\delta(I) : s) = (\sqrt{I} : s)$. Then $(\delta(I) : s) = (\delta(I) : s^2)$ and if whenever $a \in R - (\delta(I) : s)$, then $S \cap (\delta(I) : a) = \emptyset$.

Proof.
Since $\delta(I) \subseteq \sqrt{I}$, it is easy to see that if I is a ϕ-S-1-absorbing δ-primary ideal of R associated to s, then I is a ϕ-S-1-absorbing primary ideal of R associated to s. Moreover, if $\delta(I) \subseteq \sqrt{I}$ and $(\delta(I) : s) = (\sqrt{I} : s)$, then $(\delta(I) : s) = (\delta(I) : s^2)$, since $(\sqrt{I} : s) = (\sqrt{I} : s^2)$. So, if $a \in R - (\delta(I) : s)$, then $sa \notin \sqrt{I}$. Thus, by the above proposition, $S \cap (\sqrt{I} : a) = \emptyset$. Hence $S \cap (\delta(I) : a) \subseteq S \cap (\sqrt{I} : a) = \emptyset$, since $\delta(I) \subseteq \sqrt{I}$. ■

Recall that if I, J, K are ideals of R such that $K = I \cup J$, then $K = I$ or $K = J$.

Theorem 2.18. Let I be a proper ideal of R such that I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$, where $s \notin U(R)$. If $c \in R - (\delta(I) : s^2)$ such that c is not a unit in R, then $(I : s^2c) = (I : s^2)$ or $(I : s^2c) = (\phi(I) : s^2c)$.

Proof.
It is enough to show that $(I : s^2c) = (I : s^2) \cup (\phi(I) : s^2c)$. It is easy to see that $(I : s^2)$ and $(\phi(I) : s^2c)$ are subsets of $(I : s^2c)$. Let $r \in (I : s^2c)$, then $rs^2c \in I$. If $rs^2c \in \phi(I)$ then $r \in (\phi(I) : s^2c)$. So we may assume that $rs^2c \notin \phi(I)$. Thus, $rs^2c = (sr)(sc) \in I - \phi(I)$ implies that $s^2r \in I$ since $s^2c \notin \delta(I)$. So, $r \in (I : s^2)$. Thus, $(I : s^2c) = (I : s^2) \cup (\phi(I) : s^2c)$. Hence $(I : s^2c) = (I : s^2)$ or $(I : s^2c) = (\phi(I) : s^2c)$. ■

Corollary 2.19. Let I be a proper ideal of R such that I is a ϕ-S-1-absorbing primary ideal of R associated to $s \in S$, where $s \notin U(R)$. If $c \in R - (\sqrt{I} : s)$ such that c is not a unit in R, then $(I : s^2c) = (I : s^2)$ or $(I : s^2c) = (\phi(I) : s^2c)$.

Proof.
It is easy to see that $(\sqrt{I} : s) = (\sqrt{I} : s^2)$. So if c is not a unit in R such that $c \in R - (\sqrt{I} : s)$, then $c \in R - (\sqrt{I} : s^2)$. Hence, by Theorem 2.18, $(I : s^2c) = (I : s^2)$ or $(I : s^2c) = (\phi(I) : s^2c)$. ■

Definition 2.20. *Let I be a proper ideal of R such that I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ and let x, y, z be nonunit elements in R. Then (x, y, z) is said to be a ϕ-S-1-δ-triple zero of I, if $xyz \in \phi(I)$, $sxy \notin I$ and $sz \notin \delta(I)$.*

Lemma 2.21. Let I be a proper ideal of R such that I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$. If I is not an S-1-absorbing δ-primary ideal of R associated to s, then there exist nonunit elements x, y, z in R such that (x, y, z) is a ϕ-S-1-δ-triple zero of I.

Proof.
Suppose (x, y, z) is not a ϕ-S-1-δ-triple zero of I for each nonunit elements x, y, z in R. We show that I is an S-1-absorbing δ-primary ideal of R associated to s. Let a, b, c be nonunit elements in R such that $abc \in I$. If $abc \notin \phi(I)$, then $sab \in I$ or $sc \in \delta(I)$. So we may assume that $abc \in \phi(I)$. Since (a, b, c) is not a ϕ-S-1-δ-triple zero of I, we have $sab \in I$ or $sc \in \delta(I)$. Hence, we conclude that I is an S-1-absorbing δ-primary ideal of R associated to s. ■

Proposition 2.22. Let I be a proper ideal of R such that I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$. If I is not an S-1-absorbing δ-primary associated to s and (x, y, z) is a ϕ-S-1-δ-triple zero of I. Then
(1) $xyI \subseteq \phi(I)$.
(2) If $sxz, syz \notin I$, then $I^3 \subseteq \phi(I)$. In this case $\sqrt{I} = \sqrt{\phi(I)}$.

Proof.
(1) Suppose that (x, y, z) is a ϕ-S-1-δ-triple zero of I, then, by the lemma above, $xyz \in \phi(I)$ such that $sxy \notin I$ and $sz \notin \delta(I)$. Assume that $xyI \nsubseteq \phi(I)$. Then there exists $a \in I$ such that $xya \notin \phi(I)$. So, $xy(z + a) \in I - \phi(I)$. If $(z + a)$ is a unit in R, then $xy \in I$ implies that $sxy \in I$, a contradiction. So, $(z + a)$ is not a unit implies that $sxy \in I$ or $s(z + a) \in \delta(I)$ which is a contradiction. Thus, $xyI \subseteq \phi(I)$.
(2) If $sxz, syz \notin I$, then by using similar argument above we get $xzI \subseteq \phi(I)$ and $yzI \subseteq \phi(I)$. Now, we show that $xI^2 \subseteq \phi(I)$. If $xI^2 \nsubseteq \phi(I)$ then there exist $a, b \in I$ such that $xab \notin \phi(I)$ which implies that $x(y+a)(z+b) \in I-\phi(I)$. If $(y + a)$ is a unit in R, then $x(z + b) \in I$ implies that $xz \in I$ and so $sxz \in I$, a contradiction. Hence, $(y + a)$ is not a unit. Similarly, $(z + b)$ is not a unit in R, since $sxy \notin I$. Thus, either $sx(y + a) \in I$ or $s(z + b) \in \delta(I)$ implies that $sxy \in I$ or $sz \in \delta(I)$, a contradiction. So, we conclude that $xI^2 \subseteq \phi(I)$. Again by using the same argument above we get $yI^2 \subseteq \phi(I)$ and $zI^2 \subseteq \phi(I)$. Now, we show that $I^3 \subseteq \phi(I)$. If $I^3 \nsubseteq \phi(I)$, then there exist $a, b, c \in I$ such that $abc \notin \phi(I)$. Hence we conclude that $(x + a)(y + b)(z + c) \in I - \phi(I)$. If $(x + a)$ is a unit in R, then $(y+b)(z+c) = yz+yc+bz+bc \in I$ implies that $yz \in I$, so $syz \in I$, a contradiction. Similarly, $(y + b)$, $(z + c)$ are nonunit elements in R, since $sxz \notin I$ and $sxy \notin I$. Thus we have $s(x+a)(y+b) \in I$ or $s(z+c) \in \delta(I)$ which implies that $sxy \in I$ or $sz \in \delta(I)$, a contradiction. Accordingly, $I^3 \subseteq \phi(I)$. Hence $\sqrt{I} \subseteq \sqrt{\phi(I)} \subseteq \sqrt{I}$. Thus, $\sqrt{I} = \sqrt{\phi(I)}$. ■

Let R be a commutative ring with unity, I a proper ideal of R. Then, by Proposition 2.22 and by taking $S = \{1\}$, the following results hold.

Remark 2.23.
(1) If I is a weakly 1-absorbing prime ideal of R such that I is not a 1-absorbing prime ideal where (x, y, z) is a weakly 1-triple zero of I with $xz, yz \notin I$, then $I^3 = 0$ (it suffices to take $\delta = \delta_0$, $\phi = \phi_0$).
(2) If I is a weakly 1-absorbing primary ideal of R such that I is not a 1-absorbing primary ideal where (x, y, z) is a weakly 1-δ_1-triple zero of I with $xz, yz \notin I$, then $I^3 = 0$ (it suffices to take $\delta = \delta_1$, $\phi = \phi_0$).
(3) If I is an n-almost 1-absorbing primary ideal of R such that I is not a 1-absorbing primary ideal where (x, y, z) is an almost 1-δ_1-triple zero of I with $xz, yz \notin I$, then $I^3 = I^n$ (it suffices to take $\delta = \delta_1$, $\phi = \phi_n$), where $n \geq 3$.

Corollary 2.24. Let I be a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$, where s is not a unit in R, such that if x, y, z are nonunit elements in R with $xyz \in \phi(I)$, then $sxz \notin I$ and $syz \notin I$. Then $I^3 \subseteq \phi(I)$ or $s\sqrt{\phi(I)} \subseteq \delta(I)$.

Proof.
Suppose that $I^3 \nsubseteq \phi(I)$. If I is not an S-1-absorbing δ-primary ideal of R associated to s, then there exist nonunit elements $x, y, z \in R$ such that (x, y, z) is a ϕ-S-1-δ-triple zero of I. By part(2) of Proposition 2.22, we have $sxz \in I$ or $syz \in I$, since $I^3 \nsubseteq \phi(I)$, a contradiction with the assumption. Thus, I has no ϕ-S-1-δ-triple zero and this implies that I is an S-1-absorbing δ-primary ideal of R associated to s. In this case we show that $s\sqrt{\phi(I)} \subseteq \delta(I)$. Suppose on the contrary that $s\sqrt{\phi(I)} \nsubseteq \delta(I)$, then there exists $c \in \sqrt{\phi(I)}$ such that $sc \notin \delta(I)$. Let k be the minimal positive integer such that $c^k \in \phi(I) \subseteq I$. If $c \in I$, then $sc \in \delta(I)$, a contradiction. So, we may assume that $c \notin I$. Therefore, $k \geq 2$. If $k = 2$, then $sc^2 \in I$. Since c, s are not units in R and I is an S-1-absorbing δ-primary ideal of R associated to s, we get $s^2c \in I$, since $sc \notin \delta(I)$. Again, $s^2c \in I$ implies that $s^3 \in I$ or $sc \in \delta(I)$, a contradiction. Thus, $sc^2 \notin I$. If $k \geq 3$, then $c^k = c^{k-1}c \in I$ implies that $sc^{k-1} \in I$, since $sc \notin \delta(I)$. Continuing in this process to get that $sc^2 \in I$, a contradiction. Hence we conclude that $s\sqrt{\phi(I)} \subseteq \delta(I)$. ■

Theorem 2.25. Let I be a proper ideal of R. Then the following statements are equivalent.
(1) I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$.
(2) For each nonunit elements $a, b \in R$ such that $ab \notin (I : s)$ we have either $(I : ab) \subseteq (\delta(I) : s)$ or $(I : ab) = (\phi(I) : ab)$.
(3) For each proper ideal J of R such that $abJ \subseteq I$ but $abJ \nsubseteq \phi(I)$, either $sab \in I$ or $sJ \subseteq \delta(I)$.
(4) For each proper ideals J, K of R such that $aJK \subseteq I$ but $aJK \nsubseteq \phi(I)$, either $saJ \subseteq I$ or $sK \subseteq \delta(I)$.
(5) For each proper ideals J, K, L of R such that $JKL \subseteq I$ but $JKL \nsubseteq \phi(I)$, either $sJK \subseteq I$ or $sL \subseteq \delta(I)$.

Proof.
$(1 \to 2)$: Let a, b be nonunit elements in R such that $ab \notin (I : s)$, then $sab \notin I$. Suppose that $(I : ab) \neq (\phi(I) : ab)$,

then there exists $c \in (I : ab)$ such that $c \notin (\phi(I) : ab)$. So, $cab \in I - \phi(I)$ implies that $sc \in \delta(I)$, since $sab \notin I$. (note that c is not a unit in R, since if c is a unit, then $ab \in I$ implies that $sab \in I$, which is a contradiction). We show that $(I : ab) \subseteq (\delta(I) : s)$. Let $x \in (I : ab)$, then $xab \in I$. If $x \in (\phi(I) : ab)$, then $(x + c)ab \in I - \phi(I)$ and $(x + c)$ is not a unit in R, since $ab \notin I$. This implies that $s(x + c) \in \delta(I)$ and hence $sx \in \delta(I)$, since $sc \in \delta(I)$ and $sab \notin I$. Thus, $(I : ab) \subseteq (\delta(I) : s)$

$(2 \to 3)$: Suppose that $abJ \subseteq I$ but $abJ \nsubseteq \phi(I)$. If $ab \notin (I : s)$, then $(I : ab) = (\phi(I) : ab)$ or $(I : ab) \subseteq (\delta(I) : s)$. If $(I : ab) = (\phi(I) : ab)$, then $abJ \subseteq \phi(I)$, a contradiction. So, $(I : ab) \subseteq (\delta(I) : s)$, which implies that $J \subseteq (I : ab) \subseteq (\delta(I) : s)$. Thus, $sJ \subseteq \delta(I)$.

$(3 \to 4)$: Let a be a nonunit element in R such that $aJK \subseteq I$ but $aJK \nsubseteq \phi(I)$. Suppose $sK \nsubseteq \delta(I)$. We show that $saJ \subseteq I$. Let $y \in J$ be fixed such that $ayK \nsubseteq \phi(I)$. If $ay \notin (I : s)$, then either $(I : ay) \subseteq (\delta(I) : s)$ or $(I : ay) = (\phi(I) : ay)$. Since $K \subseteq (I : ay)$ and $sK \nsubseteq \delta(I)$ we have $(I : ay) = (\phi(I) : ay)$ which implies that $ayK \subseteq \phi(I)$, a contradiction. Thus, $say \in I$. Now, let $x \in J$, then $axK \subseteq I$. If $axK \subseteq \phi(I)$, then $a(x + y)K \nsubseteq \phi(I)$ and $(x + y)$ is not a unit in R, since $(x + y) \in J$. So, by using the same argument above, we have $sa(x + y) \in I$ implies that $sax \in I$, since $say \in I$. If $axK \nsubseteq \phi(I)$, then $K \subseteq (I : ax)$ and $K \nsubseteq (\delta(I) : s)$. If $sax \notin I$, then $(I : ax) = (\phi(I) : ax)$ which implies that $axK \subseteq \phi(I)$, a contradiction. Thus, $sax \in I$. Hence we conclude that $saJ \subseteq I$.

$(4 \to 5)$: Let J, K, L be proper ideals of R such that $JKL \subseteq I$ but $JKL \nsubseteq \phi(I)$. Suppose $sL \nsubseteq \delta(I)$. We show that $sJK \subseteq I$. Let $a \in J$ be fixed such that $aKL \subseteq I$ but $aKL \nsubseteq \phi(I)$. Then, by (4), $saK \subseteq I$, since $sL \nsubseteq \delta(I)$. Now, let $x \in J$. If $xKL \nsubseteq \phi(I)$, then, by the same argument above, we have $sxK \subseteq I$. If $xKL \subseteq \phi(I)$, then $(a + x)KL \nsubseteq \phi(I)$, again by the same argument above, we have $s(a + x)K \subseteq I$ and since $saK \subseteq I$, we get $sxK \subseteq I$. Consequently, we conclude that $sJK \subseteq I$.

$(5 \to 1)$: Let a, b, c be nonunit elements in R such that $abc \in I - \phi(I)$. Then $< a >< b >< c > \subseteq I$ and $< a >< b >< c > \nsubseteq \phi(I)$ implies that $s < a >< b > \subseteq I$ or $s < c > \subseteq \delta(I)$. Thus, $sab \in I$ or $sc \in \delta(I)$. Accordingly, I is a ϕ-S-1-absorbing δ-primary ideal of R associated to s. ■

Theorem 2.26. Let P be a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$. If $(\phi(P) : d) \subseteq \phi(P : d)$ and $(\delta(P) : d) \subseteq \delta(P : d)$ for each nonunit element $d \in R - P$, then $(P : d)$ is also ϕ-S-1-absorbing δ-primary ideal of R associated to s.

Proof.
Let x, y, z be nonunit elements in R such that $xyz \in (P : d) - \phi(P : d)$. So, $xydz \in P - \phi(P)$ implies that $sxyd \in P$ or $sz \in \delta(P)$. Hence, $sxy \in (P : d)$ or $sz \in \delta(P) \subseteq (\delta(P) : d) \subseteq \delta(P : d)$ Thus, $(P : d)$ is a ϕ-S-1-absorbing δ-primary ideal of R associated to s. ■

Corollary 2.27. Let P be a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ and J a proper ideal in R such that $J \nsubseteq P$. If $(\phi(P) : J) \subseteq \phi(P : J)$ and $(\delta(P) : J) \subseteq \delta(P : J)$, then $(P : J)$ is a ϕ-S-1-absorbing δ-primary ideal of R associated to s.

Proof.
Let a, b, c be nonunit elements in R such that $abc \in (P : J) - \phi(P : J)$. Then $abcJ \subseteq P$ and $abcJ \nsubseteq \phi(P)$, since $(\phi(P) : J) \subseteq \phi(P : J)$. Thus, $< a >< b >< c > J \subseteq P$ and $< a >< b >< c > J \nsubseteq \phi(P)$ implies that, by Theorem 2.25, $s < a >< b > \subseteq P \subseteq (P : J)$ or $s < c > J \subseteq \delta(P)$. So, $sab \in P \subseteq (P : J)$ or $sc \in (\delta(P) : J) \subseteq \delta(P : J)$. Hence $(P : J)$ is a ϕ-S-1-absorbing δ-primary ideal of R associated to s. ■

Suppose that I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ such that $\phi \neq \phi_\emptyset$. If $(I : s) = (\delta(I) : s)$, then the following result holds.

Proposition 2.28. Let I be a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ such that $(I : s) = (\delta(I) : s)$. If I is not an S-1-absorbing δ-primary such that (x, y, z) is a ϕ-S-1-δ-triple zero of I with $sxz, syz \notin I$, then $sI^2(\sqrt{\phi(I)})^2 \subseteq \phi(I)$.

Proof.
Suppose I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$. If I is not an S-1-absorbing δ-primary such that (x, y, z) is a ϕ-S-1-δ-triple zero of I with $sxz, syz \notin I$ then, by Proposition 2.22, $I^3 \subseteq \phi(I)$. Let

$a, b \in \sqrt{\phi(I)}$, if $ab \in (I : s)$ then $sab \in I$ which implies that $sabI^2 \subseteq I^3 \subseteq \phi(I)$. Therefore we may assume that $ab \notin (I : s) = (\delta(I) : s)$, then by Theorem 2.25, $(I : ab) \subseteq (I : s) = (\delta(I) : s)$ or $(I : ab) = (\phi(I) : ab)$. Now, if $(I : ab) = (\phi(I) : ab)$, then $abI \subseteq \phi(I)$ implies that $sabI^2 \subseteq \phi(I)$. So we may assume that $(I : ab) \subseteq (I : s)$. Let $n \geq 1$ be the minimal integer such that $(ab)^n \in \phi(I)$. Then $(ab)^{n-1} \in (I : ab) \subseteq (I : s)$ implies that $s(ab)^{n-1} \in I$. Clearly, $n - 1 \geq 2$, since $sab \notin I$. If $s(ab)^{n-1} \notin \phi(I)$, then $s(ab)^{n-1} \in I - \phi(I)$ and s is not a unit in R, since if s is a unit, then $I = (I : s) = (I : ab)$ implies that $(ab)^{n-2} \in I = (I : ab)$, so continuing in this process to get that $ab \in I$ which is a contradiction. Thus, $sa^{n-1}b^{n-1} = a^{n-2}b^{n-2}(sab) \in I - \phi(I)$ implies that $s(ab)^{n-2} \in I$ or $s^2ab \in \delta(I)$. But, if $s^2ab \in \delta(I)$, then $sab \in (\delta(I) : s) = (I : s)$ implies that $s^2ab \in I$, which implies that $s^2 \in (I : ab) \subseteq (I : s)$, a contradiction. So, $s(ab)^{n-2} \in I$ implies that $s(ab)^{n-2} \in I - \phi(I)$, since $s(ab)^{n-1} \notin \phi(I)$. Continuing in this process to get that $sab \in I$ which is a contradiction. Therefore, $s(ab)^{n-1} \in \phi(I)$. Let j be the minimal integer such that $s(ab)^j \in \phi(I)$. Then $j > 1$, since $sab \notin \phi(I)$. Suppose there exist $x, y \in I$ such that $sabxy \notin \phi(I)$. Then $sab((ab)^{j-1} + xy) \in I - \phi(I)$ and $((ab)^{j-1} + xy)$ is not a unit in R, since $sab \notin I$. Thus, $sab((ab)^{j-1} + xy) \in I - \phi(I)$ implies that $s^2ab \in I$ or $s((ab)^{j-1} + xy) \in \delta(I)$. Since $s^2ab \notin I$, we have $s((ab)^{j-1} + xy) \in \delta(I)$ which implies that $(ab)^{j-1} + xy \in (\delta(I) : s) = (I : s)$. Thus, $s(ab)^{j-1} + sxy \in I$ implies that $s(ab)^{j-1} \in I$, since $sxy \in I$. Since $j > 1$ is the minimal integer such that $s(ab)^j \in \phi(I)$, we get $s(ab)^{j-1} \in I - \phi(I)$. Again continuing in this process to get that $sab \in I$ which is a contradiction. Hence, $sabxy \in \phi(I)$ for each $x, y \in I$ and for each $a, b \in \sqrt{\phi(I)}$. Consequently, we conclude that $sI^2(\sqrt{\phi(I)})^2 \subseteq \phi(I)$. ■

Proposition 2.29. Let δ be an expansion function of ideals of R satisfies the intersection property, and let I be a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ such that $\phi(J) = \phi(I)$ for each ideal $J \subseteq I$. If P is an ideal in R such that $P \cap S \neq \emptyset$, then $I \cap P$ and IP are ϕ-S-1-absorbing δ-primary ideals of R.

Proof.
It is clear that $(P \cap I) \cap S = PI \cap S = \emptyset$. Pick $t \in P \cap S$. We show that $I \cap P$ is a ϕ-S-1-absorbing δ-primary ideal of R associated to ts. Let a, b, c be nonunit elements in R such that $abc \in I \cap P - \phi(I \cap P)$, then $abc \in I \cap P - \phi(I) \subseteq I - \phi(I)$. Thus, $sab \in I$ or $sc \in \delta(I)$ implies that $tsab \in I \cap P$ or $tsc \in \delta(I) \cap \delta(P) = \delta(I \cap P)$. Consequently, $I \cap P$ is a ϕ-S-1-absorbing δ-primary ideal of R associated to ts. We have the similar proof for IP. ■

Theorem 2.30. Let $n \geq 2$ and let a be a nonunit element in R with $(0 : a) \subseteq (a)$. Then $(a) = aR$ is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ with $\phi \leq \phi_n$ if and only if (a) is an S-1-absorbing δ-primary ideal of R associated to s.

Proof.
Every S-1-absorbing δ-primary ideal of R associated to $s \in S$ is a ϕ-S-1-absorbing δ-primary ideal of R associated to s. Conversely, let x, y, z be nonunit elements in R such that $xyz \in (a)$. If $xyz \notin (a)^n$, then $sxy \in (a)$ or $sz \in \delta((a))$. Suppose $xyz \in (a)^n$. Then $xy(z + a) \in (a)$. If $(z + a)$ is a unit in R, then $xy \in (a)$ implies that $sxy \in (a)$. Therefore, we may assume that $(z + a)$ is not a unit in R. If $xy(z + a) \notin (a)^n$, then $xy(z + a) \in (a) - (a)^n$ implies that $sxy \in (a)$ or $sz \in \delta((a))$, since $sa \in (a) \subseteq \delta((a))$. Assume that $xy(z + a) \in (a)^n$, then $xya \in (a)^n$, since $xyz \in (a)^n$. So there exists $t \in R$ such that $xya = ta^n$ implies that $(sxy - sta^{n-1})a = 0$. Thus, $sxy - sta^{n-1} \in (0 : a) \subseteq (a)$. Hence, we conclude that $sxy \in (a)$, since $sta^{n-1} \in (a)$. Consequently, (a) is an S-1-absorbing δ-primary ideal of R associated to s. ■

Remark 2.31. Let $S_1 \subseteq S_2$ be multiplicative subsets of R and I an ideal of R disjoint with S_2. Clearly, if I is a ϕ-S_1-1-absorbing δ-primary ideal of R associated to $s \in S_1$, then I is a ϕ-S_2-1-absorbing δ-primary ideal of R associated to $s \in S_2$. However, the converse is not true in general.

Proposition 2.32. Let $S_1 \subseteq S_2$ be multiplicative subsets of R such that for any $s \in S_2$, there exists $t \in S_2$ with $st \in S_1$. If I is a ϕ-S_2-1-absorbing δ-primary ideal of R associated to $s \in S_2$, then I is a ϕ-S_1-1-absorbing δ-primary ideal of R.

Proof.
Let $t \in S_2$ such that $st \in S_1$. We show that I is a ϕ-S_1-1-absorbing δ-primary ideal of R associated to $st \in S_1$. Let a, b, c be nonunit elements in R such that $abc \in I - \phi(I)$, then $sab \in I$ implies that $stab \in I$ or $sc \in \delta(I)$

implies that $stc \in \delta(I)$. Consequently, I is a ϕ-S_1-1-absorbing δ-primary ideal of R associated to $st \in S_1$. ■

Recall that if S is a multiplicative subset of R with $1 \in S$, then $S^* = \{r \in R : \frac{r}{1} \in U(S^{-1}R)\}$ is said to be the saturation of S. One can easily see that S^* is a multiplicative subset of R containing S. If $S = S^*$, then S is called saturated. Moreover, it is clear that $S^{**} = S^*$. (See [9]).

Proposition 2.33. I is a ϕ-S-1-absorbing δ-primary ideal of R if and only if I is a ϕ-S^*-1-absorbing δ-primary ideal of R.

Proof.
First we show that $S^* \cap I = \emptyset$. Let $r \in S^* \cap I$, then $\frac{r}{1}$ is a unit in $S^{-1}R$, so there exist $a \in R$, $s \in S$ such that $(\frac{r}{1})(\frac{a}{s}) = 1$. Hence, there exists $t \in S$ such that $tra = ts$ which implies that $tra \in I \cap S$, a contradiction. Therefore, $S^* \cap I = \emptyset$. Since $S \subseteq S^*$, I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ implies that I is a ϕ-S^*-1-absorbing δ-primary ideal of R associated to s. Conversely, suppose that I is a ϕ-S^*-1-absorbing δ-primary ideal of R associated to $s \in S^*$. Let $r \in S^*$, then $\frac{r}{1} \in U(S^{-1}R)$ implies that $(\frac{r}{1})(\frac{a}{x}) = 1$, where $a \in R$, $x \in S$. Hence, there exists $t \in S$ such that $tra = tx \in S$. Take $r' = ta$, then $r' \in S^*$ with $r'r = tx \in S$. Let $S_1 = S$, $S_2 = S^*$, then, by Proposition 2.32, I is a ϕ-S-1-absorbing δ-primary ideal of R. ■

Theorem 2.34. Let I be a proper ideal of R such that $I \cap S = \emptyset$. If $\delta(I) \cap S = \emptyset$, then the following statements are equivalent.
(1) I is a ϕ-S-1-absorbing δ-primary ideal of R associated to a nonunit $s \in S$.
(2) $(I : s)$ is a ϕ-1-absorbing δ-primary ideal of R.

Proof.
$(1 \rightarrow 2)$: Since $I \cap S = \emptyset$, $(I : s) \neq R$. Let a, b, c be nonunit elements in R such that $abc \in (I : s) - (\phi(I) : s)$, then $(sa)bc \in I - \phi(I)$ and sa is not a unit in R, since a is not a unit. So, $s^2ab \in I$ or $sc \in \delta(I)$. If $s^2ab \in \phi(I)$, then $sab \in (\phi(I) : s) = \phi(I)$ implies that $ab \in (\phi(I) : s) = \phi(I)$ which is a contradiction. Thus, $s^2ab \in I - \phi(I)$ and s^2 is not a unit implies that $sab \in I$, since $s^3 \notin \delta(I)$. Consequently, we conclude that $ab \in (I : s)$ or $c \in (\delta(I) : s)$. Hence, $(I : s)$ is a ϕ-1-absorbing δ-primary ideal of R.
$(2 \rightarrow 1)$: Let $abc \in I - \phi(I)$ for some nonunits a, b, c in R. Since $I \subseteq (I : s)$ and $\phi(I : s) = \phi(I)$, we have $abc \in (I : s) - \phi(I : s)$. As $(I : s)$ is a ϕ-1-absorbing δ-primary ideal of R, we get $ab \in (I : s)$ or $c \in \delta(I : s) = (\delta(I) : s)$. Which implies that $sab \in I$ or $sc \in \delta(I)$, as needed. ■

Recall that $\delta_S(S^{-1}J) = S^{-1}\delta(J)$ and $\phi_S(S^{-1}J) = S^{-1}\phi(J)$ for each $J \in \mathfrak{I}(R)$. Let I be a proper ideal of R such that $\phi(I : a) = (\phi(I) : a)$, $\delta(I : a) = (\delta(I) : a)$ for each $a \in R$. Moreover, assume that $\delta(S^{-1}I \cap R) = S^{-1}\delta(I) \cap R$. Let s be a nonunit element in S. Then under the two conditions $(I : t) \subseteq (I : s)$ and $\phi(I) = (\phi(I) : t)$ for each $t \in S$, the following result holds.

Theorem 2.35. Let I be a proper ideal of R such that $I \cap S = \emptyset$. Suppose that $\delta_S(S^{-1}I) \neq S^{-1}R$ whenever $S^{-1}I \neq S^{-1}R$. Then, if $(I : s)$ is a ϕ-1-absorbing δ-primary ideal of R, then $S^{-1}I$ is a ϕ_S-1-absorbing δ_S-primary ideal of $S^{-1}R$ with $S^{-1}I \cap R = (I : s)$.

Proof.
Let I be a proper ideal of R such that $I \cap S = \emptyset$, then $S^{-1}I$ is a proper ideal of $S^{-1}R$. Assume that $S^{-1}I \neq S^{-1}R$ implies that $\delta_S(S^{-1}I) \neq S^{-1}R$, then it is easy to check that $\delta(I) \cap S = \emptyset$. Moreover, $\phi_S(S^{-1}I) \neq S^{-1}I$, since $I \neq \phi(I)$ and $\phi(I) = (\phi(I) : t)$ for each $t \in S$. Let $\frac{a}{r}, \frac{b}{x}, \frac{c}{t}$ be nonunit elements in $S^{-1}R$ such that $\frac{a}{r}\frac{b}{x}\frac{c}{t} \in S^{-1}I - \phi_S(S^{-1}I)$. Then $\frac{abc}{rxt} = \frac{u}{q} \in S^{-1}I - \phi_S(S^{-1}I)$ for some $u \in I$ and $q \in S$. So there exists $p \in S$ such that $pqabc = purxt \in I$. If $purxt \in \phi(I)$, then $\frac{u}{q} \in \phi_S(S^{-1}I)$, a contradiction. Hence, $pqabc \in I - \phi(I)$ which implies that $pqabc \in (I : s) - (\phi(I) : s)$, since $\phi(I) = (\phi(I) : s)$. Moreover, (pqa), b, c are nonunit elements in R, since $\frac{a}{r}, \frac{b}{x}, \frac{c}{t}$ are nonunit elements in $S^{-1}R$. Thus, $pqabc \in I - \phi(I) \subseteq (I : s) - \phi(I : s)$ implies that $pqab \in (I : s)$ or $c \in (\delta(I) : s)$. Hence, $spqab \in I$ or $sc \in \delta(I)$ which implies that $\frac{a}{r}\frac{b}{x} \in S^{-1}I$ or $\frac{c}{t} \in \delta_S(S^{-1}I)$. Consequently, we conclude that $S^{-1}I$ is a ϕ_S-1-absorbing δ_S-primary ideal of $S^{-1}R$. Now, let $t \in (I : s)$, then $ts \in I$ implies that $t = \frac{ts}{s} \in S^{-1}I \cap R$. So, $(I : s) \subseteq S^{-1}I \cap R$. For the converse, let $a \in S^{-1}I \cap R$, then $a = \frac{a}{1} \in S^{-1}I$. So, $\frac{a}{1} = \frac{b}{x}$ for some $b \in I$, $x \in S$. Hence, there exists $y \in S$ such that $yax = yb \in I$ implies that $a \in (I : xy) \subseteq (I : s)$. Thus, $(I : s) = S^{-1}I \cap R$. ■

Corollary 2.36. Let I be a proper ideal of R such that $I \cap S = \emptyset$. Suppose that $\delta_S(S^{-1}I) \neq S^{-1}R$ whenever $S^{-1}I \neq S^{-1}R$. Then, if I is a ϕ-S-1-absorbing δ-primary ideal of R associated to a nonunit $s \in S$, then $S^{-1}I$ is a ϕ_S-1-absorbing δ_S-primary ideal of $S^{-1}R$ with $S^{-1}I \cap R = (I : s)$.

Proof.
It follows from Theorem 2.34 and Theorem 2.35. ■

Let R be a ring, $S \subseteq R$ be a multiplicative subset of R. Next we give an example of a proper ideal P of R with $P \cap S = \emptyset$ such that $\phi(P) = \phi(P : s) \neq (\phi(P) : s)$ for some nonunit $s \in S$. Then P is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ but $(P : s)$ is not a ϕ-1-absorbing δ-primary ideal of R.

Example 2.37. Let $R = \mathbb{Z}_{18}[x]$, $P = \{0\}$. Let $\phi(P) = \phi_0(P) = (0)$ and $\delta(P) = P$. Let $S = \{2^k : k \geq 0\} = \{1, 2, 4, 8, 16, 14, 10\}$. Then it is easy to check that $P \cap S = \emptyset$. Moreover, P is a weakly S-1-absorbing prime ideal of R associated to $s = 2 \in S$. Also, it is easy to check that $(P : 2) = (0 : 2) = 9\mathbb{Z}_{18}[x]$ is not a weakly 1-absorbing prime ideal of R, since $0 \neq (3)(x)(3) \in (P : 2)$ and neither $3x \in (P : 2)$ nor $3 \in (P : 2)$. Accordingly, we conclude that P is a weakly S-1-absorbing prime ideal of R associated to $s = 2$ and $(P : 2)$ is not a weakly 1-absorbing prime ideal of R.

3. $(\phi, \delta) - (\psi, \gamma)$-Ring Homomorphisms

Following to [18], let X, Y be commutative rings with unities and let $f : X \to Y$ be a ring homomorphism. Suppose δ, ϕ are expansion and reduction functions of ideals of X and γ, ψ are expansion and reduction functions of ideals of Y, respectively. Then f is said to be (δ, ϕ)-(γ, ψ)-homomorphism if $\delta(f^{-1}(J)) = f^{-1}(\gamma(J))$ and $\phi(f^{-1}(J)) = f^{-1}(\psi(J))$ for all $J \in \mathfrak{J}(Y)$.

Remark 3.1.
(1) If $f : X \to Y$ is a nonzero surjective homomorphism and 1 is the unity of X, then $f(1)$ is the unity of Y.
(2) Suppose $f : X \to Y$ is a nonzero (δ, ϕ)-(γ, ψ)-surjective homomorphism and let I be a proper ideal of X containing $\ker(f)$. Then it is easy to see that $\gamma(f(I)) = f(\delta(I))$ and $\psi(f(I)) = f(\phi(I))$. ([18, Remark 2.11])
(3) If S is a multiplicative subset of X containing 1, then $f(S)$ is a multiplicative subset of Y containing $f(1)$.

Theorem 3.2. Let $f : X \to Y$ be a nonzero (δ, ϕ)-(γ, ψ)-surjective homomorphism such that if whenever $a \in X$, then a is a nonunit in X if and only if $f(a)$ is a nonunit in Y. Then the following statements are satisfied.
(1) If J is a ψ-$f(S)$-1-absorbing γ-primary ideal of Y associated to $f(s) \in f(S)$, then $f^{-1}(J)$ is a ϕ-S-1-absorbing δ-primary ideal of X associated to $s \in S$.
(2) If I is a ϕ-S-1-absorbing δ-primary ideal of X associated to $s \in S$ containing $\ker(f)$ and f is surjective then $f(I)$ is a ψ-$f(S)$-1-absorbing γ-primary ideal of Y associated to $f(s) \in f(S)$.

Proof.
(1) If S is a multiplicative subset of X with $1 \in S$, then $f(S)$ is a multiplicative subset of Y with $1 = f(1) \in f(S)$, since f is a nonzero surjective homomorphism. Let J be a ψ-$f(S)$-1-absorbing γ-primary ideal of Y associated to $f(s) \in f(S)$. Choose a, b, c to be nonunit elements in X such that $abc \in f^{-1}(J) - \phi(f^{-1}(J))$. Then we have $f(a)f(b)f(c) \in J - \psi(J)$, where $f(a)$, $f(b)$, $f(c)$ are nonunit elements in Y by assumption. Since J is a ψ-$f(S)$-1-absorbing γ-primary ideal of Y associated to $f(s) \in f(S)$ we conclude that $f(s)f(a)f(b) \in J$ or $f(s)f(c) \in \gamma(J)$, which implies that $sab \in f^{-1}(J)$ or $sc \in f^{-1}(\gamma(J)) = \delta(f^{-}(J))$. Hence $f^{-1}(J)$ is a ϕ-S-1-absorbing δ-primary ideal of X associated to s.
(2) Let I be a ϕ-S-1-absorbing δ-primary ideal of X associated to s containing $\ker(f)$, then the unity in Y is $f(1) \in f(S)$, since f is a nonzero (δ, ϕ)-(γ, ψ)-surjective homomorphism. Choose x, y, z to be nonunit elements in Y such that $xyz \in f(I) - \psi(f(I))$. Since f is onto map, we can choose $a, b, c \in I$ such that $f(a) = x$, $f(b) = y$ and $f(c) = z$. This implies that $f(a)f(b)f(c) = f(abc) \in f(I)$. Since $\ker(f) \subseteq I$, we conclude that $abc \in I$. If $abc \in \phi(I)$, then $xyz = f(abc) \in f(\phi(I)) = \psi(f(I))$, which is a contradiction. So, $abc \in I - \phi(I)$, where a, b, c are nonunit elements in R by assumption. As I is a ϕ-S-1-absorbing δ-primary ideal of X associated to s, we have $sab \in I$ or $sc \in \delta(I)$. Thus, we conclude that $f(s)xy \in f(I)$ or $f(s)z \in f(\delta(I)) = \gamma(f(I))$. Therefore,

$f(I)$ is a ψ-$f(S)$-1-absorbing γ-primary ideal of Y associated to $f(s)$. ■

From the above theorem we obtain the following result.

Theorem 3.3. *(Correspondence Theorem)* Let $f : X \to Y$ be a nonzero (δ, ϕ)-(γ, ψ)-surjective homomorphism such that if whenever $a \in X$, then a is a nonunit in X if and only if $f(a)$ is a nonunit in Y. Then f induces one-to-one correspondence between the ϕ-S-1-absorbing δ-primary ideals of X associated to $s \in S$ containing $\ker(f)$ and the ψ-$f(S)$-1-absorbing γ-primary ideals of Y associated to $f(s) \in f(S)$ in such a way that if I is a ϕ-S-1-absorbing δ-primary ideal of X associated to $s \in S$ containing $\ker(f)$, then $f(I)$ is the corresponding ψ-$f(S)$-1-absorbing γ-primary ideal of Y associated to $f(s) \in f(S)$, and if J is a ψ-$f(S)$-1-absorbing γ-primary ideal of Y associated to $f(s) \in f(S)$, then $f^{-1}(J)$ is the corresponding ϕ-S-1-absorbing δ-primary ideal of X associated to $s \in S$ containing $\ker(f)$. ■

Example 3.4. Let $f : \mathbb{Z}_{24} \to \mathbb{Z}_{12}$ be a map defined by $f(m) = m(\mod 12)$ for all $m \in \mathbb{Z}_{24}$. Then, one can easily check that f is a (δ_1,ϕ_0)-(δ_1,ϕ_0)-surjective homomorphism with $\ker(f) = \{0, 12\}$ and a is a unit in $\mathbb{Z}_{24}$ if and only if $f(a)$ is a unit in $\mathbb{Z}_{12}$.

(1) Let $J = 6\mathbb{Z}_{12}$ be a proper ideal of $\mathbb{Z}_{12}$, then $\delta_1(J) = \sqrt{J} = J$. Let $f(S) = \{1, 4\}$ be a multiplicative subset of $\mathbb{Z}_{12}$, where $S = \{1, 13, 4, 16\}$ is the multiplicative subset of $\mathbb{Z}_{24}$. Then, one can easily check that J is a weakly $f(S)$-1-absorbing primary ideal of $\mathbb{Z}_{12}$ associates to $f(s) = 4$, where $s = 16$. Moreover, by Theorem 3.2(1), $f^{-1}(J)$ is a weakly S-1-absorbing primary ideal of $\mathbb{Z}_{24}$ associates to $s = 16$, where $f^{-1}(J) = 6\mathbb{Z}_{24}$.

(2) Let $I = 12\mathbb{Z}_{24}$ be a proper ideal of $\mathbb{Z}_{24}$, then $\ker(f) \subseteq I$ with $\delta_1(I) = \sqrt{I} = 6\mathbb{Z}_{24}$. Let $S = \{1, 4, 16\}$ be a multiplicative subset of $\mathbb{Z}_{24}$. We show that I is a weakly S-1-absorbing primary ideal of $\mathbb{Z}_{24}$ associates to $s = 4$. Let a, b, c be nonunit elements in $\mathbb{Z}_{24}$ such that $0 \neq abc \in I$. If $3|c$, then $12|4c$ implies that $4c \in \sqrt{I}$. If not, then $3|ab$ which implies that $4ab \in I$. Hence, I is a weakly S-1-absorbing primary ideal of $\mathbb{Z}_{24}$ associates to $s = 4$. Moreover, $f(I) = \{0\}$, $f(\sqrt{I}) = 6\mathbb{Z}_{12} = \sqrt{f(I)}$, $\phi_0(I) = \{0\}$ and $f(\phi_0(I)) = f(\{0\}) = \{0\} = \phi_0(f(I))$. Therefore, by Theorem 3.2(2), $f(I) = \{0\}$ is a weakly $f(S)$-1-absorbing primary ideal of $\mathbb{Z}_{12}$ associates to $f(s) = 4$, where $f(S) = \{1, 4\}$.

Assume that δ, ϕ are expansion and reduction functions of ideals of R, respectively. Let J be a proper ideal of R such that $J = \phi(J)$. Then $\gamma : \mathfrak{I}(R/J) \to \mathfrak{I}(R/J)$ defined by $\gamma(I/J) = \delta(I)/J$, and $\psi : \mathfrak{I}(R/J) \to \mathfrak{I}(R/J)$, defined by $\psi(I/J) = \phi(I)/J$ and $\psi(I/J) = \emptyset$ if $\phi(I) = \emptyset$, are expansion and reduction functions of ideals of R/J, respectively. Moreover, if S is a multiplicative subset of R, then $\bar{S} = S/J$ is a multiplicative subset of R/J, where $S/J = \{\bar{s} = s + J \in R/J : s \in S\}$.

Let Q be a proper ideal of R, and let S be a multiplicative subset of R. Recall that Q is said to be a weakly S-1-absorbing δ-primary ideal of R associated to $s \in S$, if whenever $0 \neq abc \in Q$ for some nonunit elements $a, b, c \in R$ then $sab \in Q$ or $sc \in \delta(Q)$.

Theorem 3.5. Let δ, ϕ, where $\phi \neq \phi_\emptyset$, be expansion and reduction functions of ideals of R and let J be a proper ideal of R such that $J = \phi(J)$. For every $L \in \mathfrak{I}(R)$ let $\gamma : \mathfrak{I}(R/J) \to \mathfrak{I}(R/J)$ be an expansion function of ideals of R/J defined by $\gamma(L + J/J) = \delta(L + J)/J$ and $\psi : \mathfrak{I}(R/J) \to \mathfrak{I}(R/J)$ be a reduction function of ideals of R/J defined by $\psi(L + J/J) = \phi(L + J)/J$. Assume that $a + J$ is a unit in R/J if and only if a is a unit in R. Then the followings statements hold.

(1) A map $f : R \to R/J$ defined by $f(r) = r + J$ for every $r \in R$ is a (δ, ϕ)-(γ, ψ)-surjective homomorphism.

(2) Let I be a proper ideal of R such that $J \subseteq I$, S a multiplicative subset of R. Then I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ if and only if I/J is a ψ-$\bar{S}$-1-absorbing γ-primary ideal of R/J associated to $\bar{s} \in \bar{S}$.

(3) Let I be a nonzero proper ideal of R such that $\phi^2(I) = \phi(I)$. Then I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ if and only if $I/\phi(I)$ is a weakly $\bar{S}$-1-absorbing γ-primary ideal of $R/\phi(I)$ associated to $\bar{s} \in \bar{S}$.

Proof.

(1) It is easy to see that f is a ring-surjective homomorphism with $\ker(f) = J$. Let K be an ideal in R/J, then

$K = L + J/J$ for some ideal $L \in \Im(R)$. Therefore,

$$f^{-1}(\gamma(K)) = f^{-1}(\delta(L + J/J)) = \delta(L + J) = \delta(f^{-1}(K)),$$

$$f^{-1}(\psi(K)) = f^{-1}(\phi(L + J/J)) = \phi(L + J) = \phi(f^{-1}(K)),$$

since f is onto. Thus, f is a (δ, ϕ)-(γ, ψ)-surjective homomorphism.
(2) Let I be a proper ideal of R such that $J \subseteq I$, S a multiplicative subset of R. Since the map f defined in (1) is a (δ, ϕ)-(γ, ψ)-surjective homomorphism with $\ker(f) = J$ and $f(I) = I/J$. Then, by the correspondence theorem, I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ if and only if I/J is a ψ-$\bar{S}$-1-absorbing γ-primary ideal of R/J associated to $\bar{s} \in \bar{S}$.
(3) Let $J = \phi(I)$, then $J = \phi(J)$. Moreover, $f(I) = I/\phi(I)$ and $\psi(I/\phi(I)) = \phi(I)/\phi(I) = 0 \in R/\phi(I)$. Hence, by the correspondence theorem, I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ if and only if $I/\phi(I)$ is a weakly $\bar{S}$-1-absorbing γ-primary ideal of $R/\phi(I)$ associated to $\bar{s} \in \bar{S}$. ∎

Recall that if I is a proper ideal of R such that I is a ϕ-S-1-absorbing δ-primary ideal of R associated to $s \in S$ and x, y, z are nonunit elements in R. Then (x, y, z) is said to be a ϕ-S-1-δ-triple zero of I, if $xyz \in \phi(I)$, $sxy \notin I$ and $sz \notin \delta(I)$.

Lemma 3.6. Let $f : X \to Y$ be a nonzero (δ, ϕ)-(γ, ψ)-surjective homomorphism and let I a ϕ-S-1-absorbing δ-primary ideal of X associated to $s \in S$ such that $\ker(f) \subseteq I$. Assume that a is nonunit in X if and only if $f(a)$ is nonunit in Y. Let a, b, c be nonunit elements in X, then (a, b, c) is a ϕ-S-1-δ-triple zero of I if and only if $(f(a), f(b), f(c))$ is a ψ-$f(S)$-1-γ-triple zero of $f(I)$.

Proof.
By Theorem 3.2, $f(I)$ is a ψ-$f(S)$-1-absorbing -γ-primary ideal of Y associated to $f(s) \in f(S)$. Let a, b, c be nonunit elements in R such that (a, b, c) is a ϕ-S-1-δ-triple zero of I. Then $abc \in \phi(I)$ with $sab \notin I$ and $sc \notin \delta(I)$. So, $f(a)f(b)f(c) = f(abc) \in \psi(f(I))$ with $f(s)f(a)f(b) \notin f(I)$, since $\ker(f) \subseteq I$ and $sab \notin I$. Similarly, $f(s)f(c) \notin \gamma(f(I))$. Which implies that $(f(a), f(b), f(c))$ is a ψ-$f(S)$-1-γ-triple zero of $f(I)$. Conversely, let $a, b, c \in R$ such that $(f(a), f(b), f(c))$ is a ψ-$f(S)$-1-γ-triple zero of $f(I)$. Then a, b, c are nonunit elements in R such that $f(a)f(b)f(c) = f(abc) \in \psi(f(I)) = f(\phi(I))$ with $f(sab) \notin f(I)$ and $f(sc) \notin \gamma(f(I)) = f(\delta(I))$. Thus, $abc \in f^{-1}(\psi(f(I))) = \phi(f^{-1}(f(I))) = \phi(I)$, since $\ker(f) \subseteq I$. Moreover, $sab \notin f^{-1}(f(I)) = I$ and $sc \notin f^{-1}(\gamma(f(I))) = \delta(I)$. Consequently, we conclude that (a, b, c) is a ϕ-S-1-δ-triple zero of I. ∎

Corollary 3.7. Let $\delta, \phi \neq \phi_\emptyset$ be expansion and reduction functions of ideals of R and let J be a proper ideal of R such that $J = \phi(J)$. For every $L \in \Im(R)$ let $\gamma : \Im(R/J) \to \Im(R/J)$ be an expansion function of ideals of R/J defined by $\gamma(L + J/J) = \delta(L + J)/J$ and $\psi : \Im(R/J) \to \Im(R/J)$ be a reduction function of ideals of R/J defined by $\psi(L + J/J) = \phi(L + J)/J$. Assume that a is nonunit in R if and only if $a + J$ is nonunit in R/J. Let a, b, c be nonunit elements in R. Then the followings statements hold.
(1) (a, b, c) is a ϕ-S-1-δ-triple zero of I if and only if $(a + J, b + J, c + J)$ is a ψ-$\bar{S}$-1-γ-triple zero of I/J.
(2) If $\phi^2(I) = \phi(I)$, then (a, b, c) is a ϕ-S-1-δ-triple zero of I if and only if $(a + \phi(I), b + \phi(I), c + \phi(I))$ is a ψ-$\bar{S}$-1-γ-triple zero of $I/\phi(I)$.

Proof.
(1) It follows from Theorem 3.5(2) and Lemma 3.6.
(2) It follows from Theorem 3.5(3) and Lemma 3.6. ∎

4. ϕ-S-1-absorbing δ-primary in direct product of rings

Let R_i be commutative rings with unity for each $i = 1, 2$ and $R = R_1 \times R_2$ denote the direct product of rings R_1, R_2. Also, let S_1, S_2 be multiplicative subsets of R_1, R_2 respectively, then $S = S_1 \times S_2$ is a multiplicative subset of R. Suppose that ϕ_i, δ_i are reduction and expansion functions of ideals of R_i for each $i = 1, 2$ respectively. Following to [18], we define the following two functions:

$$\hat{\delta}(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2),$$

$$\hat{\phi}(I_1 \times I_2) = \phi_1(I_1) \times \phi_2(I_2).$$

Then it is easy to see that $\hat{\delta}$, $\hat{\phi}$ are expansion and reduction functions of ideals of R, respectively.

Theorem 4.1. Let R_1 and R_2 be commutative rings with $1 \neq 0$, $R = R_1 \times R_2$ a direct product ring, and $S = S_1 \times S_2$ a multiplicative subset of R. Suppose that δ_i is an expansion function of ideals of R_i and ϕ_i is a reduction function of ideals of R_i for each $i = 1, 2$ such that $\phi_2(R_2) \neq R_2$. Let $s = (s_1, s_2) \in S$, and let I_1 be a proper ideal of R_1 such that if whenever x, y, z are nonunit elements in R with $xyz \in \hat{\phi}(I_1 \times R_2)$, then $sxz, syz \notin I_1 \times R_2$. Then the following statements are equivalent
(1) $I_1 \times R_2$ is a $\hat{\phi}$-S-1-absorbing $\hat{\delta}$-primary ideal of R associated to s.
(2) I_1 is an S_1-1-absorbing δ_1-primary ideal of R_1 associated to s_1 and $I_1 \times R_2$ is an S-1-absorbing $\hat{\delta}$-primary ideal of R associated to s.

Proof.
$(1 \to 2)$: Suppose that $I_1 \times R_2$ is a $\hat{\phi}$-S-1-absorbing $\hat{\delta}$-primary ideal of R associated to $s = (s_1, s_2)$ and let a, b, c be nonunit elements in R_1 such that $abc \in I_1$. Then we have $(a, 1)(b, 1)(c, 1) = (abc, 1) \in I_1 \times R_2 - \hat{\phi}(I_1 \times R_2)$, where $(a, 1), (b, 1), (c, 1)$ are nonunit elements in R, since a, b, c are nonunit elements in R_1 and $\phi_2(R_2) \neq R_2$. This implies that $(s_1, s_2)(a, 1)(b, 1) \in I_1 \times R_2$ or $(s_1, s_2)(c, 1) \in \hat{\delta}(I_1 \times R_2)$. Hence we conclude that $s_1ab \in I_1$ or $s_1c \in \delta_1(I_1)$ and thus, I_1 is an S_1-1-absorbing δ_1-primary ideal of R_1 associated to s_1. If $I_1 \times R_2$ is not an S-1-absorbing $\hat{\delta}$-primary ideal of R, then there exist x, y, z nonunit elements in R such that (x, y, z) is a $\hat{\phi}$-S-1-$\hat{\delta}$-triple zero of $I_1 \times R_2$. So, $xyz \in \hat{\phi}(I_1 \times R_2)$ with $sxy \notin I_1 \times R_2$ and $sz \notin \hat{\delta}(I_1 \times R_2)$. Since $sxz, syz \notin I_1 \times R_2$, by part(2) of Proposition 2.22, we have $(I_1 \times R_2)^3 \subseteq \hat{\phi}(I_1 \times R_2)$ which implies that $R_2 = \phi_2(R_2)$, a contradiction. Thus, $I_1 \times R_2$ is an S-1-absorbing $\hat{\delta}$-primary ideal of R associated to (s_1, s_2).
$(2 \to 1)$: It is clear, since every S-1-absorbing $\hat{\delta}$-primary ideal of R associated to (s_1, s_2) is a $\hat{\phi}$-S-1-absorbing $\hat{\delta}$-primary ideal. ∎

Corollary 4.2. Let R_1 and R_2 be commutative rings with $(1 \neq 0)$, $R = R_1 \times R_2$ a direct product ring, and $S = S_1 \times S_2$ a multiplicative subset of R. Suppose that δ_i is an expansion function of ideals of R_i and ϕ_i is a reduction function of ideals of R_i for each $i = 1, 2$. Let $s = (s_1, s_2) \in S$, and let I_1 be a proper ideal of R_1 such that if whenever x, y, z are nonunit elements in R with $xyz \in \hat{\phi}(I_1 \times R_2)$, then $sxz, syz \notin I_1 \times R_2$. If $I_1 \times R_2$ is a $\hat{\phi}$-S-1-absorbing $\hat{\delta}$-primary ideal of R associated to $(s_1, s_2) \in S$ that is not S-1-absorbing $\hat{\delta}$-primary. Then $\hat{\phi}(I_1 \times R_2) \neq \emptyset$, $\phi_2(R_2) = R_2$ and I_1 is a ϕ_1-S_1-1-absorbing δ_1-primary ideal of R_1 associated to s_1 that is not S_1-1-absorbing δ_1-primary.

Proof.
Suppose that $I_1 \times R_2$ is a $\hat{\phi}$-S-1-absorbing $\hat{\delta}$-primary ideal of R associated to (s_1, s_2) that is not S-1-absorbing $\hat{\delta}$-primary, then there exist x, y, z nonunit elements in R such that (x, y, z) is a $\hat{\phi}$-S-1-$\hat{\delta}$-triple zero of $I_1 \times R_2$. So, $xyz \in \hat{\phi}(I_1 \times R_2)$ with $sxy \notin I_1 \times R_2$ and $sz \notin \hat{\delta}(I_1 \times R_2)$. Since $sxz, syz \notin I_1 \times R_2$, by Proposition 2.22, we have $(I_1 \times R_2)^3 \subseteq \hat{\phi}(I_1 \times R_2)$ which implies that $\hat{\phi}(I_1 \times R_2) \neq \emptyset$. If $\phi_2(R_2) \neq R_2$, then by Theorem 4.1, $I_1 \times R_2$ is an S-1-absorbing $\hat{\delta}$-primary ideal of R associated to (s_1, s_2) which is a contradiction. Thus, $\phi_2(R_2) = R_2$. Moreover, it is easy to see that I_1 is a ϕ_1-S_1-1-absorbing δ_1-primary ideal of R_1 associated to s_1, since $I_1 \times R_2$ is a $\hat{\phi}$-S-1-absorbing $\hat{\delta}$-primary ideal of R associated to (s_1, s_2) and $\phi_2(R_2) = R_2$. If I_1 is an S_1-1-absorbing δ_1-primary ideal of R_1 associated to s_1, then one can easily prove that $I_1 \times R_2$ is an S-1-absorbing $\hat{\delta}$-primary ideal of R associated to (s_1, s_2) which is a contradiction. Hence I_1 is a ϕ_1-S_1-1-absorbing δ_1-primary ideal of R_1 associated to s_1 that is not S_1-1-absorbing δ_1-primary. ∎

Remark 4.3. If $I_1 \times R_2$ is a $\hat{\phi}$-S-1-absorbing $\hat{\delta}$-primary ideal of R associated to (s_1, s_2) such that $\phi_1(I_1) \neq I_1$, then $S_1 \cap I_1 = \emptyset$ since $S \cap I = \emptyset$ and $S_2 \cap I_2 \neq \emptyset$. Thus, I_1 is a ϕ_1-δ_1-S_1-primary ideal of R_1 associated to s_1. To see this, let $a, b \in R_1$ such that $ab \in I_1 - \phi_1(I_1)$, we may assume that a, b are nonunit elements in R_1, since if a or b is a unit then we are done. Then $(a, 1)(1, 0)(b, 1) \in I_1 \times R_2 - \hat{\phi}(I_1 \times R_2)$ implies that $(s_1, s_2)(a, 0) = (s_1a, 0) \in I_1 \times R_2$ or $(s_1, s_2)(b, 1) = (s_1b, s_2) \in \hat{\delta}(I_1 \times R_2)$. Thus, $s_1a \in I_1$ or $s_1b \in \delta_1(I_1)$. Hence we conclude that I_1 is a ϕ_1-δ_1-S_1-primary ideal of R_1 associated to s_1. By using the same argument above, if

$R_1 \times I_2$ is a $\hat{\phi}$-S-1-absorbing $\hat{\delta}$-primary ideal of R associated to (s_1, s_2) such that $\phi_2(I_2) \neq I_2$, then I_2 is also a ϕ_2-δ_2-S_2-primary ideal of R_2 associated to s_2.

Recall that a commutative ring R is said to be a quasi-local ring if it has a unique maximal ideal. Otherwise, we say R is a non-quasi-local ring.

Suppose that for each $i = 1, 2$, if $I_i \neq \phi_i(I_i)$, then $S_i \cap \phi_i(I_i) = \emptyset$ and if $S_i \cap \delta_i(I_i) \neq \emptyset$, then $S_i \cap I_i = S_i \cap \delta_i(I_i)$. Then we obtain the following result.

Theorem 4.4. Let R_1, R_2 be commutative rings with $(1 \neq 0)$, and let $R = R_1 \times R_2$ be a direct product ring and $S = S_1 \times S_2$ a multiplicative subset of R. Suppose that δ_i is an expansion function of ideals of R_i and ϕ_i is a reduction function of ideals of R_i for each $i = 1, 2$. Let $I = I_1 \times I_2$ be a proper ideal of R, for some ideals $I_1 \neq \phi_1(I_1)$, $I_2 \neq \phi_2(I_2)$ of R_1, R_2, respectively, such that for every $i \in \{1, 2\}$, if $S_i \cap \delta_i(I_i) \neq \emptyset$, then $S_i \cap I_i = S_i \cap \delta_i(I_i)$. Let I be a $\hat{\phi}$-S-1-absorbing $\hat{\delta}$-primary ideal of R associated to $(s_1, s_2) \in S$. Then one of the following statements must be hold.
(1) $I_1 = R_1$, I_2 is a ϕ_2-δ_2-S_2-primary ideal of R_2 associated to s_2 and if R_1 is a non-quasi-local ring, then I_2 is a δ_2-S_2-primary ideal of R_2 associated to s_2.
(2) $I_2 = R_2$, I_1 is a ϕ_1-δ_1-S_1-primary ideal of R_1 associated to s_1 and if R_2 is a non-quasi-local ring, then I_1 is a δ_1-S_1-primary ideal of R_1 associated to s_1.
(3) $I_2 \cap S_2 \neq \emptyset$, I_1 is a δ_1-S_1-primary ideal of R_1 associated to s_1.
(4) $I_1 \cap S_1 \neq \emptyset$, I_2 is a δ_2-S_2-primary ideal of R_2 associated to s_2.

Proof.
First, we show that $S_1 \cap I_1 \neq \emptyset$ or $S_2 \cap I_2 \neq \emptyset$. Let $a, b \in I_1$, choose $c \in I_2 - \phi_2(I_2)$. Then $(a, 1)(b, 1)(1, c) = (ab, c) \in I - \hat{\phi}(I)$. As I is a $\hat{\phi}$-S-1-absorbing $\hat{\delta}$-primary ideal of R associated to (s_1, s_2), we have

$$(s_1, s_2)(a, 1)(b, 1) = (s_1 ab, s_2) \in I = I_1 \times I_2 \text{ or } (s_1, s_2)(1, c) = (s_1, s_2 c) \in \hat{\delta}(I) = \delta_1(I_1) \times \delta_2(I_2).$$

Thus, $s_2 \in S_2 \cap I_2$ or $s_1 \in S_1 \cap \delta_1(I_1) = S_1 \cap I_1$. Hence, $S_1 \cap I_1 \neq \emptyset$ or $S_2 \cap I_2 \neq \emptyset$.
(1) If $I_1 = R_1$, then $S_2 \cap I_2 = \emptyset$, since $S \cap I = \emptyset$ and $S_1 \cap I_1 \neq \emptyset$. Thus, by the remark above, I_2 is a ϕ_2-δ_2-S_2-primary ideal of R_2 associated to s_2, since $\phi_2(I_2) \neq I_2$. Suppose that R_1 is a non-quasi-local ring, we show that I_2 is a δ_2-S_2-primary ideal of R_2 associated to s_2. Let $a, b \in R_2$ such that $ab \in I_2$. If a or b is a unit in R_2, then we are done. Therefore, we may assume that a, b are nonunit elements in R_2. Since R_1 is a non-quasi-local ring and $R_1 \neq \phi_1(R_1)$, choose a nonunit $x \in R_1 - \phi_1(R_1)$. Then $(x, 1)$, $(1, a)$, $(1, b)$ are nonunit elements in R such that $(x, 1)(1, a)(1, b) \in R_1 \times I_2 - \hat{\phi}(R_1 \times I_2)$ which implies that $(s_1, s_2)(x, 1)(1, a) = (s_1 x, s_2 a) \in R_1 \times I_2$ or $(s_1, s_2)(1, b) = (s_1, s_2 b) \in \hat{\delta}(R_1 \times I_2)$. So, $s_2 a \in I_2$ or $s_2 b \in \delta_2(I_2)$. Hence, we conclude that I_2 is a δ_2-S_2-primary ideal of R_2 associated to s_2.
(2) If $I_2 = R_2$, then by using the same argument above $I_1 \cap S_1 = \emptyset$, I_1 is a ϕ_1-δ_1-S_1-primary ideal of R_1 associated to s_1 and if R_2 is a non-quasi-local ring, then I_1 is a δ_1-S_1-primary ideal of R_1 associated to s_1.
(3) Assume that $I_2 \cap S_2 \neq \emptyset$. Then $I_1 \cap S_1 = \emptyset$, since $I \cap S = \emptyset$. Suppose that $I = I_1 \times I_2$ such that $I_i \neq R_i$ for each $i = 1, 2$. We show that I_1 is a δ_1-S_1-primary ideal of R_1 associated to s_1. Let $a, b \in R_1$ such that $ab \in I_1$. If a or b is a unit in R_1, then we are done. Therefore, we may assume that a, b are nonunit elements in R_1. Since $S_2 \cap I_2 \neq \emptyset$ and $S_2 \cap \phi_2(I_2) = \emptyset$, choose $t \in S_2 \cap I_2 - \phi_2(I_2)$. Then $(a, 1)$, $(1, t)$, $(b, 1)$ are nonunit elements in R such that $(a, 1)(1, t)(b, 1) \in I - \hat{\phi}(I)$ which implies that $(s_1, s_2)(a, 1)(1, t) = (s_1 a, s_2 t) \in I$ or $(s_1, s_2)(b, 1) = (s_1 b, s_2) \in \hat{\delta}(I)$. Thus, $s_1 a \in I_1$ or $s_1 b \in \delta_1(I_1)$. Accordingly, we conclude that I_1 is a δ_1-S_1-primary ideal of R_1 associated to s_1.
(4) Assume that $I_1 \cap S_1 \neq \emptyset$, then by using the same argument above $I_2 \cap S_2 = \emptyset$ and I_2 is a δ_2-S_2-primary ideal of R_2 associated to s_2. ■

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