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On a difference equation whose solution is related to Fibonacci numbers

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Abstract. In this paper, we consider the following fifth-order non-linear difference equation

$$z_n = \frac{z_{n-2}^s z_{n-3} z_{n-4}}{z_{n-1}\left(a z_{n-5}^s + b z_{n-3} z_{n-4}\right)}, \ s, n \in \mathbb{N},$$

where the initial values z_{-j}, $j = \overline{0,4}$ and the parameters a, b are non-zero real numbers. In addition, the form of the solution of a more general difference equation defined by one to one continuous function is obtained. We will show that both the solution of the above mentioned equation and the solution of the general difference equation are related to a generalized Fibonacci sequences.

1. Introduction

Non-linear difference equations and their systems appeared in some scientific areas such as engineering, biology, economics, physics. Especially, they are used in modeling in biology (see, e.g., [5, 13, 25, 26]). So, to understand these models, it is worthy to solve the corresponding non-linear difference equations in closed-form. Sometimes, non-linear difference equations can be transformed to linear ones by using some convenient transformations. Hence closed-form formulas for the solutions of the non-linear difference equations can be deduced from the corresponding linear ones. Other related difference equations or system of difference equations can be found in Refs. [1–3, 6–12, 14, 15, 18–24, 28–37].

Noting that the solutions of some difference equations are related to very known number sequences, for example, the Fibonacci sequence $\{f_n\}_{n=0}^{\infty}$ defined by

$$f_{n+1} = f_n + f_{n-1}, \ n \in \mathbb{N}, \tag{1}$$

with the initial conditions $f_0 = 1$ and $f_1 = 1$. In the present work, as in the references [16, 17], we will use the following generalized $s-$Fibonacci sequence defined by

$$F_{n+2} = F_{n+1} + sF_n, \ F_0 = 1 = F_1, \ n \in \mathbb{N}_0, \tag{2}$$

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the first eleven terms of it are

$$\begin{aligned}
F_0 &= 1,\\
F_1 &= 1,\\
F_2 &= s+1,\\
F_3 &= 2s+1,\\
F_4 &= s^2+3s+1,\\
F_5 &= 3s^2+4s+1,\\
F_6 &= s^3+6s^2+5s+1,\\
F_7 &= 4s^3+10s^2+6s+1,\\
F_8 &= s^4+10s^3+15s^2+7s+1,\\
F_9 &= 5s^4+20s^3+21s^2+8s+1,\\
F_{10} &= s^5+15s^4+35s^3+28s^2+9s+1,\\
&\vdots
\end{aligned}$$

The following special non-linear difference equations

$$y_{n+1} = \frac{y_{n-2}y_{n-3}}{y_n\left(\pm 1 \pm y_{n-2}y_{n-3}\right)},\ n \in \mathbb{N}_0, \tag{3}$$

where the initial conditions y_{-p}, $p = \overline{0,3}$ are arbitrary real numbers, was considered in [4]. Motivated by this equations and inspired by [16, 17], our aim in this paper is to obtain the solutions form of the following difference equation

$$z_n = \frac{z_{n-2}^s z_{n-3} z_{n-4}}{z_{n-1}\left(a z_{n-5}^s + b z_{n-3} z_{n-4}\right)},\ s, n \in \mathbb{N}, \tag{4}$$

where the initial values $z_{-\psi}$, $\psi = \overline{0,4}$ and the parameters a, b are non-zero real numbers. Note that, equation (4) can be seen as a generalization of equation (3), when $s = 0$, $a = \pm 1$, $b = \pm 1$. In addition, we will also determine the form of the solution of the following difference equation, which is a generalization of equation (4), it suffices to take $h(x) = x$,

$$z_n = h^{-1}\left(\frac{\left(h\left(z_{n-2}\right)\right)^s h\left(z_{n-3}\right) h\left(z_{n-4}\right)}{h\left(z_{n-1}\right)\left[a\left(h\left(z_{n-5}\right)\right)^s + b h\left(z_{n-3}\right) h\left(z_{n-4}\right)\right]}\right),\ s, n \in \mathbb{N},$$

where $h : B \to \mathbb{R}$ is one to one continuous function on $B \subseteq \mathbb{R}$, the initial values $z_{-\psi}$, $\psi = \overline{0,4}$ are non-zero real numbers in B and the parameters a, b are non-zero real numbers. The solution of equation (4) is also related to the $s-$Fibonacci sequence defined by (2).

A very well-known linear difference equation, which will be an important key role in solving our difference equations, is

$$u_n = \alpha u_{n-l} + \beta,\ n \geq l,\ l, n \in \mathbb{N}, \tag{5}$$

where the parameter α, β and the initial conditions u_j, for $j \in \{1-l, 2-l, \ldots, 0\}$ are real numbers. The equation (5) is solved in [27]. The general solution of equation (5) is

$$u_{lm+j} = \begin{cases} \alpha^m u_j + \beta\frac{\alpha^m - 1}{\alpha - 1}, & \text{if } \alpha \neq 1,\\ u_j + \beta m, & \text{if } \alpha = 1, \end{cases} \tag{6}$$

for each fixed $j \in \{1-l, 2-l, \ldots, 0\}$ and $m \in \mathbb{N}_0$.

Through this paper, the standard notations $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$ are symbolized for the sets of positive natural numbers, non-negative integers, integers, real numbers respectively. In addition, the notation of $\beta = \overline{\gamma, \delta}$ stands for $\{\beta \in \mathbb{Z} : \gamma \leq \beta \leq \delta\}$, if $\gamma, \delta \in \mathbb{Z}$, $\gamma \leq \delta$. In the sequel, as usual, we suppose that $\prod_{j=k}^{r} C_j = 1$ and $\sum_{j=k}^{r} C_j = 0$, for all $r < k$.
From here on, by a solution, we mean a well defined solution.

2. Solving in closed form the equation (4)

Let us give the definition of well-defined solution of equation (4).

Definition 2.1. *A solution* $\{z_n\}_{n\geq -4}$ *of equation* (4) *is said to be well-defined if*

$$z_{n-1}\left(az_{n-5}^{s} + bz_{n-3}z_{n-4}\right) \neq 0, \ n \in \mathbb{N}.$$

Let $\{z_n\}_{n\geq -4}$ be a well-defined solution of equation (4). We get

$$\begin{aligned} z_n &= \frac{z_{n-2}^{s} z_{n-3} z_{n-4}}{z_{n-1}\left(az_{n-5}^{s} + bz_{n-3}z_{n-4}\right)}, \\ \frac{z_n z_{n-1}}{z_{n-2}^{s}} &= \frac{z_{n-3}z_{n-4}}{az_{n-5}^{s} + bz_{n-3}z_{n-4}}, \\ \frac{z_{n-2}^{s}}{z_n z_{n-1}} &= \frac{az_{n-5}^{s} + bz_{n-3}z_{n-4}}{z_{n-3}z_{n-4}}, \\ \frac{z_{n-2}^{s}}{z_n z_{n-1}} &= a\frac{z_{n-5}^{s}}{z_{n-3}z_{n-4}} + b, \ s, n \in \mathbb{N}. \end{aligned} \tag{7}$$

By using following the change of variable,

$$w_n = \frac{z_{n-2}^{s}}{z_n z_{n-1}}, \ n = -2, -1, \ldots \tag{8}$$

from equation (7), we obtain

$$w_n = aw_{n-3} + b, \ n \in \mathbb{N}. \tag{9}$$

By using (6), we have for all $m \in \mathbb{N}_0$, $j \in \{-2, -1, 0\}$, the solution of equation (9) is

$$w_{3m+j} = \begin{cases} a^m w_j + b\frac{a^m-1}{a-1}, & \text{if } a \neq 1, \\ w_j + bm, & \text{if } a = 1. \end{cases} \tag{10}$$

From (8) and (10), it follows that for all $m \in \mathbb{N}_0$,

$$w_{3m-2} = \begin{cases} \frac{a^m z_{-4}^{s}}{z_{-2}z_{-3}} + b\frac{a^m-1}{a-1}, & \text{if } a \neq 1, \\ \frac{z_{-4}^{s}}{z_{-2}z_{-3}} + bm, & \text{if } a = 1, \end{cases} \tag{11}$$

$$w_{3m-1} = \begin{cases} \frac{a^m z_{-3}^{s}}{z_{-1}z_{-2}} + b\frac{a^m-1}{a-1}, & \text{if } a \neq 1, \\ \frac{z_{-3}^{s}}{z_{-1}z_{-2}} + bm, & \text{if } a = 1, \end{cases} \tag{12}$$

$$w_{3m} = \begin{cases} \frac{a^m z_{-2}^{s}}{z_0 z_{-1}} + b\frac{a^m-1}{a-1}, & \text{if } a \neq 1, \\ \frac{z_{-2}^{s}}{z_0 z_{-1}} + bm, & \text{if } a = 1. \end{cases} \tag{13}$$

Now, from (8) it follows that

$$z_n = \frac{z_{n-2}^s}{z_{n-1}w_n},\ n = -2, -1, \ldots \tag{14}$$

So we have,

$$z_{-2} = \frac{z_{-4}^s}{z_{-3}w_{-2}} = \frac{z_{-4}^{sF_0}}{z_{-3}^{F_1}w_{-2}^{F_0}},$$

$$z_{-1} = \frac{z_{-3}^s}{z_{-2}w_{-1}} = \frac{z_{-3}^{s+1}w_{-2}}{z_{-4}^s w_{-1}} = \frac{z_{-3}^{F_2}w_{-2}^{F_1}}{z_{-4}^{sF_1}w_{-1}^{F_0}},$$

$$z_0 = \frac{z_{-2}^s}{z_{-1}w_0} = \frac{z_{-4}^{s^2+s}w_{-1}}{z_{-3}^{2s+1}w_{-2}^{s+1}w_0} = \frac{z_{-4}^{sF_2}w_{-1}^{F_1}}{z_{-3}^{F_3}w_{-2}^{F_2}w_0^{F_0}},$$

$$z_1 = \frac{z_{-1}^s}{z_0 w_1} = \frac{z_{-3}^{s^2+3s+1}w_{-2}^{2s+1}w_0}{z_{-4}^{2s^2+s}w_{-1}^{s+1}w_1} = \frac{z_{-3}^{F_4}w_{-2}^{F_3}w_0^{F_1}}{z_{-4}^{sF_3}w_{-1}^{F_2}w_1^{F_0}},$$

$$z_2 = \frac{z_0^s}{z_1 w_2} = \frac{z_{-4}^{s^3+3s^2+s}w_{-1}^{2s+1}w_1}{z_{-3}^{3s^2+4s+1}w_{-2}^{s^2+3s+1}w_0^{s+1}w_2} = \frac{z_{-4}^{sF_4}w_{-1}^{F_3}w_1^{F_1}}{z_{-3}^{F_5}w_{-2}^{F_4}w_0^{F_2}w_2^{F_0}},$$

$$z_3 = \frac{z_{-3}^{F_6}w_{-2}^{F_5}w_0^{F_3}w_2^{F_1}}{z_{-4}^{sF_5}w_{-1}^{F_4}w_1^{F_2}w_3^{F_0}} = \frac{z_{-3}^{F_6}}{z_{-4}^{sF_5}}\prod_{i=0}^{2}\frac{w_{2(i-1)}^{F_{2(2-i)+1}}}{w_{2i-1}^{F_{2(2-i)}}},$$

$$z_4 = \frac{z_{-4}^{sF_6}w_{-1}^{F_5}w_1^{F_3}w_3^{F_1}}{z_{-3}^{F_7}w_{-2}^{F_6}w_0^{F_4}w_2^{F_2}w_4^{F_0}} = \frac{z_{-4}^{sF_6}}{z_{-3}^{F_7}}\frac{\prod_{i=0}^{2}w_{2i-1}^{F_{2(2-i)+1}}}{\prod_{i=0}^{3}w_{2(i-1)}^{F_{2(3-i)}}},$$

$$z_5 = \frac{z_{-3}^{F_8}w_{-2}^{F_7}w_0^{F_5}w_2^{F_3}w_4^{F_1}}{z_{-4}^{sF_7}w_{-1}^{F_6}w_1^{F_4}w_3^{F_2}w_5^{F_0}} = \frac{z_{-3}^{F_8}}{z_{-4}^{sF_7}}\prod_{i=0}^{3}\frac{w_{2(i-1)}^{F_{2(3-i)+1}}}{w_{2i-1}^{F_{2(3-i)}}},$$

$$z_6 = \frac{z_{-4}^{sF_8}w_{-1}^{F_7}w_1^{F_5}w_3^{F_3}w_5^{F_1}}{z_{-3}^{F_9}w_{-2}^{F_8}w_0^{F_6}w_2^{F_4}w_4^{F_2}w_6^{F_0}} = \frac{z_{-4}^{sF_8}}{z_{-3}^{F_9}}\frac{\prod_{i=0}^{3}w_{2i-1}^{F_{2(3-i)+1}}}{\prod_{i=0}^{4}w_{2(i-1)}^{F_{2(4-i)}}}.$$

It follows that

$$\begin{cases} z_{2n} = \dfrac{z_{-4}^{sF_{2(n+1)}}}{z_{-3}^{F_{2n+3}}}\left(\dfrac{\prod_{i=0}^{n}w_{2i-1}^{F_{2(n-i)+1}}}{\prod_{i=0}^{n+1}w_{2(i-1)}^{F_{2(n+1-i)}}}\right), \\ z_{2n+1} = \dfrac{z_{-3}^{F_{2(n+2)}}}{z_{-4}^{sF_{2n+3}}}\prod_{i=0}^{n+1}\left(\dfrac{w_{2(i-1)}^{F_{2(n+1-i)+1}}}{w_{2i-1}^{F_{2(n+1-i)}}}\right), \end{cases} n = -1, 0, \ldots \tag{15}$$

We consider three cases: $n \equiv 0(mod3)$, $n \equiv 1(mod3)$ and $n \equiv 2(mod3)$.

(i) If $n \equiv 0(mod3)$ equivalent $n = 3m$, $m \in \mathbb{N}_0$. Then, from (15), we have

$$\begin{cases} z_{6m} = \dfrac{z_{-4}^{sF_{6m+2}}}{z_{-3}^{F_{6m+3}}} \left(\dfrac{\prod_{i=0}^{m} w_{3(2i)-1}^{F_{6(m-i)+1}} \prod_{i=0}^{m-1} w_{3(2i+1)-2}^{F_{6(m-i)-1}} \prod_{i=0}^{m-1} w_{3(2i+1)}^{F_{6(m-i)-3}}}{\prod_{i=0}^{m} w_{3(2i)-2}^{F_{6(m-i)+2}} \prod_{i=0}^{m} w_{3(2i)}^{F_{6(m-i)}} \prod_{i=0}^{m-1} w_{3(2i+1)-1}^{F_{6(m-i)-2}}} \right), \\ z_{6m+1} = \dfrac{z_{-3}^{F_{6m+4}}}{z_{-4}^{sF_{6m+3}}} \prod_{i=0}^{m} \left(\dfrac{w_{3(2i)-2}^{F_{6(m-i)+3}} w_{3(2i)}^{F_{6(m-i)+1}}}{w_{3(2i)-1}^{F_{6(m-i)+2}} w_{3(2i+1)-2}^{F_{6(m-i)}}} \right) \prod_{i=0}^{m-1} \left(\dfrac{w_{3(2i+1)-1}^{F_{6(m-i)-1}}}{w_{3(2i+1)}^{F_{6(m-i)-2}}} \right). \end{cases} \tag{16}$$

(ii) If $n \equiv 1(mod3)$ equivalent $n = 3m+1$, $m \in \mathbb{N}_0$. Then, from (15), we get

$$\begin{cases} z_{6m+2} = \dfrac{z_{-4}^{sF_{6m+4}}}{z_{-3}^{F_{6m+5}}} \left(\dfrac{\prod_{i=0}^{m} w_{3(2i)-1}^{F_{6(m-i)+3}} \prod_{i=0}^{m} w_{3(2i+1)-2}^{F_{6(m-i)+1}} \prod_{i=0}^{m-1} w_{3(2i+1)}^{F_{6(m-i)-1}}}{\prod_{i=0}^{m} w_{3(2i)-2}^{F_{6(m-i)+4}} \prod_{i=0}^{m} w_{3(2i)}^{F_{6(m-i)+2}} \prod_{i=0}^{m} w_{3(2i+1)-1}^{F_{6(m-i)}}} \right), \\ z_{6m+3} = \dfrac{z_{-3}^{F_{6m+6}}}{z_{-4}^{sF_{6m+5}}} \prod_{i=0}^{m} \left(\dfrac{w_{3(2i)-2}^{F_{6(m-i)+5}} w_{3(2i)}^{F_{6(m-i)+3}} w_{3(2i+1)-1}^{F_{6(m-i)+1}}}{w_{3(2i)-1}^{F_{6(m-i)+4}} w_{3(2i+1)-2}^{F_{6(m-i)+2}} w_{3(2i+1)}^{F_{6(m-i)}}} \right). \end{cases} \tag{17}$$

(iii) If $n \equiv 2(mod3)$ equivalent $n = 3m+2$, $m \geq -1$. Then, from (15), we obtain

$$\begin{cases} z_{6m+4} = \dfrac{z_{-4}^{sF_{6m+6}}}{z_{-3}^{F_{6m+7}}} \left(\dfrac{\prod_{i=0}^{m} w_{3(2i)-1}^{F_{6(m-i)+5}} \prod_{i=0}^{m} w_{3(2i+1)-2}^{F_{6(m-i)+3}} \prod_{i=0}^{m} w_{3(2i+1)}^{F_{6(m-i)+1}}}{\prod_{i=0}^{m+1} w_{3(2i)-2}^{F_{6(m-i+1)}} \prod_{i=0}^{m} w_{3(2i)}^{F_{6(m-i)+4}} \prod_{i=0}^{m} w_{3(2i+1)-1}^{F_{6(m-i)+2}}} \right), \\ z_{6m+5} = \dfrac{z_{-3}^{F_{6m+8}}}{z_{-4}^{sF_{6m+7}}} \prod_{i=0}^{m+1} \left(\dfrac{w_{3(2i)-2}^{F_{6(m-i+1)+1}}}{w_{3(2i)-1}^{F_{6(m-i+1)}}} \right) \prod_{i=0}^{m} \left(\dfrac{w_{3(2i)}^{F_{6(m-i)+5}} w_{3(2i+1)-1}^{F_{6(m-i)+3}}}{w_{3(2i+1)-2}^{F_{6(m-i)+4}} w_{3(2i+1)}^{F_{6(m-i)+2}}} \right). \end{cases} \tag{18}$$

Case $a \neq 1$.

(i) If $n \equiv 0(mod3)$ equivalent $n = 3m$, $m \in \mathbb{N}_0$. Then, from (10) and (16), we have

$$\begin{cases} z_{6m} = \dfrac{z_{-4}^{sF_{6m+2}}}{z_{-3}^{F_{6m+3}}} \left(\dfrac{\prod_{i=0}^{m} \left(\frac{a^{2i}((a-1)w_{-1}+b)-b}{a-1}\right)^{F_{6(m-i)+1}} \prod_{i=0}^{m-1} \left(\frac{a^{2i+1}((a-1)w_{-2}+b)-b}{a-1}\right)^{F_{6(m-i)-1}}}{\prod_{i=0}^{m} \left(\frac{a^{2i}((a-1)w_{-2}+b)-b}{a-1}\right)^{F_{6(m-i)+2}} \prod_{i=0}^{m} \left(\frac{a^{2i}((a-1)w_{0}+b)-b}{a-1}\right)^{F_{6(m-i)}}} \right) \\ \qquad \times \dfrac{\prod_{i=0}^{m-1} \left(\frac{a^{2i+1}((a-1)w_{0}+b)-b}{a-1}\right)^{F_{6(m-i)-3}}}{\prod_{i=0}^{m-1} \left(\frac{a^{2i+1}((a-1)w_{-1}+b)-b}{a-1}\right)^{F_{6(m-i)-2}}}, \\ z_{6m+1} = \dfrac{z_{-3}^{F_{6m+4}}}{z_{-4}^{sF_{6m+3}}} \prod_{i=0}^{m} \left(\dfrac{\left(\frac{a^{2i}((a-1)w_{-2}+b)-b}{a-1}\right)^{F_{6(m-i)+3}} \left(\frac{a^{2i}((a-1)w_{0}+b)-b}{a-1}\right)^{F_{6(m-i)+1}}}{\left(\frac{a^{2i}((a-1)w_{-1}+b)-b}{a-1}\right)^{F_{6(m-i)+2}} \left(\frac{a^{2i+1}((a-1)w_{-2}+b)-b}{a-1}\right)^{F_{6(m-i)}}} \right) \\ \qquad \times \prod_{i=0}^{m-1} \left(\dfrac{\left(\frac{a^{2i+1}((a-1)w_{-1}+b)-b}{a-1}\right)^{F_{6(m-i)-1}}}{\left(\frac{a^{2i+1}((a-1)w_{0}+b)-b}{a-1}\right)^{F_{6(m-i)-2}}} \right). \end{cases} \tag{19}$$

(ii) If $n \equiv 1(mod3)$ equivalent $n = 3m+1$, $m \in \mathbb{N}_0$. Then, from (10) and (17), we get

$$\begin{cases} z_{6m+2} = \dfrac{z_{-4}^{sF_{6m+4}}}{z_{-3}^{F_{6m+5}}} \left(\dfrac{\prod_{i=0}^{m} \left(\frac{a^{2i}((a-1)w_{-1}+b)-b}{a-1}\right)^{F_{6(m-i)+3}} \prod_{i=0}^{m} \left(\frac{a^{2i+1}((a-1)w_{-2}+b)-b}{a-1}\right)^{F_{6(m-i)+1}}}{\prod_{i=0}^{m} \left(\frac{a^{2i}((a-1)w_{-2}+b)-b}{a-1}\right)^{F_{6(m-i)+4}} \prod_{i=0}^{m} \left(\frac{a^{2i}((a-1)w_{0}+b)-b}{a-1}\right)^{F_{6(m-i)+2}}} \right) \\ \qquad \times \dfrac{\prod_{i=0}^{m-1} \left(\frac{a^{2i+1}((a-1)w_{0}+b)-b}{a-1}\right)^{F_{6(m-i)-1}}}{\prod_{i=0}^{m} \left(\frac{a^{2i+1}((a-1)w_{-1}+b)-b}{a-1}\right)^{F_{6(m-i)}}}, \\ z_{6m+3} = \dfrac{z_{-3}^{F_{6m+6}}}{z_{-4}^{sF_{6m+5}}} \prod_{i=0}^{m} \left(\dfrac{\left(\frac{a^{2i}((a-1)w_{-2}+b)-b}{a-1}\right)^{F_{6(m-i)+5}} \left(\frac{a^{2i}((a-1)w_{0}+b)-b}{a-1}\right)^{F_{6(m-i)+3}}}{\left(\frac{a^{2i}((a-1)w_{-1}+b)-b}{a-1}\right)^{F_{6(m-i)+4}} \left(\frac{a^{2i+1}((a-1)w_{-2}+b)-b}{a-1}\right)^{F_{6(m-i)+2}}} \right) \\ \qquad \times \prod_{i=0}^{m} \left(\dfrac{\left(\frac{a^{2i+1}((a-1)w_{-1}+b)-b}{a-1}\right)^{F_{6(m-i)+1}}}{\left(\frac{a^{2i+1}((a-1)w_{0}+b)-b}{a-1}\right)^{F_{6(m-i)}}} \right). \end{cases} \tag{20}$$

(iii) If $n \equiv 2(mod3)$ equivalent $n = 3m + 2$, $m \geq -1$. Then, from (10) and (18), we obtain

$$\begin{cases} z_{6m+4} = & \frac{z_{-4}^{sF_{6m+6}}}{z_{-3}^{F_{6m+7}}} \left(\frac{\prod_{i=0}^{m} \left(\frac{a^{2i}((a-1)w_{-1}+b)-b}{a-1} \right)^{F_{6(m-i)+5}} \prod_{i=0}^{m} \left(\frac{a^{2i+1}((a-1)w_{-2}+b)-b}{a-1} \right)^{F_{6(m-i)+3}}}{\prod_{i=0}^{m+1} \left(\frac{a^{2i}((a-1)w_{-2}+b)-b}{a-1} \right)^{F_{6(m-i+1)}} \prod_{i=0}^{m} \left(\frac{a^{2i}((a-1)w_{0}+b)-b}{a-1} \right)^{F_{6(m-i)+4}}} \right), \\ & \times \prod_{i=0}^{m} \left(\frac{\left(\frac{a^{2i+1}((a-1)w_{0}+b)-b}{a-1} \right)^{F_{6(m-i)+1}}}{\left(\frac{a^{2i+1}((a-1)w_{-1}+b)-b}{a-1} \right)^{F_{6(m-i)+2}}} \right) \\ z_{6m+5} = & \frac{z_{-3}^{F_{6m+8}}}{z_{-4}^{sF_{6m+7}}} \prod_{i=0}^{m+1} \left(\frac{\left(\frac{a^{2i}((a-1)w_{-2}+b)-b}{a-1} \right)^{F_{6(m-i+1)+1}}}{\left(\frac{a^{2i}((a-1)w_{-1}+b)-b}{a-1} \right)^{F_{6(m-i+1)}}} \right) \\ & \times \prod_{i=0}^{m} \left(\frac{\left(\frac{a^{2i}((a-1)w_{0}+b)-b}{a-1} \right)^{F_{6(m-i)+5}} \left(\frac{a^{2i+1}((a-1)w_{-1}+b)-b}{a-1} \right)^{F_{6(m-i)+3}}}{\left(\frac{a^{2i+1}((a-1)w_{-2}+b)-b}{a-1} \right)^{F_{6(m-i)+4}} \left(\frac{a^{2i+1}((a-1)w_{0}+b)-b}{a-1} \right)^{F_{6(m-i)+2}}} \right), \end{cases} \quad (21)$$

where $w_{-2} = \frac{z_{-4}^{s}}{z_{-2}z_{-3}}$, $w_{-1} = \frac{z_{-3}^{s}}{z_{-1}z_{-2}}$ and $w_{0} = \frac{z_{-2}^{s}}{z_{0}z_{-1}}$.

Case $a = 1$.

(i) If $n \equiv 0(mod3)$ equivalent $n = 3m$, $m \in \mathbb{N}_0$. Then, from (10) and (16), we have

$$\begin{cases} z_{6m} = & \frac{z_{-4}^{sF_{6m+2}}}{z_{-3}^{F_{6m+3}}} \left(\frac{\prod_{i=0}^{m} (w_{-1}+2ib)^{F_{6(m-i)+1}} \prod_{i=0}^{m-1} (w_{-2}+(2i+1)b)^{F_{6(m-i)-1}}}{\prod_{i=0}^{m} (w_{-2}+2ib)^{F_{6(m-i)+2}} \prod_{i=0}^{m} (w_{0}+2ib)^{F_{6(m-i)}}} \right) \\ & \times \frac{\prod_{i=0}^{m-1} (w_{0}+(2i+1)b)^{F_{6(m-i)-3}}}{\prod_{i=0}^{m-1} (w_{-1}+(2i+1)b)^{F_{6(m-i)-2}}}, \\ z_{6m+1} = & \frac{z_{-3}^{F_{6m+4}}}{z_{-4}^{sF_{6m+3}}} \prod_{i=0}^{m} \left(\frac{(w_{-2}+2ib)^{F_{6(m-i)+3}} (w_{0}+2ib)^{F_{6(m-i)+1}}}{(w_{-1}+2ib)^{F_{6(m-i)+2}} (w_{-2}+(2i+1)b)^{F_{6(m-i)}}} \right) \\ & \times \prod_{i=0}^{m-1} \left(\frac{(w_{-1}+(2i+1)b)^{F_{6(m-i)-1}}}{(w_{0}+(2i+1)b)^{F_{6(m-i)-2}}} \right). \end{cases} \quad (22)$$

(ii) If $n \equiv 1(mod3)$ equivalent $n = 3m + 1$, $m \in \mathbb{N}_0$. Then, from (10) and (17), we get

$$\begin{cases} z_{6m+2} = & \frac{z_{-4}^{sF_{6m+4}}}{z_{-3}^{F_{6m+5}}} \left(\frac{\prod_{i=0}^{m} (w_{-1}+2ib)^{F_{6(m-i)+3}} \prod_{i=0}^{m} (w_{-2}+(2i+1)b)^{F_{6(m-i)+1}}}{\prod_{i=0}^{m} (w_{-2}+2ib)^{F_{6(m-i)+4}} \prod_{i=0}^{m} (w_{0}+2ib)^{F_{6(m-i)+2}}} \right) \\ & \times \frac{\prod_{i=0}^{m-1} (w_{0}+(2i+1)b)^{F_{6(m-i)-1}}}{\prod_{i=0}^{m} (w_{-1}+(2i+1)b)^{F_{6(m-i)}}}, \\ z_{6m+3} = & \frac{z_{-3}^{F_{6m+6}}}{z_{-4}^{sF_{6m+5}}} \prod_{i=0}^{m} \left(\frac{(w_{-2}+2ib)^{F_{6(m-i)+5}} (w_{0}+2ib)^{F_{6(m-i)+3}}}{(w_{-1}+2ib)^{F_{6(m-i)+4}} (w_{-2}+(2i+1)b)^{F_{6(m-i)+2}}} \right) \\ & \times \prod_{i=0}^{m} \left(\frac{(w_{-1}+(2i+1)b)^{F_{6(m-i)+1}}}{(w_{0}+(2i+1)b)^{F_{6(m-i)}}} \right). \end{cases} \quad (23)$$

(iii) If $n \equiv 2(mod3)$ equivalent $n = 3m + 2$, $m \geq -1$. Then, from (10) and (18), we obtain

$$\begin{cases} z_{6m+4} = & \frac{z_{-4}^{sF_{6m+6}}}{z_{-3}^{F_{6m+7}}} \left(\frac{\prod_{i=0}^{m} (w_{-1}+2ib)^{F_{6(m-i)+5}} \prod_{i=0}^{m} (w_{-2}+(2i+1)b)^{F_{6(m-i)+3}}}{\prod_{i=0}^{m+1} (w_{-2}+2ib)^{F_{6(m-i+1)}} \prod_{i=0}^{m} (w_{0}+2ib)^{F_{6(m-i)+4}}} \right), \\ & \times \prod_{i=0}^{m} \left(\frac{(w_{0}+(2i+1)b)^{F_{6(m-i)+1}}}{(w_{-1}+(2i+1)b)^{F_{6(m-i)+2}}} \right) \\ z_{6m+5} = & \frac{z_{-3}^{F_{6m+8}}}{z_{-4}^{sF_{6m+7}}} \prod_{i=0}^{m+1} \left(\frac{(w_{-2}+2ib)^{F_{6(m-i+1)+1}}}{(w_{-1}+2ib)^{F_{6(m-i+1)}}} \right) \\ & \times \prod_{i=0}^{m} \left(\frac{(w_{0}+2ib)^{F_{6(m-i)+5}} (w_{-1}+(2i+1)b)^{F_{6(m-i)+3}}}{(w_{-2}+(2i+1)b)^{F_{6(m-i)+4}} (w_{0}+(2i+1)b)^{F_{6(m-i)+2}}} \right), \end{cases} \quad (24)$$

where $w_{-2} = \frac{z_{-4}^s}{z_{-2}z_{-3}}$, $w_{-1} = \frac{z_{-3}^s}{z_{-1}z_{-2}}$ and $w_0 = \frac{z_{-2}^s}{z_0 z_{-1}}$.

Theorem 2.2. *Consider equation* (4) *where the initial conditions* z_{-q}, $q = \overline{0,4}$ *and the parameters* a, b *are non-zero real numbers . Then, the following statements hold:*

(a) *If* $n = 3m$, $m \in \mathbb{N}_0$, *and* $a \neq 1$, *then the solution of equation* (4) *is given by* (19).

(b) *If* $n = 3m$, $m \in \mathbb{N}_0$, *and* $a = 1$, *then the solution of equation* (4) *is given by* (22).

(c) *If* $n = 3m+1$, $m \in \mathbb{N}_0$, *and* $a \neq 1$, *then the solution of equation* (4) *is given by* (20).

(d) *If* $n = 3m+1$, $m \in \mathbb{N}_0$, *and* $a = 1$, *then the solution of equation* (4) *is given by* (23).

(e) *If* $n = 3m+2$, $m \geq -1$, *and* $a \neq 1$, *then the solution of equation* (4) *is given by* (21).

(f) *If* $n = 3m+2$, $m \geq -1$, *and* $a = 1$, *then the solution of equation* (4) *is given by* (24).

3. The solutions of a general difference equation defined by a one to one continuous function

In this section, we solve in explicit form the following general difference equation

$$z_n = h^{-1}\left(\frac{(h(z_{n-2}))^s h(z_{n-3}) h(z_{n-4})}{h(z_{n-1})\left[a(h(z_{n-5}))^s + bh(z_{n-3}) h(z_{n-4})\right]}\right),\ s, n \in \mathbb{N}, \tag{25}$$

where $h : B \to \mathbb{R}$ is one to one continuous function on $B \subseteq \mathbb{R}$, the initial values $z_{-\psi}$, $\psi = \overline{0,4}$ are non-zero real numbers in B and the parameters a, b are non-zero real numbers.

Definition 3.1. *A solution* $\{z_n\}_{n\geq -4}$ *of equation* (25) *is said to be well-defined if*

$$h(z_{n-1})\left[a(h(z_{n-5}))^s + bh(z_{n-3}) h(z_{n-4})\right] \neq 0,\ n \in \mathbb{N},$$

and

$$\frac{(h(z_{n-2}))^s h(z_{n-3}) h(z_{n-4})}{h(z_{n-1})\left[a(h(z_{n-5}))^s + bh(z_{n-3}) h(z_{n-4})\right]} \in A_{h^{-1}}.$$

Theorem 3.2. *Let* $\{z_n\}_{n\geq -4}$ *be a well-defined solution of equation* (25). *Then, the system is solvable in closed form.*

Proof. Since h is one to one continuous function, then, from equation (25), we obtain

$$h(z_n) = \frac{(h(z_{n-2}))^s h(z_{n-3}) h(z_{n-4})}{h(z_{n-1})\left[a(h(z_{n-5}))^s + bh(z_{n-3}) h(z_{n-4})\right]},\ s, n \in \mathbb{N}. \tag{26}$$

By using following the change of variable,

$$Z_n = h(z_n),\ n = -4, -3, \ldots \tag{27}$$

it follows that equation (25) can be transformed to the following equation

$$Z_n = \frac{Z_{n-2}^s Z_{n-3} Z_{n-4}}{Z_{n-1}\left(aZ_{n-5}^s + bZ_{n-3}Z_{n-4}\right)},\ s, n \in \mathbb{N}, \tag{28}$$

which is in the form of equation (4). By using the following transform,

$$\widehat{w}_n = \frac{Z_{n-2}^s}{Z_n Z_{n-1}},\ n = -2, -1, \ldots \tag{29}$$

from equation (28), we get

$$\widehat{w}_n = a\widehat{w}_{n-3} + b,\ n \in \mathbb{N}. \tag{30}$$

By using (6), we have for all $m \in \mathbb{N}_0$, $j \in \{-2, -1, 0\}$, the solution of equation (30) is

$$\widehat{w}_{3m+j} = \begin{cases} a^m \widehat{w}_j + b\frac{a^m-1}{a-1}, & \text{if } a \neq 1, \\ \widehat{w}_j + bm, & \text{if } a = 1. \end{cases} \tag{31}$$

Moreover, using (27), (29) and (31), for all $m \in \mathbb{N}_0$, we have

$$\widehat{w}_{3m-2} = \begin{cases} \frac{a^m (h(z_{-4}))^s}{h(z_{-2})h(z_{-3})} + b\frac{a^m-1}{a-1}, & \text{if } a \neq 1, \\ \frac{(h(z_{-4}))^s}{h(z_{-2})h(z_{-3})} + bm, & \text{if } a = 1, \end{cases} \tag{32}$$

$$\widehat{w}_{3m-1} = \begin{cases} \frac{a^m (h(z_{-3}))^s}{h(z_{-1})h(z_{-2})} + b\frac{a^m-1}{a-1}, & \text{if } a \neq 1, \\ \frac{(h(z_{-3}))^s}{h(z_{-1})h(z_{-2})} + bm, & \text{if } a = 1, \end{cases} \tag{33}$$

$$\widehat{w}_{3m} = \begin{cases} \frac{a^m (h(z_{-2}))^s}{h(z_0)h(z_{-1})} + b\frac{a^m-1}{a-1}, & \text{if } a \neq 1, \\ \frac{(h(z_{-2}))^s}{h(z_0)h(z_{-1})} + bm, & \text{if } a = 1. \end{cases} \tag{34}$$

By using (27), we obtain

$$z_n = h^{-1}(Z_n),\ n = -4, -3, \ldots \tag{35}$$

In addition, by using (16)-(18), (27) and (35), we get

$$\begin{aligned}
z_{6m} &= h^{-1}\left(\frac{(h(z_{-4}))^{sF_{6m+2}}}{(h(z_{-3}))^{F_{6m+3}}}\left(\frac{\prod_{i=0}^{m}\widehat{w}_{3(2i)-1}^{F_{6(m-i)+1}}\prod_{i=0}^{m-1}\widehat{w}_{3(2i+1)-2}^{F_{6(m-i)-1}}\prod_{i=0}^{m-1}\widehat{w}_{3(2i+1)}^{F_{6(m-i)-3}}}{\prod_{i=0}^{m}\widehat{w}_{3(2i)-2}^{F_{6(m-i)+2}}\prod_{i=0}^{m}\widehat{w}_{3(2i)}^{F_{6(m-i)}}\prod_{i=0}^{m-1}\widehat{w}_{3(2i+1)-1}^{F_{6(m-i)-2}}}\right)\right), m \in \mathbb{N}_0, \\
z_{6m+1} &= h^{-1}\left(\frac{(h(z_{-3}))^{F_{6m+4}}}{(h(z_{-4}))^{sF_{6m+3}}}\prod_{i=0}^{m}\left(\frac{\widehat{w}_{3(2i)-2}^{F_{6(m-i)+3}}\widehat{w}_{3(2i)}^{F_{6(m-i)+1}}}{\widehat{w}_{3(2i)-1}^{F_{6(m-i)+2}}\widehat{w}_{3(2i+1)-2}^{F_{6(m-i)}}}\right)\prod_{i=0}^{m-1}\left(\frac{\widehat{w}_{3(2i+1)-1}^{F_{6(m-i)-1}}}{\widehat{w}_{3(2i+1)}^{F_{6(m-i)-2}}}\right)\right), m \in \mathbb{N}_0, \\
z_{6m+2} &= h^{-1}\left(\frac{(h(z_{-4}))^{sF_{6m+4}}}{(h(z_{-3}))^{F_{6m+5}}}\left(\frac{\prod_{i=0}^{m}\widehat{w}_{3(2i)-1}^{F_{6(m-i)+3}}\prod_{i=0}^{m}\widehat{w}_{3(2i+1)-2}^{F_{6(m-i)+1}}\prod_{i=0}^{m-1}\widehat{w}_{3(2i+1)}^{F_{6(m-i)-1}}}{\prod_{i=0}^{m}\widehat{w}_{3(2i)-2}^{F_{6(m-i)+4}}\prod_{i=0}^{m}\widehat{w}_{3(2i)}^{F_{6(m-i)+2}}\prod_{i=0}^{m}\widehat{w}_{3(2i+1)-1}^{F_{6(m-i)}}}\right)\right), m \in \mathbb{N}_0, \\
z_{6m+3} &= h^{-1}\left(\frac{(h(z_{-3}))^{F_{6m+6}}}{(h(z_{-4}))^{sF_{6m+5}}}\prod_{i=0}^{m}\left(\frac{\widehat{w}_{3(2i)-2}^{F_{6(m-i)+5}}\widehat{w}_{3(2i)}^{F_{6(m-i)+3}}\widehat{w}_{3(2i+1)-1}^{F_{6(m-i)+1}}}{\widehat{w}_{3(2i)-1}^{F_{6(m-i)+4}}\widehat{w}_{3(2i+1)-2}^{F_{6(m-i)+2}}\widehat{w}_{3(2i+1)}^{F_{6(m-i)}}}\right)\right), m \in \mathbb{N}_0, \\
z_{6m+4} &= h^{-1}\left(\frac{(h(z_{-4}))^{sF_{6m+6}}}{(h(z_{-3}))^{F_{6m+7}}}\left(\frac{\prod_{i=0}^{m}\widehat{w}_{3(2i)-1}^{F_{6(m-i)+5}}\prod_{i=0}^{m}\widehat{w}_{3(2i+1)-2}^{F_{6(m-i)+3}}\prod_{i=0}^{m}\widehat{w}_{3(2i+1)}^{F_{6(m-i)+1}}}{\prod_{i=0}^{m+1}\widehat{w}_{3(2i)-2}^{F_{6(m-i+1)}}\prod_{i=0}^{m}\widehat{w}_{3(2i)}^{F_{6(m-i)+4}}\prod_{i=0}^{m}\widehat{w}_{3(2i+1)-1}^{F_{6(m-i)+2}}}\right)\right), m \geq -1, \\
z_{6m+5} &= h^{-1}\left(\frac{(h(z_{-3}))^{F_{6m+8}}}{(h(z_{-4}))^{sF_{6m+7}}}\prod_{i=0}^{m+1}\left(\frac{\widehat{w}_{3(2i)-2}^{F_{6(m-i+1)+1}}}{\widehat{w}_{3(2i)-1}^{F_{6(m-i+1)}}}\right)\prod_{i=0}^{m}\left(\frac{\widehat{w}_{3(2i)}^{F_{6(m-i)+5}}\widehat{w}_{3(2i+1)-1}^{F_{6(m-i)+3}}}{\widehat{w}_{3(2i+1)-2}^{F_{6(m-i)+4}}\widehat{w}_{3(2i+1)}^{F_{6(m-i)+2}}}\right)\right), m \geq -1.
\end{aligned} \tag{36}$$

As a consequence, by using (32)-(34) and (36), the solution of equation (25) can be obtained if $a \neq 1$, or if $a = 1$. The proof of theorem is finished. □

4. Conclusion

In this study, we have solved a non-linear difference equation of fifth-order in closed form. In addition, we have obtained the solutions of a more general difference equation defined by a one to one continuous function. In both of these difference equations, the solutions are presented by using a generalized Fibonacci sequence.

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