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Congruences with sums of powers of quadrinomial coefficients

Zhongyan Shen[a,*], Jiajia Guo[a]

[a] *Department of Mathematics, Zhejiang International Studies University, Hangzhou 310023, P. R. China*

Abstract. Quadrinomial coefficients $\binom{n}{k}_3$ are defined by

$$(1+x+x^2+x^3)^n = \sum_{k=0}^{3n} \binom{n}{k}_3 x^k.$$

Let $p > 3$ be a prime number and n, m be positive integers, we obtained the congruences modulo p^2 with partial sums of powers of quadrinomial coefficients

$$\sum_{0\le 4k+i\le p-1} \binom{np-1}{4k+i}_3^m \text{ and } \sum_{0\le 4k+i\le \frac{p-1}{2}} \binom{np-1}{4k+i}_3^m (0\le i\le 3).$$

We also studied the congruences modulo p^2 with sums and alternating sums of powers of quadrinomial coefficients

$$\sum_{k=0}^{p-1}\binom{np-1}{k}_3^m, \sum_{k=0}^{\frac{p-1}{2}}\binom{np-1}{k}_3^m, \sum_{k=0}^{p-1}(-1)^k\binom{np-1}{k}_3^m \text{ and } \sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{np-1}{k}_3^m.$$

1. Introduction

In 1819, Babbage [1] showed the congruence

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}$$

for any odd prime number p. In 1862, Wolstenholme [15] proved the above congruence about modulo p^3,

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$$

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* Corresponding author: Zhongyan Shen
Email addresses: huanchenszyan@163.com (Zhongyan Shen), 1146060846@qq.com (Jiajia Guo)

and in 1900, Glaisher [9] extended the congruence

$$\binom{np-1}{p-1} \equiv 1 \pmod{p^3}$$

for any prime number $p > 3$ and positive integer n. In 1895, Morley [11] showed that for any prime $p \geq 5$,

$$(-1)^{\frac{p-1}{2}}\binom{p-1}{(p-1)/2} \equiv 4^{p-1} \pmod{p^3}.$$

And in 1953, Carlitz [5],[6] extended Morley's congruence and showed that, for any prime number $p \geq 5$,

$$(-1)^{\frac{p-1}{2}}\binom{p-1}{(p-1)/2} \equiv 4^{p-1} + \frac{p^3}{12} \pmod{p^4}.$$

In 2002, Cai and Granville [2] showed several arithmetic properties on the residues of binomial coefficients and their products modulo primes powers, they also proved that if $p \geq 5$ is a prime and m is an integer, then

$$\sum_{s=0}^{p-1}\binom{p-1}{s}^m \equiv \begin{cases} 2^{m(p-1)} \pmod{p^3}, \text{ if } 2 \nmid m, \\ \binom{mp-2}{p-1} \pmod{p^4}, \text{ if } 2 \mid m; \end{cases}$$

and

$$\sum_{s=0}^{p-1}(-1)^s\binom{p-1}{s}^m \equiv \begin{cases} 2^{m(p-1)} \pmod{p^3}, \text{ if } 2 \mid m, \\ \binom{mp-2}{p-1} \pmod{p^4}, \text{ if } 2 \nmid m. \end{cases}$$

In 2018, for any prime $p \geq 7$, integer $l \geq 0$ and positive integers $k,\ m$, the second author and Cai [12] proved that

$$\sum_{s=lp}^{(l+1)p-1}\binom{kp-1}{s}^m \equiv \begin{cases} \binom{k-1}{l}^m 2^{km(p-1)} \pmod{p^3}, \text{ if } 2 \nmid m, \\ \binom{k-1}{l}^m\binom{kmp-2}{p-1} \pmod{p^4}, \text{ if } 2 \mid m; \end{cases}$$

and

$$\sum_{s=lp}^{(l+1)p-1}(-1)^s\binom{kp-1}{s}^m \equiv \begin{cases} \binom{k-1}{l}^m 2^{km(p-1)} \pmod{p^3}, \text{ if } 2 \mid m, \\ \binom{k-1}{l}^m\binom{kmp-2}{p-1} \pmod{p^4}, \text{ if } 2 \nmid m. \end{cases}$$

In 2014, Sun [13] gave some properties and congruences involving the trinomial coefficients $\binom{n}{k}_2$ defined by

$$(1+x+x^2)^n = \sum_{k=0}^{2n}\binom{n}{k}_2 x^k,$$

also see [3] and [4]. In 2019, for any prime number $p > 3$ and positive integer n, Elkhiri and Mihoubi [7] studied congruences modulo p^2 for the trinomial coefficients $\binom{np-1}{p-1}_2$ and $\binom{np-1}{\frac{p-1}{2}}_2$. They also proved following congruences involving the sum of trinomial coefficients

$$\sum_{k=0}^{p-1}\binom{np-1}{k}_2 \equiv \begin{cases} 1 + npq_p(3) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 3, \\ 0 \pmod{p^2}, \text{ if } p \equiv 2 \pmod 3; \end{cases}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{np-1}{k}_2 \equiv \begin{cases} 1 + np\left(\frac{4}{3}q_p(2) + q_p(3)\right) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 6, \\ -\frac{2np}{3}q_p(2) \pmod{p^2}, \text{ if } p \equiv 5 \pmod 6. \end{cases}$$

Here and in what follows $q_p(a)$ is the Fermat quotient defined for a given prime number $p > 3$ by $q_p(a) = \frac{a^{p-1}-1}{p}(\gcd(a,p) = 1)$. Recently, for any prime number $p > 3$ and positive integer n, Mechacha [10] studied congruences modulo p^2 for the quadrinomial coefficients $\binom{np-1}{p-1}_3$ and $\binom{np-1}{\frac{p-1}{2}}_3$, where quadrinomial coefficients $\binom{n}{k}_3$ defined by

$$(1 + x + x^2 + x^3)^n = \sum_{k=0}^{3n} \binom{n}{k}_3 x^k.$$

He also proved following congruences involving the sum of quadrinomial coefficients

$$\sum_{k=0}^{p-1} \binom{np-1}{k}_3 \equiv \begin{cases} 1 + \frac{9np}{4}q_p(2) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 4; \\ -\frac{np}{4}q_p(2) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 4; \end{cases}$$

and

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{np-1}{k}_3 \equiv \begin{cases} 1 + np\left(3q_p(2) + \frac{3}{2}\chi_p\right) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 8; \\ -np\left(\frac{1}{4}q_p(2) - \frac{1}{2}\chi_p\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 8; \\ -np\left(\frac{1}{2}q_p(2) - \frac{1}{2}\chi_p\right) \pmod{p^2}, \text{ if } p \equiv 5 \pmod 8; \\ -np\left(\frac{1}{4}q_p(2) + \frac{1}{2}\chi_p\right) \pmod{p^2}, \text{ if } p \equiv 7 \pmod 8; \end{cases}$$

where $\left(\frac{2}{p}\right)$ is the Legendre Symbol, $\chi_p = \frac{P_{p-\left(\frac{2}{p}\right)}}{p}$ is the Pell quotient and P_n is the Pell sequence (OEIS A000129),the first 16 Pell numbers are

$$1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, 470832.$$

This paper mainly considered the congruences modulo p^2 with partial sums of powers of quadrinomial coefficients $\sum_{0\le 4k+i\le p-1} \binom{np-1}{4k+i}_3^m$ and $\sum_{0\le 4k+i\le \frac{p-1}{2}} \binom{np-1}{4k+i}_3^m (0 \le i \le 3)$. We also studied the congruences modulo p^2 with sums and alternating sums of powers of quadrinomial coefficients $\sum_{k=0}^{p-1} \binom{np-1}{k}_3^m$, $\sum_{k=0}^{\frac{p-1}{2}} \binom{np-1}{k}_3^m$, $\sum_{k=0}^{p-1} (-1)^k \binom{np-1}{k}_3^m$, $\sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{np-1}{k}_3^m$ and obtained the following theorems.

Theorem 1.1. *Let $p > 3$ be a prime number and n, m be positive integers. Then we have*

$$\sum_{k=0}^{\left[\frac{p}{4}\right]} \binom{np-1}{4k}_3^m \equiv \begin{cases} \frac{p+3}{4} + mnp\left(\frac{25}{16}q_p(2) - \frac{1}{4}\right) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 4; \\ \frac{p+1}{4} + mnp\left(\frac{9}{16}q_p(2) - \frac{3}{4}\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 4, \end{cases}$$

where $[x]$ is the greatest integer not greater than x.

Theorem 1.2. *Let $p > 3$ be a prime number and n, m be positive integers. Then we have*

$$\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+1}_3^m \equiv \begin{cases} (-1)^m\left(\frac{p-1}{4} - mnp\left(\frac{9}{16}q_p(2) + \frac{1}{4}\right)\right) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 4; \\ (-1)^m\left(\frac{p+1}{4} + mnp\left(\frac{7}{16}q_p(2) - \frac{1}{2}\right)\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 4. \end{cases}$$

Theorem 1.3. *Let $p > 3$ be a prime number and n, m be positive integers. Then we have*

$$\sum_{k=0}^{[\frac{p-3}{4}]} \binom{np-1}{4k+2}_3^m \equiv \begin{cases} 0 \pmod{p^2}, \text{ if } m > 1; \\ -\frac{np}{16} q_p(2) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 4 \text{ and } m = 1; \\ -\frac{5np}{16} q_p(2) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 4 \text{ and } m = 1. \end{cases}$$

Theorem 1.4. *Let $p > 3$ be a prime number and n, m be positive integers. Then we have*

$$\sum_{k=0}^{[\frac{p}{4}]-1} \binom{np-1}{4k+3}_3^m \equiv \begin{cases} 0 \pmod{p^2}, \text{ if } m > 1; \\ \frac{3np}{16} q_p(2) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 4 \text{ and } m = 1; \\ -\frac{np}{16} (q_p(2) - 4) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 4 \text{ and } m = 1. \end{cases}$$

Theorem 1.5. *Let $p > 3$ be a prime number and n, m be positive integers. Then we have*

$$\sum_{k=0}^{p-1} \binom{np-1}{k}_3 \equiv \begin{cases} 1 + \frac{9np}{4} q_p(2) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 4; \\ -\frac{np}{4} q_p(2) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 4. \end{cases}$$

When even $m > 1$, we have

$$\sum_{k=0}^{p-1} \binom{np-1}{k}_3^m \equiv \begin{cases} \frac{p+1}{2} + mnp\left(q_p(2) - \frac{1}{2}\right) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 4; \\ \frac{p+1}{2} + mnp\left(q_p(2) - \frac{5}{4}\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 4. \end{cases}$$

When odd $m > 1$, we have

$$\sum_{k=0}^{p-1} \binom{np-1}{k}_3^m \equiv \begin{cases} 1 + \frac{17mnp}{8} q_p(2) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 4; \\ \frac{mnp}{8} \left(q_p(2) - 2\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 4. \end{cases}$$

Theorem 1.6. *Let $p > 3$ be a prime number and n, m be positive integers. Then we have*

$$\sum_{k=0}^{p-1} (-1)^k \binom{np-1}{k}_3 \equiv \begin{cases} \frac{p+1}{2} + np\left(\frac{3}{4} q_p(2) - \frac{1}{2}\right) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 4; \\ \frac{p+1}{2} + np\left(\frac{3}{4} q_p(2) - \frac{3}{2}\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 4. \end{cases}$$

When even $m > 1$, we have

$$\sum_{k=0}^{p-1} (-1)^k \binom{np-1}{k}_3^m \equiv \begin{cases} 1 + \frac{17mnp}{8} q_p(2) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 4; \\ \frac{mnp}{8} \left(q_p(2) - 2\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 4. \end{cases}$$

When odd $m > 1$, we have

$$\sum_{k=0}^{p-1} (-1)^k \binom{np-1}{k}_3^m \equiv \begin{cases} \frac{p+1}{2} + mnp\left(q_p(2) - \frac{1}{2}\right) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 4; \\ \frac{p+1}{2} + mnp\left(q_p(2) - \frac{5}{4}\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 4. \end{cases}$$

Theorem 1.7. *Let $p > 3$ be a prime number and n, m be positive integers. Then we have*

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{np-1}{k}_3 \equiv \begin{cases} 1 + np\left(3q_p(2) + \frac{3}{2}\chi_p\right) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 8; \\ -np\left(\frac{1}{4} q_p(2) - \frac{1}{2}\chi_p\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 8; \\ -np\left(\frac{1}{2} q_p(2) - \frac{1}{2}\chi_p\right) \pmod{p^2}, \text{ if } p \equiv 5 \pmod 8; \\ -np\left(\frac{1}{4} q_p(2) + \frac{1}{2}\chi_p\right) \pmod{p^2}, \text{ if } p \equiv 7 \pmod 8. \end{cases}$$

When even $m > 1$, we have

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{np-1}{k}_3^m \equiv \begin{cases} \frac{p+3}{4} + mnp\left(\frac{39}{16}q_p(2) + \frac{7}{8}\chi_p - \frac{1}{4}\right) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 8; \\ \frac{p+5}{4} + mnp\left(\frac{65}{16}q_p(2) + \frac{13}{8}\chi_p - \frac{1}{2}\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 8; \\ \frac{p+3}{4} + mnp\left(\frac{39}{16}q_p(2) + \frac{11}{8}\chi_p - 1\right) \pmod{p^2}, \text{ if } p \equiv 5 \pmod 8; \\ \frac{p+1}{4} + mnp\left(\frac{13}{32}q_p(2) + \frac{1}{8}\chi_p - \frac{5}{4}\right) \pmod{p^2}, \text{ if } p \equiv 7 \pmod 8. \end{cases}$$

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{np-1}{k}_3^m \equiv \begin{cases} 1 + mnp\left(\frac{25}{4}q_p(2) + \frac{9}{8}\chi_p\right) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 8; \\ -mnp\left(\frac{1}{8}q_p(2) - \frac{5}{8}\chi_p + \frac{1}{4}\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 8; \\ -mnp\left(\frac{1}{8}q_p(2) + \frac{3}{8}\chi_p + \frac{1}{4}\right) \pmod{p^2}, \text{ if } p \equiv 5 \pmod 8; \\ -mnp\left(\frac{1}{8}q_p(2) - \frac{1}{8}\chi_p\right) \pmod{p^2}, \text{ if } p \equiv 7 \pmod 8. \end{cases}$$

When odd $m > 1$, we have

$$\sum_{k=0}^{\frac{p-1}{2}} \binom{np-1}{k}_3^m \equiv \begin{cases} 1 + mnp\left(\frac{25}{4}q_p(2) + \frac{9}{8}\chi_p\right) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 8; \\ -mnp\left(\frac{1}{8}q_p(2) - \frac{5}{8}\chi_p + \frac{1}{4}\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 8; \\ -mnp\left(\frac{1}{8}q_p(2) + \frac{3}{8}\chi_p + \frac{1}{4}\right) \pmod{p^2}, \text{ if } p \equiv 5 \pmod 8; \\ -mnp\left(\frac{1}{8}q_p(2) - \frac{1}{8}\chi_p\right) \pmod{p^2}, \text{ if } p \equiv 7 \pmod 8. \end{cases}$$

$$\sum_{k=0}^{\frac{p-1}{2}} (-1)^k \binom{np-1}{k}_3^m \equiv \begin{cases} \frac{p+3}{4} + mnp\left(\frac{39}{16}q_p(2) + \frac{7}{8}\chi_p - \frac{1}{4}\right) \pmod{p^2}, \text{ if } p \equiv 1 \pmod 8; \\ \frac{p+5}{4} + mnp\left(\frac{65}{16}q_p(2) + \frac{13}{8}\chi_p - \frac{1}{2}\right) \pmod{p^2}, \text{ if } p \equiv 3 \pmod 8; \\ \frac{p+3}{4} + mnp\left(\frac{39}{16}q_p(2) + \frac{11}{8}\chi_p - 1\right) \pmod{p^2}, \text{ if } p \equiv 5 \pmod 8; \\ \frac{p+1}{4} + mnp\left(\frac{13}{32}q_p(2) + \frac{1}{8}\chi_p - \frac{5}{4}\right) \pmod{p^2}, \text{ if } p \equiv 7 \pmod 8. \end{cases}$$

2. Auxiliary Results

Let H_n be the n-th harmonic number defined by

$$H_0 = 0, H_n = \sum_{j=1}^{n} \frac{1}{j}.$$

Lemma 2.1 ([8],[14]). *Let $p > 3$ be a prime number. Then we have*

$$H_{[\frac{p}{4}]} \equiv -3q_p(2) \pmod p$$

and

$$H_{[\frac{p}{8}]} \equiv -4q_p(2) - 2\chi_p \pmod p.$$

Lemma 2.2 ([10]). *Let $p > 3$ be a prime number. Then we have*

$$\sum_{j=0}^{[\frac{p-3}{4}]} \frac{1}{4j+1} \equiv \begin{cases} \frac{3}{4}q_p(2) \pmod p, \text{ if } p \equiv 1 \pmod 4; \\ \frac{1}{4}q_p(2) \pmod p, \text{ if } p \equiv 3 \pmod 4; \end{cases}$$

$$\sum_{j=0}^{[\frac{p-3}{4}]} \frac{1}{4j+2} \equiv -\frac{1}{4}q_p(2) \pmod p$$

and

$$\sum_{j=0}^{[\frac{p}{4}]-1} \frac{1}{4j+3} \equiv \begin{cases} \frac{1}{4}q_p(2) \pmod p, \text{ if } p \equiv 1 \pmod 4; \\ \frac{3}{4}q_p(2) \pmod p, \text{ if } p \equiv 3 \pmod 4. \end{cases}$$

Lemma 2.3 ([10]). *Let $p > 3$ be a prime number. Then we have*

$$\sum_{j=0}^{\left[\frac{p-3}{8}\right]} \frac{1}{4j+1} \equiv \begin{cases} -\frac{1}{4}q_p(2) - \frac{1}{2}\chi_p \pmod p, \text{ if } p \equiv 1 \pmod 4; \\ -\frac{1}{4}q_p(2) + \frac{1}{2}\chi_p \pmod p, \text{ if } p \equiv 3 \pmod 4; \end{cases}$$

$$\sum_{j=0}^{\left[\frac{p-5}{8}\right]} \frac{1}{4j+2} \equiv -\frac{1}{2}q_p(2) + \frac{1}{2}\chi_p \pmod p$$

and

$$\sum_{j=0}^{\left[\frac{p-7}{8}\right]} \frac{1}{4j+3} \equiv \begin{cases} -\frac{1}{4}q_p(2) + \frac{1}{2}\chi_p \pmod p, \text{ if } p \equiv 1 \pmod 4; \\ -\frac{1}{4}q_p(2) - \frac{1}{2}\chi_p \pmod p, \text{ if } p \equiv 3 \pmod 4. \end{cases}$$

Lemma 2.4 ([10]). *Let $p > 3$ be a prime number and $n > 0, k$ be integers. Then we have*

$$\binom{np-1}{4k}_3 \equiv 1 - np\left(\frac{3}{4}H_k + \sum_{j=0}^{k-1} \frac{1}{4j+3}\right) \pmod{p^2}, 1 \le 4k \le p-1,$$

$$\binom{np-1}{4k+1}_3 \equiv -1 + np\left(\frac{3}{4}H_k + \sum_{j=0}^{k} \frac{1}{4j+1}\right) \pmod{p^2}, 1 \le 4k+1 \le p-1,$$

$$\binom{np-1}{4k+2}_3 \equiv np\left(\sum_{j=0}^{k} \frac{1}{4j+2} - \sum_{j=0}^{k} \frac{1}{4j+1}\right) \pmod{p^2}, 1 \le 4k+2 \le p-1$$

and

$$\binom{np-1}{4k+3}_3 \equiv np\left(\sum_{j=0}^{k} \frac{1}{4j+3} - \sum_{j=0}^{k} \frac{1}{4j+2}\right) \pmod{p^2}, 1 \le 4k+3 \le p-1.$$

3. Proofs

3.1. Proof of Theorem 1.1

Proof. By Lemma 2.4, we have

$$\begin{aligned} \sum_{k=0}^{\left[\frac{p}{4}\right]} \binom{np-1}{4k}_3^m &= 1 + \sum_{k=1}^{\left[\frac{p}{4}\right]} \binom{np-1}{4k}_3^m \\ &\equiv 1 + \sum_{k=1}^{\left[\frac{p}{4}\right]} \left(1 - mnp\left(\frac{3}{4}H_k + \sum_{j=0}^{k-1} \frac{1}{4j+3}\right)\right) \\ &\equiv 1 + \left[\frac{p}{4}\right] - mnp\sum_{k=1}^{\left[\frac{p}{3}\right]} \left(\frac{3}{4}H_k + \sum_{j=0}^{k-1} \frac{1}{4j+3}\right) \pmod{p^2}. \end{aligned} \tag{1}$$

Changing the sum order of j and k in (1), we get

$$\sum_{k=0}^{[\frac{p}{4}]}\binom{np-1}{4k}_3^m \equiv 1+\left[\frac{p}{4}\right]-mnp\left(\frac{3}{4}\sum_{j=1}^{[\frac{p}{4}]}\frac{1}{j}\sum_{k=j}^{[\frac{p}{4}]}1+\sum_{j=0}^{[\frac{p}{4}]-1}\frac{1}{4j+3}\sum_{k=j+1}^{[\frac{p}{4}]}1\right)$$
$$\equiv 1+\left[\frac{p}{4}\right]-mnp\left(\frac{3}{4}\sum_{j=1}^{[\frac{p}{4}]}\frac{\left[\frac{p}{4}\right]-j+1}{j}+\sum_{j=0}^{[\frac{p}{4}]-1}\frac{\left[\frac{p}{4}\right]-j}{4j+3}\right) \pmod{p^2}. \tag{2}$$

For $p\equiv 1 \pmod 4$ in (2), we have $\left[\frac{p}{4}\right]=\frac{p-1}{4}$, then

$$\sum_{k=0}^{[\frac{p}{4}]}\binom{np-1}{4k}_3^m \equiv 1+\frac{p-1}{4}-mnp\left(\frac{3}{4}\sum_{j=1}^{[\frac{p}{4}]}\frac{\frac{p-1}{4}-j+1}{j}+\sum_{j=0}^{[\frac{p}{4}]-1}\frac{\frac{p-1}{4}-j}{4j+3}\right)$$
$$\equiv \frac{p+3}{4}-mnp\left(\frac{9}{16}H_{[\frac{p}{4}]}-\frac{3}{4}\left[\frac{p}{4}\right]-\frac{1}{4}\sum_{j=0}^{[\frac{p}{4}]-1}\frac{(4j+3)-2}{4j+3}\right)$$
$$\equiv \frac{p+3}{4}-mnp\left(\frac{9}{16}H_{[\frac{p}{4}]}-\left[\frac{p}{4}\right]+\frac{1}{2}\sum_{j=0}^{[\frac{p}{4}]-1}\frac{1}{4j+3}\right)$$
$$\equiv \frac{p+3}{4}-mnp\left(\frac{9}{16}H_{[\frac{p}{4}]}+\frac{1}{4}+\frac{1}{2}\sum_{j=0}^{[\frac{p}{4}]-1}\frac{1}{4j+3}\right) \pmod{p^2}. \tag{3}$$

By Lemma 2.1 and Lemma 2.2, for $p\equiv 1 \pmod 4$, (3) is congruent to

$$\sum_{k=0}^{[\frac{p}{4}]}\binom{np-1}{4k}_3^m \equiv \frac{p+3}{4}+mnp\left(\frac{25}{16}q_p(2)-\frac{1}{4}\right) \pmod{p^2}. \tag{4}$$

For $p\equiv 3 \pmod 4$ in (2), we have $\left[\frac{p}{4}\right]=\frac{p-3}{4}$, then

$$\sum_{k=0}^{[\frac{p}{4}]}\binom{np-1}{4k}_3^m \equiv 1+\frac{p-3}{4}-mnp\left(\frac{3}{4}\sum_{j=1}^{[\frac{p}{3}]}\frac{\frac{p-3}{4}-j+1}{j}+\sum_{j=0}^{[\frac{p}{4}]-1}\frac{\frac{p-3}{4}-j}{4j+3}\right)$$
$$\equiv \frac{p+1}{4}-mnp\left(\frac{3}{16}H_{[\frac{p}{4}]}-\frac{3}{4}\left[\frac{p}{4}\right]-\frac{1}{4}\sum_{j=0}^{[\frac{p}{4}]-1}\frac{4j+3}{4j+3}\right)$$
$$\equiv \frac{p+1}{4}-mnp\left(\frac{3}{16}H_{[\frac{p}{3}]}+\frac{3}{4}\right) \pmod{p^2}. \tag{5}$$

By Lemma 2.1, for $p\equiv 3 \pmod 4$, (5) is congruent to

$$\sum_{k=0}^{[\frac{p}{4}]}\binom{np-1}{4k}_3^m \equiv \frac{p+1}{4}+mnp\left(\frac{9}{16}q_p(2)-\frac{3}{4}\right) \pmod{p^2}. \tag{6}$$

Combining (4) and (6), we obtain Theorem 1.1. □

3.2. Proof of Theorem 1.2

Proof. By Lemma 2.4, we have

$$\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+1}_3^m \equiv (-1)^m \sum_{k=0}^{\left[\frac{p-3}{4}\right]} (1 - mnp(\frac{3}{4}H_k + \sum_{j=0}^{k} \frac{1}{4j+1})) \pmod{p^2}. \tag{7}$$

Changing the sum order of j and k in (7), we get

$$\begin{aligned}\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+1}_3^m &\equiv (-1)^m (\left[\frac{p-3}{4}\right] + 1 - mnp(\frac{3}{4}\sum_{j=1}^{\left[\frac{p-3}{4}\right]} \frac{1}{j} \sum_{k=j}^{\left[\frac{p-3}{4}\right]} 1 \\ &+ \sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{1}{4j+1} \sum_{k=j}^{\left[\frac{p-3}{4}\right]} 1)) \equiv (-1)^m (\left[\frac{p-3}{4}\right] + 1 - mnp(\frac{3}{4}\sum_{j=1}^{\left[\frac{p-3}{4}\right]} \frac{\left[\frac{p-3}{4}\right] - j + 1}{j} \\ &+ \sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{\left[\frac{p-3}{4}\right] - j + 1}{4j+1})) \pmod{p^2}.\end{aligned} \tag{8}$$

For $p \equiv 1 \pmod 4$ in (8), we have $\left[\frac{p-3}{4}\right] = \frac{p-5}{4}$, then

$$\begin{aligned}\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+1}_3^m &\equiv (-1)^m \left(\frac{p-1}{4} - mnp\left(\frac{3}{4}\sum_{j=1}^{\left[\frac{p-3}{4}\right]} \frac{\frac{p-1}{4} - j}{j} + \sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{\frac{p-1}{4} - j}{4j+1}\right)\right) \\ &\equiv (-1)^m \left(\frac{p-1}{4} - mnp\left(-\frac{3}{16}H_{\left[\frac{p-3}{4}\right]} - \frac{3}{4}\left[\frac{p-3}{4}\right] - \frac{1}{4}\sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{4j+1}{4j+1}\right)\right) \\ &\equiv (-1)^m \left(\frac{p-1}{4} - mnp\left(-\frac{3}{16}H_{\left[\frac{p-3}{4}\right]} + 1\right)\right) \pmod{p^2}.\end{aligned} \tag{9}$$

By Lemma 2.1, for $p \equiv 1 \pmod 4$, (9) is congruent to

$$\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+1}_3^m \equiv (-1)^m \left(\frac{p-1}{4} - mnp\left(\frac{9}{16}q_p(2) + \frac{1}{4}\right)\right) \pmod{p^2}. \tag{10}$$

For $p \equiv 3 \pmod 4$ in (8), we have $\left[\frac{p-3}{4}\right] = \frac{p-3}{4}$, then

$$\begin{aligned}\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+1}_3^m &\equiv (-1)^m \left(\frac{p+1}{4} - mnp\left(\frac{3}{4}\sum_{j=1}^{\left[\frac{p-3}{4}\right]} \frac{\frac{p+1}{4} - j}{j} + \sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{\frac{p+1}{4} - j}{4j+1}\right)\right) \\ &\equiv (-1)^m \left(\frac{p+1}{4} - mnp\left(\frac{3}{16}H_{\left[\frac{p-3}{4}\right]} - \frac{3}{4}\left[\frac{p-3}{4}\right] - \frac{1}{4}\sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{(4j+1)-2}{4j+1}\right)\right) \\ &\equiv (-1)^m \left(\frac{p+1}{4} - mnp\left(\frac{3}{16}H_{\left[\frac{p-3}{4}\right]} - \frac{3}{4}\left[\frac{p-3}{4}\right] - \frac{1}{4}(\left[\frac{p-3}{4}\right] + 1) + \frac{1}{2}\sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{1}{4j+1}\right)\right)\end{aligned}$$

$$\equiv (-1)^m \left(\frac{p+1}{4} - mnp \left(\frac{3}{16} H_{\left[\frac{p-3}{4}\right]} + \frac{1}{2} + \frac{1}{2} \sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{1}{4j+1} \right) \right) \pmod{p^2}. \tag{11}$$

By Lemma 2.1 and Lemma 2.2, for $p \equiv 3 \pmod 4$, (11) is congruent to

$$\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+1}_3^m \equiv (-1)^m \left(\frac{p+1}{4} + mnp \left(\frac{7}{16} q_p(2) - \frac{1}{2} \right) \right) \pmod{p^2}. \tag{12}$$

Combining (10) and (12), we obtain Theorem 1.2. □

3.3. Proof of Theorem 1.3

Proof. By Lemma 2.4, if $m > 1$, we obtain

$$\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+2}_3^m \equiv 0 \pmod{p^2}. \tag{13}$$

By Lemma 2.4, if $m = 1$, changing the sum order of j and k, we obtain

$$\begin{aligned}
\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+2}_3 &\equiv \sum_{k=0}^{\left[\frac{p-3}{4}\right]} np \left(\sum_{j=0}^{k} \frac{1}{4j+2} - \sum_{j=0}^{k} \frac{1}{4j+1} \right) \\
&\equiv np \left(\sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{1}{4j+2} \sum_{k=j}^{\left[\frac{p-3}{4}\right]} 1 - \sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{1}{4j+1} \sum_{k=j}^{\left[\frac{p-3}{4}\right]} 1 \right) \\
&\equiv np \left(\sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{\left[\frac{p-3}{4}\right] - j + 1}{4j+2} - \sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{\left[\frac{p-3}{4}\right] - j + 1}{4j+1} \right) \pmod{p^2}.
\end{aligned} \tag{14}$$

For $p \equiv 1 \pmod 4$ in (14), we have $\left[\frac{p-3}{4}\right] = \frac{p-5}{4}$, then

$$\begin{aligned}
\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+2}_3 &\equiv np \left(\sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{\frac{p-1}{4} - j}{4j+2} - \sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{\frac{p-1}{4} - j}{4j+1} \right) \\
&\equiv np \left(\frac{1}{4} \sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{1+4j}{4j+1} - \frac{1}{4} \sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{(2+4j)-1}{4j+2} \right) \\
&\equiv \frac{np}{4} \sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{1}{4j+2} \pmod{p^2}.
\end{aligned} \tag{15}$$

By Lemma 2.2, for $p \equiv 1 \pmod 4$, (15) is congruent to

$$\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+2}_3 \equiv -\frac{np}{16} q_p(2) \pmod{p^2}. \tag{16}$$

For $p \equiv 3 \pmod 4$ in (14), we have $\left[\frac{p-3}{4}\right] = \frac{p-3}{4}$, then

$$\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+2}_3 \equiv np\left(\sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{\frac{p+1}{4}-j}{4j+2} - \sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{\frac{p+1}{4}-j}{4j+1}\right)$$
$$\equiv \frac{np}{4}\left(3\sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{1}{4j+2} - 2\sum_{j=0}^{\left[\frac{p-3}{4}\right]} \frac{1}{4j+1}\right) \pmod{p^2}. \tag{17}$$

By Lemma 2.2, for $p \equiv 3 \pmod 4$, (15) is congruent to

$$\sum_{k=0}^{\left[\frac{p-3}{4}\right]} \binom{np-1}{4k+2}_3 \equiv -\frac{5np}{16} q_p(2) \pmod{p^2}. \tag{18}$$

Combining (16) and (18), we obtain Theorem 1.3. □

3.4. Proof of Theorem 1.4

Proof. By Lemma 2.4, if $m > 1$, we obtain

$$\sum_{k=0}^{\left[\frac{p}{4}\right]-1} \binom{np-1}{4k+3}_3^m \equiv 0 \pmod{p^2}. \tag{19}$$

By Lemma 2.4, if $m = 1$, changing the sum order of j and k, we obtain

$$\sum_{k=0}^{\left[\frac{p}{4}\right]-1} \binom{np-1}{4k+3}_3 \equiv \sum_{k=0}^{\left[\frac{p}{4}\right]-1} np\left(\sum_{j=0}^{k} \frac{1}{4j+3} - \sum_{j=0}^{k} \frac{1}{4j+2}\right)$$
$$\equiv np\left(\sum_{j=0}^{\left[\frac{p}{4}\right]-1} \frac{1}{4j+3} \sum_{k=j}^{\left[\frac{p}{4}\right]-1} 1 - \sum_{j=0}^{\left[\frac{p}{4}\right]-1} \frac{1}{4j+2} \sum_{k=j}^{\left[\frac{p}{4}\right]-1} 1\right)$$
$$\equiv np\left(\sum_{j=0}^{\left[\frac{p}{4}\right]-1} \frac{\left[\frac{p}{4}\right]-j}{4j+3} - \sum_{j=0}^{\left[\frac{p}{4}\right]-1} \frac{\left[\frac{p}{4}\right]-j}{4j+2}\right) \pmod{p^2}. \tag{20}$$

For $p \equiv 1 \pmod 4$ in (20), we have $\left[\frac{p}{4}\right] = \frac{p-1}{4}$, then

$$\sum_{k=0}^{\left[\frac{p}{4}\right]-1} \binom{np-1}{4k+3}_3 \equiv np\left(\sum_{j=0}^{\left[\frac{p}{4}\right]-1} \frac{\frac{p-1}{4}-j}{4j+3} - \sum_{j=0}^{\left[\frac{p}{4}\right]-1} \frac{\frac{p-1}{4}-j}{4j+2}\right)$$
$$\equiv \frac{np}{4}\left(2\sum_{j=0}^{\left[\frac{p}{4}\right]-1} \frac{1}{4j+3} - \sum_{j=0}^{\left[\frac{p}{4}\right]-1} \frac{1}{4j+2}\right) \pmod{p^2}. \tag{21}$$

By Lemma 2.2, for $p \equiv 1 \pmod 4$, (21) is congruent to

$$\sum_{k=0}^{\left[\frac{p}{4}\right]-1} \binom{np-1}{4k+3}_3 \equiv \frac{3np}{16} q_p(2) \pmod{p^2}. \tag{22}$$

For $p \equiv 3 \pmod 4$ in (20), we have $\left[\frac{p}{3}\right] = \frac{p-3}{4}$, then

$$\sum_{k=0}^{[\frac{p}{4}]-1} \binom{np-1}{4k+3}_3 \equiv np\left(\sum_{j=0}^{[\frac{p}{4}]-1} \frac{\frac{p-3}{4}-j}{4j+3} - \sum_{j=0}^{[\frac{p}{4}]-1} \frac{\frac{p-3}{4}-j}{4j+2}\right)$$
$$\equiv \frac{np}{4}\sum_{j=0}^{[\frac{p}{4}]-1} \frac{1}{4j+2} \pmod{p^2}. \tag{23}$$

By Lemma 2.2, for $p \equiv 3 \pmod 4$, (23) is congruent to

$$\sum_{k=0}^{[\frac{p}{4}]-1} \binom{np-1}{4k+3}_3 \equiv -\frac{np}{16}(q_p(2)-4) \pmod{p^2}. \tag{24}$$

Combining (22) and (24), we obtain Theorem 1.4. □

3.5. Proof of Theorem 1.5

Proof.

$$\sum_{k=0}^{p-1} \binom{np-1}{k}_3^m = \sum_{k=0}^{[\frac{p}{4}]} \binom{np-1}{4k}_3^m + \sum_{k=0}^{[\frac{p-3}{4}]} \binom{np-1}{4k+1}_3^m + \sum_{k=0}^{[\frac{p-3}{4}]} \binom{np-1}{4k+2}_3^m + \sum_{k=0}^{[\frac{p}{4}]-1} \binom{np-1}{4k+3}_3^m. \tag{25}$$

By Theorem 1.1-1.4, when $m = 1$ and $p \equiv 1 \pmod 4$, we obtain

$$\sum_{k=0}^{p-1} \binom{np-1}{k}_3 \equiv 1 + \frac{9np}{4}q_p(2) \pmod{p^2}. \tag{26}$$

When $m = 1$ and $p \equiv 3 \pmod 4$, we obtain

$$\sum_{k=0}^{p-1} \binom{np-1}{k}_3 \equiv -\frac{np}{4}q_p(2) \pmod{p^2}. \tag{27}$$

(26) and (27) has be proved in [10]. In (25), by Theorem 1.1-1.4, when $m > 1$ and $p \equiv 1 \pmod 4$, we obtain

$$\sum_{k=0}^{p-1} \binom{np-1}{k}_3^m \equiv \frac{((-1)^m+1)p+3-(-1)^m}{4}$$
$$+ mnp\left(\frac{25-(-1)^m 9}{16}q_p(2) - \frac{1+(-1)^m}{4}\right) \pmod{p^2}. \tag{28}$$

In (25), by Theorem 1.1-1.4, when $m > 1$ and $p \equiv 3 \pmod 4$, we obtain

$$\sum_{k=0}^{p-1} \binom{np-1}{k}_3^m \equiv \frac{((-1)^m+1)(p+1)}{4}$$
$$+ mnp\left(\frac{9+(-1)^m 7}{16}q_p(2) - \frac{3+(-1)^m 2}{4}\right) \pmod{p^2}. \tag{29}$$

Combining (25)-(29), we obtain Theorem 1.5. □

3.6. Proof of Theorem 1.6

Proof. Since

$$\sum_{k=0}^{p-1}(-1)^k\binom{np-1}{k}_3^m=\sum_{k=0}^{[\frac{p}{4}]}\binom{np-1}{4k}_3^m-\sum_{k=0}^{[\frac{p-3}{4}]}\binom{np-1}{4k+1}_3^m+\sum_{k=0}^{[\frac{p-3}{4}]}\binom{np-1}{4k+2}_3^m-\sum_{k=0}^{[\frac{p}{4}]-1}\binom{np-1}{4k+3}_3^m,$$

by Theorem 1.1-1.4 and similar to the proof of Theorem 1.5, we obtain Theorem 1.6. □

3.7. Proof of Theorem 1.7

Proof. By Lemma 2.4, Lemma 2.1 and Lemma 2.3, similar to the proof of Theorem 1.1, we obtain that

$$\sum_{k=0}^{[\frac{p}{8}]}\binom{np-1}{4k}_3^m\equiv\begin{cases}\frac{p+7}{8}+mnp\left(\frac{89}{32}q_p(2)+\chi_p-\frac{1}{8}\right) \pmod{p^2}, \text{ if } p\equiv 1 \pmod 8;\\ \frac{p+5}{8}+mnp\left(\frac{63}{32}q_p(2)+\frac{9}{8}\chi_p-\frac{3}{8}\right) \pmod{p^2}, \text{ if } p\equiv 3 \pmod 8;\\ \frac{p+3}{8}+mnp\left(\frac{37}{32}q_p(2)+\frac{1}{2}\chi_p-\frac{5}{8}\right) \pmod{p^2}, \text{ if } p\equiv 5 \pmod 8;\\ \frac{p+1}{8}+mnp\left(\frac{11}{32}q_p(2)+\frac{1}{8}\chi_p-\frac{5}{8}\right) \pmod{p^2}, \text{ if } p\equiv 7 \pmod 8.\end{cases}$$

By Lemma 2.4, Lemma 2.1 and Lemma 2.3, similar to the proof of Theorem 1.2, we obtain that

$$\sum_{k=0}^{[\frac{p-3}{8}]}\binom{np-1}{4k+1}_3^m\equiv\begin{cases}(-1)^m\left(\frac{p-1}{8}-mnp\left(\frac{11}{32}q_p(2)+\frac{1}{8}\chi_p+\frac{1}{8}\right)\right) \pmod{p^2}, \text{ if } p\equiv 1 \pmod 8;\\ (-1)^m\left(\frac{p+5}{8}+mnp\left(\frac{67}{32}q_p(2)+\frac{1}{2}\chi_p-\frac{1}{8}\right)\right) \pmod{p^2}, \text{ if } p\equiv 3 \pmod 8;\\ (-1)^m\left(\frac{p+3}{8}+mnp\left(\frac{41}{32}q_p(2)+\frac{7}{8}\chi_p-\frac{3}{8}\right)\right) \pmod{p^2}, \text{ if } p\equiv 5 \pmod 8;\\ (-1)^m\left(\frac{p+1}{8}+mnp\left(\frac{15}{32}q_p(2)-\frac{5}{8}\right)\right) \pmod{p^2}, \text{ if } p\equiv 7 \pmod 8.\end{cases}$$

By Lemma 2.4, Lemma 2.1 and Lemma 2.3, similar to the proof of Theorem 1.3, we obtain that

$$\sum_{k=0}^{[\frac{p-5}{8}]}\binom{np-1}{4k+2}_3^m\equiv\begin{cases}0 \pmod{p^2}, \text{ if } m>1;\\ -\frac{np}{8}\left(\frac{5}{4}q_p(2)-2\chi_p\right) \pmod{p^2}, \text{ if } p\equiv 1 \pmod 8 \text{ and } m=1;\\ -\frac{np}{8}\left(\frac{3}{4}q_p(2)-\chi_p-2\right) \pmod{p^2}, \text{ if } p\equiv 3 \pmod 8 \text{ and } m=1;\\ -\frac{np}{8}\left(\frac{9}{4}q_p(2)-6\chi_p\right) \pmod{p^2}, \text{ if } p\equiv 5 \pmod 8 \text{ and } m=1;\\ -\frac{np}{8}\left(\frac{7}{4}q_p(2)-\chi_p\right) \pmod{p^2}, \text{ if } p\equiv 7 \pmod 8 \text{ and } m=1.\end{cases}$$

By Lemma 2.4, Lemma 2.1 and Lemma 2.3, similar to the proof of Theorem 1.4, we obtain that

$$\sum_{k=0}^{[\frac{p-7}{8}]}\binom{np-1}{4k+3}_3^m\equiv\begin{cases}0 \pmod{p^2}, \text{ if } m>1;\\ \frac{np}{8}\left(\frac{1}{4}q_p(2)+\chi_p\right) \pmod{p^2}, \text{ if } p\equiv 1 \pmod 8 \text{ and } m=1;\\ -\frac{np}{8}\left(\frac{1}{4}q_p(2)+2\chi_p\right) \pmod{p^2}, \text{ if } p\equiv 3 \pmod 8 \text{ and } m=1;\\ -\frac{np}{8}\left(\frac{3}{4}q_p(2)-\chi_p-2\right) \pmod{p^2}, \text{ if } p\equiv 5 \pmod 8 \text{ and } m=1;\\ \frac{np}{8}\left(\frac{3}{4}q_p(2)-6\chi_p\right) \pmod{p^2}, \text{ if } p\equiv 7 \pmod 8 \text{ and } m=1.\end{cases}$$

Since

$$\sum_{k=0}^{\frac{p-1}{2}}\binom{np-1}{k}_3^m=\sum_{k=0}^{[\frac{p}{8}]}\binom{np-1}{4k}_3^m+\sum_{k=0}^{[\frac{p-3}{8}]}\binom{np-1}{4k+1}_3^m+\sum_{k=0}^{[\frac{p-5}{8}]}\binom{np-1}{4k+2}_3^m+\sum_{k=0}^{[\frac{p-7}{8}]}\binom{np-1}{4k+3}_3^m$$

and

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k\binom{np-1}{k}_3^m=\sum_{k=0}^{[\frac{p}{8}]}\binom{np-1}{4k}_3^m-\sum_{k=0}^{[\frac{p-3}{8}]}\binom{np-1}{4k+1}_3^m+\sum_{k=0}^{[\frac{p-5}{8}]}\binom{np-1}{4k+2}_3^m-\sum_{k=0}^{[\frac{p-7}{8}]}\binom{np-1}{4k+3}_3^m,$$

similar to the proof of Theorem 1.5 and Theorem 1.6, we obtain Theorem 1.7. □

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