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Global well-posedness for the 3D rotating fractional Boussinesq equations in Fourier-Besov-Morrey spaces with variable exponent

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Abstract. This paper considers the Cauchy problem of the 3D rotating fractional Boussinesq equations in Fourier-Besov-Morrey spaces with variable exponent. By using the contraction mapping method, Littlewood-Paley theory and the Fourier localization argument, we get, with small initial data in the critical Fourier-Besov-Morrey spaces with variable exponent $\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}$, the global well-posedness result.

1. Introduction and main result

In this article, we would like to study the 3D rotating fractional Boussinesq equations given by

$$\begin{cases} v_t + \nu(-\Delta)^\beta v + (v\cdot\nabla)v + \Omega e_3 \times v + \nabla P = g\theta e_3 & \text{for} \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ \theta_t + \kappa(-\Delta)^\beta \theta + (v\cdot\nabla)\theta = 0 & \text{for} \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ \nabla\cdot v = 0 & \text{for} \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ v(x,0) = v_0(x),\ \ \theta(x,0) = \theta_0(x) & \text{for} \quad x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

Here, $\frac{1}{2} < \beta \leq 1$, $v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))$ represents the fluid's velocity field, while $\theta(x,t)$ signifies the fluctuation, the operator $(-\Delta)^\beta$ is the Fourier Transform defined by $(-\Delta)^\beta := \mathcal{F}^{-1}\left(|\xi|^{2\beta}\mathcal{F}\right)$, and P denotes the pressure. The thermal diffusivity, the kinetic viscosity, and the gravity are respectively denoted by the positive constants κ, ν and g. $\Omega \in \mathbb{R}$ represents the Coriolis parameter, signifying twice the rotational speed around the vertical unit vector $e_3 = (0,0,1)$. The expression $g\theta e_3$ symbolizes the buoyancy force in accordance with the Boussinesq approximation, a method that involves disregarding the density dependency in all terms except the gravitational one. As the specific values of ν and κ do not hold particular significance in our analysis, we simplify by assigning $\nu = \kappa = 1$ for convenience in our discussion.

when $\beta = 1$ and $\Omega = 0$, Equation (1.1) corresponds to to the classical Boussinesq equations, which are a set of simplified fluid dynamics equations used to study buoyancy-induced flows in stratified fluids [7]. These equations describe the effect of temperature variations on fluid motion, making them fundamental for modeling natural convection phenomena in oceans, atmospheres, and other geophysical systems (see,

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e.g. [21, 24]). The global well-posedness of these equations has been studied by many researchers in various spaces, see [1, 7–9, 11, 14, 16, 26] and the references therein.

when $\beta = 1$, $\theta \equiv 0$ and $\Omega = 0$, Equation (1.1) leads to the following classical Navier-Stokes equations:

$$\begin{cases} v_t + \nu(-\Delta)v + (v\cdot\nabla)v + \nabla P = 0 & \text{for} \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ \nabla\cdot v = 0 & \text{for} \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ v(x,0) = v_0(x) & \text{for} \quad x \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

There are many studies on well-posedness results of Equation (1.2), which can be consulted in [15, 17, 19, 20, 22] and related references cited therein.

when $\theta \equiv 0$, but $\Omega \neq 0$, Equation (1.1) corresponds to fractional Navier-Stokes equations with Coriolis force. We have global well-posedness results for these equations with small initial data in various critical Fourier transform-based functional spaces. For the case where $\beta = 1$, that is, the classical Navier-Stokes equations with Coriolis force, we have them for example, in the Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^3)$ [13], in $\mathcal{FM}_0^{-1}(\mathbb{R}^3)$ which can be identified with the Fourier-Besov space $\mathcal{F}\dot{\mathcal{B}}^{-1}_{1,1}(\mathbb{R}^3)$ [12], in $\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{r}}_{r,\infty}(\mathbb{R}^3)$ with $1 < r \leq \infty$ [18], in $\mathcal{F}\dot{\mathcal{B}}^{-1}_{1,2}(\mathbb{R}^3)$ [15], in the Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{2-\frac{3-\lambda}{r}}_{r,\lambda,\infty}(\mathbb{R}^3)$ [4]. Furthermore, we have ill-posedness in $\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{r}}_{r,\infty}(\mathbb{R}^3)$ with $1 < r \leq \infty$ [15]. For the general case $\frac{1}{2} < \beta \leq 1$, Wang and Wu established global well-posedness in the Lei-Lin-type spaces $\mathcal{X}^{1-2\beta}\left(\mathbb{R}^3\right)$ with $\frac{1}{2} \leq \beta \leq 1$, and in the Fourier-Besov spaces $\mathcal{F}\dot{\mathcal{B}}^{4-2\beta-\frac{3}{r}}_{r,h}(\mathbb{R}^3)$ with $\frac{2}{3} < \beta < \frac{5}{3} - \frac{1}{r}$, $2 \leq r \leq \infty$ and $1 \leq h \leq \infty$, in [29] and [30], respectively. El Baraka1 and Toumlilin [10] demonstrate the global well-posedness in the Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3-\lambda}{r}}_{r,\lambda,h}(\mathbb{R}^3)$ with $0 \leq \lambda < 3$, $\frac{1}{2} < \beta < \frac{5}{2} - \frac{3-\lambda}{r}$, $1 \leq r < \infty$ and $1 \leq h \leq 2$.

In general, variable exponent function spaces have garnered significant attention from researchers in recent times. This interest extends beyond theoretical aspects, encompassing their pivotal role in various applications, such as resolving specific equations. However, there are many challenges in addressing the well-posedness of equations in these spaces. In our case, certain classical theories, like the multiplier theorem and Young's inequality, are inapplicable within Fourier-Besov-Morrey spaces with variable exponents, unlike classical Fourier-Besov-Morrey spaces. To overcome these challenges, the present paper primarily relies on the properties described in Section 2 to look at the global well-posedness result. Moreover, variable exponent function spaces have different structures from each other. In particular, an analysis of the structure of a variable exponent Fourier-Besov-Morrey space reveals its notable distinctions from a variable exponent Besov space. In contrast to the latter, this specific space is better suited for studying the boundedness of semigroup operators and for estimating nonlinear terms. For an in-depth exploration of these variable exponent function spaces, we direct the reader to [2, 3, 5, 23, 25, 28] and the associated references therein.

Very recently, Ru & Abidin [25] and Abidin & Chen [2] demonstrated the global well-posedness to Equation (1.1) with $\theta \equiv 0$ and $\Omega = 0$, for small initial data in the variable exponent function spaces $\mathcal{F}\dot{\mathcal{B}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),h}$ and $\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}$ with $2 \leq q(\cdot) < \infty$, respectively, with $\frac{1}{2} < \beta \leq 1$, $2 \leq r(\cdot) \leq \frac{6}{5-4\beta}$ and $1 \leq h \leq \frac{3}{2\beta-1}$. Sun, Wu and Xu [28] proved the global well-posedness to Equation (1.1) with $\beta = 1$, for small initial data in the critical variable exponent Fourier-Besov spaces $\mathcal{F}\dot{\mathcal{B}}^{2-\frac{3}{r(\cdot)}}_{r(\cdot),h}$ with $2 \leq r(\cdot) \leq 6$ and $1 \leq h \leq \infty$. Inspired by These works, we aim to establish the global existence of solutions to Equation (1.1), for small initial data in the critical variable exponent Fourier-Besov-Morrey space $\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}$ with $\frac{1}{2} < \beta \leq 1$, $2 \leq r(\cdot) \leq \frac{6}{5-4\beta}$, $2 \leq q(\cdot) \leq \infty$ and $1 \leq h \leq \infty$.

Throughout this paper, $E \lesssim F$ denotes the existence of a constant $C > 0$ such that $E \leq CF$ and $E \sim F$ denotes the existence of constants $C_1, C_2 > 0$ such that $C_1F \leq E \leq C_2F$. We define, for a Banach space Y and $v, w \in Y$, the norm of vector (v, w) as

$$\|(v,w)\|_Y := \|v\|_Y + \|w\|_Y ,$$

for $v \in S(\mathbb{R}^d)$, the Fourier transform as

$$\mathcal{F}v(\xi) = \widehat{v}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} \varphi(x) dx,$$

and its inverse Fourier transform as

$$\mathcal{F}^{-1}v(x) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} v(\xi) d\xi.$$

Now we state our main theorem. Please see Section 2 for the definitions of variable exponent function spaces.

Theorem 1.1. *Let $\frac{1}{2} < \beta \leq 1$, $\Omega \in \mathbb{R}$, $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ such that $2 \leq r(\cdot) \leq \frac{6}{5-4\beta}$, $r(\cdot) \leq q(\cdot) \leq \infty$ and let $1 \leq h, \rho \leq \infty$. Then there exists a sufficiently small $\epsilon > 0$ such that for any $v_0, \theta_0 \in \mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}$ satisfy $\nabla \cdot v_0 = 0$ and*

$$\|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} + \|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} < \epsilon,$$

Equation (1.1) has a unique global solution $(v, \theta) \in Y$, where

$$Y := \left\{ (v,\theta)/\ v, \theta \in \widetilde{L}^\rho(0,\infty; \mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}) \cap \widetilde{L}^\rho(0,\infty; \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta+\frac{2\beta}{\rho}}_{2,q(\cdot),h}) \cap \widetilde{L}^\infty(0,\infty; \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta}_{2,q(\cdot),h}),\ \ \|(v,\theta)\|_Y \leq \delta \right\},$$

with

$$\|(v,\theta)\|_Y := \|(v,\theta)\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}) \cap \widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta+\frac{2\beta}{\rho}}_{2,q(\cdot),h}) \cap \widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta}_{2,q(\cdot),h})},$$

and C_0 is a constant depending on δ. Moreover, let $r_0(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$, $s_0(\cdot) \in C_{\log}(\mathbb{R}^d)$ with $s_0(\cdot) = 4-2\beta+\frac{2\beta}{\rho}-\frac{3}{r_0(\cdot)}$, if there exists $c > 0$ such that $2 \leq r_0(\cdot) \leq c \leq r(\cdot)$, we then also get that $v, \theta \in \widetilde{L}^\rho\left(0,\infty; \mathcal{F}\dot{\mathcal{N}}^{s_0(\cdot)}_{r_0(\cdot),q(\cdot),h}\right)$.

Remark 1.2.
1. *This theorem extends the corresponding result of [28], where the authors considered Equation (1.1) with $\beta = 1$, in variable exponent Fourier-Besov spaces.*
2. *We will use in this paper, to prove the global solution, a different method compared to [23], where the authors studied the primitive equations, which have a considerable difference with the rotating fractional Boussinesq equations (1.1) (see [27]).*

Remark 1.3. *The variable exponent Fourier-Besov-Morrey $\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}$ is critical for Equation (1.1). In fact, if $(v(x,t), \theta(x,t))$ is the solution of Equation (1.1), then*

$$(v_\lambda(x,t), \theta_\lambda(x,t)) := \left(\lambda^{2\beta-1} v(\lambda x, \lambda^{2\beta} t), \lambda^{2\beta-1}\theta(\lambda x, \lambda^{2\beta} t)\right)$$

is also a solution of the same equation and

$$\|(v(\cdot,0),\theta(\cdot,0))\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} \sim \|(v_\lambda(\cdot,0),\theta_\lambda(\cdot,0))\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}}.$$

The remainder of this article is structured as follows: In Section 2, we present some basic background information on the Littlewood-Paley theory and some different laws on products in variable exponent Fourier-Besov-Morrey spaces. In Section 3, we introduce the semigroup $S_{\Omega,\beta}(\cdot)$ and then prove its estimates. In Section 4, we establish the main theorem.

2. Preliminaries

This section presents some definitions of different variable exponent function spaces, basic knowledge of Littlewood-Paley theory and some propositions that are pertinent to our purposes.

Definition 2.1. *[5] For the measurable function $r(\cdot)$, let*

$$\mathcal{P}_0(\mathbb{R}^d) := \left\{ r(\cdot) : \mathbb{R}^d \to (0,\infty);\ 0 < r_- = \operatorname*{essinf}_{x\in\mathbb{R}^d} r(x), \operatorname*{esssup}_{x\in\mathbb{R}^d} r(x) = r_+ < \infty \right\}.$$

The Lebesgue space with variable exponent is defined by

$$L^{r(\cdot)}\left(\mathbb{R}^d\right) = \left\{ u : \mathbb{R}^d \to \mathbb{R} \text{ is measurable, } \int_{\mathbb{R}^d} |u(x)|^{r(x)} dx < \infty \right\}.$$

with norm

$$\begin{aligned} \|u\|_{L^{r(\cdot)}} &:= \inf\left\{\lambda > 0 : \varrho_{r(\cdot)}(u/\lambda) \le 1\right\} \\ &= \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|u(x)|}{\lambda}\right)^{r(x)} dx \le 1\right\}. \end{aligned}$$

We use the following notation to separate variable exponents from constant exponents: $r(\cdot)$ for variable exponents, r for constant exponents. Also $(L^{r(\cdot)}\left(\mathbb{R}^d\right), \|u\|_{L^{r(\cdot)}})$ is a Banach space.

$L^{r(\cdot)}$ doesn't have the same features as L^r. Therefore, to assure the boundedness of the maximal Hardy-Littlewood operator M on $L^{r(\cdot)}(\mathbb{R}^d)$, the following standard conditions are assumed:

1. (Locally log-Hölder's continuous)[5] There exists a constant $C_{\log}(r)$ such that

$$|r(x) - r(y)| \le \frac{C_{\log}(r)}{\log\left(e + |x-y|^{-1}\right)}, \text{ for all } x, y \in \mathbb{R}^d,\ x \ne y.$$

2. (Globally log-Hölder's continuous)[5] There exist two constants $C_{\log}(r)$ and r_∞ such that

$$|r(x) - r_\infty| \le \frac{C_{\log}(r)}{\log(e + |x|)}, \text{ for all } x \in \mathbb{R}^n.$$

$C_{\log}(\mathbb{R}^d)$ denotes the set of all functions $r(\cdot) : \mathbb{R}^d \to \mathbb{R}$ that satisfy 1 and 2.

Definition 2.2. *[3] Let $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d)$ with $0 < r_- \le r(\cdot) \le q(\cdot) \le \infty$, the variable exponent Morrey space $\mathcal{M}^{q(\cdot)}_{r(\cdot)} := \mathcal{M}^{q(\cdot)}_{r(\cdot)}(\mathbb{R}^d)$ is defined as the set of all measurable functions on $\mathbb{R}^d$ such that*

$$\|u\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} := \sup_{x_0\in\mathbb{R}^d, R>0} \left\| R^{\frac{d}{q(x)} - \frac{d}{r(x)}} u \right\|_{L^{r(\cdot)}(B(x_0,R))} < \infty.$$

From the definition of $L^{r(\cdot)}$-norm, the quasinorm $\|\cdot\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}$ is also expressed as follows:

$$\|u\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} := \sup_{x_0\in\mathbb{R}^d, R>0} \inf\left\{\lambda > 0 : \varrho_{r(\cdot)}\left(R^{\frac{d}{q(x)} - \frac{d}{r(x)}} \frac{u}{\lambda} \chi_{B(x_0,R)}\right) \le 1\right\}.$$

Here we give an important lemma.

Lemma 2.3. *[3] Let $r(\cdot) \in \mathcal{P}_0(\mathbb{R}^d)$, then*

1. *If $q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d)$ satisfies $r(\cdot) \le q(\cdot)$. Then for any measurable function u, we have*

$$\|u\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} := \inf\left\{\lambda > 0 : \sup_{x_0\in\mathbb{R}^d, R>0} \varrho_{r(\cdot)}\left(R^{\frac{d}{q(x)}-\frac{d}{r(x)}}\frac{u}{\lambda}\chi_{B(x_0,R)}\right) \le 1\right\}.$$

2. *For any measurable function u*

$$\sup_{x_0\in\mathbb{R}^d, R>0} \varrho_{r(\cdot)}\left(u\chi_{B(x_0,R)}\right) = \varrho_{r(\cdot)}(u),$$

and $\|u\|_{\mathcal{M}^{r(\cdot)}_{r(\cdot)}} = \|u\|_{L^{r(\cdot)}}$.

We now recall the Littlewood-Paley decomposition (see [6] for more details). Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ be a smooth radial function such that

$$0 \le \varphi \le 1,$$
$$\operatorname{supp} \varphi \subset \left\{\xi \in \mathbb{R}^d : \frac{3}{4} \le |\xi| \le \frac{8}{3}\right\},$$
$$\sum_{j\in Z}\varphi\left(2^{-j}\xi\right) = 1, \quad \text{for all } \xi \neq 0,$$

and we denote $\varphi_j(\xi) = \varphi(2^{-j}\xi)$. Then for every $u \in \mathcal{S}'(\mathbb{R}^d)$, we define the frequency localization operators for all $j \in \mathbb{Z}$, as follows

$$\Delta_j u = \mathcal{F}^{-1}\varphi_j * u \qquad \text{and} \qquad S_j u = \sum_{k\le j-1}\Delta_k u, \tag{2.1}$$

Here, we observe that Δ_j is a frequency to $\{|\xi| \sim 2^j\}$ and S_j is a frequency to $\{|\xi| \lesssim 2^j\}$, and we denote also that the almost orthogonality property of the Littlewood-Paley decomposition is satisfied, i.e. for any $u, v \in \mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$,

$$\Delta_i\Delta_j u = 0 \quad \text{if} \quad |i-j| \ge 2 \quad \text{and} \quad \Delta_i\left(S_{j-1}u\Delta_j v\right) = 0 \quad \text{if} \quad |i-j| \ge 5, \tag{2.2}$$

where $\mathcal{P}$ is the set of all polynomials on $\mathbb{R}^d$.

Throughout this paper, the following Bony paraproduct decomposition will be used:

$$uv = T_u v + T_v u + R(u,v), \tag{2.3}$$

with

$$T_u v = \sum_j S_{j-1}u\Delta_j v, \quad R(u,v) = \sum_j\sum_{|j-l|\le 1}\Delta_j u\Delta_l v.$$

Definition 2.4. *[3] Let $r(\cdot), q(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^d)$ with $r(\cdot) \le q(\cdot)$, the mixed Morrey-sequence space $\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})$ is the set of all sequences $\left\{a_j\right\}_{j\in\mathbb{Z}}$ of measurable functions on $\mathbb{R}^d$ such that*

$$\|\left\{a_j\right\}_{j\in\mathbb{Z}}\|_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})} := \inf\left\{\lambda > 0 : \varrho_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}(\left\{a_j/\lambda\right\}_{j\in\mathbb{Z}}) \le 1\right\},$$

where

$$\varrho_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}(\left\{a_j\right\}_{j\in\mathbb{Z}}) := \sum_{j\in\mathbb{Z}}\inf\left\{\nu > 0 : \int_{\mathbb{R}^n}\left(\frac{|R^{\frac{d}{q(x)}-\frac{d}{r(x)}}a_j\chi_{B(x_0,R)}|}{\nu^{\frac{1}{h(x)}}}\right)^{r(x)} dx \le 1\right\}.$$

Notice that if $h_+ < \infty$ and $r(\cdot) \le h(\cdot)$, then

$$\varrho_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})}(\left\{a_j\right\}_{j\in\mathbb{Z}}) := \sum_{j\in\mathbb{Z}}\sup_{x_0\in\mathbb{R}^d, R>0}\left\|\left(R^{\frac{d}{q(x)}-\frac{d}{r(x)}}u\right)^{h(x)}\right\|_{L^{\frac{r(\cdot)}{h(\cdot)}}(B(x_0,R))}.$$

Definition 2.5. *[3] Let $s(\cdot) \in C_{\log}(\mathbb{R}^d)$ and $r(\cdot), q(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ with $0 < r_- \le r(\cdot) \le q(\cdot) \le \infty$. The variable exponent homogeneous Besov-Morrey space $\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}$ is defined by*

$$\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)} := \left\{ u \in \mathcal{D}'\left(\mathbb{R}^d\right) : \|u\|_{\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}} < \infty \right\},$$

with norm

$$\|u\|_{\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}} := \left\| \left\{ 2^{js(\cdot)} \Delta_j u'' \right\}_{j\in\mathbb{Z}} \right\|_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})},$$

and $\mathcal{D}'\left(\mathbb{R}^d\right)$ represents the dual space of

$$\mathcal{D}\left(\mathbb{R}^d\right) = \left\{ u \in \mathcal{S}\left(\mathbb{R}^d\right) : \left(D^\beta u\right)(0) = 0, \ \ \forall \beta \right\}.$$

Definition 2.6. *[2] Let $s(\cdot) \in C_{\log}(\mathbb{R}^d)$ and $r(\cdot), q(\cdot), h(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ with $0 < r_- \le r(\cdot) \le q(\cdot) \le \infty$. The variable exponent homogeneous Fourier-Besov-Morrey space $\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}$ is defined by*

$$\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)} := \left\{ u \in \mathcal{D}'\left(\mathbb{R}^d\right) : \|u\|_{\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}} < \infty \right\},$$

with norm

$$\|u\|_{\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}} := \left\| \left\{ 2^{js(\cdot)} \widehat{\Delta_j u} \right\}_{j\in\mathbb{Z}} \right\|_{\ell^{h(\cdot)}(\mathcal{M}^{q(\cdot)}_{r(\cdot)})},$$

and $\mathcal{D}'\left(\mathbb{R}^d\right)$ represents the dual space of

$$\mathcal{D}\left(\mathbb{R}^d\right) = \left\{ u \in \mathcal{S}\left(\mathbb{R}^d\right) : \left(D^\beta u\right)(0) = 0, \ \ \forall \beta \right\}.$$

Definition 2.7. *[2] Let $T > 0$, $s(\cdot) \in C_{\log}(\mathbb{R}^d)$, $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ with $0 < r_- \le r(\cdot) \le q(\cdot) \le \infty$, and $1 \le h, \rho \le \infty$. The variable exponent Chemin-Lerner type homogeneous Fourier-Besov-Morrey space $\widetilde{L}^\rho\left(0, T; \mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}\right)$ is defined by*

$$\widetilde{L}^\rho\left(0, T; \mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}\right) := \left\{ u \in \mathcal{D}'\left(\mathbb{R}^d\right) : \|u\|_{\widetilde{L}^\rho\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}\right)} < \infty \right\},$$

with norm

$$\|u\|_{\widetilde{L}^\rho\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s(\cdot)}_{r(\cdot),q(\cdot),h(\cdot)}\right)} := \left(\sum_{j\in\mathbb{Z}} \left\| 2^{js(\cdot)} \widehat{\Delta_j u} \right\|^h_{L^\rho\left(0,T;\mathcal{M}^{q(\cdot)}_{r(\cdot)}\right)} \right)^{\frac{1}{h}},$$

where $\|u\|_{L^\rho\left(0,T;\mathcal{M}^{q(\cdot)}_{r(\cdot)}\right)} := \left(\int_0^T \|u(\cdot, t)\|^\rho_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \, \mathrm{d}t \right)^{\frac{1}{\rho}}$.

Proposition 2.8. *The following inclusions hold for variable exponent Morrey spaces.*

1. *(Hölder's inequality)[2] Let $r(\cdot), r_1(\cdot), r_2(\cdot), q(\cdot), q_1(\cdot), q_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^d)$ satisfying $r(\cdot) \le q(\cdot)$, $r_i(\cdot) \le q_i(\cdot) (i = 1, 2)$, $\frac{1}{r(\cdot)} = \frac{1}{r_1(\cdot)} + \frac{1}{r_2(\cdot)}$ and $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$. Then for all $u \in \mathcal{M}^{q_1(\cdot)}_{r_1(\cdot)}$ and $v \in \mathcal{M}^{q_2(\cdot)}_{r_2(\cdot)}$, there is a constant C depending only on r_- and r_+ such that*

$$\|uv\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \le C \|u\|_{\mathcal{M}^{q_1(\cdot)}_{r_1(\cdot)}} \|v\|_{\mathcal{M}^{q2(\cdot)}_{r2(\cdot)}}. \tag{2.4}$$

2. *[2] Let* $r_1(\cdot), r_2(\cdot), q_1(\cdot), q_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^d)$, $0 < h < \infty$ *and* $s_1(\cdot), s_2(\cdot) \in L^\infty \cap C_{\log}(\mathbb{R}^d)$ *with* $s_1(\cdot) > s_2(\cdot)$. *If* $\frac{1}{h}$ *and*

$$s_1(\cdot) - \frac{d}{r_1(\cdot)} = s_2(\cdot) - \frac{d}{r_2(\cdot)}$$

are locally log-Hölder continuous, then

$$\dot{\mathcal{N}}^{s_1(\cdot)}_{r_1(\cdot),q_1(\cdot),h(\cdot)} \hookrightarrow \dot{\mathcal{N}}^{s_2(\cdot)}_{r_2(\cdot),q_2(\cdot),h(\cdot)}. \tag{2.5}$$

3. *[3] Let* $r(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ *and* $\phi \in L^1(\mathbb{R}^d)$, *suppose* $\Phi(y) = \sup_{x \notin B(0,|y|)} \left|\phi(x)\right|$ *is integrable. Then for all* $u \in \mathcal{M}^{q(\cdot)}_{r(\cdot)}$, *there is a constant C depending only on d such that*

$$\left\|u * \phi_\varepsilon\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \leq C \left\|u\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}} \left\|\Phi\right\|_{L^1},$$

where $\phi_\varepsilon = \frac{1}{\varepsilon^d}\phi(\varepsilon)$.

Proposition 2.9. *[2] Let* $T > 0$, $s > 0$, $r_1(\cdot), r_2(\cdot), q(\cdot), q_1(\cdot), q_2(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ *and* $1 \leq r, h, \rho, \rho_1, \rho_2 \leq \infty$ *satisfying* $\frac{1}{r} = \frac{1}{r_1(\cdot)} + \frac{1}{r_2(\cdot)}$, $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)}$ *and* $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$. *Then there holds*

$$\begin{aligned}\|uv\|_{\widetilde{L}^{\rho}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{r,q(\cdot),h}\right)} \lesssim & \|u\|_{\widetilde{L}^{\rho_1}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{r_1(\cdot),q_1(\cdot),h}\right)}\|v\|_{\widetilde{L}^{\rho_2}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{0}_{r_2(\cdot),q_2(\cdot),h}\right)} \\ & + \|v\|_{\widetilde{L}^{\rho_1}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{s}_{r_1(\cdot),q_1(\cdot),h}\right)}\|u\|_{\widetilde{L}^{\rho_2}\left(0,T;\mathcal{F}\dot{\mathcal{N}}^{0}_{r_2(\cdot),q_2(\cdot),h}\right)}.\end{aligned}$$

3. Main estimates

In this part, we first give a brief presentation of the generalized Stokes-Coriolis semigroup $S_{\Omega,\beta}(\cdot)$, that is closely related to the rotating fractional Boussinesq equations. For $\theta = 0$, Equation (1.1) corresponds to the fractional Navier-Stokes equations in the rotational framework. Indeed, we have to consider the following generalized linear problem

$$\begin{cases} v_t + (-\Delta)^\beta v + \Omega e_3 \times v + \nabla P = 0 & \text{for} \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ \nabla \cdot v = 0 & \text{for} \quad (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ v(x,0) = v_0(x) & \text{for} \quad x \in \mathbb{R}^3. \end{cases} \tag{3.1}$$

The solution of (3.1) can be obtained by the generalized Stokes-Coriolis semigroup $S_{\Omega,\beta}(\cdot)$, which is given explicitly by

$$S_{\Omega,\beta}(t)v = \mathcal{F}^{-1}\left[\cos\left(\Omega\frac{\xi_3}{|\xi|}t\right)I + \sin\left(\frac{\xi_3}{|\xi|}t\right)R(\xi)\right] * e^{-t(-\Delta)^\beta}v,$$

for $t \geq 0$, and divergence-free vector fields $v \in S(\mathbb{R}^3)$. Where I is the identity matrix in $\mathbb{R}^3$, $e^{-t(-\Delta)^\beta} := \mathcal{F}^{-1}\left(e^{-t|\xi|^{2\beta}}\mathcal{F}\right)$, and $R(\xi)$ is the skew symmetric matrix defined by

$$R(\xi) := \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}, \ \xi \in \mathbb{R}^3 \setminus \{0\}.$$

For the derivation of explicit form of $S_{\Omega,\beta}(\cdot)$, please see [13, 29].

Next, we establish the estimates of the semigroup $S_{\Omega,\beta}(\cdot)$.

Lemma 3.1. *Let* $\frac{1}{2} < \beta \leq 1$, $s_0(\cdot) \in C_{\log}(\mathbb{R}^d)$, $r(\cdot), r_0(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ *such that* $2 \leq r_0(\cdot) \leq c \leq r(\cdot) \leq q(\cdot) \leq \infty$ *and* $s_0(\cdot) = 4 - 2\beta + \frac{2\beta}{\rho} - \frac{3}{r_0(\cdot)}$, *and let* $1 \leq h, \rho \leq \infty$. *Then there holds*

$$\left\|S_{\Omega,\beta}(t)f_0\right\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{s_0(\cdot)}_{r_0(\cdot),q(\cdot),h}\right)} \lesssim \left\|f_0\right\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}}, \tag{3.2}$$

for all $\Omega \in \mathbb{R}$, $f_0 \in \mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}$. *Moreover, we have*

$$\left\|S_{\Omega,\beta}(t)f_0\right\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta+\frac{2\beta}{\rho}}_{2,q(\cdot),h}\right)} \lesssim \left\|f_0\right\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}}, \tag{3.3}$$

and

$$\left\|S_{\Omega,\beta}(t)f_0\right\|_{\widetilde{L}^\infty\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}\right)} \lesssim \left\|f_0\right\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}}. \tag{3.4}$$

Proof. Set $\widetilde{r_c}(\cdot) = \frac{c r_0(\cdot)}{c - r_0(\cdot)}$. According to Proposition and with $r_0(\cdot) \leq c \leq r(\cdot)$, we obtain

$$\begin{aligned}
\left\|S_{\Omega,\beta}(t)f_0\right\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{s_0(\cdot)}_{r_0(\cdot),q(\cdot),h}\right)} &\leq \left\{\sum_{j\in\mathbb{Z}}\left(\left\|2^{js_0(\cdot)}\varphi_j e^{-t|\xi|^{2\beta}}\hat{f}_0\right\|_{L^\rho\left(0,\infty;\mathcal{M}^{q(\cdot)}_{r_0(\cdot)}\right)}\right)^h\right\}^{\frac{1}{h}} \\
&\lesssim \left\{\sum_{j\in\mathbb{Z}}\left(\sum_{k=0,\pm1}\left\|2^{j(4-2\beta-\frac{3}{c})}\varphi_j\hat{f}_0\right\|_{\mathcal{M}^{q(\cdot)}_c}\right.\right. \\
&\qquad \left.\left.\left\|R^{-\frac{3}{\widetilde{r_c}(\cdot)}}2^{j(\frac{3}{c}+\frac{2\beta}{\rho}-\frac{3}{r_0(\cdot)})}\varphi_{j+k}e^{-t2^{2\beta(j+k)}}\right\|_{L^\rho\left(0,\infty;L^{\widetilde{r_c}(\cdot)}\right)}\right)^h\right\}^{\frac{1}{h}} \\
&\lesssim \left\{\sum_{j\in\mathbb{Z}}\left(\sum_{k=0,\pm1}\left\|2^{j(4-2\beta-\frac{3}{r(\cdot)})}\varphi_j\hat{f}_0\right\|_{\mathcal{M}^{q(\cdot)}_{r(\cdot)}}\right)^h\right\}^{\frac{1}{h}} \\
&\lesssim \left\|f_0\right\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}},
\end{aligned}$$

where we have utilized the following estimate

$$\begin{aligned}
\left\|R^{-\frac{3}{\widetilde{r_c}(\cdot)}}2^{j(\frac{3}{c}+\frac{2\beta}{\rho}-\frac{3}{r_0(\cdot)})}\varphi_{j+k}e^{-t2^{2\beta(j+k)}}\right\|_{L^\rho\left(0,\infty;L^{\widetilde{r_c}(\cdot)}\right)} &\leq \left\|2^{j\frac{2\beta}{\rho}}e^{-t2^{2\beta(j+k)}}\right\|_{L^\rho([0,\infty))}\left\|R^{-\frac{3}{\widetilde{r_c}(\cdot)}}2^{-3j\frac{1}{\widetilde{r_c}(x)}}\varphi_{j+k}\right\|_{L^\rho\left(0,\infty;L^{\widetilde{r_c}(\cdot)}\right)} \\
&\lesssim \left\|R^{-\frac{3}{\widetilde{r_c}(\cdot)}}2^{-3j\frac{1}{\widetilde{r_c}(x)}}\varphi_{j+k}\right\|_{L^{\widetilde{r_c}(\cdot)}} \\
&\lesssim \inf\left\{\lambda > 0 : \int_{\mathbb{R}^3}\left|\frac{R^{-\frac{3}{\widetilde{r_c}(x)}}2^{-3j\frac{1}{\widetilde{r_c}(x)}}\varphi_{j+k}}{\lambda}\right|^{\widetilde{r_c}(x)}dx \leq 1\right\} \\
&\lesssim \inf\left\{\lambda > 0 : \int_{\mathbb{R}^3}\left|\frac{R^{-\frac{3}{\widetilde{r_c}(x)}}\varphi_{j+k}}{\lambda}\right|^{\widetilde{r_c}(x)}2^{-3j}dx \leq 1\right\} \\
&\lesssim \inf\left\{\lambda > 0 : \int_{\mathbb{R}^3}\left|\frac{R^{-\frac{3}{\widetilde{r_c}(2^jx)}}\varphi_k}{\lambda}\right|^{\widetilde{r_c}(2^jx)}dx \leq 1\right\} \lesssim C.
\end{aligned}$$

Thus the inequality (3.2) is obtained. Similarly for the other inequalities. We just need to replace both c and $r_0(\cdot)$ with 2 (since $2 \leq r(\cdot)$), and replace them again with $r(\cdot)$ in the above proof for the inequalities (3.3) and (3.4), respectively. Then we get the results. And this completes the proof of Lemma 3.1. □

Lemma 3.2. *Let* $\frac{1}{2} < \beta \le 1$, $s(\cdot) \in C_{\log}(\mathbb{R}^d)$, $r(\cdot), r_0(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^d) \cap C_{\log}(\mathbb{R}^d)$ *such that* $r_0(\cdot) \le r(\cdot) \le q(\cdot) \le \infty$, $r(\cdot) \le \frac{6}{5-4\beta}$ *and* $s(\cdot) = 4 - 2\beta + \frac{2\beta}{\rho} - \frac{3}{r_0(\cdot)}$, *and let* $1 \le h, \rho \le \infty$. *Then there holds*

$$\left\| \int_0^t S_{\Omega,\beta}(t-\tau) f d\tau \right\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{s_0(\cdot)}_{r_0(\cdot),q(\cdot),h}\right)} \lesssim \|f\|_{\widetilde{L}^\rho\left(0,\infty;\dot{\mathcal{N}}^{\frac{3}{2}-2\beta+\frac{2\beta}{\rho}}_{\frac{6}{4\beta+1},q(\cdot),h}\right)}, \tag{3.5}$$

for all $\Omega \in \mathbb{R}$, $f \in \widetilde{L}^\rho\left(0,\infty;\dot{\mathcal{N}}^{\frac{3}{2}-2\beta+\frac{2\beta}{\rho}}_{\frac{6}{4\beta+1},q(\cdot),h}\right)$.

Proof. Set $\widetilde{r}_\beta(\cdot) = \frac{6r_0(\cdot)}{6-(5-4\beta)r_0(\cdot)}$. $\frac{1}{2} < \beta \le 1$ with Proposition 2.8, and Proposition 2.9 give us

$$\begin{aligned}
\left\| \int_0^t S_{\Omega,\beta}(t-\tau) f d\tau \right\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{s_0(\cdot)}_{r_0(\cdot),q(\cdot),h}\right)} &\lesssim \left\| \left\| \int_0^t 2^{js_0(\cdot)} \varphi_j e^{-(t-\tau)|\xi|^{2\beta}} \widehat{f} d\tau \right\|_{L^\rho\left(0,\infty;\mathcal{M}^{q(\cdot)}_{r_0(\cdot)}\right)} \right\|_{\ell^h} \\
&\lesssim \left\| \left\| \int_0^t \left\| 2^{j(4-2\beta+\frac{2\beta}{\rho}-\frac{3}{r_0(\cdot)})} R^{-\frac{3}{\widetilde{r}_\beta(\cdot)}} \varphi_j e^{-(t-\tau)|\xi|^{2\beta}} \right\|_{L^{\widetilde{r}_\beta(\cdot)}} \|\Delta_j f\|_{\mathcal{M}^{q(\cdot)}_{\frac{6}{4\beta+1}}} d\tau \right\|_{L^\rho([0,\infty))} \right\|_{\ell^h} \\
&\lesssim \left\| \left\| \int_0^t 2^{j(\frac{2\beta}{\rho}+\frac{3}{2})} e^{-(t-\tau)|\xi|^{2\beta}} \left\| R^{-\frac{3}{\widetilde{r}_\beta(\cdot)}} 2^{-3j\frac{1}{\widetilde{r}_\beta(\cdot)}} \varphi_j \right\|_{L^{\widetilde{r}_\beta(\cdot)}} \|\Delta_j f\|_{\mathcal{M}^{q(\cdot)}_{\frac{6}{4\beta+1}}} d\tau \right\|_{L^\rho([0,\infty))} \right\|_{\ell^h} \\
&\lesssim \left\| \left\| 2^{j(\frac{2\beta}{\rho}+\frac{3}{2}-2\beta)} \|\Delta_j f\|_{\mathcal{M}^{q(\cdot)}_{\frac{6}{4\beta+1}}} \right\|_{L^\rho([0,\infty))} \left\| 2^{2\beta j} e^{-t2^{2\beta j}} \right\|_{L^1([0,\infty))} \right\|_{\ell^h} \\
&\lesssim \left\| 2^{j(\frac{2\beta}{\rho}+\frac{3}{2}-2\beta)} \|\Delta_j f\|_{L^\rho\left(0,\infty;\mathcal{M}^{q(\cdot)}_{\frac{6}{4\beta+1}}\right)} \right\|_{\ell^h} \\
&\lesssim \|f\|_{\widetilde{L}^\rho\left(0,\infty;\dot{\mathcal{N}}^{\frac{3}{2}-2\beta+\frac{2\beta}{\rho}}_{\frac{6}{4\beta+1},q(\cdot),h}\right)}.
\end{aligned}$$

□

4. Proof of main result

In this part we establish global well-posedness for Equation (1.1) by applying the contraction mapping argument. Due to Duhamel's principle, the solution of Equation (1.1) can be written as

$$\begin{cases} v = S_{\Omega,\beta}(t)v_0 - \int_0^t S_{\Omega,\beta}(t-\tau)\mathbb{P}[\nabla \cdot (v \otimes v)]d\tau + \int_0^t S_{\Omega,\beta}(t-\tau)\mathbb{P}g\theta e_3 d\tau := \phi_1(v,\theta), \\ \theta = e^{-t(-\Delta)^\beta}\theta_0 - \int_0^t e^{-(t-\tau)(-\Delta)^\beta}\mathbb{P}[\nabla \cdot (v \otimes \theta)]d\tau := \phi_2(v,\theta), \end{cases} \tag{4.1}$$

where $S_{\Omega,\beta}(\cdot)$ is the generalized Stokes-Coriolis semigroup presented previously and $\mathbb{P} = I - \nabla\Delta^{-1}\nabla$is the Leray-Hopf projector. We define the map

$$\Psi : (v,\theta) \to (\phi_1(v,\theta), \phi_2(v,\theta)), \tag{4.2}$$

in the following metric space, for $\delta > 0$ which will be chosen later:

$$Y := \left\{ (v,\theta)/\ v,\theta \in \widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}) \cap \widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta+\frac{2\beta}{\rho}}_{2,q(\cdot),h}) \cap \widetilde{L}^\infty(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta}_{2,q(\cdot),h}),\ \ \|(v,\theta)\|_Y \le \delta \right\},$$

equipped with the distance

$$d\left(\begin{pmatrix} v \\ \theta \end{pmatrix}, \begin{pmatrix} w \\ \vartheta \end{pmatrix}\right) := \left\| \begin{pmatrix} v - w \\ \theta - \vartheta \end{pmatrix} \right\|_Y .$$

Where

$$\|(v,\theta)\|_Y := \|v\|_Y + \|\theta\|_Y ,$$

and

$$\|v\|_Y := \|v\|_{\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h})\cap\widetilde{L}^\rho(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta+\frac{2\beta}{\rho}}_{2,q(\cdot),h})\cap\widetilde{L}^\infty\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta}_{2,q(\cdot),h}\right)} .$$

We will prove that $\Psi : (Y,d) \to (Y,d)$ is a contraction mapping. First we estimate $S_{\Omega,\beta}(t)v_0$ and $e^{-t(-\Delta)^\beta}\theta_0$. From Lemma 3.1, we have for $v_0 \in \mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}$,

$$\begin{aligned}
\left\|S_{\Omega,\beta}(t)v_0\right\|_{\widetilde{L}^\infty\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}\right)} &\lesssim \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} \\
\left\|S_{\Omega,\beta}(t)v_0\right\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta+\frac{2\beta}{\rho}}_{2,q(\cdot),h}\right)} &\lesssim \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} \\
\left\|S_{\Omega,\beta}(t)v_0\right\|_{\widetilde{L}^\infty\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta}_{2,q(\cdot),h}\right)} &\lesssim \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} .
\end{aligned}$$

Then

$$\left\|S_{\Omega,\beta}(t)v_0\right\|_Y \lesssim \|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} . \tag{4.3}$$

And similarly for $\theta_0 \in \mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}$ and $\Omega = 0$. From Lemma 3.1 again, one has

$$\left\|e^{-t(-\Delta)^\beta}\theta_0\right\|_Y \lesssim \|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} . \tag{4.4}$$

Next, we estimate the remaining terms. According to Lemma 3.2 and Proposition 2.9, since $\frac{1}{2} < \beta \leq 1$, one has

$$\begin{aligned}
\left\|\int_0^t S_{\Omega,\beta}(t-\tau)\mathbb{P}[\nabla\cdot(v_1\otimes v_2)]d\tau\right\|_{\widetilde{L}^\rho\left(0,\infty;\mathcal{F}\dot{\mathcal{N}}^{s_0(\cdot)}_{r_0(\cdot),q(\cdot),h}\right)} &\lesssim \|v_1v_2\|_{\widetilde{L}^\rho\left(0,\infty;\dot{\mathcal{N}}^{\frac{5}{2}-2\beta+\frac{2\beta}{\rho}}_{\frac{6}{4\beta+1},q(\cdot),h}\right)} \\
&\lesssim \|v_1\|_{\widetilde{L}^\rho\left(0,\infty;\dot{\mathcal{N}}^{\frac{5}{2}-2\beta+\frac{2\beta}{\rho}}_{2,q(\cdot),h}\right)}\|v_2\|_{\widetilde{L}^\infty\left(0,\infty;\dot{\mathcal{N}}^0_{\frac{3}{2\beta-1},\infty,h}\right)} \\
&\quad + \|v_2\|_{\widetilde{L}^\rho\left(0,\infty;\dot{\mathcal{N}}^{\frac{5}{2}-2\beta+\frac{2\beta}{\rho}}_{2,q(\cdot),h}\right)}\|v_1\|_{\widetilde{L}^\infty\left(0,\infty;\dot{\mathcal{N}}^0_{\frac{3}{2\beta-1},\infty,h}\right)} \\
&\lesssim \|v_1\|_{\widetilde{L}^\rho\left(0,\infty;\dot{\mathcal{N}}^{\frac{5}{2}-2\beta+\frac{2\beta}{\rho}}_{2,q(\cdot),h}\right)}\|v_2\|_{\widetilde{L}^\infty\left(0,\infty;\dot{\mathcal{N}}^{\frac{5}{2}-2\beta}_{2,q(\cdot),h}\right)} \\
&\quad + \|v_2\|_{\widetilde{L}^\rho\left(0,\infty;\dot{\mathcal{N}}^{\frac{5}{2}-2\beta+\frac{2\beta}{\rho}}_{2,q(\cdot),h}\right)}\|v_1\|_{\widetilde{L}^\infty\left(0,\infty;\dot{\mathcal{N}}^{\frac{5}{2}-2\beta}_{2,q(\cdot),h}\right)} \\
&\lesssim \|v_1\|_Y\|v_2\|_Y .
\end{aligned}$$

Then, we obtain

$$\left\|\int_0^t S_{\Omega,\beta}(t-\tau)\mathbb{P}[\nabla\cdot(v\otimes v)]d\tau\right\|_Y \lesssim \|v\|_Y\|v\|_Y . \tag{4.5}$$

Similarly, we can obtain

$$\left\| \int_0^t e^{-(t-\tau)(-\Delta)^\beta} \mathbb{P}[\nabla \cdot (v \otimes \theta)] d\tau \right\|_Y \lesssim \|v\|_Y \|\theta\|_Y , \tag{4.6}$$

and

$$\left\| \int_0^t S_{\Omega,\beta}(t-\tau) \mathbb{P} g \theta e_3 d\tau \right\|_Y \lesssim \|\theta\|_Y . \tag{4.7}$$

therefore, (4.3), (4.4), (4.5), (4.6) and (4.7) give us that there exist $C > 0$ such that

$$\begin{aligned} \|\Psi(v,\theta)\|_Y &= \left\|\phi_1(v,\theta)\right\|_Y + \left\|\phi_2(v,\theta)\right\|_Y \\ &\leq C \left(\|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} + \|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} \right) + C\|v\|_Y \|v\|_Y + C\|\theta\|_Y + C\|v\|_Y \|\theta\|_Y . \end{aligned}$$

Put

$$\delta = 2C \left(\|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} + \|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} \right) < 2C\varepsilon.$$

If ε is sufficiently small, then one has

$$\|\Psi(v,\theta)\|_Y \leq \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Similarly, we obtain

$$d(\Psi(v,\theta),\ \Psi(w,\vartheta)) \leq \frac{1}{2} d(\begin{pmatrix} v \\ \theta \end{pmatrix}, \begin{pmatrix} w \\ \vartheta \end{pmatrix}).$$

Finally, by applying Banach's contraction mapping principle, we get a unique fixed point $(v,\theta) \in Y$ of Ψ, which is the global solution of Equation (1.1).

On the other hand, let

$$\begin{aligned} \widetilde{Y} := \Big\{ (v,\theta)/\ v,\theta \in \widetilde{L}^\rho \left(0,\infty; \mathcal{F}\dot{\mathcal{N}}^{s_0(\cdot)}_{r_0(\cdot),q(\cdot),h}\right) \cap \widetilde{L}^\rho(0,\infty; \mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}) \\ \cap \widetilde{L}^\rho(0,\infty; \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta+\frac{2\beta}{\rho}}_{2,q(\cdot),h}) \cap \widetilde{L}^\infty \left(0,\infty; \mathcal{F}\dot{\mathcal{N}}^{\frac{5}{2}-2\beta}_{2,q(\cdot),h}\right) : \|(v,\theta)\|_{\widetilde{Y}} \leq \delta \Big\}. \end{aligned}$$

If we follow a procedure similar to the one described above, we get

$$\|\Psi(v,\theta)\|_{\widetilde{Y}} \leq C \left(\|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} + \|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} \right) + C\|v\|_{\widetilde{Y}} \|v\|_{\widetilde{Y}} + C\|\theta\|_{\widetilde{Y}} + C\|v\|_{\widetilde{Y}} \|\theta\|_{\widetilde{Y}} .$$

Analogously to the case of the space Y, one can demonstrate that Equation (1.1) admits a unique global solution $(v,\theta) \in \widetilde{Y}$, for a sufficiently small ϵ and $\|v_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} + \|\theta_0\|_{\mathcal{F}\dot{\mathcal{N}}^{4-2\beta-\frac{3}{r(\cdot)}}_{r(\cdot),q(\cdot),h}} < \epsilon$. And this completes the proof of Theorem 1.1, as desired.

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