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A study on pseudoparallel submanifolds of generalized Lorentz-Sasakian space forms

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Abstract. In this article, pseudoparallel submanifolds for generalized Lorentz-Sasakian space forms are investigated. Submanifolds of these manifolds with properties such as pseudoparallel, 2-pseudoparallel, Ricci generalized pseudoparallel, and 2-Ricci generalized pseudoparallel have been investigated and the conditions under which these pseudoparallel submanifolds are totally geodesic are shown. In addition, necessary and sufficient conditions have been obtained for these submanifolds to be totally geodesic by means of the concircular, projective and quasi-conformally curvature tensors. At last, we provide an example for such manifold.

1. Introduction

$\phi-$sectional curvature plays the an important role for Sasakian manifold. If the $\phi-$sectional curvature of a Sasakian manifold is constant, then the manifold is called a Sasakian-space-form [10]. Alegre and Blair described generalized Sasakian space forms [1]. Alegre and Blair obtained important properties of generalized Sasakian space forms in their studies and gave some examples. Alegre and Carriazo later discussed generalized indefinite Sasakian space forms [2]. Pandey and Prasad et al. are also studied on Sasakian manifolds for different properties in ([12], [13], [17], [18], [19]). Generalized Sasakian space forms studied for concircular curvature tensor in [4]. The Lorentzian manifolds are of great importance for Einstein's theory of relativity. Sasakian space forms, generalized Sasakian space forms and Lorentz-Sasakian space forms have been discussed by many scientists and important properties of these manifolds have been obtained in ([9]-[14]). Again, some researcher have considered invariant submanifolds in different spaces ([5], [6], [21]).

In this paper, pseudoparallel submanifolds for generalized Lorentz-Sasakian space forms are investigated. Submanifolds of these manifolds with properties such as pseudoparallel, 2-pseudoparallel, Ricci generalized pseudoparallel, and 2-Ricci generalized pseudoparallel have been investigated and the conditions under which these pseudoparallel submanifolds are totally geodesic are shown. In addition, necessary and sufficient conditions have been obtained for these submanifolds to be totally geodesic on the concircular and projective and quasi-conformally curvature tensors.

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2. Preliminary

Let $\tilde{M}$ be a $(2n+1)-$dimensional semi-Riemannian manifold. If the $\tilde{M}$ semi-Riemannian manifold with $\left(\phi, \xi, \eta, g\right)$ structure tensors satisfies the following conditions, this manifold is called a $\varepsilon-$ almost contact metric manifold and $\left(\phi, \xi, \eta\right)$ triple is called almost cotact structure.

$$\phi\xi = 0, \eta \circ \phi = 0, \eta(\xi) = 1, \phi^2 = -id + \eta \otimes \xi,$$

$$g(\vartheta_1, \vartheta_2) = g\left(\phi\vartheta_1, \phi\vartheta_2\right) + \varepsilon\eta(\vartheta_1)\eta(\vartheta_2), \eta(\vartheta_1) = \varepsilon g(\vartheta_1, \xi),$$

where,

$$\varepsilon = g(\xi, \xi) = \pm 1.$$

If $d\eta$ and g provide the relation

$$d\eta(\vartheta_1, \vartheta_2) = g\left(\vartheta_1, \phi\vartheta_2\right),$$

$\tilde{M}$ is called a contact pseudometric manifold and the $\left(\phi, \xi, \eta\right)$ triple is called a contact structure.

Let's define a $\left(h\left(\frac{d}{d\vartheta_1}\right), \vartheta_4\right)$ vector field on $\mathbb{R}\times\tilde{M}$, where ϑ_1 is a coordinate on $\mathbb{R}$ and h is a C^∞ function on $\mathbb{R}\times\tilde{M}$. The structure defined as

$$J\left(h\frac{d}{d\vartheta_1}, \vartheta_4\right) = \left(\eta(\vartheta_4)\frac{d}{d\vartheta_1}, \phi\vartheta_4 - h\xi\right),$$

on $\mathbb{R}\times\tilde{M}$ is called a almost complex structure and $J^2 = -id$.If J is integrable, the almost contact structure $\left(\phi, \xi, \eta\right)$ is said to be normal.

If ϑ_1 is perpendicular to ξ, the plane spanned by ϑ_1 and $\phi\vartheta_1$, is called the $\phi-$section. The curvature of the ϕ-section is called the ϕ-sectional curvature. The curvature of the indefinite Sasakian manifold defined in this way is precisely determined by the $\phi-$section curvature. If the $\phi-$section curvature of the indefinit Sasakian manifold is equal to a constant c, the curvature tensor of this manifold is defined as

$$\tilde{R}(\vartheta_1, \vartheta_2)\vartheta_3 = \left(\tfrac{c+3\varepsilon}{4}\right)\{g(\vartheta_2, \vartheta_3)\vartheta_1 - g(\vartheta_1, \vartheta_3)\vartheta_2\}$$

$$+\left(\tfrac{c-\varepsilon}{4}\right)\left\{g\left(\vartheta_1, \phi\vartheta_3\right)\phi\vartheta_2 - g\left(\vartheta_2, \phi\vartheta_3\right)\phi\vartheta_1\right.$$

$$\left.+2g\left(\vartheta_1, \phi\vartheta_2\right)\phi\vartheta_3\right\} + \left(\tfrac{c-\varepsilon}{4}\right)\{\eta(\vartheta_2)\eta(\vartheta_3)\vartheta_1 - \eta(\vartheta_1)\eta(\vartheta_3)\vartheta_2$$

$$+\varepsilon g(\vartheta_1, \vartheta_3)\eta(\vartheta_2)\xi - \varepsilon g(\vartheta_2, \vartheta_3)\eta(\vartheta_1)\xi\}.$$

For an $\varepsilon-$almost contact metric manifold $\tilde{M}$, if there are $F_1, F_2, F_3 \in C^\infty\left(\tilde{M}\right)$ functions such that

$$\tilde{R}(\vartheta_1, \vartheta_2)\vartheta_3 = F_1\{g(\vartheta_2, \vartheta_3)\vartheta_1 - g(\vartheta_1, \vartheta_3)\vartheta_2\}$$

$$+F_2\left\{g\left(\vartheta_1, \phi\vartheta_3\right)\phi\vartheta_2 - g\left(\vartheta_2, \phi\vartheta_3\right)\phi\vartheta_1\right.$$

$$\left.+2g\left(\vartheta_1, \phi\vartheta_2\right)\phi\vartheta_3\right\} + F_3\{\eta(\vartheta_2)\eta(\vartheta_3)\vartheta_1 - \eta(\vartheta_1)\eta(\vartheta_3)\vartheta_2$$

$$+\varepsilon g(\vartheta_1, \vartheta_3)\eta(\vartheta_2)\xi - \varepsilon g(\vartheta_2, \vartheta_3)\eta(\vartheta_1)\xi\},$$

then manifold $\tilde{M}$ is called an indefinit Sasakian space form.

In this article, only the Lorentzian case, which corresponds to the $\varepsilon = -1$, where the index of the metric is 1, will be discussed. Such manifolds are called generalized Lorenzt-Sasakian space forms and are denoted by $M_1^{2n+1}(F_1,F_2,F_3)$. Thus, the curvature tensor of a $(2n+1)-$dimensional generalized Lorentz Sasakian space form is defined as

$$\begin{aligned}&\tilde{R}(\vartheta_1,\vartheta_2)\vartheta_3 = F_1\{g(\vartheta_2,\vartheta_3)\vartheta_1 - g(\vartheta_1,\vartheta_3)\vartheta_2\}\\&+F_2\left\{g\left(\vartheta_1,\phi\vartheta_3\right)\phi\vartheta_2 - g\left(\vartheta_2,\phi\vartheta_3\right)\phi\vartheta_1\right.\\&\left.+2g\left(\vartheta_1,\phi\vartheta_2\right)\phi\vartheta_3\right\} + F_3\{-\eta(\vartheta_2)\eta(\vartheta_3)\vartheta_1 + \eta(\vartheta_1)\eta(\vartheta_3)\vartheta_2\\&-g(\vartheta_1,\vartheta_3)\eta(\vartheta_2)\xi + g(\vartheta_2,\vartheta_3)\eta(\vartheta_1)\xi\}.\end{aligned} \tag{1}$$

Lemma 2.1. *Let $M_1^{2n+1}(F_1,F_2,F_3)$ be the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form. The following relations are provided for $M_1^{2n+1}(F_1,F_2,F_3)$.*

$$\tilde{\nabla}_{\vartheta_1}\xi = (F_1+F_3)\phi\vartheta_1, \tag{2}$$

$$\tilde{R}(\xi,\vartheta_2)\vartheta_3 = (F_1+F_3)[g(\vartheta_2,\vartheta_3)\xi + \eta(\vartheta_3)\vartheta_2], \tag{3}$$

$$\tilde{R}(\xi,\vartheta_2)\xi = (F_1+F_3)[-\eta(\vartheta_2)\xi + \vartheta_2], \tag{4}$$

$$\tilde{R}(\vartheta_1,\vartheta_2)\xi = (F_1+F_3)[\eta(\vartheta_1)\vartheta_2 - \eta(\vartheta_2)\vartheta_1], \tag{5}$$

$$S(\vartheta_1,\xi) = -2n(F_1+F_3)\eta(\vartheta_1), \tag{6}$$

$$Q\xi = 2n(F_1+F_3)\xi, \tag{7}$$

where $\tilde{R}$, S and Q are the Riemann curvature tensor, Ricci curvature tensor and Ricci operator of $M_1^{2n+1}(F_1,F_2,F_3)$, respectively.

Let M be the immersed submanifold of the $(2n+1)-$dimensional genetalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1,F_2,F_3)$. Let the tangent and normal subspaces of M in $M_1^{2n+1}(F_1,F_2,F_3)$ be $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, respectively. Gauss and Weingarten formulas for $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$ are

$$\tilde{\nabla}_{\vartheta_1}\vartheta_2 = \nabla_{\vartheta_1}\vartheta_2 + \sigma(\vartheta_1,\vartheta_2), \tag{8}$$

$$\tilde{\nabla}_{\vartheta_1}\vartheta_5 = -A_{\vartheta_5}\vartheta_1 + \nabla^{\perp}_{\vartheta_1}\vartheta_5, \tag{9}$$

respectively, for all $\vartheta_1,\vartheta_2 \in \Gamma(TM)$ and $\vartheta_5 \in \Gamma(T^{\perp}M)$, where ∇ and $\nabla^{\perp}$ are the connections on M and $\Gamma(T^{\perp}M)$, respectively, σ and A are the second fundamental form and the shape operator of M. There is a relation

$$g(A_{\vartheta_5}\vartheta_1,\vartheta_2) = g(\sigma(\vartheta_1,\vartheta_2),\vartheta_5), \tag{10}$$

between the second basic form and shape operator defined as above. The covariant derivative of the second fundamental form σ is defined as

$$(\tilde{\nabla}_{\vartheta_1}\sigma)(\vartheta_2,\vartheta_3) = \nabla^{\perp}_{\vartheta_1}\sigma(\vartheta_2,\vartheta_3) - \sigma(\nabla_{\vartheta_1}\vartheta_2,\vartheta_3) - \sigma(\vartheta_2,\nabla_{\vartheta_1}\vartheta_3). \tag{11}$$

Specifically, if $\tilde{\nabla}\sigma = 0$, M is said to be is parallel second fundamental form.

Let R be the Riemann curvature tensor of M. In this case, the Gauss equation can be expressed as

$$\begin{aligned}&\tilde{R}(\vartheta_1,\vartheta_2)\vartheta_3 = R(\vartheta_1,\vartheta_2)\vartheta_3 + A_{\sigma(\vartheta_1,\vartheta_3)}\vartheta_2 - A_{\sigma(\vartheta_2,\vartheta_3)}\vartheta_1\\&+(\tilde{\nabla}_{\vartheta_1}\sigma)(\vartheta_2,\vartheta_3) - (\tilde{\nabla}_{\vartheta_2}\sigma)(\vartheta_1,\vartheta_3).\end{aligned} \tag{12}$$

Let M be a Riemannian manifold, T is $(0,k)-$type tensor field and A is $(0,2)-$type tensor field. In this case, Tachibana tensor field $Q(A,T)$ is defined as

$$Q(A,T)(\vartheta_{11},...,X_k;X,Y) = -T((X\wedge_A Y)X_1,...,X_k)$$
$$-... - T(X_1,...,X_{k-1},(X\wedge_A Y)X_k), \tag{13}$$

where,

$$(X\wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y, \tag{14}$$

$k \geq 1, X_1, X_2, ..., X_k, X, Y \in \Gamma(TM)$.

3. Invariant Pseudoparalel Submanifolds of Generalized Lorentz-Sasakian Space Forms

Let M be the immersed submanifold of a $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1,F_2,F_3)$. If $\phi(T_{\vartheta_1}M) \subset T_{\vartheta_1}M$ in every ϑ_1 point, the M manifold is called invariant submanifold. From this section of the article, we will assume that the manifold M is the invariant submanifold of the generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1,F_2,F_3)$. So, it is clear that

$$\sigma(\vartheta_1,\xi) = 0, \sigma\left(\phi\vartheta_1,\vartheta_2\right) = \sigma\left(\vartheta_1,\phi\vartheta_2\right) = \phi\sigma(\vartheta_1,\vartheta_2), \tag{15}$$

for all $\vartheta_1, \vartheta_2 \in \Gamma(TM)$.

Lemma 3.1. *Let M be the invariant submanifold of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1,F_2,F_3)$. The second fundamental form σ of M is parallel if and only if M is the totally geodesic submanifold provided $F_1 \neq -F_3$.*

Proof. The proof of the theorem is easily obtained if we choose $\vartheta_3 = \xi$ in (11) and make the necessary adjustments. □

Definition 3.2. *Let M be the invariant submanifold of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1,F_2,F_3)$. If $\tilde{R}.\sigma$ and $Q(g,\sigma)$ are linearly dependent, M is called pseudoparallel submanifold [7].*

Equivalent to this definition, it can be said that there is a function λ_1 on the set $M_1 = \{x \in M | \sigma(x) \neq g(x)\}$ such that

$$\tilde{R}.\sigma = \lambda_1 Q(g,\sigma).$$

If $\lambda_1 = 0$ specifically, M is called a semiparallel submanifold.

Let us now examine the case of pseudoparallel for the submanifold M of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1,F_2,F_3)$.

Theorem 3.3. *Let M be the invariant submanifold of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1,F_2,F_3)$. If M is pseudoparallel submanifold, then M is either a totally geodesic or $\lambda_1 = F_1 + F_3$.*

Proof. Let's assume that M is a pseudoparallel submanifold. So, we can write

$$\left(\tilde{R}(\vartheta_1,\vartheta_2)\sigma\right)(\vartheta_4,\vartheta_5) = \lambda_1 Q(g,\sigma)(\vartheta_4,\vartheta_5,\vartheta_1,\vartheta_2), \tag{16}$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(TM)$. From (16), it is clear that

$$R^{\perp}(\vartheta_1,\vartheta_2)\sigma(\vartheta_4,\vartheta_5) - \sigma(R(\vartheta_1,\vartheta_2)\vartheta_4,\vartheta_5)$$
$$-\sigma(\vartheta_4,R(\vartheta_1,\vartheta_2)\vartheta_5) = -\lambda_1\left\{\sigma\left(\left(\vartheta_1\wedge_g\vartheta_2\right)\vartheta_4,\vartheta_5\right)\right.$$
$$\left.+\sigma\left(\vartheta_4,\left(\vartheta_1\wedge_g\vartheta_2\right)\vartheta_5\right)\right\}.$$

Easily from here, we can write

$$R^{\perp}(\vartheta_1,\vartheta_2)\sigma(\vartheta_4,\vartheta_5)-\sigma(R(\vartheta_1,\vartheta_2)\vartheta_4,\vartheta_5)$$

$$-\sigma(\vartheta_4,R(\vartheta_1,\vartheta_2)\vartheta_5)=-\lambda_1\{g(\vartheta_2,\vartheta_4)\sigma(\vartheta_1,\vartheta_5)$$

$$-g(\vartheta_1,\vartheta_4)\sigma(\vartheta_2,\vartheta_5)+g(\vartheta_2,\vartheta_5)\sigma(\vartheta_4,\vartheta_1)$$

$$-g(\vartheta_1,\vartheta_5)\sigma(\vartheta_4,\vartheta_2)\}.$$

If we choose $\vartheta_5=\xi$ in from the last equality and make use of (5) , (15), we get

$$[\lambda_1-(F_1+F_3)]\{\eta(\vartheta_1)\sigma(\vartheta_4,\vartheta_2)-\eta(\vartheta_2)\sigma(\vartheta_4,\vartheta_2)\}=0. \tag{17}$$

IIf we choose $\vartheta_2=\xi$ in (17) , we obtain

$$-[\lambda_1-(F_1+F_3)]\sigma(\vartheta_4,\vartheta_1)=0.$$

This completes the proof. □

Definition 3.4. *Let M be the invariant submanifold of the* $(2n+1)-$*dimensional generalized Lorentz-Sasakian space form* $M_1^{2n+1}(F_1,F_2,F_3)$*. If* $\tilde{R}.\tilde{\nabla}\sigma$ *and* $Q(g,\tilde{\nabla}\sigma)$ *are linearly dependent, then M is called* $2-$*pseudoparallel submanifold [8], [15].*

In this case, it can be said that there is a function λ_2 on the set $M_2=\{x\in M|\,\tilde{\nabla}\sigma(x)\neq g(x)\}$ such that

$$\tilde{R}.\tilde{\nabla}\sigma=\lambda_2 Q(g,\tilde{\nabla}\sigma).$$

If $\lambda_2=0$ specifically, M is called a $2-$semiparallel submanifold.

Let us now examine the case of $2-$pseudoparallel for the submanifold M of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1,F_2,F_3)$.

Theorem 3.5. *Let M be the invariant submanifold of the* $(2n+1)-$*dimensional generalized Lorentz-Sasakian space form* $M_1^{2n+1}(F_1,F_2,F_3)$*. If M is* $2-$*pseudoparallel submanifold, then M is a totally geodesic submanifold provided* $F_1\neq -F_3$.

Proof. Let's assume that M is a $2-$pseudoparallel submanifold. So, we can write

$$\left(\tilde{R}(\vartheta_1,\vartheta_2)\tilde{\nabla}\sigma\right)(\vartheta_4,\vartheta_5,\vartheta_3)=\lambda_2 Q(g,\tilde{\nabla}\sigma)(\vartheta_4,\vartheta_5,\vartheta_3,\vartheta_1,\vartheta_2), \tag{18}$$

for all $\vartheta_1,\vartheta_2,\vartheta_4,\vartheta_5,\vartheta_3\in\Gamma(TM)$. If we choose $\vartheta_1=\vartheta_3=\xi$ in (18) , we can write

$$\begin{aligned}&R^{\perp}(\xi,\vartheta_2)(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_5,\xi)-\left(\tilde{\nabla}_{R(\xi,\vartheta_2)\vartheta_4}\sigma\right)(\vartheta_5,\xi)\\&-(\tilde{\nabla}_{\vartheta_4}\sigma)(R(\xi,\vartheta_2)\vartheta_5,\xi)-(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_5,R(\xi,\vartheta_2)\xi)\\&=-\lambda_2\left\{\left(\tilde{\nabla}_{(\xi\wedge_g\vartheta_2)\vartheta_4}\sigma\right)(\vartheta_5,\xi)+(\tilde{\nabla}_{\vartheta_4}\sigma)\left(\left(\xi\wedge_g\vartheta_2\right)\vartheta_5,\xi\right)\right.\\&\left.+(\tilde{\nabla}_{\vartheta_4}\sigma)\left(\vartheta_5,\left(\xi\wedge_g\vartheta_2\right)\xi\right)\right\}.\end{aligned} \tag{19}$$

Let's calculate all the expressions in (19). So, we can write

$$\begin{aligned}&R^{\perp}(\xi,\vartheta_2)(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_5,\xi)=R^{\perp}(\xi,\vartheta_2)\left\{\nabla^{\perp}_{\vartheta_4}\sigma(\vartheta_5,\xi)\right.\\&\left.-\sigma(\nabla_{\vartheta_4}\vartheta_5,\xi)-\sigma(\vartheta_5,\nabla_{\vartheta_4}\xi)\right\}\\&=R^{\perp}(\xi,\vartheta_2)(F_1+F_3)\phi h(\vartheta_5,\vartheta_4),\end{aligned} \tag{20}$$

$$\begin{aligned}
&\left(\tilde{\nabla}_{R(\xi,\vartheta_2)\vartheta_4}\sigma\right)(\vartheta_5,\xi) = \nabla^{\perp}_{R(\xi,\vartheta_2)\vartheta_4}\sigma(\vartheta_5,\xi) \\
&-\sigma\left(\nabla_{R(\xi,\vartheta_2)\vartheta_4}\vartheta_5,\xi\right) - \sigma\left(\vartheta_5,\nabla_{R(\xi,\vartheta_2)\vartheta_4}\xi\right) \\
&= -\sigma\left(\vartheta_5,(F_1+F_3)\,\phi R(\xi,\vartheta_2)\,\vartheta_4\right) \\
&= -(F_1+F_3)^2\,\phi\eta(\vartheta_4)\,\sigma(\vartheta_5,\vartheta_2)\,,
\end{aligned} \tag{21}$$

$$\begin{aligned}
&(\tilde{\nabla}_{\vartheta_4}\sigma)(R(\xi,\vartheta_2)\,\vartheta_5,\xi) = \nabla^{\perp}_{\vartheta_4}\sigma(R(\xi,\vartheta_2)\,\vartheta_5,\xi) \\
&-\sigma(\nabla_{\vartheta_4}R(\xi,\vartheta_2)\,\vartheta_5,\xi) - \sigma(R(\xi,\vartheta_2)\,\vartheta_5,\nabla_{\vartheta_4}\xi) \\
&= -\sigma((F_1+F_3)\,[g(\vartheta_2,\vartheta_5)\,\xi + \eta(\vartheta_5)\,\vartheta_2] \\
&,(F_1+F_3)\,\phi\vartheta_4\Big) \\
&= -(F_1+F_3)^2\,\phi\eta(\vartheta_5)\,\sigma(\vartheta_2,\vartheta_4)\,,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_5,R(\xi,\vartheta_2)\,\xi) = (\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_5,(F_1+F_3)\,[-\eta(\vartheta_2)\,\xi + \vartheta_2]) \\
&= -(F_1+F_3)\,(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_5,\eta(\vartheta_2)\,\xi) \\
&+ (F_1+F_3)\,(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_5,\vartheta_2) \\
&= (F_1+F_3)\,\sigma(\vartheta_5,\vartheta_4\eta(\vartheta_2)\,\xi + \eta(\vartheta_2)\,\nabla_{\vartheta_4}\xi) \\
&+ (F_1+F_3)\,(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_5,\vartheta_2) \\
&= (F_1+F_3)^2\,\eta(\vartheta_2)\,\phi\sigma(\vartheta_5,\vartheta_4) \\
&+ (F_1+F_3)\,(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_5,\vartheta_2)\,,
\end{aligned} \tag{23}$$

$$\begin{aligned}
&\left(\tilde{\nabla}_{(\xi\wedge_g\vartheta_2)\vartheta_4}\sigma\right)(\vartheta_5,\xi) = \nabla^{\perp}_{(\xi\wedge_g\vartheta_2)\vartheta_4}\sigma(\vartheta_5,\xi) \\
&-\sigma\left(\nabla_{(\xi\wedge_g\vartheta_2)\vartheta_4}\vartheta_5,\xi\right) - \sigma\left(\vartheta_5,\nabla_{(\xi\wedge_g\vartheta_2)\vartheta_4}\xi\right) \\
&= -\sigma\Big(\vartheta_5,(F_1+F_3)\,\phi\,[g(\vartheta_2,\vartheta_4)\,\xi \\
&-g(\xi,\vartheta_4)\,\vartheta_2]) \\
&= -(F_1+F_3)\,\phi\eta(\vartheta_4)\,\sigma(\vartheta_5,\vartheta_2)\,,
\end{aligned} \tag{24}$$

$$
\begin{aligned}
&(\tilde{\nabla}_{\vartheta_4}\sigma)\left(\left(\xi \wedge_g \vartheta_2\right)\vartheta_5, \xi\right) = \nabla^{\perp}_{\vartheta_4}\sigma\left(\left(\xi \wedge_g \vartheta_2\right)\vartheta_5, \xi\right)\\
&-\sigma\left(\nabla_{\vartheta_4}\left(\xi \wedge_g \vartheta_2\right)\vartheta_5, \xi\right) - \sigma\left(\left(\xi \wedge_g \vartheta_2\right)\vartheta_5, \nabla_{\vartheta_4}\xi\right)\\
&= -\sigma\left(g\left(\vartheta_2, \vartheta_5\right)\xi - g\left(\xi, \vartheta_5\right)\vartheta_2,\right.\\
&\left.\left(F_1 + F_3\right)\phi\vartheta_4\right)\\
&= -\left(F_1 + F_3\right)\phi\eta\left(\vartheta_5\right)\sigma\left(\vartheta_2, \vartheta_4\right),
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
&(\tilde{\nabla}_{\vartheta_4}\sigma)\left(\vartheta_5, \left(\xi \wedge_g \vartheta_2\right)\xi\right) = (\tilde{\nabla}_{\vartheta_4}\sigma)\left(\vartheta_5, -\eta\left(\vartheta_2\right)\xi + \vartheta_2\right)\\
&= (\tilde{\nabla}_{\vartheta_4}\sigma)\left(\vartheta_5, -\eta\left(\vartheta_2\right)\xi\right) - (\tilde{\nabla}_{\vartheta_4}\sigma)\left(\vartheta_5, \vartheta_2\right)\\
&= \left(F_1 + F_3\right)\phi\eta\left(\vartheta_2\right)\sigma\left(\vartheta_5, \vartheta_4\right) - (\tilde{\nabla}_{\vartheta_4}\sigma)\left(\vartheta_5, \vartheta_2\right).
\end{aligned}
\tag{26}
$$

If we substitute (20) , (21) , (22) , (23) , (24) , (25) , (26) in (19) , we obtain

$$
\begin{aligned}
&R^{\perp}\left(\xi, \vartheta_2\right)\left(F_1 + F_3\right)\phi\sigma\left(\vartheta_5, \vartheta_4\right) + \left(F_1 + F_3\right)^2\phi\eta\left(\vartheta_4\right)\sigma\left(\vartheta_5, \vartheta_2\right)\\
&+ \left(F_1 + F_3\right)^2\phi\eta\left(\vartheta_5\right)\sigma\left(\vartheta_2, \vartheta_4\right) - \left(F_1 + F_3\right)^2\eta\left(\vartheta_2\right)\phi\sigma\left(\vartheta_5, \vartheta_4\right)\\
&- \left(F_1 + F_3\right)(\tilde{\nabla}_{\vartheta_4}\sigma)\left(\vartheta_5, \vartheta_2\right) = -\lambda_2\left\{-\left(F_1 + F_3\right)\phi\eta\left(\vartheta_5\right)\sigma\left(\vartheta_4, \vartheta_2\right)\right.\\
&- \left(F_1 + F_3\right)\phi\eta\left(\vartheta_4\right)\sigma\left(\vartheta_2, \vartheta_5\right) + \left(F_1 + F_3\right)\phi\eta\left(\vartheta_2\right)\sigma\left(\vartheta_5, \vartheta_4\right)\\
&\left.- (\tilde{\nabla}_{\vartheta_4}\sigma)\left(\vartheta_5, \vartheta_2\right)\right\}.
\end{aligned}
\tag{27}
$$

If we choose $\vartheta_5 = \xi$ in (27) and by using (15) , we get

$$
\begin{aligned}
&\left(F_1 + F_3\right)^2\phi\sigma\left(\vartheta_2, \vartheta_4\right) - \left(F_1 + F_3\right)(\tilde{\nabla}_{\vartheta_4}\sigma)\left(\xi, \vartheta_2\right)\\
&= -\lambda_2\left\{\left(F_1 + F_3\right)\phi\sigma\left(\vartheta_2, \vartheta_4\right) + (\tilde{\nabla}_{\vartheta_4}\sigma)\left(\xi, \vartheta_2\right)\right\}.
\end{aligned}
\tag{28}
$$

On the other hand, it is clear that

$$
(\tilde{\nabla}_{\vartheta_4}\sigma)\left(\xi, \vartheta_2\right) = -\left(F_1 + F_3\right)\phi\sigma\left(\vartheta_2, \vartheta_4\right).
\tag{29}
$$

If (29) is written instead of (28) , we obtain

$$
2\left(F_1 + F_3\right)^2\phi\sigma\left(\vartheta_2, \vartheta_4\right) = 0.
$$

This completes of the proof. □

Definition 3.6. *Let M be the invariant submanifold of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1, F_2, F_3)$. If $\tilde{R}.\sigma$ and $Q(S,\sigma)$ are linearly dependent, M is called Ricci generalized pseudoparallel submanifold [16].*

In this case, there is a function λ_3 on the set $M_3 = \{x \in M|\,\sigma(x) \neq S(x)\}$ such that

$$
\tilde{R}.\sigma = \lambda_3 Q(S,\sigma).
$$

Let us now examine the case of Ricci generalized pseudoparallel for the submanifold M of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1, F_2, F_3)$.

Theorem 3.7. *Let M be the invariant submanifold of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1, F_2, F_3)$. If M is Ricci generalized pseudoparallel submanifold, then M is either a totally geodesic or $\lambda_3 = \frac{1}{2n}$.*

Proof. Let's assume that M is a Ricci generalized pseudoparallel submanifold. So, we can write

$$\left(\tilde{R}(\vartheta_1, \vartheta_2)\sigma\right)(\vartheta_4, \vartheta_5) = \lambda_3 Q(S, \sigma)(\vartheta_4, \vartheta_5, \vartheta_1, \vartheta_2), \tag{30}$$

that is

$$R^{\perp}(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) - \sigma(R(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5)$$
$$-\sigma(\vartheta_4, R(\vartheta_1, \vartheta_2)\vartheta_5) = -\lambda_3\{\sigma((\vartheta_1 \wedge_S \vartheta_2)\vartheta_4, \vartheta_5)$$
$$+\sigma(\vartheta_4, (\vartheta_1 \wedge_S \vartheta_2)\vartheta_5)\},$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(TM)$. If we choose $\vartheta_1 = \vartheta_5 = \xi$ in from the last equality and make use of (8), (15), we get

$$[-(F_1 + F_3) + 2n(F_1 + F_3)\lambda_3]\sigma(\vartheta_4, \vartheta_2) = 0.$$

It is clear from the last equation that either

$$\sigma(\vartheta_4, \vartheta_2) = 0,$$

or

$$\lambda_3 = \frac{1}{2n}.$$

This completes the proof. □

Definition 3.8. *Let M be the invariant submanifold of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1, F_2, F_3)$. If $\tilde{R}.\tilde{\nabla}\sigma$ and $Q(S, \tilde{\nabla}\sigma)$ are linearly dependent, M is called $2-$Ricci generalized pseudoparallel submanifold.*

Then, there is a function λ_4 on the set $M_4 = \{x \in M | \tilde{\nabla}\sigma(x) \neq S(x)\}$ such that

$$\tilde{R}.\tilde{\nabla}\sigma = \lambda_4 Q(S, \tilde{\nabla}\sigma).$$

Let us now examine the case of $2-$Ricci generalized pseudoparallel for the submanifold M of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1, F_2, F_3)$.

Theorem 3.9. *Let M be the invariant submanifold of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1, F_2, F_3)$. If M is $2-$Ricci generalized pseudoparallel submanifold, then M is either a totally geodesic, $F_1 = F_3$, or $\lambda_4 = \frac{1}{2n}$.*

Proof. Let's assume that M is a $2-$Ricci generalized pseudoparallel submanifold. So, we can write

$$\left(\tilde{R}(\vartheta_1, \vartheta_2)\tilde{\nabla}\sigma\right)(\vartheta_4, \vartheta_5, \vartheta_3) = \lambda_4 Q(S, \tilde{\nabla}\sigma)(\vartheta_4, \vartheta_5, \vartheta_3, \vartheta_1, \vartheta_2), \tag{31}$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5, \vartheta_3 \in \Gamma(TM)$. If we choose $\vartheta_1 = \vartheta_5 = \xi$ in (31), we can write

$$\begin{aligned} & R^{\perp}(\xi, \vartheta_2)(\tilde{\nabla}_{\vartheta_4}\sigma)(\xi, \vartheta_3) - \left(\tilde{\nabla}_{R(\xi,\vartheta_2)\vartheta_4}\sigma\right)(\xi, \vartheta_3) \\ & -(\tilde{\nabla}_{\vartheta_4}\sigma)(R(\xi, \vartheta_2)\xi, \vartheta_3) - (\tilde{\nabla}_{\vartheta_4}\sigma)(\xi, R(\xi, \vartheta_2)\vartheta_3) \\ & = -\lambda_4\left\{\left(\tilde{\nabla}_{(\xi\wedge_S\vartheta_2)\vartheta_4}\sigma\right)(\xi, \vartheta_3) + (\tilde{\nabla}_{\vartheta_4}\sigma)((\xi \wedge_S \vartheta_2)\xi, \vartheta_3)\right. \\ & \left. + (\tilde{\nabla}_{\vartheta_4}\sigma)(\xi, (\xi \wedge_S \vartheta_2)\vartheta_3)\right\}, \end{aligned} \tag{32}$$

Let's calculate all the expressions in (32). Firstly, we can write

$$\begin{aligned} & R^{\perp}(\xi,\vartheta_2)(\tilde{\nabla}_{\vartheta_4}\sigma)(\xi,\vartheta_3) = R^{\perp}(\xi,\vartheta_2)\left\{\nabla^{\perp}_{\vartheta_4}\sigma(\xi,\vartheta_3)\right. \\ & -\sigma(\nabla_{\vartheta_4}\vartheta_3,\xi) - \sigma(\vartheta_3,\nabla_{\vartheta_4}\xi)\} \\ & = -R^{\perp}(\xi,\vartheta_2)(F_1+F_3)\phi\sigma(\vartheta_3,\vartheta_4), \end{aligned} \tag{33}$$

$$\begin{aligned} & \left(\tilde{\nabla}_{R(\xi,\vartheta_2)\vartheta_4}\sigma\right)(\xi,\vartheta_3) = \nabla^{\perp}_{R(\xi,\vartheta_2)\vartheta_4}\sigma(\xi,\vartheta_3) \\ & -\sigma\left(\nabla_{R(\xi,\vartheta_2)\vartheta_4}\xi,\vartheta_3\right) - \sigma\left(\xi,\nabla_{R(\xi,\vartheta_2)\vartheta_4}\vartheta_3\right) \\ & = -\sigma\left(\phi(F_1+F_3)R(\xi,\vartheta_2)\vartheta_4,\vartheta_3\right) \\ & = -(F_1+F_3)^2\phi\eta(\vartheta_4)\sigma(\vartheta_2,\vartheta_3), \end{aligned} \tag{34}$$

$$\begin{aligned} & (\tilde{\nabla}_{\vartheta_4}\sigma)(R(\xi,\vartheta_2)\xi,\vartheta_3) = (\tilde{\nabla}_{\vartheta_4}\sigma)((F_1+F_3)[-\eta(\vartheta_2)\xi+\vartheta_2],\vartheta_3) \\ & = -(F_1+F_3)(\tilde{\nabla}_{\vartheta_4}\sigma)(\eta(\vartheta_2)\xi,\vartheta_3) \\ & +(F_1+F_3)(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_2,\vartheta_3) \\ & = (F_1+F_3)^2\phi\eta(\vartheta_2)\sigma(\vartheta_4,\vartheta_3) \\ & +(F_1+F_3)(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_2,\vartheta_3), \end{aligned} \tag{35}$$

$$\begin{aligned} & (\tilde{\nabla}_{\vartheta_4}\sigma)(\xi,R(\xi,\vartheta_2)\vartheta_3) = \nabla^{\perp}_{\vartheta_4}\sigma(\xi,R(\xi,\vartheta_2)\vartheta_3) \\ & -\sigma(\nabla_{\vartheta_4}\xi,R(\xi,\vartheta_2)\vartheta_3) - \sigma(\xi,\nabla_{\vartheta_4}R(\xi,\vartheta_2)\vartheta_3) \\ & = -(F_1+F_3)\sigma\left(\phi\vartheta_4,[g(\vartheta_2,\vartheta_3)\xi\right. \\ & +\eta(\vartheta_3)\vartheta_2]) \\ & = -(F_1+F_3)^2\phi\eta(\vartheta_3)\sigma(\vartheta_4,\vartheta_2) \end{aligned} \tag{36}$$

$$\begin{aligned} & \left(\tilde{\nabla}_{(\xi\wedge_S\vartheta_2)\vartheta_4}\sigma\right)(\xi,\vartheta_3) = \nabla^{\perp}_{(\xi\wedge_S\vartheta_2)\vartheta_4}\sigma(\xi,\vartheta_3) \\ & -\sigma\left(\nabla_{(\xi\wedge_S\vartheta_2)\vartheta_4}\xi,\vartheta_3\right) - \sigma\left(\xi,\nabla_{(\xi\wedge_S\vartheta_2)\vartheta_4}\vartheta_3\right) \\ & = -S(\vartheta_2,\vartheta_4)\sigma(\nabla_{\xi}\xi,\vartheta_3) + S(\xi,\vartheta_4)\sigma(\nabla_{\vartheta_2}\xi,\vartheta_3) \\ & = -2n(F_1+F_3)^2\phi\eta(\vartheta_4)\sigma(\vartheta_2,\vartheta_3), \end{aligned} \tag{37}$$

$$
\begin{aligned}
&(\tilde{\nabla}_{\vartheta_4}\sigma)((\xi \wedge_S \vartheta_2)\xi, \vartheta_3) = (\tilde{\nabla}_{\vartheta_4}\sigma)(S(\vartheta_2, \xi)\xi - S(\xi, \xi)\vartheta_2, \vartheta_3) \\
&= (\tilde{\nabla}_{\vartheta_4}\sigma)(-2n(F_1 + F_3)\eta(\vartheta_2)\xi + 2n(F_1 + F_3)\vartheta_2, \vartheta_3) \\
&= 2n(F_1 + F_3)\Big\{-\nabla^{\perp}_{\vartheta_4}\sigma(\eta(\vartheta_2)\xi, \vartheta_3) + \sigma(\nabla_{\vartheta_4}\eta(\vartheta_2)\xi, \vartheta_3) \\
&+\sigma(\eta(\vartheta_2)\xi, \nabla_{\vartheta_4}\vartheta_3) + (\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_2, \vartheta_3)\} \\
&= 2n(F_1 + F_3)(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_2, \vartheta_3) \\
&+2n(F_1 + F_3)^2\phi\eta(\vartheta_2)\sigma(\vartheta_4, \vartheta_3),
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
&(\tilde{\nabla}_{\vartheta_4}\sigma)(\xi, (\xi \wedge_S \vartheta_2)\vartheta_3) = (\tilde{\nabla}_{\vartheta_4}\sigma)(\xi, S(\vartheta_2, \vartheta_3)\xi - S(\xi, \vartheta_3)\vartheta_2) \\
&= (\tilde{\nabla}_{\vartheta_4}\sigma)(\xi, S(\vartheta_2, \vartheta_3)\xi) + 2n(F_1 + F_3)(\tilde{\nabla}_{\vartheta_4}\sigma)(\xi, \eta(\vartheta_3)\vartheta_2) \\
&= -2n(F_1 + F_3)^2\phi\eta(\vartheta_3)\sigma(\vartheta_4, \vartheta_2).
\end{aligned}
\tag{39}
$$

If we substitute (33), (34), (35), (36), (37), (38), (39) in (32), we obtain

$$
\begin{aligned}
&-R^{\perp}(\xi, \vartheta_2)(F_1 + F_3)\phi\sigma(\vartheta_3, \vartheta_4) + (F_1 + F_3)^2\phi\eta(\vartheta_4)\sigma(\vartheta_2, \vartheta_3) \\
&-(F_1 + F_3)^2\phi\eta(\vartheta_2)\sigma(\vartheta_4, \vartheta_3) + (F_1 + F_3)^2\eta(\vartheta_3)\phi\sigma(\vartheta_4, \vartheta_2) \\
&-(F_1 + F_3)(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_2, \vartheta_3) = -\lambda_4\Big\{-2n(F_1 + F_3)^2\phi\eta(\vartheta_4)\sigma(\vartheta_2, \vartheta_3) \\
&+2n(F_1 + F_3)\phi\eta(\vartheta_2)\sigma(\vartheta_4, \vartheta_3) - 2n(F_1 + F_3)^2\phi\eta(\vartheta_3)\sigma(\vartheta_4, \vartheta_2) \\
&+2n(F_1 + F_3)(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_2, \vartheta_3)\}.
\end{aligned}
\tag{40}
$$

If we choose $\vartheta_3 = \xi$ in (40) and using (15), we get

$$
\begin{aligned}
&-(F_1 + F_3)(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_2, \xi) + (F_1 + F_3)^2\phi\sigma(\vartheta_4, \vartheta_2) \\
&= -\lambda_4\{2n(F_1 + F_3)(\tilde{\nabla}_{\vartheta_4}\sigma)(\vartheta_2, \xi) \\
&-2n(F_1 + F_3)^2\phi\sigma(\vartheta_4, \vartheta_2)\Big\}.
\end{aligned}
\tag{41}
$$

On the other hand, it is clear that

$$
(\tilde{\nabla}_{\vartheta_4}\sigma)(\xi, \vartheta_2) = -(F_1 + F_3)\phi\sigma(\vartheta_2, \vartheta_4). \tag{42}
$$

If (42) is written instead of (41), we obtain

$$
2(F_1 + F_3)^2(1 - 2n\lambda_4)\phi\sigma(\vartheta_2, \vartheta_4) = 0.
$$

It is clear from the last equality

$$
\sigma(\vartheta_2, \vartheta_4) = 0, F_1 = -F_3 \text{ or } \lambda_4 = \frac{1}{2n}.
$$

This completes the proof. □

4. Invariant Submanifolds of M_1^{2n+1} (F_1, F_2, F_3) On Some Special Curvature Tensors

In this section, the invariant submanifold M of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form M_1^{2n+1} will be considered with the concircular, projective, quasi-conformally curvature tensor.

The concircular curvature tensor is defined as

$$\tilde{Z}(\vartheta_1, \vartheta_2)\vartheta_3 = R(\vartheta_1, \vartheta_2)\vartheta_3 - \frac{r}{2n(2n+1)}\left[g(\vartheta_2, \vartheta_3)\vartheta_1 - g(\vartheta_1, \vartheta_3)\vartheta_2\right], \tag{43}$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \Gamma(M)$. If we choose $\vartheta_1 = \vartheta_3 = \xi$ in (43) and using (3), we get

$$\tilde{Z}(\xi, \vartheta_2)\xi = \left[(F_1 + F_3) - \frac{r}{2n(2n+1)}\right]\left[-\eta(\vartheta_2)\xi + \vartheta_2\right]. \tag{44}$$

Theorem 4.1. *Let M be the invariant submanifold of the* $(2n+1)-$*dimensional generalized Lorentz-Sasakian space form* $M_1^{2n+1}(F_1, F_2, F_3)$*. If* $\tilde{Z}.\sigma = \lambda_5 Q(g, \sigma)$*, then M is either totally geodesic or* $\lambda_5 = (F_1 + F_3) - \frac{r}{2n(2n+1)}$.

Proof. Let's assume that M satisfies the condition

$$\left(\tilde{Z}(\vartheta_1, \vartheta_2)\sigma\right)(\vartheta_4, \vartheta_5) = \lambda_5 Q(g, \sigma)(\vartheta_4, \vartheta_5, \vartheta_1, \vartheta_2), \tag{45}$$

that is

$$\tilde{Z}^{\perp}(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) - \sigma\left(\tilde{Z}(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5\right)$$
$$-\sigma\left(\vartheta_4, \tilde{Z}(\vartheta_1, \vartheta_2)\vartheta_5\right) = -\lambda_5\left\{\sigma\left(\left(\vartheta_1 \wedge_g \vartheta_2\right)\vartheta_4, \vartheta_5\right)\right.$$
$$\left.+\sigma\left(\vartheta_4, \left(\vartheta_1 \wedge_g \vartheta_2\right)\vartheta_5\right)\right\},$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(TM)$. If we choose $\vartheta_1 = \vartheta_5 = \xi$ in the last equality and using (15), we get

$$-\sigma\left(\vartheta_4, \tilde{Z}(\xi, \vartheta_2)\xi\right) = -\lambda_5\sigma(\vartheta_4, \vartheta_2). \tag{46}$$

If we use (44) out of (46), we obtain

$$\left[(F_1 + F_3) - \frac{r}{2n(2n+1)} - \lambda_5\right]\sigma(\vartheta_4, \vartheta_2) = 0.$$

This completes the proof. □

Theorem 4.2. *Let M be the invariant submanifold of the* $(2n+1)-$*dimensional generalized Lorentz-Sasakian space form* $M_1^{2n+1}(F_1, F_2, F_3)$*. If* $\tilde{Z}.\sigma = \lambda_6 Q(S, \sigma)$*, then M is either totally geodesicor* $\lambda_6 = \frac{2n(2n+1)(F_1+F_3)-r}{4n^2(2n+1)(F_1+F_3)}$ *and* $F_1 \neq -F_3$.

Proof. Let's assume that M satisfies the condition

$$\left(\tilde{Z}(\vartheta_1, \vartheta_2)\sigma\right)(\vartheta_4, \vartheta_5) = \lambda_6 Q(S, \sigma)(\vartheta_4, \vartheta_5, \vartheta_1, \vartheta_2), \tag{47}$$

that is

$$\tilde{Z}^{\perp}(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) - \sigma\left(\tilde{Z}(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5\right)$$
$$-\sigma\left(\vartheta_4, \tilde{Z}(\vartheta_1, \vartheta_2)\vartheta_5\right) = -\lambda_6\{\sigma((\vartheta_1 \wedge_S \vartheta_2)\vartheta_4, \vartheta_5)$$
$$+\sigma(\vartheta_4, (\vartheta_1 \wedge_S \vartheta_2)\vartheta_5)\},$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(TM)$. If we choose $\vartheta_1 = \vartheta_5 = \xi$ in the last equation and using (15), we get

$$-\sigma\left(\vartheta_4, \tilde{Z}(\xi, \vartheta_2)\xi\right) = \lambda_6 S(\xi, \xi)\sigma(\vartheta_4, \vartheta_2). \tag{48}$$

If we use (44) and (6) out of (48), we obtain

$$\left[-(F_1 + F_3) + \frac{r}{2n(2n+1)} + 2n(F_1 + F_3)\lambda_6\right]\sigma(\vartheta_4, \vartheta_2) = 0.$$

This completes the proof. □

The projective curvature tensor is defined as

$$P(\vartheta_1, \vartheta_2)\vartheta_3 = R(\vartheta_1, \vartheta_2)\vartheta_3 - \frac{1}{2n}[S(\vartheta_2, \vartheta_3)\vartheta_1 - S(\vartheta_1, \vartheta_3)\vartheta_2], \tag{49}$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \Gamma(M)$. If we choose $\vartheta_1 = \vartheta_3 = \xi$ in (49) and using (3), (6), we get

$$P(\xi, \vartheta_2)\xi = 0. \tag{50}$$

Theorem 4.3. *Let M be the invariant submanifold of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1, F_2, F_3)$. If $P.\sigma = \lambda_7 Q(g, \sigma)$, then M is either totally geodesic or $L_7 = 0$.*

Proof. Let's assume that M satisfies the condition

$$(P(\vartheta_1, \vartheta_2)\sigma)(\vartheta_4, \vartheta_5) = \lambda_7 Q(g, \sigma)(\vartheta_4, \vartheta_5, \vartheta_1, \vartheta_2),$$

that is

$$\begin{aligned} &P^{\perp}(\vartheta_1, \vartheta_2)\sigma(\vartheta_4, \vartheta_5) - \sigma(P(\vartheta_1, \vartheta_2)\vartheta_4, \vartheta_5) \\ &-\sigma(\vartheta_4, P(\vartheta_1, \vartheta_2)\vartheta_5) = -\lambda_7\left\{\sigma\left(\left(\vartheta_1 \wedge_g \vartheta_2\right)\vartheta_4, \vartheta_5\right)\right. \\ &\left.+\sigma\left(\vartheta_4, \left(\vartheta_1 \wedge_g \vartheta_2\right)\vartheta_5\right)\right\}, \end{aligned} \tag{51}$$

for all $\vartheta_1, \vartheta_2, \vartheta_4, \vartheta_5 \in \Gamma(TM)$. If we choose $\vartheta_1 = \vartheta_5 = \xi$ in (51) and using (15), (50), we get

$$-\lambda_7\sigma(\vartheta_4, \vartheta_2) = 0. \tag{52}$$

This completes the proof. □

In the same way, it can be easily seen that the following theorem is satisfied.

Theorem 4.4. *Let M be the invariant submanifold of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1, F_2, F_3)$. If $P.\sigma = \lambda_8 Q(S, \sigma)$, then M is either totally geodesic or $\lambda_8 = 0$ and $F_1 = -F_3$.*

The quasi-conformally curvature tensor is defined as

$$\begin{aligned} &\tilde{C}(\vartheta_1, \vartheta_2)\vartheta_3 = aR(\vartheta_1, \vartheta_2)\vartheta_3 + b[S(\vartheta_2, \vartheta_3)\vartheta_1 - S(\vartheta_1, \vartheta_3)\vartheta_2 \\ &+g(\vartheta_2, \vartheta_3)Q\vartheta_1 - g(\vartheta_1, \vartheta_3)Q\vartheta_2] - \tfrac{r}{2n+1}\left[\tfrac{a}{2n} + b\right] \\ &[g(\vartheta_2, \vartheta_3)\vartheta_1 - g(\vartheta_1, \vartheta_3)\vartheta_2], \end{aligned} \tag{53}$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \Gamma(M)$. If we choose $\vartheta_1 = \vartheta_3 = \xi$ in (53) and using (3), (6), (7), we get

$$\begin{aligned} &\tilde{C}(\xi, \vartheta_2)\xi = \{-(F_1 + F_3)(a + 4nb) + [(2n-1)F_3 - 3F_2]b \\ &-\tfrac{r}{2n+1}\left(\tfrac{a}{2n} + b\right)\}\eta(\vartheta_2)\xi + \{(F_1 + F_3)(a + 2nb) \\ &+[2nF_1 + 3F_2 + F_3]b - \tfrac{r}{2n+1}\left(\tfrac{a}{2n} + b\right)\}\vartheta_2. \end{aligned} \tag{54}$$

Theorem 4.5. *Let M be the invariant submanifold of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1,F_2,F_3)$. If $\tilde{C}.\sigma = \lambda_9 Q(g,\sigma)$, then M is either totally geodesic or $\lambda_9 \neq H_1$.*

Proof. Let's assume that M satisfies the condition

$$\left(\tilde{C}(\vartheta_1,\vartheta_2)\sigma\right)(\vartheta_4,\vartheta_5) = \lambda_9 Q(g,\sigma)(\vartheta_4,\vartheta_5,\vartheta_1,\vartheta_2),$$

that is

$$\begin{aligned} &\tilde{C}^{\perp}(\vartheta_1,\vartheta_2)\sigma(\vartheta_4,\vartheta_5) - \sigma\left(\tilde{C}(\vartheta_1,\vartheta_2)\vartheta_4,\vartheta_5\right) \\ &-\sigma\left(\vartheta_4,\tilde{C}(\vartheta_1,\vartheta_2)\vartheta_5\right) = -\lambda_9\left\{\sigma\left(\left(\vartheta_1\wedge_g\vartheta_2\right)\vartheta_4,\vartheta_5\right)\right. \\ &\left.+\sigma\left(\vartheta_4,\left(\vartheta_1\wedge_g\vartheta_2\right)\vartheta_5\right)\right\}. \end{aligned} \tag{55}$$

for all $\vartheta_1,\vartheta_2,\vartheta_4,\vartheta_5 \in \Gamma(TM)$. If we choose $\vartheta_1 = \vartheta_5 = \xi$ in (55) and using (15), we get

$$\sigma\left(\vartheta_4,\tilde{C}(\xi,\vartheta_2)\xi\right) = \lambda_9\sigma(\vartheta_4,\vartheta_2). \tag{56}$$

If we use (54) out of (56), we obtain

$$[\lambda_9 - H_1]\sigma(\vartheta_4,\vartheta_2) = 0,$$

where

$$H_1 = (F_1+F_3)(a+2nb) + [2nF_1 + 3F_2 + F_3]b - \frac{r}{2n+1}\left(\frac{a}{2n}+b\right).$$

This completes the proof. □

Theorem 4.6. *Let M be the invariant submanifold of the $(2n+1)-$dimensional generalized Lorentz-Sasakian space form $M_1^{2n+1}(F_1,F_2,F_3)$. If $\tilde{C}.\sigma = \lambda_{10} Q(S,\sigma)$, then M is either totally geodesicor $\lambda_{10} = \frac{H_2}{2n(F_1+F_3)}$ and $F_1 \neq -F_3$.*

Proof. Let's assume that M satisfies the condition

$$\left(\tilde{C}(\vartheta_1,\vartheta_2)\sigma\right)(\vartheta_4,\vartheta_5) = \lambda_{10} Q(S,\sigma)(\vartheta_4,\vartheta_5,\vartheta_1,\vartheta_2),$$

that is

$$\begin{aligned} &\tilde{C}^{\perp}(\vartheta_1,\vartheta_2)\sigma(\vartheta_4,\vartheta_5) - \sigma\left(\tilde{C}(\vartheta_1,\vartheta_2)\vartheta_4,\vartheta_5\right) \\ &-\sigma\left(\vartheta_4,\tilde{C}(\vartheta_1,\vartheta_2)\vartheta_5\right) = -\lambda_{10}\{\sigma((\vartheta_1\wedge_S\vartheta_2)\vartheta_4,\vartheta_5) \\ &+\sigma(\vartheta_4,(\vartheta_1\wedge_S\vartheta_2)\vartheta_5)\}, \end{aligned} \tag{57}$$

for all $\vartheta_1,\vartheta_2,\vartheta_4,\vartheta_5 \in \Gamma(TM)$. If we choose $\vartheta_1 = \vartheta_5 = \xi$ in (57) and using (15), we get

$$-\sigma\left(\vartheta_4,\tilde{C}(\xi,\vartheta_2)\xi\right) = \lambda_{10} S(\xi,\xi)\sigma(\vartheta_4,\vartheta_2). \tag{58}$$

If we use (54) and (6) out of (58), we obtain

$$[2n(F_1+F_3)\lambda_{10} - H_2]\sigma(\vartheta_4,\vartheta_2) = 0,$$

where

$$H_2 = -(F_1+F_3)(a+4nb) + [(2n-1)F_3 - 3F_2]b - \frac{r}{2n+1}\left(\frac{a}{2n}+b\right).$$

This completes the proof. □

Example 4.7. *Let us consider on* $\mathbb{R}^7$ *the following normal Lorentzian Sasakian structure* (φ, ξ, η, g), *given by*

$$\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{3} y_i dx_i\right), \xi = \frac{\partial}{\partial z},$$

$$g = -\eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{3}\left(dx_i \otimes dx_i + dy_i \otimes dy_i\right),$$

$$\varphi\left(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i} + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^{3} Y_i \frac{\partial}{\partial x_i} - X_i \frac{\partial}{\partial y_i} + \sum_{i=1}^{3} Y_i y_i \frac{\partial}{\partial z},$$

where $\{x_1, y_1, x_2, y_2, x_3, y_3, z\}$ *are the chartezian coordinates of* $\mathbb{R}^7$.
Now we construct the of subbasis

$$e_1 = 2\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} + y_1 \frac{\partial}{\partial z}\right),$$

$$e_2 = 2\left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_3}\right),$$

$$e_3 = 2\left(\frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_4}\right),$$

$$e_4 = 2\left(-\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_4} + 2\frac{\partial}{\partial z}\right),$$

$$e_5 = \xi = 2\frac{\partial}{\partial z}.$$

These vwctor fields are integrable and involutive. One can easily to see that its integral manifold 5*-dimensional an invariant submanifold of the usual Lorentzian Sasakian manifold* $\mathbb{R}^7\left(\varphi, \xi, \eta, g\right)$.

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