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On ψ_g–interpolative Hardy–Rogers type contractions over rectangular quasi–partial b–metric space with an application

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Abstract. In the present study, the concept of a new type of contraction namely ψ_g–interpolative Hardy–Rogers contraction is introduced which is a unification of g-interpolation and Hardy-Rogers contraction. By utilizing this concept, unique results of the existence of fixed points in the extent of rectangular quasi–partial b–metric space are proved. The validity of the obtained results is verified with the help of comparative examples with vivid representations. The existence of a solution to the Fredholm integral equation is also provided here via a fixed point for such mappings.

1. Introduction

The theory of fixed point first emerged in the solution of differential equations in 1837 when Liouville[28] solved such equations by applying successive approximation. Later on, in the year 1890, Picard[27] introduced the applicability of the acclaimed method and developed solutions of corresponding differential equations. In 1906, Frechet[11] defined the metric space by observing the notion of distance between the points and their images. After that, Banach[6] in 1922, derived the most prominent technique to prove a fixed point theorem in complete metric space. This theorem is characterized by the Banach Contraction Principle which evidenced an imperative role in nonlinear functional analysis. Numerous generalizations were given by means of different types of contractive mappings such as Chatterjea[9], Bhaktin[7], and Czerwik[8]. Among these generalizations, one of the propitious theorems was given by Kannan[15] in 1968, in which the continuity condition was removed from the contraction map to obtain a fixed point. i.e.,

Theorem 1.1 ([15]). *Let (X, d) be a complete metric space and a self map $T : X \to X$ be a Kannan contraction mapping. i.e.,*

$$d(T\sigma, T\eta) \leq \varrho[d(\sigma, T\sigma) + d(\eta, T\eta)]$$

for all $\sigma, \eta \in X$, where $\varrho \in [0, \frac{1}{2})$. Then T admits a unique fixed point in X.

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Karapınar gave the definition of interpolative contraction to investigate the metric fixed point theory in 2018.i.e.

Theorem 1.2 ([17]). *Let (X, d) be a metric space, a self-mapping $T\colon X \to X$ is said to be an interpolative Kannan-type contraction map if there exists a constant $\varrho \in (0,1)$ and $\alpha \in (0,1)$ such that*

$$d(T\sigma, T\eta) \leq \varrho[d(\sigma, T\sigma)]^{\alpha} \cdot [d(\eta, T\eta)]^{1-\alpha}$$

for all $\sigma, \eta \in X \setminus \ Fix(T)$, where $Fix(T) = \{z \in X\colon Tz = z\}$.

Along the line, one more result was established by Karapinar[19] on Hardy–Rogers contraction map.i.e.,

Theorem 1.3 ([19]). *Let (X, d) be a metric space. If the self-mapping $T : X \to X$ is an interpolative Hardy–Rogers type contraction i.e., there exist $\rho \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, such that*

$$\begin{aligned} d(T\sigma, T\eta) &\leq \rho[d(\sigma, \eta)]^{\beta}[d(\sigma, T\sigma)]^{\alpha} \cdot d(\eta, T\eta)]^{\gamma} \\ &\quad \cdot \left[\frac{1}{2}(d(\sigma, T\eta) + d(\eta, T\sigma))\right]^{1-\alpha-\beta-\gamma} \qquad \text{for all } \sigma, \eta \in X \setminus Fix(T), \end{aligned}$$

then T has a fixed point in X.

A new refinement was done in 2019 by Gaba et.al.[14] as the inequality in Theorem 1.2 increases the degree of freedom of the powers appearing on the right-hand side in the framework of standard metric spaces. It is observed that the aforementioned result satisfies the contractive condition for all $\sigma, \eta \in X$ with $\sigma \neq T\sigma$. In this case, if T has a fixed point in X then it will be a constant map, and therefore T has a unique fixed point trivially. To remove such triviality, the contraction condition with $\sigma, \eta \in X \setminus Fix(T)$, where Fix(T) is the set of all fixed points of T is preassumed which leads to obtain non-unique fixed points and possesses more than one fixed point. Motivated by interpolation theory Debnath et.al.[10] and Aydi et.al.[5] presented results on set valued interpolative maps and ω–interpolative maps. One of the interesting results appears when Karapinar[18] indicated the gap in the proof of the uniqueness in Theorem 1.2. In the last decade, the notion of multivalued interpolative contractions via an auxiliary function called as simulation function came into existence. See [20, 22, 24] which given numerous applications in data dependence and homotopy.

In 2000, the concept of rectangular metric space was introduced by Branciari [1] in which quadrilateral inequality is used. Suzuki [3] in his research discovered that the comparison of topological properties of the standard metric space and the rectangular metric space is not feasible. The generalized results in b-metric space as graphical b–metric space provided by Younis[32] and b–Branciari by Samani[29, 30] are the prominent results in fixed point theory. Recently, Karapinar[21, 23] established some new fixed point theorems for Meir-keeler modified versions in the context of interpolative theory. Subsequently, other interesting versions of the Banach are presented by [2, 13, 25, 26, 31].

Throughout this paper, we have denoted $\mathbb{N}$ the set of all positive integers, and rqp_b denotes the rectangular quasi–partial b–metric space.

2. Preliminaries

In this section, we present the basic definitions and results that are required to obtain the main results.

Definition 2.1 ([1]). *A rectangular metric on a non-empty set X is a function $r\colon X \times X \to \mathbb{R}^+$ such that for all $\sigma, \upsilon \in X$ and $u, v \in X$:*

1. $r(\sigma, \eta) = 0$ *iff* $\sigma = \eta$ *(identification),*
2. $r(\sigma, \eta) = r(\eta, \sigma)$ *(symmetry),*
3. $r(\sigma, \eta) \leq r(\sigma, u) + r(u, v) + r(v, \eta)$ *(quadrilateral inequality).*

(X, r) is called a rectangular metric space.

Definition 2.2 ([1]). *Let (X, r) be a rectangular metric. Then*

1. *A sequence $\{\sigma_n\} \subset X$ converges to $\sigma \in M$ if $r(\sigma, \sigma) = \lim_{n\to\infty} r(\sigma, \sigma_n)$.*
2. *A sequence $\{\sigma_n\} \subset X$ is called a Cauchy sequence if for every $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon)$ such that $r(\sigma_n, \sigma_m) < \varepsilon$ for all $n, m > N$.*
3. *(X, r) is said to be a complete rectangular metric if each Cauchy sequence in X is convergent.*

Example 2.3. *Let $X = R^+$. For the metric, $r\colon X\times X \to \mathbb{R}^+$ defined by $r(\sigma, \eta) = (\sigma - \eta)^2$. Then, (X,r) is a complete rectangular metric space.*

Definition 2.4 ([12]). *A rectangular quasi-partial b-metric on a non-empty set X is a function $rqp_b\colon\ X\times X \to\ \mathbb{R}^+$ such that for some real number $s \geq 1$ and all $\sigma, \eta, u, v \in X$:*

1. $rqp_b(\sigma, \sigma) = rqp_b(\sigma, \eta) = rqp_b(v, \eta) \Rightarrow \sigma = \eta$,
2. $rqp_b(\sigma, \sigma) \leq rqp_b(\sigma, \eta)$,
3. $rqp_b(\sigma, \sigma) \leq rqp_b(\eta, \sigma)$,
4. $rqp_b(\sigma, \eta) \leq s[rqp_b(\sigma, u) + rqp_b(u, v) + rqp_b(v, \eta)] - rqp_b(u, u) - rqp_b(v, v)$.

(X, rqp_b) is called a rectangular quasi-partial b-metric space. The number s is called the coefficient of (X, rqp_b).

Example 2.5. *Let $X = \left[0, \frac{\pi}{4k}\right]$ equipped with the metric $rqp_b(\sigma, \eta) = \sin k|\sigma - \eta| + \sigma$ for any $(\sigma, \eta) \in X \times X$ and $k \geq 2$. It is easy to verify that (X, rqp_b) is a rectangular quasi-partial b-metric space. It has been observed that if $rqp_b(\sigma, \sigma) = rqp_b(\sigma, \eta) = rqp_b(\eta, \eta)$, that is,*
$\sigma = \sin k|\sigma - \eta| + \sigma = \eta$, then (1) holds trivially for any $(\sigma, \eta) \in X \times X$.
Furthermore, using the property of the sine function:
$\sin k|\sigma - \eta| \geq 0$ and $\sin k|\sigma - \eta| \geq |\sigma - \eta|$ when $|\sigma - \eta| \in \left[0, \frac{\pi}{4k}\right]$, then
$rqp_b(\sigma, \sigma) = \sigma \leq \sin k|\sigma - \eta| + \sigma = rqp_b(\sigma, \eta)$.
We have,

$$\begin{aligned} rqp_b(\sigma, \sigma) &= \sigma \\ &= |\sigma - \eta + \eta| \\ &\leq |\sigma - \eta| + |\eta| \\ &\leq \sin k|\eta - \sigma| + \eta \\ &\leq rqp_b(\eta, \sigma) \end{aligned}$$

Moreover, for any $\sigma, \eta, u, v \in X, |\sigma - u| \leq \frac{\pi}{4k} \leq \frac{\pi}{2k}$ and $[|\sigma - u| + |u - v| + |v - \eta|] \leq \frac{\pi}{2k}$ when $k(|\sigma - u| + |u - v| + |v - \eta|) \in \left[0, \frac{\pi}{2k}\right]$, or
$k(|\sigma - \delta| + |\delta - \sigma|) \leq \frac{\pi}{2}$, and since $\sin \sigma$ is increasing on $\left[0, \frac{\pi}{2}\right]$, we get the (4) of Definition 2.4 :

$$\begin{aligned} rqp_b(\sigma, \eta) + rqp_b(u, u) + rqp_b(v, v) &= \sin k|\sigma - \eta| + \sigma + u + v \\ &\leq \sin k(|\sigma - u| + |u - v| + |v - \eta|) + \sigma + u + v \\ &\leq k(|\sigma - u| + |u - v| + |v - \eta|) + \sigma + u + v \\ &\leq k \sin k|\sigma - u| + k \sin k|u - v| + k \sin k|v - \eta| + \sigma + u + v \\ &= k(\sin k|\sigma - u| + \sin k|u - v| + \sin k|v - \eta| + \sigma + u + v) \\ &\leq s(rqp_b(\sigma, u) + rqp_b(u, v) + rqp_b(v, \eta)) \text{ for all } \sigma, \eta, u, v \in X \end{aligned}$$

and $s \geq k$, hence (X, rqp_b) is a rectangular quasi-partial b-metric space with $s \geq k$ as shown in figure 1.

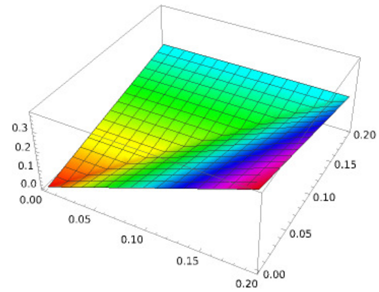

Figure 1: The shaded region demonstrates the visualisation of the function $rqp_b(\sigma, \eta) = \sin k|\sigma - \eta| + \sigma$ in $X = \left[0, \dfrac{\pi}{4k}\right]$.

Definition 2.6 ([12]). *Let (X, rqp_b) be a rectangular quasi-partial b-metric space. Then, the following hold:*

1. *If $rqp_b\ (\sigma, \eta) = 0$, then $\sigma = \eta$.*
2. *If $\sigma = \eta$, then $rqp_b\ (\sigma, \eta) > 0$ and $rqp_b\ (\eta, \sigma) > 0$.*

Definition 2.7. *Let (X, rqp_b) be a rectangular quasi-partial b-metric space. Then for $x_0 \in X$, $\epsilon > 0$, the $rqp_b -$ ball with centre x_0 and radius ϵ is defined as:*

$$B_{rqp_b}(x_0, \epsilon) = \{y \in X : rqp_b(x_0, y) < \epsilon, rqp_b(y, x_0) < \epsilon\}.$$

Definition 2.8 ([12]). *Let (X, rqp_b) be a rectangular quasi-partial b-metric. Then:*

1. *A sequence $\{\sigma_n\} \subset X$ converges to $\sigma \in X$ if and only if $rqp_b(\sigma, \sigma) = \lim_{n\to\infty} rqp_b(\sigma, \sigma_n)$.*
2. *A sequence $\{\sigma_n\} \subset X$ is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} rqp_b(\sigma_n, \sigma_m)$ exists.*
3. *A rectangular quasi-partial b-metric space (X, rqp_b) is said to be complete if every Cauchy sequence $\{\sigma_n\} \subset X$ converges with respect to τ_{rqp_b} to a point $\sigma \in X$ such that*

$$rqp_b(\sigma, \sigma) = \lim_{n,m\to\infty} rqp_b(\sigma_n, \sigma_m).$$

4. *A mapping $f : X \to X$ is said to be continuous at $\sigma_0 \in X$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$f(B(\sigma_0, \delta)) \subset B(f(\sigma_0), \varepsilon).$$

Definition 2.9. *Let Ψ be denoted as the set of all non-decreasing function $\psi\colon [0, \infty] \to [0, \infty]$ such that $\sum_{k=0}^{\infty} \psi^k(t) < \infty$ for each $t > 0$. Then :*

1. *$\psi(0) = 0$,*
2. *$\psi(t) < t$ for each $t > 0$.*

Definition 2.10. *Let σ_n be a sequence in a (X, rqp_b). Consider $g, h\colon X \to X$ are self mappings and $\sigma \in X$. σ is said to be the coincidence point of pair g, h if $g\sigma = h\sigma$.*

3. *Main Results*

In this section, the concept of Hardy-Rogers type interpolative contraction in rectangular quasi-partial b-metric space is discussed. Here Φ denotes the set of functions $\phi\colon [0, \infty) \to [0, \infty)$ such that $\phi(t) < t$ for every $t > 0$.

Theorem 3.1. *Let (X, rqp_b) be a complete rectangular quasi-partial b- metric space and T be a self–mapping on X such that*

$$rqp_b(T\sigma, T\eta) \leqslant \phi([rqp_b(\sigma,\eta)]^{\alpha} . [rqp_b(\sigma, T\,\sigma)]^{\beta} . [rqp_b(\eta, T\,\eta)]^{\gamma} . \frac{1}{3s}\left[rqp_b(\sigma, T\,\eta) + rqp_b(\eta, T\,\sigma)\right]^{\delta} \tag{1}$$

is satisfied for all $\sigma, \eta \in X \setminus Fix(T)$: where $Fix(T) = \{a \in X | Ta = a\}, \alpha, \beta, \gamma, \delta \in (0, 1)$ such that $\alpha + \beta + \gamma + \delta > 1$ and $\phi \in \Phi$.

If there exists $\sigma \in X$ such that $rqp_b(\sigma, T\sigma) < 1$, then T has a fixed point in X.

Proof. Let $\sigma_0 \in X$ be arbitrary. Define a sequence σ_n by $\sigma_{n+1} = T\sigma_n$ for all integers n, and we assume that $\sigma_n \neq T\sigma_n$, for all n. From equation 1, we have

$$rqp_b(T\sigma_{n-1}, T\sigma_n) \leqslant \phi([rqp_b(\sigma_{n-1}, \sigma_n)]^{\alpha} . [rqp_b(\sigma_{n-1}, T\sigma_{n-1})]^{\beta} . [rqp_b(\sigma_n, T\sigma_n)]^{\gamma} . \frac{1}{3s}\left[rqp_b(\sigma_{n-1}, T\sigma_n) + rqp_b(\sigma_n, T\sigma_{n-1})\right]^{\delta} \tag{2}$$

$$rqp_b(\sigma_n, \sigma_{n+1}) \leqslant \phi([rqp_b(\sigma_{n-1}, \sigma_n)]^{\alpha} . [rqp_b(\sigma_{n-1}, \sigma_n)]^{\beta} . [rqp_b(\sigma_n, \sigma_{n+1})]^{\gamma} . \frac{1}{3s}\left[rqp_b(\sigma_{n-1}, \sigma_{n+1}) + rqp_b(\sigma_n, \sigma_n)\right]^{\delta} . \tag{3}$$

Since $\phi(t) < t$ for each t, equation 3 yields,

$$rqp_b(\sigma_n, \sigma_{n+1}) < ([rqp_b(\sigma_{n-1}, \sigma_n)]^{\alpha} . [rqp_b(\sigma_{n-1}, \sigma_n)]^{\beta} . [rqp_b(\sigma_n, \sigma_{n+1})]^{\gamma} . \frac{1}{3s}\left[rqp_b(\sigma_n, \sigma_{n+1}) + rqp_b(\sigma_n, \sigma_n)\right]^{\delta}$$

$$rqp_b(\sigma_n, \sigma_{n+1}) < ([rqp_b(\sigma_{n-1}, \sigma_n)]^{\alpha} . [rqp_b(\sigma_{n-1}, \sigma_n)]^{\beta} . [rqp_b(\sigma_n, \sigma_{n+1})]^{\gamma} . \frac{1}{3s}\left[rqp_b(\sigma_n, \sigma_{n+1}) + rqp_b(\sigma_n, \sigma_{n-1})\right]^{\delta},$$

$$rqp_b(\sigma_n, \sigma_{n+1}) < ([rqp_b(\sigma_{n-1}, \sigma_n)]^{\alpha} . [rqp_b(\sigma_{n-1}, \sigma_n)]^{\beta} . [rqp_b(\sigma_n, \sigma_{n+1})]^{\gamma} . \left[rqp_b(\sigma_n, \sigma_{n+1})\right]^{\delta} . \tag{4}$$

Suppose that $rqp_b(\sigma_n, \sigma_{n-1}) < rqp_b(\sigma_n, \sigma_{n+1})$

$$\frac{1}{3s}(rqp_b(\sigma_{n+2}, \sigma_{n+1}) + rqp_b(\sigma_{n+1}, \sigma_n) + (rqp_b(\sigma_n, \sigma_{n-1})) \leq rqp_b(\sigma_n, \sigma_{n+1}).$$

or

$$[rqp_b(\sigma_{n+1}, \sigma_n)] \leq \psi[rqp_b(\sigma_n, \sigma_{n-1})]$$

Therefore, we obtain $[rqp_b(\sigma_{n+1}, \sigma_n)] \leq [rqp_b(\sigma_n, \sigma_{n-1})]$, which is a contradiction. Thus, we have

$$\frac{1}{3s}(rqp_b(\sigma_{n+2}, \sigma_{n+1}) + rqp_b(\sigma_{n+1}, \sigma_n) + (rqp_b(\sigma_n, \sigma_{n-1})) \leq rqp_b(\sigma_{n-1}, \sigma_n).$$

$$[rqp_b(\sigma_n, \sigma_{n+1})]^{1-\gamma-\delta} < [rqp_b(\sigma_{n-1}, \sigma_n)]^{\alpha+\beta} .$$

Now using the fact that $rqp_b(\sigma_0, \sigma_1) < 1$, so there exists a real $\lambda \in (0, 1)$ such that $rqp_b(\sigma_0, \sigma_1) \leqslant \lambda$ and $\lambda = \frac{rqp_b(\sigma_0,\sigma_1)+1}{2}$

$$rqp_b(\sigma_1, \sigma_2)([rqp_b(\sigma_0, \sigma_1)]^{\frac{\alpha+\beta}{1-\gamma-\delta}} \leqslant \lambda^{\frac{\alpha+\beta}{1-\gamma-\delta}} .$$

By taking $\epsilon = \frac{\alpha+\beta}{1-\gamma-\delta}$ for all n,

$$rqp_b(\sigma_{n+1}, \sigma_n) \leqslant rqp_b(\sigma_n, \sigma_{n-1})^{1+\epsilon}$$

$$rqp_b(\sigma_{n+1}, \sigma_n) \leqslant \lambda^{(1+\epsilon)^n},$$

where $0 < \lambda < 1$ for $n = 1$, this is the inequality at the bottom. By induction

$$rqp_b(\sigma_{n+2}, \sigma_{n+1}) \leqslant rqp_b(\sigma_{n+1}, \sigma_n)^{1+\epsilon} \leqslant (\lambda^{(1+\epsilon)^n})^{1+\epsilon} = \lambda^{(1+\epsilon)(n+1)}$$

Since $(1+\epsilon)^n \geqslant 1 + n\epsilon$ and since $\lambda < 1$.

$$rqp_b(\sigma_{n+1}, \sigma_n) \leqslant \lambda^{1+n\epsilon} = \lambda^n$$

for all n, where $p = \lambda^\epsilon < 1$. This implies :

$$\begin{aligned} rqp_b(\sigma_{n+k}, \sigma_n) &\leqslant \lambda(e^{n+k-1} + e^{n+k-2} + \cdots + e^{'}) \\ &= \lambda e^n(\frac{1-e^k}{1-e}) = cp^n, \end{aligned}$$

where $c = \lambda e^n(\frac{1-e^k}{1-e})$ for some integer k, from which it follows that σ_n forms a Cauchy sequence in (X, rqp_b) and then it converge to some $z \in X$. Assume that $z \neq Tz$.

By letting $\sigma = \sigma_n$ and $\eta = z$, we obtain for all n, which leads to $rqp_b(z, Tz) = 0$. Then $Tz = z$. Thus T has a fixed point in X. □

In our next result, we will prove the existence of the fixed point using ψ_g–interpolative Hardy–Rogers type contraction in the framework of rqp_b space.

Definition 3.2. *Let (X, rqp_b, s) be a rectangular quasi partial b–metric space and $T, g\colon X \longrightarrow X$ be a self–mappings on X. We say that T is ψ_g–interpolative Hardy Rogers type contraction if there exists a continuous $\psi \in \Psi$ and $\alpha, \beta, \gamma \in (0, 1)$ such that*

$$\begin{aligned} rqp_b(T\sigma, T\eta) \leqslant \psi(&[rqp_b(g\sigma, g\eta)]^\alpha \,.\, [rqp_b(g\sigma, T\sigma)]^\beta \,.\, [rqp_b(g\eta, T\eta)]^\gamma \,. \\ &\frac{1}{3s}\left[rqp_b(g\sigma, T\eta) + rqp_b(g\eta, T\sigma)\right]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{5}$$

is satisfied for all $\sigma, \eta \in X$ such that $T\sigma \neq g\sigma, T\eta \neq gy$ and $g\sigma \neq g\eta$.

Theorem 3.3. *Let (X, rqp_b, s) be a complete rectangular quasi–partial b–metric space and T is a g–interpolative Hardy-Rogers type contraction. Suppose that $T\sigma \subseteq g\sigma$ such that $g\sigma$ is closed. Then, T and g have a coincidence point in X.*

Proof. Let $\sigma_0 \in X$, since $T\sigma \subseteq g\sigma$, we can define inductively a sequence σ_n such that $\sigma_0 = \sigma$, and $g\sigma_{n+1} = T\sigma_n$ for all integer n. If there exist $n \in 0, 1, 2, 3, ..$ such that $g\sigma_n = Tx_n$ then σ_n is a coincidence point of g and T.

Assume that $g\sigma_n \neq T\sigma_n$ for all n. By equation 5, we obtain

$$\begin{aligned} rqp_b(T\sigma_{n+1}, T\sigma_n) &\leqslant \psi([rqp_b(g\sigma_{n+1}, g\sigma_n)]^{\alpha} . [rqp_b(g\sigma_{n+1}, T\sigma_{n+1})]^{\beta} . [rqp_b(g\sigma_n, T\sigma_n)]^{\gamma} . \\ &\quad \frac{1}{3s}\left[rqp_b(g\sigma_{n+1}, T\sigma_n) + rqp_b(g\sigma_n, T\sigma_{n+1})\right]^{1-\alpha-\beta-\gamma} \\ &\leqslant \psi([rqp_b(T\sigma_n, T\sigma_{n-1})]^{\alpha} . [rqp_b(T\sigma_n, T\sigma_{n+1})]^{\beta} . [rqp_b(T\sigma_{n-1}, T\sigma_n)]^{\gamma} . \\ &\quad \frac{1}{3s}\left[rqp_b(T\sigma_n, T\sigma_n) + rqp_b(T\sigma_{n-1}, T\sigma_{n+1})\right]^{1-\alpha-\beta-\gamma} \\ &\leqslant \psi(([rqp_b(T\sigma_n, T\sigma_{n-1})]^{\alpha} . [rqp_b(T\sigma_{n-1}, T\sigma_n)]^{\gamma} . [rqp_b(T\sigma_n, T\sigma_{n+1})]^{\beta} . \\ &\quad \frac{1}{3s}\left[srqp_b(\sigma_{n+1}, \sigma_{n+2}) + srqp_b(\sigma_{n+2}, \sigma_n) + srqp_b(\sigma_n, \sigma_{n-1})\right]^{1-\alpha-\beta-\gamma}) \\ &\leqslant \psi(([rqp_b(T\sigma_n, T\sigma_{n-1})]^{\alpha} . [rqp_b(T\sigma_{n-1}, T\sigma_n)]^{\gamma} . [rqp_b(T\sigma_n, T\sigma_{n+1})]^{\beta} . \\ &\quad [rqp_b(T\sigma_n, T\sigma_{n-1})]^{1-\alpha-\beta-\gamma}). \end{aligned} \tag{6}$$

Using the fact $\psi(t) < t$ for each $t > 0$,

$$\begin{aligned} rqp_b(T\sigma_{n+1}, T\sigma_n) &\leqslant \psi([rqp_b(T\sigma_n, T\sigma_{n-1})]^{\alpha} . [rqp_b(T\sigma_{n-1}, T\sigma_n)]^{\gamma} . \\ &\quad [rqp_b(T\sigma_n, T\sigma_{n+1})]^{\beta} . [rqp_b(T\sigma_n, T\sigma_{n-1})]^{1-\alpha-\beta-\gamma}). \end{aligned}$$

By equation 6, we have

$$\begin{aligned} &\leqslant ([rqp_b(T\sigma_n, T\sigma_{n-1})]^{\alpha} . [rqp_b(T\sigma_{n-1}, T\sigma_n)]^{\gamma} . [rqp_b(T\sigma_n, T\sigma_{n+1})]^{\beta} . \\ &\quad [rqp_b(T\sigma_n, T\sigma_{n+1})]^{\beta} . [rqp_b(T\sigma_n, T\sigma_{n-1})]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{7}$$

$$\begin{aligned} &[rqp_b(T\sigma_{n+1}, T\sigma_n)]^{1-\beta} \leqslant [rqp_b(T\sigma_n, T\sigma_{n-1})]^{1-\beta} . \\ &\quad rqp_b(T\sigma_{n+1}, T\sigma_n) \leqslant rqp_b(T\sigma_n, T\sigma_{n-1}) \text{ for all } n \geqslant 1. \end{aligned}$$

That is, the positive sequence $\{rqp_b(T\sigma_{n+1}, T\sigma_n)\}$ is monotone decreasing and consequently, there exists $c \geqslant 0$ such that $\lim_{n\to\infty} rqp_b(T\sigma_{n+1}, T\sigma_n) = c$

$$\begin{aligned} [rqp_b(T\sigma_n, T\sigma_{n-1})]^{1-\beta}\,[rqp_b(T\sigma_n, T\sigma_{n+1})]^{\beta} &\leqslant [rqp_b(T\sigma_n, T\sigma_{n-1})]^{1-\beta}\,[rqp_b(T\sigma_n, T\sigma_{n-1})]^{\beta} \\ &= rqp_b(T\sigma_n, T\sigma_{n-1}). \end{aligned}$$

Therefore, with the equation together with the non-decreasing character of ψ, we get

$$\begin{aligned} rqp_b(T\sigma_{n+1}, T\sigma_n) &\leqslant \psi\,[rqp_b(T\sigma_n, T\sigma_{n-1})]^{1-\beta} . [rqp_b(T\sigma_n, T\sigma_{n+1})]^{\beta} \\ &\leqslant \psi\,[rqp_b(T\sigma_n, T\sigma_{n-1})] \end{aligned}$$

By repeating this arrangement, we get

$$\begin{aligned} rqp_b(T\sigma_{n+1}, T\sigma_n) &\leqslant \psi\,[rqp_b(T\sigma_n, T\sigma_{n-1}] \leqslant \psi^2\,[rqp_b(T\sigma_n, T\sigma_{n-1}] \\ &\leqslant \cdots \leqslant \psi^n\,[rqp_b(T\sigma_1, T\sigma_0]\,. \end{aligned}$$

Taking $n \to \infty$ in equations and using the fact $\lim_{n\to\infty} \psi^n(t) = 0$ for each $t > 0$, we deduce that $c = 0$. That is,

$$\lim_{n\to\infty} rqp_b(T\sigma_{n+1}, T\sigma_n) = 0, \tag{8}$$

We want to show that, $T\sigma_n$ is a cauchy sequence. Suppose on the contrary that there exists an $\epsilon > 0$ and subsequence $\{T\sigma_{m_k}\}$ and $\{T\sigma_{n_k}\}$ of $\{T\sigma_n\}$ such that n_k is the smallest integers for which :

$$n_k > m_k > k, rqp_b(T\sigma_{n_k}, T\sigma_{n_k}) \geqslant \text{ and } rqp_b(T\sigma_{n_{k-1}}, T\sigma_{m_k}) < \epsilon$$

Consequently, we arrive

$$\begin{aligned} rqp_b(g\sigma_{n_k}, g\sigma_{m_k}) =& rqp_b(T\sigma_{n_k-1}, T\sigma_{m_k-1}) \\ &\leqslant srqp_b(T\sigma_{n_k-1}, T\sigma_{m_k}) + rqp_b(T\sigma_{m_k}, T\sigma_{m_k-1}) \\ &\leqslant s\epsilon + srqp_b(T\sigma_{m_k}, T\sigma_{m_k-1}) \end{aligned}$$

by the inequality above, we obtain

$$\begin{aligned} \lim_{k\to\infty} sup\ rqp_b(T\sigma_{n_k-1}, T\sigma_{m_k-1}) &= \lim_{k\to\infty} sup\ rqp_b(g\sigma_{n_k}, g\sigma_{m_k}) \\ &\leqslant s\epsilon \end{aligned}$$

Substituting $\sigma = \sigma_{n_k}$ and $\eta = \sigma_{m_k}$ in equation 5,

$$\begin{aligned} \epsilon \leq rqp_b(T\sigma_{n_k}, T\sigma_{m_k}) &\leqslant \psi([rqp_b(g\sigma_{n_k}, g\sigma_{m_k})]^{\alpha} . [rqp_b(g\sigma_{n_k}, T\sigma_{n_k})]^{\beta} . [rqp_b(g\sigma_{m_k}, T\sigma_{m_k})]^{\gamma} . \\ &\quad \frac{1}{3s}\left[rqp_b(g\sigma_{n_k}, T\sigma_{m_k}) + rqp_b(g\sigma_{m_k}, T\sigma_{n_k})\right]^{1-\alpha-\beta-\gamma} \qquad (9) \\ &\leqslant \psi([rqp_b(T\sigma_{n_k-1}, T\sigma_{m_k-1})]^{\alpha} . [rqp_b(T\sigma_{n_k-1}, T\sigma_{n_k})]^{\beta} . [rqp_b(T\sigma_{m_k-1}, T\sigma_{m_k})]^{\gamma} . \\ &\quad \frac{1}{3s}\left[rqp_b(T\sigma_{n_k-1}, T\sigma_{m_k}) + rqp_b(T\sigma_{m_k-1}, T\sigma_{n_k})\right]^{1-\alpha-\beta-\gamma} \end{aligned} \qquad (10)$$

Letting the limit as $k \to \infty$ in equation 10 and Definition 2.9,

$$\epsilon \leq \lim_{k\to\infty} sup\ rqp_b(T\sigma_{n_k}, T\sigma_{m_k}) \leq \psi(0) = 0.$$

Therefore $\epsilon = 0$ which is a contradiction. Since $T\sigma_n$ and $g\sigma_n$ are Cauchy sequence. Let $u \in X$ such that,

$$\lim_{n\to\infty} rqp_b(T\sigma_n, Tz) = \lim_{n\to\infty} rqp_b(g\sigma_{n+1}, z) = 0.$$

As $z \in gX$, there exist $u \in X$ such that $z = gu$. We shall prove that u is a coincidence point of g and T. By equation 5

$$\begin{aligned} rqp_b(T\sigma_n, Tu) &\leqslant \psi([rqp_b(g\sigma_n, gu)]^{\alpha} . [rqp_b(g\sigma_n, T\sigma_n)]^{\beta} . [rqp_b(gu, Tu)]^{\gamma} . \\ &\quad \frac{1}{3s}\left[rqp_b(g\sigma_n, Tu) + rqp_b(gu, T\sigma_n)\right]^{1-\alpha-\beta-\gamma} \\ &\leq ([rqp_b(g\sigma_n, gu)]^{\alpha} . [rqp_b(g\sigma_n, T\sigma_n)]^{\beta} . [rqp_b(gu, Tu)]^{\gamma} . \\ &\quad \frac{1}{3s}\left[rqp_b(g\sigma_n, Tu) + rqp_b(gu, T\sigma_n)\right]^{1-\alpha-\beta-\gamma} \end{aligned} \qquad (11)$$

Letting $n \to \infty$ in equation 11, we conclude that $Tu = z = gu$. □

Corollary 3.4. *Let (X, rqp_b, s) be a complete rectangular quasi partial b–metric space and $T, g \colon X \longrightarrow X$ be a self–mappings on X. Consider T is ψ_g–interpolative Reich-Rus-Ćirić type contraction if there exists a continuous $\psi \in \Psi$ and $\alpha, \beta \in (0, 1)$ such that*

$$rqp_b(T\sigma, T\eta) \leqslant \psi([rqp_b(g\sigma, g\eta)]^{\alpha} . [rqp_b(g\sigma, T\ \sigma)]^{\beta} . [rqp_b(g\eta, T\eta)])^{1-\alpha-\beta}$$

is satisfied for all $\sigma, \eta \in X$ such that $T\sigma \neq g\sigma, T\eta \neq gy$ and $g\sigma \neq g\eta$. Suppose that $T\sigma \subseteq g\sigma$ such that $g\sigma$ is closed. Then, T and g have a coincidence point in X.

The existence of a fixed point is simple to prove for continuous maps. The following example can justify this result with a discontinuous map by visual illustration.

Example 3.5. *Let us consider $X = [0, \infty)$ equipped with a complete rectangular quasi-partial b-metric as $rqp_b(\sigma, \eta) = (\sigma - \eta)^2 + \sigma$.*

We define two self mappings T and g as shown in Figure 2 on X as $g(\sigma) = \sigma$ for all $\sigma \in X$ and

$$T\sigma = \begin{cases} 1 & , \sigma \in [0,3] \\ \frac{1}{\sigma^2} & , \sigma \in (3, \infty). \end{cases}$$

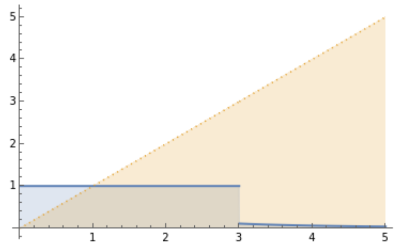

Figure 2: The intersection of T and g map at the point $\sigma = 1$ demonstrates the coincidence point on $X = [0, 3]$.

T is a ψ_g-interpolative Hardy–Rogers type contraction for $\alpha = \frac{3}{10}$, $\beta = \frac{1}{10}$ and $\gamma = \frac{4}{10}$. Taking $\psi(t) = \frac{t}{2}$ for all $t \in [0, \infty)$. Consider the following cases:

Case 1: *If $(\sigma, \eta) = [0, 3]$ or $\sigma = \eta$ for all $\sigma \in [0, \infty)$.It is obvious.*
Case 2:*If $\sigma, \eta \in (3, \infty)$ and $\sigma \neq \eta$. We have,*

$$rqp_b(T\sigma, T\eta) = (\frac{1}{\sigma^2} - \frac{1}{\eta^2})^2 + \frac{1}{\sigma^2} \leq 1. \tag{12}$$

From equation 5,

$$\begin{aligned} &= \psi([rqp_b(g\sigma, g\eta)]^\alpha . [rqp_b(g\sigma, T\sigma)]^\beta . [rqp_b(g\eta, T\eta)]^\gamma . \\ &\quad \frac{1}{3s} [rqp_b(g\sigma, T\eta) + rqp_b(g\eta, T\sigma)]^{1-\alpha-\beta-\gamma} \\ &= \psi([rqp_b(\sigma, \eta)]^\alpha . \left[rqp_b(\sigma, \frac{1}{\sigma^2})\right]^\beta . \left[rqp_b(\eta, \frac{1}{\eta^2})\right]^\gamma . \\ &\quad \frac{1}{3s} \left[rqp_b(\sigma, \frac{1}{\eta^2}) + rqp_b(\eta, \frac{1}{\sigma^2})\right]^{1-\alpha-\beta-\gamma} \\ &= \psi(\left[(\sigma - \eta)^2 + \sigma\right]^\alpha . \left[(\sigma - \frac{1}{\sigma^2})^2 + \sigma\right]^\beta . \left[(\eta - \frac{1}{\eta^2})^2 + \eta\right]^\gamma . \\ &\quad \frac{1}{3s} \left[(\sigma - \frac{1}{\eta^2})^2 + \sigma + (\eta - \frac{1}{\sigma^2})^2 + \eta\right]^{1-\alpha-\beta-\gamma} \\ &\geq 1 \end{aligned}$$

Thus, the inequality holds.

$$\begin{aligned} rqp_b(T\sigma, T\eta) \leqslant \psi([rqp_b(g\sigma, g\eta)]^\alpha . [rqp_b(g\sigma, T\sigma)]^\beta . [rqp_b(g\eta, T\eta)]^\gamma . \\ \frac{1}{3s} [rqp_b(g\sigma, T\eta) + rqp_b(g\eta, T\sigma)]^{1-\alpha-\beta-\gamma} \end{aligned}$$

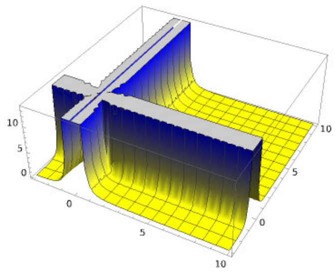

Figure 3: Dominance of right-hand side of equation 5 is visually checked in Figure 3. Thus the inequality holds for $\sigma, \eta \in (3, \infty)$.

Case 3:*If* $\sigma \in [0,3] \setminus \{1\}$ *and* $\eta \in (3, \infty)$. *We have,*

$$rqp_b(T\sigma, T\eta) = (1 - \frac{1}{\eta^2})^2 + 1 \leq 1.79. \tag{13}$$

From equation 5,

$$\begin{aligned}
&= \psi([rqp_b(g\sigma, g\eta)]^{\alpha} . [rqp_b(g\sigma, T\sigma)]^{\beta} . [rqp_b(g\eta, T\eta)]^{\gamma} . \\
&\qquad \frac{1}{3s} [rqp_b(g\sigma, T\eta) + rqp_b(g\eta, T\sigma)]^{1-\alpha-\beta-\gamma} \\
&= \psi([rqp_b(\sigma, \eta)]^{\alpha} . \left[rqp_b(\sigma, \frac{1}{\sigma^2})\right]^{\beta} . \left[rqp_b(\eta, \frac{1}{\eta^2})\right]^{\gamma} . \\
&\qquad \frac{1}{3s} \left[rqp_b(\sigma, \frac{1}{\eta^2}) + rqp_b(\eta, \frac{1}{\sigma^2})\right]^{1-\alpha-\beta-\gamma} \\
&= \psi(\left[(\sigma - \eta)^2 + \sigma\right]^{\alpha} . \left[(\sigma - \frac{1}{\sigma^2})^2 + \sigma\right]^{\beta} . \left[(\eta - \frac{1}{\eta^2})^2 + \eta\right]^{\gamma} . \\
&\qquad \frac{1}{3s} \left[(\sigma - \frac{1}{\eta^2})^2 + \sigma + (\eta - \frac{1}{\sigma^2})^2 + \eta\right]^{1-\alpha-\beta-\gamma} \\
&\geq 1.79.
\end{aligned}$$

Hence, one is the coincidence point of g and T.

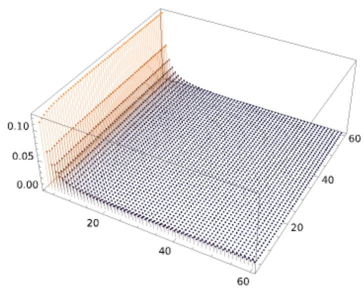

Figure 4: Dominance of right-hand side of equation 5 is visually checked in Figure 4. Thus inequality holds for $\sigma \in [0,3] \setminus \{1\}$ and $\eta \in (3, \infty)$. Here Figure 4 demonstrates at the point $\eta = 1$, mappings g and T satisfy $g\sigma = T\sigma$.

4. Application in solving non-linear Fredholm integral equation

In this section, the existence of a unique solution for the non-linear Fredholm integral equation has been proved. To apply our result, consider $X = C[0,1]$ to be a set of all real continuous functions on $[a,b]$ equipped with metric $rqp_b(f,g)$=$|f-g|$=$\max_{t\in[a,b]}|f(t)-g(t)|$ for all $f,g \in C[a,b]$. Then (X, rqp_b) is a complete rectangular quasi-partial b- metric space.
Let us consider the non-linear Fredholm integral equation:

$$\sigma(t) = v(t) + \tfrac{1}{b-a}\int_0^t K(t,s,\sigma(s))\,ds \tag{14}$$

for all $t,s \in [a,b]$ and assume that $K\colon [a,b]\times[a,b]\times X \to X$ and $v\colon [a,b] \to \mathbb{R}$ is a continuous function where $v(t)$ is a given function in X.

Theorem 4.1. *Suppose (X, rqp_b) be a rectangular quasi-partial b-metric space $rqp_b(f,g) = |f-g| = \max_{t\in[a,b]}|f(t)-g(t)|$ for all $f,g \in X$ and $T,g\colon X \to X$ be an operator on X defined by*

$$T\sigma(t) = v(t) + \tfrac{1}{b-a}\int_0^t K(t,s,\sigma(s))\,ds \tag{15}$$

If there exist $k \in [0,1)$, $\alpha,\beta,\gamma \in (0,1)$ with $\alpha+\beta+\gamma \le 1$ such that for all $\sigma,\eta \in X, s,t \in [a,b]$ satisfying the following inequality

$$0 \le K(t,s,\sigma(s)) - K(t,s,\eta(s)) \le kM(\sigma(s)-\eta(s))$$

where $M = |g\sigma(s)-g\eta(s)|^{\alpha}.|g\sigma(s)-T\sigma(s)|^{\beta}.|g\eta(s)-T\eta(s)|^{\gamma}.\frac{1}{3s}(|g\sigma(s)-T\eta(s)|+|g\eta(s)-T\sigma(s)|)^{1-\alpha-\beta-\gamma}$
Then the integral equation has a unique solution in X.

Proof. Since,

$$|T\sigma(t)-T\eta(t)| \le \frac{1}{|b-a|}\int_0^t |K(t,s,\sigma(s))-K(t,s,\eta(s))|ds \tag{16}$$

$$\le \frac{1}{|b-a|}\int_0^t M|\sigma(s)-\eta(s)|ds$$

$$\le \frac{k}{|b-a|}\int_0^t (|g\sigma(s)-g\eta(s)|^{\alpha}.|g\sigma(s)-T\sigma(s)|^{\beta}.|g\eta(s)-T\eta(s)|^{\gamma}. \frac{1}{3s}(|g\sigma(s)-T\eta(s)|+|g\eta(s)-T\sigma(s)|)^{1-\alpha-\beta-\gamma})ds$$

$$rqp_b(T\sigma,T\eta) = \max_{t\in[a,b]}|T\sigma(t)-T\eta(t)|$$

$$\le \frac{k}{|b-a|}\max_{t\in[a,b]}\int_0^t (|g\sigma(s)-g\eta(s)|^{\alpha}.|g\sigma(s)-T\sigma(s)|^{\beta}.|g\eta(s)-T\eta(s)|^{\gamma}. \frac{1}{3s}(|g\sigma(s)-T\eta(s)|+|g\eta(s)-T\sigma(s)|)^{1-\alpha-\beta-\gamma})ds$$

$$\le kB(\sigma,\eta)$$

Thus $X = C[a,b]$ is a complete metric space. Therefore all the conditions are satisfied by setting $\psi(t)$=kt for all $t \ge 0$, where $k \in [0,1)$ and the integral equation has a solution in X. □

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Conflict of interest Both authors declare that they do not have a conflict of interest.

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