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Global existence and decay of solutions for a delayed m-Laplacian equation with logarithmic term in variable-exponent Sobolev spaces

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Abstract. This work, we deal with the delayed m-Laplacian wave equation in variable-exponent Sobolev spaces. Firstly, we proved the global existence of solutions. Later, we considered the decay of solutions by the using the Komornik's lemma.

1. Introduction

1.1. Setting of the problem:

In this study, we considered the delayed m-Laplacian wave equation with variable exponent as follows:

$$\begin{cases} z_{tt} - \Delta_m z + z + \mu_1 z_t |z_t|^{p(\cdot)-2} & \\ +\mu_2 z_t(x,t-\tau) |z_t|^{p(\cdot)-2}(x,t-\tau) = z|z|^{q(\cdot)-2} \ln |z|^k, & (x,t) \in \Omega \times (0,\infty), \\ z(x,t) = 0, & (x,t) \in \partial\Omega \times (0,\infty), \\ z(x,0) = z_0(x), \ z_t(x,0) = z_1(x), & x \in \Omega, \\ z_t(x,t-\tau) = f_0(x,t-\tau) & in\ \Omega \times (0,\tau) \end{cases} \tag{1}$$

where $\Omega \subset R^n$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$. τ, k, $\mu_1 > 0$, $m \geq 2$, $\mu_2 \in R$ and the term

$$\Delta_m z = div\left(|\nabla z|^{m-2} \nabla z\right)$$

is called m-Laplacian. The initial value functions z_0, z_1 and f_0 will be defined later.

The exponents $p(\cdot)$ and $q(\cdot)$ are measurable functions on Ω satisfiying

$$2 \leq p_1 \leq p(x) \leq p_2 < q_1 \leq q(x) \leq q_2 \leq 2\frac{n-1}{n-2}, \ n \geq 3, \tag{2}$$

here

$$\begin{cases} p_1 = ess\inf_{x\in\Omega} p(x), \ p_2 = ess\sup_{x\in\Omega} p(x), \\ q_1 = ess\inf_{x\in\Omega} q(x), \ q_2 = ess\sup_{x\in\Omega} q(x), \end{cases}$$

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and $p(\cdot)$, $q(\cdot)$ satisfy the log-Hölder continuity condition:

$$\left|p(x) - p(y)\right| \leq -\frac{\mathcal{A}}{\ln\left|x-y\right|}, \textit{for all } x, y \in \Omega, \textit{ with } \left|x-y\right| < \delta, \tag{3}$$

where $0 < \delta < 1$ and $\mathcal{A} > 0$.

- Logarithmic source term: The problems with the logarithmic source term has a wide range of applications of physics such as nuclear physics, quantum field theory, optics, geophysics [7, 8].
- Delay term: Time delay frequently arises in various practical problems, including thermal, biological, chemical, physical, and economic phenomena [12].
- Variable exponent: Problems involving variable exponents are prevalent in diverse scientific fields such as image processing, electrorheological fluids, and nonlinear elasticity theory [9, 26].

1.2. Literature overview:

When $m = 2$ and without the third term (z), eq. (1) become the following delayed wave equation

$$z_{tt} - \Delta z + \mu_1 z_t \left|z_t\right|^{m(x)-2} + \mu_2 z_t(x, t-\tau)\left|z_t\right|^{m(x)-2}(x, t-\tau) = z\left|z\right|^{p(x)-2}\ln\left|z\right|^k. \tag{4}$$

Kafini and Noor [13] established the global existence and decay of solutions.

When $\mu_1 = 1$, $\mu_2 = 0$ and $k = 1$, eq. (4) become the following wave equation

$$z_{tt} - \Delta z + \left|z_t\right|^{m(x)-2} z_t = \left|z\right|^{p(x)-2} z \ln z.$$

Rahmoune [23] proved the local existence and blow up of solutions.

Pişkin and Yüksekkaya [22] investigated the following equation:

$$z_{tt} + \Delta^2 z - div\left(\left|\nabla z\right|^{m-2}\nabla z\right) + \mu_1 z_t(x,t) + \mu_2 z_t(x, t-\tau) = z\left|z\right|^{p(x)-2}\ln\left|z\right|^k.$$

They proved the blow up of solutions in finite time.

Yüksekkaya and Pişkin [29] studied the following Klein-Gordon equation:

$$z_{tt} - \Delta z - \Delta z_t + m^2 z + \mu_1\left|z_t\right|^{r(x)-2} z_t + \mu_2 z_t(x, t-\tau)\left|z_t\right|^{r(x)-2}(x, t-\tau) = bz\left|z\right|^{p(x)-2}.$$

They established the blow up of solutions in finite time.

Antontsev et al. [5] studied a nonlinear $p(x)$ −Laplacian equation with time delay and variable exponents as follows:

$$z_{tt} - \Delta_{p(x)} z + \mu_1\left|z_t\right|^{m(x)-2} z_t + \mu_2 z_t(x, t-\tau)\left|z_t\right|^{m(x)-2}(x, t-\tau) = z\left|z\right|^{q(x)-2}\ln\left|z\right|^k,$$

they proved the blow up and asymptotic behavior of solutions.

Also, some other authors investigated the partial differential equations with variable exponents Sobolev spaces (see [1], [3], [4], [6], [10], [15], [16], [18], [19], [20], [24], [25], [27], [28], [30]).

Inspired by the works mentioned earlier, we investigate the global existence and decay of solutions for the m-Laplacian wave equation with a variable exponent, delay term, and logarithmic source term. To our best knowledge there is no research, related to the m-Laplacian $(\Delta_m z)$ wave equation with delay term $(\mu_2 z_t(x, t-\tau)\left|z_t\right|^{p(\cdot)-2}(x, t-\tau))$ and logarithmic source term$(z\left|z\right|^{q(\cdot)-2}\ln\left|z\right|^k)$.

The remainder of our work is structured as follows: In Section 2, we present some lemmas, definitions, and theorems. Section 3 is dedicated to proving the global existence, while Section 4 focuses on establishing the decay result.

2. Preliminaries

In this part, we introduce some preliminary information about Lebesgue spaces and Sobolev spaces with variable exponents ([2],[9],[11], [21]).

Let $q:\Omega\rightarrow[1,\infty]$ be a measurable function, where Ω is a domain in R^n. We define the variable exponent Lebesgue space by

$$L^{q(x)}(\Omega)=\left\{z:\Omega\longrightarrow R;\ z \text{ measurable in } \Omega: \varrho_{q(\cdot)}(\lambda z)<\infty, \text{for some } \lambda>0\right\},$$

where

$$\varrho_{q(\cdot)}(z)=\int_\Omega \frac{1}{q(x)}|z(x)|^{q(x)}dx$$

is a modular. Endowed with the Luxembourg-type norm:

$$\|z\|_{q(\cdot)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{z(x)}{\lambda}\right|^{q(x)}dx\leq1\right\},$$

$L^{q(\cdot)}(\Omega)$ becomes a Banach space.

Next, we define the variable-exponent Sobolev space $W^{m,p(\cdot)}(\Omega)$ as

$$W^{m,q(\cdot)}(\Omega)=\left\{z\in L^{q(\cdot)}(\Omega):\ D^\alpha z\in L^{q(\cdot)}(\Omega),\ |\alpha|\leq m\right\}.$$

Lemma 2.1. *[9] Let $\Omega\subset R^n$ be a bounded domain and suppose that $q(\cdot)$ satisfies (3), then*

$$\|z\|_{q(\cdot)}\leq C\|\nabla z\|_{q(\cdot)}, \text{ for all } z\in W_0^{1,q(\cdot)}(\Omega),$$

where $C=C(q_1,q_2,\ \Omega)>0$. In particular, $\|\nabla z\|_{q(\cdot)}$ defines an equivalent norm on $W_0^{1,q(\cdot)}(\Omega)$.

Lemma 2.2. *[9] If $m(\cdot)\in C(\overline{\Omega})$ and $r:\Omega\rightarrow[1,\infty)$ is a measurable function such that*

$$ess\inf_{x\in\Omega}(m^*(x)-r(x))>0 \text{ with } m^*(x)=\begin{cases}\frac{nm(x)}{ess\sup_{x\in\Omega}(n-m(x))} & \text{if } m_2<n\\ \infty & \text{if } m_2\geq n\end{cases}.$$

Then the embedding $W_0^{1,m(\cdot)}(\Omega)\hookrightarrow L^{r(\cdot)}(\Omega)$ is continuous and compact.

Lemma 2.3. *[13] Let $s,\ p,\ q\geq1$ are measurable functions defined on Ω such that*

$$\frac{1}{s(l)}=\frac{1}{p(l)}+\frac{1}{q(l)}, \text{ for a.e. } l\in\Omega.$$

If $z\in L^{p(\cdot)}(\Omega)$ and $w\in L^{q(\cdot)}(\Omega)$, then $zw\in L^{s(\cdot)}(\Omega)$ with

$$\|zw\|_{s(\cdot)}\leq2\|z\|_{p(\cdot)}\|w\|_{q(\cdot)}.$$

Lemma 2.4. *[9] If q is measurable functions defined on Ω. Then*

$$\|f\|_{q(\cdot)}\leq1 \text{ and } \varrho_{q(\cdot)}(f)\leq1,$$

where

$$\varrho_{q(\cdot)}(f)=\int_\Omega|f(x)|^{q(x)}dx.$$

Lemma 2.5. *[9] If q is a measurable function on Ω satisfying (2), then the inequality:*

$$\min\left\{\|z\|_{q(\cdot)}^{q_1},\|z\|_{q(\cdot)}^{q_2}\right\}\leq\varrho_{q(\cdot)}(z)\leq\max\left\{\|z\|_{q(\cdot)}^{q_1},\|z\|_{q(\cdot)}^{q_2}\right\},$$

holds for any $z\in L^{q(\cdot)}(\Omega)$.

3. Global existence

In this part, our goal is to demonstrate the global existence of solutions for problem (1). To initiate this process, we introduce a new variable, similar to [17].

$$u(x,\rho,t) = z_t(x,t-\tau\rho),\ \ x\in\Omega,\ \rho\in(0,1),\ t>0.$$

So, we consider the equation

$$\tau u_t(x,\rho,t) + u_\rho(x,t-\tau\rho) = 0,\ \ x\in\Omega,\ \rho\in(0,1),\ t>0.$$

This results in the formulation of problem (1) as follows:

$$\begin{cases} z_{tt} - \Delta_m z + z + \mu_1 |z_t|^{p(\cdot)-2} z_t + \mu_2 u(x,1,t)|u(x,1,t)|^{p(\cdot)-2} = z|z|^{q(\cdot)-2}\ln|z|^k, & \Omega\times(0,\infty),\\ \tau u_t(x,\rho,t) + u_\rho(x,t-\tau\rho) = 0, & \Omega\times(0,1)\times(0,\infty),\\ u(x,\rho,0) = f_0(x,-\rho\tau), & \Omega\times(0,1),\\ z(x,t) = 0, & \partial\Omega\times[0,\infty),\\ z(x,0) = z_0(x),\ z_t(x,0) = z_1(x), & \Omega. \end{cases} \tag{5}$$

The energy functional associated with (5) for $t \geq 0$ is given by

$$E(t) = \frac{1}{2}\|z_t\|^2 + \frac{1}{m}\|\nabla z\|_m^m + \frac{1}{2}\|z\|^2 + k\int_\Omega \frac{|z|^{q(x)}}{q^2(x)}dx - k\int_\Omega \frac{|z|^{q(x)}\ln z}{q(x)}dx + \int_0^1\int_\Omega \frac{\zeta(x)\left|u(x,\rho,x)\right|^{p(x)}}{p^2(x)}dxd\rho \tag{6}$$

and the function ζ is a continuous function that satisfies the inequality

$$\tau\left|\mu_2\right|(p(x)-1) < \zeta(x) < \tau\left(\mu_1 m(x) - \left|\mu_2\right|\right),\ x\in\overline{\Omega}. \tag{7}$$

For instance, one can choose

$$\zeta(x) = \frac{\tau}{2}\left|\mu_2\right|(p(x)-1) + \mu_1 m(x) - \left|\mu_2\right| = \frac{\tau}{2}\left[\left(\mu_1 + \left|\mu_2\right|\right)p(x) - 2\left|\mu_2\right|\right] > 0 \text{ on } \overline{\Omega}.$$

The lemma below demonstrates that the associated energy of the problem is non-increasing under the condition

Lemma 3.1. *Let (z,u) be the solution of (5). Then, for some $C_0 > 0$,*

$$E'(t) \leq -C_0\left[\int_\Omega\left(|z_t|^{p(x)} + |u(x,1,t)|^{p(x)}dx\right)\right] \leq 0. \tag{8}$$

Proof. Multiplying the first equation in (5) by z_t and integrating over Ω, and multiplying the second equation in (5) by $\frac{1}{\tau}\zeta(x)|u|^{p(\cdot)-2}u$ and integrating over $\Omega\times(0,1)$, then combining the results, yields:

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|z_t\|^2 + \frac{1}{m}\|\nabla z\|_m^m + \frac{1}{2}\|z\|^2 + k\int_\Omega\frac{|z|^{q(\cdot)}}{q^2(x)}dx\right.\\ &\qquad\left. - k\int_\Omega\frac{|z|^{q(x)}\ln z}{q(x)}dx + \int_0^1\int_\Omega\frac{\zeta(x)|u|^{p(x)}}{p^2(x)}dxd\rho\right]\\ &= -\mu_1\int_\Omega|z_t|^{p(x)}dx - \frac{1}{\tau}\int_\Omega\int_0^1\zeta(x)\left|u(x,\rho,t)\right|^{p(x)-2}u(x,\rho,t)u_\rho(x,\rho,t)d\rho dx\\ &\qquad - \mu_2\int_\Omega z_t|u(x,1,t)|^{p(x)-2}u(x,1,t)dx. \end{aligned} \tag{9}$$

Now, let's estimate the second and third terms on the right-hand side of (9):

$$\begin{aligned}
&-\frac{1}{\tau}\int_{\Omega}\int_{0}^{1}\zeta(x)\left|u(x,\rho,t)\right|^{p(x)-2}uu_{\rho}(x,\rho,t)\,d\rho dx\\
=&\ \frac{1}{\tau}\int_{\Omega}\frac{\zeta(x)}{p(x)}\left(|u(x,0,t)|^{p(x)}-|u(x,1,t)|^{p(x)}\right)dx\\
=&\ \int_{\Omega}\frac{\zeta(x)}{\tau p(x)}|z_t|^{p(x)}\,dx-\int_{\Omega}\frac{\zeta(x)}{\tau p(x)}|u(x,1,t)|^{p(x)}\,dx.
\end{aligned}$$

For the final term, we apply Young's inequality with $r=\frac{p(x)}{p(x)-1}$and $r'=p(x)$ to obtain:

$$|z_t|\,|u(x,1,t)|^{p(x)-1}\leq\frac{1}{p(x)}|z_t|^{p(x)}+\frac{p(x)}{p(x)-1}|u(x,1,t)|^{p(x)}.$$

Consequently, we get

$$-\mu_2\int_{\Omega}z_t u\,|u(x,1,t)|^{p(x)-2}\,dx\leq\left|\mu_2\right|\left(\int_{\Omega}\frac{1}{p(x)}|z_t|^{p(x)}\,dx+\int_{\Omega}\frac{p(x)}{p(x)-1}|u(x,1,t)|^{p(x)}\,dx\right).$$

Thus, we arrive at the expression for

$$\frac{d}{dt}E(t)=-\int_{\Omega}\left[\mu_1-\left(\frac{\zeta(x)}{\tau p(x)}+\frac{\left|\mu_2\right|}{p(x)}\right)\right]|z_t|^{p(x)}\,dx-\int_{\Omega}\left(\frac{\zeta(x)}{\tau p(x)}-\frac{\left|\mu_2\right|(p(x)-1)}{p(x)}\right)|u(x,1,t)|^{p(x)}\,dx.$$

In the end, for all $\forall x\in\overline{\Omega}$, the relation (7) yields

$$\psi_1(x)=\mu_1-\left(\frac{\zeta(x)}{\tau p(x)}+\frac{\left|\mu_2\right|}{p(x)}\right)>0\ \text{ and }\ \psi_2(x)=\frac{\zeta(x)}{\tau p(x)}-\frac{\left|\mu_2\right|(p(x)-1)}{p(x)}>0.$$

As $p(x)$ is bounded, and consequently $\zeta(x)$ is bounded, we infer that both $\psi_1(x)$ and $\psi_2(x)$ are bounded.Therefore, if we define

$$C_0(x)=\min\{\psi_1(x),\ \psi_2(x)\}>0,\ \text{ for any } x\in\overline{\Omega}$$

and $C_0=\inf_{\overline{\Omega}}C_0(x)$, then $C_0(x)>C_0>0$.

Hence, we have

$$E'(t)\leq-C_0\left[\int_{\Omega}\left(|z_t|^{p(x)}+|u(x,1,t)|^{p(x)}\,dx\right)\right]\leq 0.$$

□

Now, we demonstrate that the solution to equation (5) maintains uniform boundedness and is globally defined over time.

To achieve this, we introduce

$$I(t)=\|\nabla z\|_m^m+\|z\|^2-\int_{\Omega}|z|^{q(x)}\ln|z|^k\,dx,$$

$$J(t)=\frac{1}{m}\|\nabla z\|_m^m+\frac{1}{2}\|z\|^2+k\int_{\Omega}\frac{|z|^{q(x)}}{q^2(x)}dx-\int_{\Omega}\frac{|z|^{q(x)}\ln|z|^k}{q(x)}dx+\int_0^1\int_{\Omega}\frac{\zeta(x)|u|^{p(x)}}{p^2(x)}dxdt.$$

Therefore, we have

$$E(t)=J(t)+\frac{1}{2}\|z_t\|^2.$$

Lemma 3.2. *Suppose that the initial data $z_0 \in W_0^{1,m}(\Omega)$ and $z_1 \in L^2(\Omega)$ satisfying $q_1 > m$, $I(0) > 0$ and*

$$\beta = C_{q_2+k}\left(\frac{q_1 E(0)}{q_1 - m}\right)^{\frac{q_2+k-m}{m}} < 1.$$

Subsequently, $I(t) > 0$, holds for any $t \in [0, T]$, and $\gamma > 0$ will be specified later.

Proof. If

$$\int_\Omega |z|^{q(x)} \ln |z|^k \, dx \le 0$$

holds, the result is clear. So, let's suppose that

$$\int_\Omega |z|^{q(x)} \ln |z|^k \, dx > 0.$$

Since $I(0) > 0$, we deduce by continuity that there exists $T^* \le T$ such that $I(t) \ge 0$ for all $t \in [0, T^*]$.
This implies that, for every $t \in [0, T^*]$,

$$\begin{aligned} J(t) &\ge \frac{1}{m}\|\nabla z\|_m^m + \frac{1}{2}\|z\|^2 + \frac{k}{q_2^2}\int_\Omega |z|^{q(x)} dx - \frac{1}{q_1}\int_\Omega |z|^{q(x)} \ln |z|^k \, dx + \int_0^1 \int_\Omega \frac{\zeta(x)|u|^{p(x)}}{p^2(x)} dx dt \\ &\ge \left(\frac{q_1 - m}{q_1}\right)\|\nabla z\|_m^m + \left(\frac{q_1 - 2}{q_1}\right)\|z\|^2 + \frac{k}{q_2^2}\int_\Omega |z|^{q(x)} dx + \int_0^1 \int_\Omega \frac{\zeta(x)|u|^{p(x)}}{p^2(x)} dx dt + \frac{1}{q_1} I(t) \\ &\ge \left(\frac{q_1 - m}{q_1}\right)\|\nabla z\|_m^m . \end{aligned}$$

Thus,

$$\|\nabla z\|_m^m \le \left(\frac{q_1}{q_1 - m}\right) J(t) \le \left(\frac{q_1}{q_1 - m}\right) E(t) \le \left(\frac{q_1}{q_1 - m}\right) E(0).$$

On the other hand, using the facts that $\ln |z| < |z|$ and $|z| > 1$, we get

$$\int_\Omega |z|^{q(x)} \ln |z|^k \, dx < \int_\Omega |z|^{q(x)+k} dx \le \int_\Omega |z|^{q_2+k} dx.$$

If we choose $0 < k < \frac{2}{n-2}$ then the embedding $W_0^{1,m}(\Omega) \hookrightarrow L^{q_2+k}(\Omega)$ yields

$$\begin{aligned} \int_\Omega |z|^{q_2+k} dx &\le C_{q_2+k}\|\nabla z\|_m^{q_2+k} \\ &= C_{q_2+k}\|\nabla z\|_m^m \|\nabla z\|_m^{q_2+k-m} \\ &= C_{q_2+k}\|\nabla z\|_m^m \left(\|\nabla z\|_m^m\right)^{\frac{q_2+k-m}{m}} \\ &\le C_{q_2+k}\|\nabla z\|_m^m \left(\frac{q_1 E(0)}{q_1 - m}\right)^{\frac{q_2+k-m}{m}} \end{aligned} \tag{10}$$

where C_{q_2+k} is the embedding constant. So,

$$\int_\Omega |z|^{q(x)} \ln |z|^k \, dx \le \beta \|\nabla z\|_m^m . \tag{11}$$

Consequently, from (10) and (11) we deduce that

$$I(t) > (1 - \beta)\|\nabla z\|_m^m > 0, \ \forall t \in [0, T^*].$$

By repeating this procedure, T^* can be extended to T. □

Theorem 3.3. *If the initial data z_0, z_1 satisfy the conditions of Lemma 3.2, then the solution of (5) is uniformly bounded and globally defined in time.*

Proof. It is sufficient to demonstrate that $\|\nabla z\|_m^m + \|z_t\|^2$ is bounded independently of t.

Clearly,

$$\begin{aligned} E(0) &\geq E(t) = \frac{1}{2}\|z_t\|^2 + J(t) \\ &\geq \frac{1}{2}\|z_t\|^2 + \left(\frac{q_1 - m}{q_1}\right)\|\nabla z\|_m^m + \left(\frac{q_1 - 2}{q_1}\right)\|z\|^2 + \frac{k}{q_2^2}\int_\Omega |z|^{q(x)}\,dx + \int_0^1\int_\Omega \frac{\zeta(x)\,|u|^{p(x)}}{p^2(x)}dxdt + \frac{1}{q_1}I(t) \\ &\geq \frac{1}{2}\|z_t\|^2 + \left(\frac{q_1 - m}{q_1}\right)\|\nabla z\|_m^m. \end{aligned}$$

Therefore,

$$\|\nabla z\|_m^m + \|z_t\|^2 \leq CE(0),$$

where $C(k,\ q_1,\ q_2) > 0$ is a positive constant. □

4. Decay

In this part, we aim to establish the decay of solutions for problem (1).

Lemma 4.1. *[14] (Komornik) Let $E : R^+ \longrightarrow R^+$ be a non-increasing function. Suppose there exist $\sigma > 0$ and $\omega > 0$ such that*

$$\int_s^\infty E^{1+\sigma}(t)\,dt \leq \frac{1}{\omega}E^\sigma(0)\,E^\sigma(s) = cE(s),\ \forall s > 0.$$

Then, $\forall t \geq 0$,

$$E(t) \leq \begin{cases} cE(0)(1+t)^{\frac{-1}{\sigma}}, & \text{if } \sigma > 0, \\ E(t) \leq cE(0)\,e^{-\omega t}, & \text{if } \sigma = 0. \end{cases}$$

For our main theorem, we need the the following lemma.

Lemma 4.2. *[13] The functional*

$$F(t) = \tau\int_0^1\int_\Omega e^{-\rho\tau}\zeta(x)\left|u(x,\rho,t)\right|^{p(x)}dxd\rho,$$

satisfies, along the solution of (5),

$$F'(t) \leq \int_\Omega \zeta(x)\,|z_t|^{p(x)}\,dx - \tau e^{-\tau}\int_0^1\int_\Omega \zeta(x)\left|u(x,\rho,t)\right|^{p(x)}dxd\rho.$$

Our main result is below:

Theorem 4.3. *Assume that the conditions (3) and (2) are satisfied. Then there exist two positive constants c and α such that any global solution of (5) satisfies*

$$E(t) \leq \begin{cases} cE(0)(1+t)^{\frac{-2}{(p_2-2)}}, & \text{if } p_2 > 0, \\ ce^{-\alpha t}, & \text{if } p(x) = 0. \end{cases}$$

Proof. Multiply (5) by $zE^r(t)$, for $r > 0$ to be specified later, and integrate over $\Omega \times (s, T)$, $s < T$, to obtain

$$\int_s^T E^r(t) \int_\Omega \Big(zz_{tt} - z div\left(|\nabla z|^{m-2} \nabla z \right) + z^2 + \mu_1 z |z_t|^{p(x)-2} z_t + \mu_2 z u(x,1,t) |u(x,1,t)|^{p(x)-2} - z^2 |z|^{q(x)-2} \ln |z|^k \Big) dxdt = 0,$$

which gives

$$\begin{aligned} & \int_s^T E^r(t) \int_\Omega \frac{d}{dt}(zz_t)\, dxdt - \int_s^T E^r(t) \int_\Omega z_t^2 dxdt + \int_s^T E^r(t) \int_\Omega |\nabla z|^m dxdt \\ & + \int_s^T E^r(t) \int_\Omega z^2 dxdt + \mu_1 \int_s^T E^r(t) \int_\Omega z |z_t|^{p(x)-2} z_t dxdt \\ & + \mu_2 \int_s^T E^r(t) \int_\Omega z u(x,1,t) |u(x,1,t)|^{p(x)-2} dxdt - \int_s^T E^r(t) \int_\Omega |z|^{q(x)} \ln |z|^k dxdt \\ = \; & 0. \end{aligned} \tag{12}$$

Recalling the definition of $E(t)$ as stated in (6), adding and subtracting certain terms, and using the relationship

$$\frac{d}{dt}\left(E^r(t) \int_\Omega zz_t dx \right) = qE^{r-1}(t) E'(t) \int_\Omega zz_t dx + E^r(t) \frac{d}{dt} \int_\Omega zz_t dx.$$

Eq (12) becomes

$$\begin{aligned} 2\int_s^T E^{r+1}(t) = & -\int_s^T \frac{d}{dt}\left(E^r(t) \int_\Omega zz_t dx \right) dt \\ & + q\int_s^T E^{r-1}(t) E'(t) \int_\Omega zz_t dxdt - \mu_1 \int_s^T E^r(t) \int_\Omega z |z_t|^{p(x)-2} z_t dxdt \\ & - \mu_2 \int_s^T E^r(t) \int_\Omega z u(x,1,t) |u(x,1,t)|^{p(x)-2} dxdt \\ & + 2\int_s^T E^r(t) \int_\Omega |z_t|^2 dxdt + \int_s^T E^r(t) \int_\Omega |z|^{q(x)} \ln |z|^k dxdt \\ & + 2\int_s^T E^r(t) \int_0^1 \int_\Omega \frac{\zeta(x) \left| z(x,\rho,t) \right|^{p(x)}}{p(x)} dxdt. \end{aligned} \tag{13}$$

The first term in the right hand side of (13) is estimated as follows.

$$\begin{aligned} \left| -\int_s^T \frac{d}{dt}\left(E^r(t) \int_\Omega zz_t dx \right) dt \right| = & \; |E^r(s) zz_t(x,s)\, dxdt - E^r(T) zz_t(x,T)\, dxdt| \\ \le & \; \frac{1}{2} E^r(s) \left[\int_\Omega z^2(x,s)\, dx + \int_\Omega z_t^2(x,s)\, dx \right] \\ & + \frac{1}{2} E^r(T) \left[\int_\Omega z^2(x,T)\, dx + \int_\Omega z_t^2(x,T)\, dx \right] \\ \le & \; \frac{1}{2} E^r(s) \left[C_* \|\nabla z\|^2 + 2E(s) \right] \\ & + \frac{1}{2} E^r(T) \left[\int_\Omega z^2(x,T)\, dx + \int_\Omega z_t^2(x,T)\, dx \right] \\ \le & \; E^r(s) \left[C_* E(s) + E(s) \right] + E^r(s) \left[C_* E(T) + E(T) \right], \end{aligned}$$

where C_* is the Poincaré constant. Exploiting the monotonicity of $E(t)$, we infer that

$$\begin{aligned}\left|-\int_s^T \frac{d}{dt}\left(E^r(t)\int_\Omega z z_t dx\right)dt\right| &\leq cE^{r+1}(s)\\ &\leq cE^r(0)E(s)\\ &\leq cE(s).\end{aligned} \tag{14}$$

Likewise, we address the term:

$$\begin{aligned}\left|q\int_s^T E^{r-1}(t)E'(t)\int_\Omega z z_t dxdt\right| &\leq -q\int_s^T E^{r-1}(t)E'(t)\left[C_*E(t)+2E(t)\right]\\ &\leq -c\int_s^T E^r(t)E'(t)dt\\ &\leq cE^{r+1}(s)\\ &\leq cE(s).\end{aligned} \tag{15}$$

To handle the next term, we set

$$\Omega_+ = \{x\in\Omega : |z_t(x,t)|\geq 1\} \text{ and } \Omega_- = \{x\in\Omega : |z_t(x,t)|<1\}$$

and use Hölder's and Young's inequalities, to get

$$\begin{aligned}\left|\int_s^T E^r(t)\int_\Omega |z_t|^2 dxdt\right| &= \left|\int_s^T E^r(t)\left[\int_{\Omega_+} z_t^2 dx + \int_{\Omega_-} z_t^2 dx\right]\right| dt\\ &\leq c\int_s^T E^r(t)\left[\left(\int_{\Omega_+}|z_t|^{p_1}dx\right)^{\frac{2}{p_1}} + \left(\int_{\Omega_-}|z_t|^{p_2}dx\right)^{\frac{2}{p_2}}\right]dt\\ &\leq c\int_s^T E^r(t)\left[\left(\int_{\Omega}|z_t|^{p(x)}dx\right)^{\frac{2}{p_1}} + \left(\int_{\Omega}|z_t|^{p(x)}dx\right)^{\frac{2}{p_2}}\right]dt\\ &\leq c\int_s^T E^r(t)\left[(-E'(t))^{\frac{2}{p_1}} + (-E'(t))^{\frac{2}{p_2}}\right]dt\\ &\leq c\varepsilon\int_s^T [E(t)]^{\frac{rp_1}{p_1-2}}dt + c(\varepsilon)\int_s^T(-E'(t))dt + c\varepsilon\int_s^T E^{r+1}(t)dt\\ &\quad +c(\varepsilon)\int_s^T(-E'(t))^{\frac{2(r+1)}{p_2}}dt.\end{aligned}$$

For $p_1>2$, the choice of $r=\frac{p_2}{2}-1$ will make $\frac{rp_1}{p_1-2}=r+1+\frac{p_2-p_1}{p_1-2}$. Hence,

$$\begin{aligned}\left|\int_s^T E^r(t)\int_\Omega z_t^2 dxdt\right| &\leq c\varepsilon\int_s^T E^{r+1}(t)dt + c\varepsilon\,[E(0)]^{\frac{p_2-p_1}{p_1-2}}\int_s^T E^{r+1}(t)dt + c(\varepsilon)E(s)\\ &\leq c\varepsilon\int_s^T E^{r+1}(t)dt + c(\varepsilon)E(s).\end{aligned} \tag{16}$$

For $p_1>2$, the choice of $r=\frac{p_2}{2}-1$ will make $\frac{rp_1}{p_1-2}=r+1+\frac{p_2-p_1}{p_1-2}$. Hence,

$$\begin{aligned}\left|\int_s^T E^r(t)\int_\Omega z_t^2 dxdt\right| &\leq c\varepsilon\int_s^T E^{r+1}(t)dt + c\varepsilon\,[E(0)]^{\frac{p_2-p_1}{p_1-2}}\int_s^T E^{r+1}(t)dt + c(\varepsilon)E(s)\\ &\leq c\varepsilon\int_s^T E^{r+1}(t)dt + c(\varepsilon)E(s).\end{aligned} \tag{17}$$

For the case $p_1 = 2$, the choice of $r = \frac{p_2}{2} - 1$, will give a similar result.

For the next term, we use Young's inequality. So, for a.e., $x \in \Omega$, we have

$$\begin{aligned}\left|-\mu_1 \int_s^T E^r(t) \int_\Omega z |z_t|^{p(x)-2} z_t dxdt\right| &\leq \varepsilon \int_s^T E^r(t) \int_\Omega |z(t)|^{p(x)} dxdt + c \int_s^T E^r(t) \int_\Omega c_\varepsilon(x) |z_t|^{p(x)} dxdt \\ &\leq \varepsilon \int_s^T E^r(t) \left[\int_{\Omega_+} |z(t)|^{p_1} dx + \int_{\Omega_-} |z_t|^{p_2} dx\right] dt \\ &\quad + c \int_s^T E^r(t) \int_\Omega c_\varepsilon(x) |z_t|^{p(x)} dxdt,\end{aligned}$$

where we used Young's inequality with

$$q(x) = \frac{p(x)}{p(x)-1} \text{ and } q'(x) = p(x)$$

and, hence,

$$c_\varepsilon(x) = \varepsilon^{1-p(x)} \left[(p(x))^{-p(x)} (p(x)-1)\right]^{p(x)-1}.$$

Therefore, using the embedding of $H_0^1(\Omega) \hookrightarrow L^{p_1}(\Omega)$ and $H_0^1(\Omega) \hookrightarrow L^{p_2}(\Omega)$ we arrive at

$$\begin{aligned}&\left|-\mu_1 \int_s^T E^r(t) \int_\Omega z |z_t|^{p(x)-2} z_t dxdt\right| \\ \leq\ & \varepsilon \int_s^T E^r(t) \left[c \|\nabla z(s)\|_2^{p_1} + c \|\nabla z(s)\|_2^{p_2}\right] dt + c \int_s^T E^r(t) \int_\Omega c_\varepsilon(x) |z_t|^{p(x)} dxdt \\ \leq\ & \varepsilon \int_s^T E^r(t) \left[cE^{\frac{p_1-2}{2}}(0) E(t) + cE^{\frac{p_2-2}{2}}(0) E(t)\right] dt + c \int_s^T E^r(t) \int_\Omega c_\varepsilon(x) |z_t|^{p(x)} dxdt \\ \leq\ & \varepsilon \int_s^T E^{r+1}(t) dt + c \int_s^T E^r(t) \int_\Omega c_\varepsilon(x) |z_t|^{p(x)} dxdt, \\ \leq\ & \varepsilon \int_s^T E^{r+1}(t) dt + c(\varepsilon) E(s),\end{aligned} \tag{18}$$

where $c(\varepsilon)$ is a finite constant depend on ε whence it is fixed because $p(x)$ is bounded.

The subsequent term in (13) can be estimated similarly, leading to

$$\begin{aligned}&\left|-\mu_2 \int_s^T E^r(t) \int_\Omega z |u(x,1,t)|^{p(x)-1} dxdt\right| \\ \leq\ & \varepsilon \int_s^T E^r(t) \left[c \|\nabla z(s)\|_2^{p_1} + c \|\nabla z(s)\|_2^{p_2}\right] dt + \int_s^T E^r(t) \int_\Omega c_\varepsilon(x) |u(x,1,t)|^{p(x)} dxdt \\ \leq\ & \varepsilon \int_s^T E^{r+1}(t) dt + c(\varepsilon) E(s).\end{aligned} \tag{19}$$

For the logarithmic term, we use the embedding. We have

$$q_2 + k < \frac{n}{n-2} + k < \frac{2n}{n-2}.$$

Thus the embedding $H_0^1(\Omega) \hookrightarrow L^{q_2+k}(\Omega)$, yields, for some $0 < \beta_* < 1$, to

$$\int_\Omega |z|^{q(x)} \ln |z|^k dx \leq \beta_* \|\nabla z\|^2.$$

Thus, we have

$$\begin{aligned}\left|\int_s^T E^r(t)\int_\Omega z^2|z|^{q(x)-2}\ln|z|^k\,dxdt\right| &\le \left|\int_s^T E^r(t)\int_\Omega |z|^{q(x)}\ln|z|^k\,dxdt\right| \\ &\le \beta_*\int_s^T E^r(t)\|\nabla z\|^2\,dt \\ &\le \beta_*\int_s^T E^{r+1}(t)\,dt. \qquad (20)\end{aligned}$$

The final term in (12) can be estimated by employing Lemma 3.2 as follows,

$$\begin{aligned}2\int_s^T E^r(t)\int_0^1\int_\Omega \frac{\zeta(x)\left|z(x,\rho,t)\right|^{p(x)}}{p(x)}dxd\rho &\le \frac{2}{p_1}\int_s^T E^r(t)\int_0^1\int_\Omega \zeta(x)\left|z(x,\rho,t)\right|^{p(x)}dxd\rho dt \\ &\le \frac{2\tau}{p_1}\left(E^r(t)\int_0^1\int_\Omega e^{-\rho\tau}\zeta(x)|z|^{p(x)}dxd\rho\right)\Bigg|_{t=s}^{t=T} \\ &\quad +\frac{2}{p_1}\int_s^T E^r(t)\int_0^1\int_\Omega e^{-\rho\tau}\zeta(x)|z|^{p(x)}dxd\rho dt.\end{aligned}$$

Since $\zeta(x)$ is bounded, we obtain, for $c>0$,

$$\begin{aligned}2\int_s^T E^r(t)\int_0^1\int_\Omega \frac{\zeta(x)\left|z(x,\rho,t)\right|^{p(x)}}{p(x)}dxd\rho dt &\le \frac{2\tau e^{-\tau}}{p_1}E^r(s)E(s)+\frac{2c}{p_1}E^{r+1}(T) \\ &\le \frac{2\tau e^{-\tau}}{p_1}E^r(0)E(s)+\frac{2c}{p_1}E^{r+1}(T)E(s) \\ &\le cE(s). \qquad (21)\end{aligned}$$

Combining (13) - (21), we arrive at

$$\int_s^T E^{r+1}(t)\,dt \le (\varepsilon+\beta_*)\int_s^T E^{r+1}(t)\,dt + cE(s).$$

Recalling that $\beta_*<1$ then the choice of ε small enough will make $\varepsilon+\beta_*<1$. Therefore,

$$\int_s^T E^{r+1}(t)\,dt \le cE(s).$$

As $T\longrightarrow\infty$, we get

$$\int_s^\infty E^{r+1}(t)\,dt \le cE(s).$$

Therefore, Komornik's lemma is satisfied with $\sigma=r=\frac{p_2}{2}-1$ which implies the desired result. □

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