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King type modification of q-Szász-Mirakjan operators

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Abstract. In this work, we define a King type modification of q-Szász-Mirakjan operators which reproduce u^2. The aim is to study the approximation properties for our operators in weighted function spaces by using the K-functional, second order and usual modulus of continuity.

1. Introduction and Preliminaries

On the q-calculus has taked an important part in the field of approximation theory since last three contracts. In 1987, A. Lupas defined the definite q-calculus to approximation theory. He applied the q-analogue and studied the approximation properties of Bernstein polynomials [16]. Another notable application of the quantum calculus was given in 1997 by Phillips [18]. He applied the q-calculus to define the q-analog of the classical Bernstein operators. Ostrovska [23] obtained more consequences on the q-Bernstein operators. Several mathematicians have studied the approximation properties behavior of q-analogue [3], [6], [7], [8], [9], [14], [17], [20], [21]. Recently the q-calculus has also been applied to study the q-analog of some summability methods in [28] and [22], respectively.

In 1950, Szász [27] introduced and exhaustively investigated the operator

$$S_m(g;u) = e^{-mu}\sum_{j=0}^{\infty}\frac{(mu)^j}{j!}g(\frac{j}{m}). \tag{1.1}$$

The King type modification [18] of positive linear operators which preserve the function $e_2(u) = u^2$ have a better rate of convergence than the classical ones.
The operators $V_m : C[0,1] \to C[0,1]$, for any $m \in \mathbb{N}$, are defined by

$$(V_m g)(u) = \sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}(s_m^*(u))^j(1-s_m^*(u))^{m-j}g\Big(\frac{j}{m}\Big), \qquad ,$$

for any function, $g \in C[0,1], u \in [0,1]$, where $s_m^* : [0,1] \to [0,1]$,

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$$s_m^*(u) = \begin{cases} u^2, & m = 1 \\ -\frac{1}{2(m-1)} + \sqrt{\frac{m}{m-1}u^2 + \frac{1}{4(m-1)^2}}, & m = 2, 3, \ldots \end{cases}$$

This sequence preserves two test functions e_0, e_2 and $(V_m e_1)(u) = s_m^*(u)$ holds.

This idea was implemented to some other familiar approximating operators in [11], [26],[25], [1], [19], [2].

Let $q \in (0, 1)$. The definition of q-integer is given by [15]

$$[m]_q := \begin{cases} \frac{1-q^m}{1-q}, & q \in \mathbb{R}^+ \setminus \{1\} \\ m, & q = 1, \end{cases}$$

$$[m]_q! := \begin{cases} [m]_q[m-1]_q \cdots [1]_q, & m \geq 1, \\ 1, & m = 0, \end{cases}$$

$$\begin{bmatrix} m \\ j \end{bmatrix}_q := \frac{[m]_q!}{[j]_q![m-j]_q!},$$

$$(1+u)_q^m := \begin{cases} (1+u)(1+qu)\cdots(1+q^{m-1}u) & m = 1, 2, 3, \cdots \\ 1 & m = 0. \end{cases}$$

$$(u;q)_0 = 1,\ (u;q)_m = \prod_{j=0}^{m-1}(1-q^j x),\ (u;q)_\infty = \prod_{j=0}^{\infty}(1-q^j u).$$

Gauss binomial is defined by

$$(u+a)_q^m = \sum_{j=0}^{m} \begin{bmatrix} m \\ j \end{bmatrix}_q q^{j(j-1)/2} a^j u^{m-j}.$$

There are two q-analogues of the exponential function e^u, see [16]
for $| u |< \frac{1}{1-q}$,

$$e_q(u) = \sum_{j=0}^{\infty} \frac{u^j}{[j]_q!} = \frac{1}{1-((1-q)u)_q^\infty},$$

for $| q |< 1$,
where $(1-u)_q^\infty = \prod_{j=0}^{\infty}(1-q^j u)$.

$$E_q(u) = \prod_{j=0}^{\infty}\left(1+(1-q)q^j u\right)_q^\infty = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}} \frac{u^j}{[j]_q!} = (1+(1-q)u)_q^\infty.$$

In [24] the classical Szász-Mirakjan operators have been changed on a closed subintervals of $[0, \infty)$, as

$$S_m(g;u) = e^{-mu} \sum_{j=0}^{\infty} \frac{(mu)^j}{j!} g\left(\frac{j}{m}\frac{m+c}{m+d}\right) \tag{1.2}$$

where $0 \leq u < \infty$, $c, d \in \mathbb{N}$ and $0 \leq c \leq d$.

In [4], the generalized Szász-Mirakjan operators based on the concept of q-integer were defined by

$$S_{m,q}(g;u) = E_q(-[m]_q u)\sum_{j=0}^{\infty}\beta_{m,j}(q;u)g\Big(\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_q}\Big) \tag{1.3}$$

for any $m \in \mathbb{N}$, $0 \leq u < \infty$, $g \in C[0,\infty)$, $c, d \in \mathbb{N}$ and $0 \leq c \leq d$, $S_{m,j}(q;u) \geq 0$. We can easily check that

$$\sum_{j=0}^{\infty} S_{m,j}(q;u) = E_q(-[m]_q u)\sum_{j=0}^{\infty}\beta_{m,j}(q;u) = 1, \tag{1.4}$$

where

$$\beta_{m,j}(q;u) = \frac{q^{\frac{j(j-1)}{2}}([m]_q u)^j}{[j]_q!}.$$

2. Construction

We propose the operators defined in (1.3) which preserve the quadratic function e_2, defining the functions

$$s_m(u) = \frac{-q^2[m+c]_q + \sqrt{q^4[m+c]_q^2 + 4q[m]_q^2[m+d]_q^2u^2}}{2q\,[m]_q\,[m+c]_q}$$

where $u \geq 0$, we consider the sequence of linear positive operators

$$S^*_{m,q}(g;u) = E_q(-[m]_q s_m(u))\sum_{j=0}^{\infty}\beta_{m,j}(q;s_m(u))g\Big(\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_q q^{j-2}}\Big). \tag{2.5}$$

Lemma 2.1. *for any $m \in \mathbb{N}$, $0 \leq u < \infty$, $c, d \in \mathbb{N}$ and $0 \leq c \leq d$,. We have*

$$S^*_{m,q}(t^{i+1};u) = \sum_{n=0}^{i}\binom{i}{n}\frac{[m+c]_q^{i+1-n}s_m(u)}{q^{2n-i-1}[m]_q^{i-n}[m+d]_q^{i+1-n}}S^*_{m,q}(t^n;u) \tag{2.6}$$

Proof. If we have

$$[j]_q = q^{j-1} + [j-1]_q$$

we can write

$$\begin{aligned}
S^*_{m,q}(t^{i+1};u) &= E_q(-[m]_q s_m(u))\sum_{j=0}^{\infty}\frac{[j]_q^{i+1}[m+c]_q^{i+1}}{q^{(j-2)(i+1)}[m]_q^{i+1}[m+d]_q^{i+1}}\\
&\quad\times\frac{q^{\frac{j(j-1)}{2}}([m]_q s_m(u))^j}{[j]_q!}\\
&= E_q(-[m]_q s_m(u))\sum_{j=1}^{\infty}\frac{q[j]_q^{i}[m+c]_q^{i+1}}{q^{(j-2)i}[m]_q^{i}[m+d]_q^{i+1}}\\
&\quad\times\frac{q^{\frac{j(j-1)}{2}-j+1}([m]_q s_m(u))^{j-1}}{[j-1]_q!}s_m(u)\\
&= E_q(-[m]_q s_m(u))\sum_{j=1}^{\infty}\frac{q\Big(q^{j-1}+[j-1]_q\Big)^i[m+c]_q^{i+1}}{q^{(k-2)i}[m]_q^{i}[m+d]_q^{i+1}}\\
&\quad\times\frac{q^{\frac{(j-2)(j-1)}{2}}([m]_q s_m(u))^{j-1}}{[j-1]_q!}s_m(u)\\
&= qE_q(-[m]_q s_m(u))\sum_{j=1}^{\infty}\sum_{n=0}^{i}\left(\begin{array}{c} i\\ n\end{array}\right)q^{(j-1)(i-n)}[j-1]_q^n\\
&\quad\times\frac{q^{\frac{(j-1)(j-2)}{2}}[m+c]_q^{i+1}}{q^{(j-2)i}[m]_q^{i}[m+d]_q^{i+1}}\frac{([m]_q s_m(u))^{j-1}}{[j-1]_q!}s_m(u)\\
&= qE_q(-[m]_q s_m(u))\sum_{n=0}^{i}\left(\begin{array}{c} i\\ n\end{array}\right)\frac{[m+c]_q^{i-n+1}}{q^{2n-i}[m]_q^{i-n}[m+d]_q^{i-n+1}}\\
&\quad\times\sum_{j=1}^{\infty}\frac{[j-1]_q^n[m+c]_q^n}{q^{(j-3)n}[m]_q^n[m+d]_q^n}q^{\frac{(j-1)(j-2)}{2}}\frac{([m]_q s_m(u))^{j-1}}{[j-1]_q!}s_m(u)\\
&= qE_q(-[m]_q s_m(u))\sum_{n=0}^{i}\left(\begin{array}{c} i\\ n\end{array}\right)\frac{[m+c]_q^{i-n+1}s_m(u)}{q^{2n-i}[m]_q^{i-n}[m+d]_q^{i-n+1}}\\
&\quad\times\sum_{j=0}^{\infty}\frac{[j]_q^n[m+c]_q^n}{q^{(j-2)n}[m]_q^n[m+d]_q^n}q^{\frac{j(j-1)}{2}}\frac{([m]_q s_m(u))^{j-1}}{[j]_q!}\\
&= \sum_{n=0}^{i}\left(\begin{array}{c} i\\ n\end{array}\right)\frac{[m+c]_q^{i+1-n}s_m(u)}{q^{2n-i-1}[m]_q^{i-n}[m+d]_q^{i-n+1}}S^*_{m,q}(t^n;x)
\end{aligned}$$

which is desired. □

3. Moments

Lemma 3.1. *For $S_{m,q}(t^i;x)$, $i=0,1,2$, one has*

$$\begin{aligned}
(i)S_{m,q}(1;u) &= 1\\
(ii)S_{m,q}(t;u) &= q\frac{[m+c]_q}{[m+d]_q}u\\
(iii)S_{m,q}(t^2;u) &= \Big(qu^2+\frac{q^2u}{[m]_q}\Big)\frac{[m+c]_q^2}{[m+d]_q^2}\\
(iv)S_{m,q}(t^3;u) &= \Big(u^3+\frac{2q^2+q}{[m]_q}u^2+\frac{q^2}{[m]_q^2}u\Big)\frac{[m+c]_q^3}{[m+d]_q^3}\\
(v)S_{m,q}(t^4;u) &= \Big(\frac{u^4}{q^2}+\frac{3q^2+2q+1}{q[m]_q}u^3+\frac{3q^3+3q+q}{[m]_q^2}u^2+\frac{q^4}{[m]_q^3}u\Big)\frac{[m+c]_q^4}{[m+d]_q^4}.
\end{aligned}$$

Lemma 3.2. *Let $S^*_{m,q}(g;u)$ be given by* (2.5)*. Then the followings hold:*

$$\begin{aligned}
(i)S^*_{m,q}(e_0;u) &= 1\\
(ii)S^*_{m,q}(e_1;u) &= q\frac{[m+c]_q}{[m+d]_q}s_m(u)\\
(iii)S^*_{m,q}(e_2;u) &= u^2\\
(iv)S^*_{m,q}(e_3;u) &= \Big(s_m^3(u)+\frac{2q^2+q}{[m]_q}s_m^2(u)+\frac{q^2}{[m]_q^2}s_m(u)\Big)\frac{[m+c]_q^3}{[m+d]_q^3}\\
(v)S^*_{m,q}(e_4;u) &= \Big(\frac{s_m^4(u)}{q^2}+\frac{3q^2+2q+1}{q[m]_q}s_m^3(u)+\frac{3q^3+3q+q}{[m]_q^2}s_m^2(u)\\
&\quad+\frac{q^4}{[m]_q^3}s_m(u)\Big)\frac{[m+c]_q^4}{[m+d]_q^4}.
\end{aligned}$$

Proof.

$$\begin{aligned}
(i)S^*_{m,q}(e_0;u) &= 1\\
(ii)S^*_{m,q}(e_1;u) &= E_q(-[m]_qs_m(u))\sum_{j=0}^{\infty}\beta_{m,j}(q;s_m(u))\Big(\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_qq^{j-2}}\Big)\\
&= q\frac{[m+c]_q}{[m+d]_q}s_m(u)S^*_{m,q}(1;u)\\
&= q\frac{[m+c]_q}{[m+d]_q}s_m(u)\\
(iii)S^*_{m,q}(e_2;u) &= E_q(-[m]_qs_m(u))\sum_{j=0}^{\infty}\beta_{m,j}(q;s_m(u))\Big(\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_qq^{j-2}}\Big)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{[m+c]_q}{[m+d]_q} s_m(u) S^*_{m,q}(t;u) + q^2 \frac{[m+c]_q^2}{[m]_q[m+d]_q^2} s_m(u) S^*_{m,q}(1;u) \\
&= \frac{q[m+c]_q^2}{[m+d]_q^2} s_m^2(u) + q^2 \frac{[m+c]_q^2}{[m]_q[m+d]_q^2} s_m(u) \\
&= u^2
\end{aligned}
$$

$$
\begin{aligned}
(iv) S^*_{m,q}(e_3;u) &= E_q(-[m]_q s_m(u)) \sum_{j=0}^{\infty} \beta_{m,j}(q; s_m(u)) \Big(\frac{[j]_q}{[m]_q} \frac{[m+c]_q}{[m+d]_q q^{j-2}} \Big)^3 \\
&= \frac{[m+c]_q}{q[m+d]_q} s_m(u) S^*_{m,q}(t^2;u) + \frac{2q[m+c]_q^2}{[m]_q[m+d]_q^2} s_m(u) S^*_{m,q}(t;u) \\
&\quad + \frac{q^3[m+c]_q^3}{[m]_q^2[m+d]_q^3} s_m(u) S^*_{m,q}(1;u) \\
&= \Big(s_m^3(u) + \frac{2q^2+q}{[m]_q} s_m^2(u) + \frac{q^2}{[m]_q^2} s_m(u) \Big) \frac{[m+c]_q^3}{[m+d]_q^3} \\
(v) S^*_{m,q}(e_4;u) &= E_q(-[m]_q s_m(u)) \sum_{j=0}^{\infty} \beta_{m,j}(q; s_m(u)) \Big(\frac{[j]_q}{[m]_q} \frac{[m+c]_q}{[m+d]_q q^{j-2}} \Big)^4 \\
&= \frac{[m+c]_q}{q^2[m+d]_q} s_m(u) S^*_{m,q}(t^3;u) + \frac{3[m+c]_q^2}{[m]_q[m+d]_q^2} s_m(u) S^*_{m,q}(t^2;u) \\
&\quad + \frac{3q^2[m+c]_q^3}{[m]_q^2[m+d]_q^3} s_m(u) S^*_{m,q}(t;u) + \frac{q^4[m+c]_q^4}{[m]_q^3[m+d]_q^4} s_m(u) S^*_{m,q}(1;u) \\
&= \Big(\frac{s_m^4(u)}{q^2} + \frac{3q^2+2q+1}{q[m]_q} s_m^3(u) + \frac{3q^3+3q+q}{[m]_q^2} s_m^2(u) \\
&\quad + \frac{q^4}{[m]_q^3} s_m(u) \Big) \frac{[m+c]_q^4}{[m+d]_q^4}..
\end{aligned}
$$

□

Lemma 3.3. *For $u \in [0,\infty)$, $c, d \in \mathbb{N}$ and $0 \le c \le d$, we have*

$$
S^*_{m,q}((t-u);u) = q \frac{[m+c]_q}{[m+d]_q} s_m(u) - u \tag{3.7}
$$

$$
S^*_{m,q}((t-u)^2;u) = 2u \Big(u - q \frac{[m+c]_q}{[m+d]_q} s_m(u) \Big) \tag{3.8}
$$

and

$$
| S^*_{m,q}((t-u);u) | \le \frac{1-q}{\sqrt{q}+1} u - \frac{q^2[m+c]_q}{[m]_q[m+d]_q(\sqrt{q}+1)}
$$

$$
S^*_{m,q}((t-u)^2;u) \le \frac{2(1-q)}{\sqrt{q}+1} u^2 + \frac{2q^2[m+c]_q u}{[m]_q[m+d]_q(\sqrt{q}+1)} := \delta_m(u). \tag{3.9}
$$

Proof. From Lemma (3.2), using inequalities, (3.7) and (3.8) we get

$$\begin{aligned}
\mid S^*_{m,q}((t-u);u)\mid &= \left|\frac{-q^2[m+c]_q+\sqrt{q^4[m+c]_q^2+4q[m]_q^2[m+d]_q^2u^2}}{2[m]_q[m+d]_q}-u\right| \\
&= \left|\frac{2(q[m]_q[m+d]_qu^2-q^2[m+c]_qu-[m]_q[m+d]_qu^2)}{q^2[m+c]_q+\sqrt{q^4[m+c]_q^2+4q[m]_q^2[m+d]_q^2u^2+2[m]_q[m+d]_qu}}\right| \\
&\leq \left|\frac{q[m]_q[m+d]_qu-q^2[m+c]_q-[m]_q[m+d]_qu}{\sqrt{q}[m]_q[m+d]_q+[m]_q[m+d]_q}\right| \\
&\leq \frac{q-1}{\sqrt{q}+1}u-\frac{q^2[m+c]_q}{[m]_q[m+d]_q(\sqrt{q}+1)}.
\end{aligned}$$

$$\begin{aligned}
S^*_{m,q}((t-u)^2;u) &= \frac{4([m]_q[m+d]_qu^3-q[m]_q[m+d]_qu^3+q^2[m+c]_qu^2)}{q^2[m+c]_q+\sqrt{q^4[m+c]_q^2+4q[m]_q^2[m+d]_q^2u^2+2[m]_q[m+d]_qu}} \\
&\leq \frac{2(q^2[m+c]_qu+[m]_q[m+d]_qu^2-q[m]_q[m+d]_qu^2)}{\sqrt{q}[m]_q[m+d]_q+[m]_q[m+d]_q} \\
&\leq \frac{2(1-q)}{\sqrt{q}+1}u^2+\frac{2q^2[m+c]_qu}{[m]_q[m+d]_q(\sqrt{q}+1)} \\
&:= \delta_m(u).
\end{aligned}$$

□

4. Approximation Properties

The weighted Korovkin-type theorems were demonstrated by Gadzhiev [12]. Let $\gamma(u)=1+u^2$ and $B_\gamma[0,\infty)$ be the set of all functions g satisfying the condition, $\mid g(u)\mid\leq M_g\gamma(u)$, where M_g is a constant depending only on g. $B_\gamma[0,\infty)$ is a normed space with the norm $\| g\|_\gamma=\sup\{\mid g(u)\mid/\gamma(u): u\geq 0\}$, for any $g\in B_\gamma[0,\infty)$. $C_\gamma[0,\infty)$ denotes the subspace of all continuous functions in $B_\gamma[0,\infty)$ and $C^*_\gamma[0,\infty)$ denotes the subspace of all functions $g\in C_\gamma[0,\infty)$ for which $\lim_{|u|\to\infty}(g(u)/\gamma(u))$ exists finitely.

Theorem 5.1. Let $q_m\in(0,1)$, such that $q_m\to 1$ as $m\to\infty$. Then, for every $g\in C^*_\gamma[0,\infty)$, we have

$$\lim_{m\to\infty}\| S^*_{m,q_m}(g;u)-g(u)\|_\gamma = 0.$$

Proof. Taking weighted Korovkin theorem given by [13], it is enough to check the following three assertions

$$\lim_{m\to\infty}\| S^*_{m,q_m}(t^r;u)-u^r\|_\gamma=0,\qquad r=0,1,2. \tag{4.10}$$

Since, $S^*_{m,q_m}(1;u)=1$, the first condition of (4.10) is satisfied for $r=0$. Now,

$$\begin{aligned}
\| S^*_{m,q_m}(t;u)-u\|_\gamma &= \sup_{u\in[0,\infty)}\frac{\mid S^*_{m,q_m}(t;u)-u\mid}{1+u^2} \\
&\leq \left|\frac{q_m-1}{\sqrt{q_m}+1}-1\right|\sup_{u\in[0,\infty)}\frac{u}{1+u^2}-\frac{q_m^2[m+c]_{q_m}}{[m]_{q_m}[m+d]_{q_m}(\sqrt{q_m}+1)} \\
&\leq \left|\frac{q_m-1}{\sqrt{q_m}+1}-1\right|-\frac{q_m^2[m+c]_{q_m}}{[m]_{q_m}[m+d]_{q_m}(\sqrt{q_m}+1)}
\end{aligned}$$

which implies that the condition in (4.10) holds for $r = 1$. In a similar fashion, we can write

$$\begin{aligned} \| S^*_{m,q_m}(t;u) - u^2 \|_\gamma &= \sup_{u\in[0,\infty)} \frac{| S^*_{m,q_m}(t;u) - u^2 |}{1+u^2} \\ &\le \left|\frac{2(1-q_m)}{\sqrt{q_m}+1} - 1\right| \sup_{u\in[0,\infty)} \frac{u^2}{1+u^2} \\ &+ \left|\frac{2q_m^2[m+c]_{q_m}}{[m]_{q_m}[m+d]_{q_m}(\sqrt{q_m}+1)}\right| \sup_{u\in[0,\infty)} \frac{u}{1+u^2} \\ &\le \left|\frac{2(1-q_m)}{\sqrt{q_m}+1} - 1\right| + \left|\frac{2q_m^2[m+c]_{q_m}}{[m]_{q_m}[m+d]_{q_m}(\sqrt{q_m}+1)}\right|. \end{aligned}$$

Which implies that

$$\lim_{m\to\infty} \| S^*_{m,q_m}(t^2;u) - u^2 \|_\gamma = 0,$$

(4.10) holds for $r = 2$. □

Theorem 5.2. Let $\alpha > 0$, $q_m \in (0,1)$, such that $q_m \to 1$ as $m \to \infty$ and $g \in C^*_\gamma[0,\infty)$. Then, we have the following result

$$\lim_{m\to\infty} \sup_{u\in[0,\infty)} \frac{| S^*_{m,q_m}(g;u) - g(u) |}{(1+u^2)^\alpha} = 0.$$

Proof. For any fixed $u_0 \in [0,\infty)$, then

$$\begin{aligned} \sup_{u\in[0,\infty)} \frac{| S^*_{m,q_m}(g;u) - g(u) |}{(1+u^2)^{1+\alpha}} &\le \sup_{u\le u_0} \frac{| S^*_{m,q_m}(g;u) - g(u) |}{(1+u^2)^{1+\alpha}} + \sup_{u>u_0} \frac{| S^*_{m,q_m}(g;u) - g(u) |}{(1+u^2)^{1+\alpha}} \\ &\le \| S^*_{m,q_m}(g) - g \|_{C[0,u_0]} + \| g \|_\gamma \sup_{u>u_0} \frac{| S^*_{m,q_m}(1+t^2;u) |}{(1+u^2)^{1+\alpha}} \end{aligned}$$

$$+ \sup_{u>u_0} \frac{| g(u) |}{(1+u^2)^{1+\alpha}}. \tag{4.11}$$

Since $| g(u) | \le \| g \|_\gamma (1+u^2)$, we have $\sup_{u>u_0} \frac{|g(u)|}{(1+u^2)^{1+\alpha}} \le \frac{\|g\|_\gamma}{(1+u_0^2)^\alpha}$. Let's choose $\varepsilon > 0$ to be arbitrary. We can have u_0 to be too large, so that

$$\frac{\| g \|_\gamma}{(1+u_0^2)^\alpha} < \frac{\varepsilon}{3}. \tag{4.12}$$

In view of Lemma (3.2) we obtain

$$\begin{aligned} \| g \|_\gamma \lim_{m\to\infty} \frac{| S^*_{m,q_m}(1+t^2;u) |}{(1+u^2)^{1+\alpha}} &= \frac{1+u^2}{(1+u^2)^{1+\alpha}} \| g \|_\gamma = \frac{\| g \|_\gamma}{(1+u^2)^\alpha} \\ &\le \frac{\| g \|_\gamma}{(1+u_0^2)^\alpha} < \frac{\varepsilon}{3}. \end{aligned} \tag{4.13}$$

Using Korovkin's theorem, we can figure out that the first term of the inequality (4.11), implies that

$$\| S^*_{m,q_m}(g) - g \|_{C[0,u_0]} < \frac{\varepsilon}{3}, \qquad as \qquad m \to \infty. \tag{4.14}$$

Therefore, combining (4.12) - (4.14), we get the wanted result. □

5. Local approximation

In this part, we obtain local approximation for the operators $S^*_{m,q}$. By $C_B[0,\infty)$, represent the space of real-valued continuous and bounded functions g on $[0,\infty)$. The norm on $C_B[0,\infty)$ is given by

$$\| g \|_\infty = \sup_{0\leq u<\infty} | g(u) | .$$

K-functional is given by

$$K_2(g,\vartheta) = \inf_{g\in W^2}\{\| g-h \|_\infty + \vartheta \| h'' \|_\infty\},$$

where $\vartheta > 0$ and $W^2 = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$. In view (2.4) of [10], there consist an absolute constant $C > 0$ such that

$$K_2(g,\vartheta) \leq C\omega_2(g, \sqrt{\vartheta}) \tag{5.15}$$

where

$$\omega_2(g,\sqrt{\vartheta}) = \sup_{0<t\leq\sqrt{\vartheta}} \sup_{u\in[0,\infty)} | g(u+2t) - 2g(u+t) + g(u) |$$

is the second-order modulus of smoothness of $g \in C_B[0,\infty)$. we state the usual modulus of continuity of $g \in C_B[0,\infty)$ is given by

$$\omega(g,\vartheta) = \sup_{0<t\leq\vartheta} \sup_{0\leq u<\infty} | g(u+t) - g(u) | .$$

Theorem 5.1. *Let $g \in C_B[0,\infty)$ and $0 < q < 1$. For all $m \in \mathbb{N}$, $c, d \in \mathbb{N}$ and $0 \leq c \leq d$, and for constant $C > 0$ we obtain*

$$| S^*_{m,q}(g;u) - g(u) | \quad \leq \quad C\omega_2(g, \gamma_{m,q}(u)) + \omega(g, \alpha_{m,q}(u)),$$

where

$$\gamma_{m,q}(u) = \sqrt{\frac{2(1-q)}{\sqrt{q}+1}u^2 + \frac{2q^2[m+c]_q u}{[m]_q[m+d]_q(\sqrt{q}+1)} + (\alpha_{m,q}(u))^2}$$

,

$$\alpha_{m,q}(u) = \frac{1-q}{\sqrt{q}+1}u - \frac{q^2[m+c]_q}{[m]_q[m+d]_q(\sqrt{q}+1)}.$$

Proof. For $u \in [0,\infty)$,such that $\bar{S}^*_m$ defined by

$$\bar{S}^*_m(g;u) \quad = \quad S^*_{m,q}(g;u) + g(u) - g(s_m(u)).$$

From Lemma (3.2), we have

$$\begin{aligned} \bar{S}^*_m(1;u) &= S^*_{m,q}(1;u) + 1 - 1 = 1 \\ \bar{S}^*_m(t;u) &= S^*_{m,q}(t;u) + u - s_m(u) = u \\ \bar{S}^*_m((t-u);u) &= \bar{S}^*_m(t;u) - u\bar{S}^*_m(1;u) = 0. \end{aligned}$$

Let $0 \leq u < \infty$ and $f \in W^2$. By Taylor's formula we have

$$g(t) = g(u) + g'(u)(t-u) + \int_u^t (t-s)g''(s)\mathrm{d}s.$$

From above equation, we have

$$\begin{aligned}\bar{S}^*_m(g;u)-g(u) &= g'(u)\bar{S}^*_m((t-u);u)+\bar{S}^*_m\left(\int_u^t (t-s)g''(s)\mathrm{d}s;u\right)\\ &= S^*_{m,q}\left(\int_u^t (t-s)g''(s)\mathrm{d}s;u\right)\\ &\quad -\int_u^{q\frac{[m+c]_q}{[m+d]_q}s_m(u)}\left(q\frac{[m+c]_q}{[m+d]_q}s_m(u)-s\right)g''(u)\mathrm{d}u.\end{aligned}$$

Therefore

$$\left|\int_u^t (t-s)g''(s)\mathrm{d}s\right| \le \int_u^t \mid t-s\parallel g''(s)\mid \mathrm{d}s \le \parallel g''\parallel \int_u^t \mid t-s\mid \mathrm{d}s \le (t-u)^2 \parallel g''\parallel$$

$$\left|\int_u^{q\frac{[m+c]_q}{[m+d]_q}s_m(u)}\left(q\frac{[m+c]_q}{[m+d]_q}s_m(u)-s\right)g''(s)\mathrm{d}s\right|$$

$$\le \left(q\frac{[m+c]_q}{[m+d]_q}s_m(u)-u\right)^2 \parallel g''\parallel .$$

We conclude that

$$\begin{aligned}\left|\bar{S}^*_m(g;u)-g(u)\right| &\le \left|S^*_{m,q}\left(\int_u^t (t-s)g''(s)\mathrm{d}s;u\right)\right.\\ &\quad \left.-\int_u^{q\frac{[m+c]_q}{[m+d]_q}s_m(u)}\left(q\frac{[m+c]_q}{[m+d]_q}s_m(u)-s\right)g''(s)\mathrm{d}s\right|\\ &\le \parallel g''\parallel S^*_{m,q}((t-u)^2;u)+\parallel g''\parallel \left(q\frac{[m+c]_q}{[m+d]_q}s_m(u)-u\right)^2\\ &= \parallel g''\parallel \gamma^2_{m,q}(u).\end{aligned}$$

Now, taking $\bar{S}^*_m$, we get

$$\mid \bar{S}^*_m(g;u)\mid \le \mid S^*_{m,q}(g;u)\mid +2\parallel g\parallel \le 3\parallel g\parallel .$$

Therefore

$$\begin{aligned}\mid S^*_{m,q}(g;u)-g(u)\mid &\le \mid \bar{S}^*_m(g-f;u)-(g-f)(u)\mid\\ &\quad +\left|g\left(q\frac{[m+c]_q}{[m+d]_q}s_m(u)\right)-g(u)\right|+\mid \bar{S}^*_m(f;u)-f(u)\mid\\ &\le \mid \bar{S}^*_m(g-f;u)\mid+\mid(g-f)(u)\mid\\ &\quad +\left|g\left(q\frac{[m+c]_q}{[m+d]_q}s_m(u)\right)-g(u)\right|+\mid \bar{S}^*_m(f;u)-f(u)\mid\\ &\le \omega(g,\alpha_{m,q}(u))+\gamma^2_{m,q}(u)\parallel f''\parallel +4\parallel g-f\parallel .\end{aligned}$$

Taking the infimum over all $f\in W^2$, we get

$$\mid S^*_{m,q}(g;u)-g(u)\mid \le \omega(g,\alpha_{m,q}(u))+4K_2(g,\gamma^2_{m,q}(u)).$$

From K-functional, we have

$$\mid S^*_{m,q}(g;u)-g(u)\mid \le \omega(g,\alpha_{m,q}(u))+C\omega_2(g,\gamma_{m,q}(u)).$$

which gives the proof. □

Theorem 5.2. *Let $q = q_m \in (0,1)$, we have $q_m \to 1$, as $m \to \infty$. For $g \in C_i^*(0,\infty]$ and $g^*(w) = g(w^2), 0 \le w < \infty$. Then for all $m \in \mathbb{N}$, $c, d \in \mathbb{N}$ and $0 \le c \le d$, $u \ge 0$, we have*

$$| S^*_{m,q_m}(g;u) - g(u) | \quad \le \quad 2\omega\Big(g^*, \sqrt{\frac{1-q_m}{\sqrt{q_m}+1}u - \frac{q_m^2[m+c]_{q_m}}{[m]_{q_m}[m+d]_{q_m}(\sqrt{q_m}+1)}}\Big).$$

Proof. Let $f \in C^*_m(0,\infty]$ is fixed. By g^* we have

$$S^*_{m,q_m}(g;u) \quad = \quad S^*_{m,q_m}(g^*(\sqrt{\cdot});u).$$

Now,

$$\begin{aligned}
| S^*_{m,q_m}(g;u) - g(u) | &= | S^*_{m,q_m}(g^*(\sqrt{\cdot});u) - g^*\sqrt{u} | \\
&= \Big| \sum_{j=0}^{\infty} \Big\{ g^*\Big(\frac{[j]_{q_m}}{[m]_{q_m}} \frac{[m+c]_{q_m}}{[m+d]_{q_m} q_m^{j-2}} \Big) - g^*(\sqrt{u}) \Big\} S_{m,j}(q_m;s_m(u)) \Big| \\
&\le \sum_{j=0}^{\infty} \Big| g^*\Big(\frac{[j]_{q_m}}{[m]_{q_m}} \frac{[m+c]_{q_m}}{[m+d]_{q_m} q_m^{j-2}} \Big) - g^*(\sqrt{u}) \Big| S_{m,j}(q_m;s_m(u)) \\
&\le \sum_{j=0}^{\infty} \omega\Big(g^*; \Big| \frac{[j]_{q_m}}{[m]_{q_m}} \frac{[m+c]_{q_m}}{[m+d]_{q_m} q_m^{j-2}} - \sqrt{u} \Big| \Big) S_{m,j}(q_m;s_m(u)) \\
&= \sum_{j=0}^{\infty} \omega\Bigg(g^*; \frac{\Big| \sqrt{\frac{[j]_{q_m}}{[m]_{q_m}} \frac{[m+c]_{q_m}}{[m+d]_{q_m} q_m^{j-2}}} - \sqrt{u} \Big|}{S^*_{m,q_m}(| \sqrt{\cdot} - \sqrt{u} |;u)} S^*_{m,q_m}(| \sqrt{\cdot} - \sqrt{u} |;u) \Bigg) \\
&\quad \times S_{m,j}(q_m;s_m(u))
\end{aligned}$$

where

$$S_{m,j}(q_m;s_m(u)) \quad = \quad E_{q_m}(-[m]_{q_m}s_m(u)) \sum_{j=0}^{\infty} \frac{q_m^{\frac{j(j-1)}{2}}([m]_{q_m}s_m(u))^j}{[j]_{q_m}!},$$

from modulus of continuity we get

$$\omega(g^*;\alpha\vartheta) \quad \le \quad (1+\alpha)\omega(g^*;\vartheta),$$

where $\alpha, \vartheta \ge 0$, we obtain

$$\begin{aligned}
| S^*_{m,q_m}(g;u) - g(u) | &\le \omega(g^*; S^*_{m,q_m}(| \sqrt{\cdot} - \sqrt{u} |;u)) \\
&\quad \times \sum_{j=0}^{\infty} \Bigg(1 + \frac{\Big| \sqrt{\frac{[j]_{q_m}}{[m]_{q_m}} \frac{[m+c]_{q_m}}{[m+d]_{q_m} q_m^{j-2}}} - \sqrt{u} \Big|}{S^*_{m,q_m}(| \sqrt{\cdot} - \sqrt{u} |;u)} \Bigg) S_{m,j}(q_m;s_m(u)) \\
&= 2\omega(g^*; S^*_{m,q_m}(| \sqrt{\cdot} - \sqrt{u} |;u)).
\end{aligned}$$

Since $\dfrac{1}{\sqrt{\frac{[j]_{q_m}}{[m]_{q_m}}\frac{[m+c]_{q_m}}{[m+d]_{q_m}q_m^{j-2}}}+\sqrt{u}} \leq \dfrac{1}{\sqrt{u}}$ and using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
S^*_{m,q_m}(|\sqrt{\cdot}-\sqrt{u}|;u) &= \sum_{j=0}^{\infty}\left|\sqrt{\frac{[j]_{q_m}}{[m]_{q_m}}\frac{[m+c]_{q_m}}{[m+d]_{q_m}q_m^{j-2}}}-\sqrt{u}\right|S_{m,j}(q_m;s_m(u))\\
&= \sum_{j=0}^{\infty}\frac{\left|\frac{[j]_{q_m}}{[m]_{q_m}}\frac{[m+c]_{q_m}}{[m+d]_{q_m}q_m^{j-2}}-u\right|}{\sqrt{\frac{[j]_{q_m}}{[m]_{q_m}}\frac{[m+c]_{q_m}}{[m+d]_{q_m}q_m^{j-2}}}+\sqrt{u}}S_{m,j}(q_m;s_m(u))\\
&\leq \frac{1}{\sqrt{u}}\sum_{j=0}^{\infty}\left|\frac{[j]_{q_m}}{[m]_{q_m}}\frac{[m+c]_{q_m}}{[m+d]_{q_m}q_m^{j-2}}-u\right|S_{m,j}(q_m;s_m(u))\\
&\leq \frac{1}{\sqrt{u}}\sqrt{\sum_{j=0}^{\infty}\left|\frac{[j]_{q_m}}{[m]_{q_m}}\frac{[m+c]_{q_m}}{[m+d]_{q_m}q_m^{j-2}}-u\right|^2 S_{m,j}(q_m;s_m(u))}\\
&\leq \frac{1}{\sqrt{u}}\sqrt{S^*_{m,q_m}((\cdot-u)^2;s_m(u))}\\
&\leq \sqrt{\frac{1-q_m}{\sqrt{q_m}+1}u-\frac{q_m^2[m+c]_{q_m}}{[m]_{q_m}[m+d]_{q_m}(\sqrt{q_m}+1)}}
\end{aligned}$$

we get the desired result. □

From Lipschitz constant is given by

$$lip_D(\varrho) = g \in C_B[0,\infty); |g(t)-g(u)| \leq D\frac{|t-u|^{\varrho}}{(t+u)^{\frac{\varrho}{2}}}, \tag{5.16}$$

we study ordinary approximation where D is a positive constant, $0 < \varrho \leq 1$.

Theorem 5.3. *Let $g \in C_B[0,\infty)$, $m \in \mathbb{N}$, $q \in (0,1)$, $c, d \in \mathbb{N}$, for any $u \in (0,\infty)$, we have*

$$|S^*_{m,q}(g;u)-g(u)| \leq D\left(\frac{\zeta_{m,q}(u)}{u}\right)^{\frac{\varrho}{2}},$$

where

$$\zeta_{m,q}(u) = S^*_{m,q}((t-u)^2;u).$$

Proof. For $\varrho = 1$ and $g \in lip_D(\varrho)$, we obtain

$$\begin{aligned}
|S^*_{m,q}(g;u)-g(u)| &\leq \sum_{j=0}^{\infty}S_{m,j}(q;s_m(u))\left|g\Big(\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_q q^{j-2}}\Big)-g(u)\right|\\
&\leq D\sum_{j=0}^{\infty}\frac{\left|\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_q q^{j-2}}-u\right|}{\sqrt{\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_q q^{j-2}}+u}}.
\end{aligned}$$

By using $\sqrt{u} < \sqrt{\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_q q^{j-2}}+u}$ from Cauchy Schwarz inequality, we obtain

$$\begin{aligned}|S^*_{m,q}(g;u)-g(u)| &\leq \frac{D}{\sqrt{u}}\sum_{j=0}^{\infty} S_{m,j}(q;s_m(u))\Big|\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_q q^{j-2}}-u\Big| \\ &= \frac{D}{\sqrt{u}}S^*_{m,q}((t-u)^2;u) \leq D\sqrt{\frac{\zeta_{m,q}(u)}{u}}.\end{aligned}$$

The result for $\varrho = 1$, is true. Now prove that $0 < \varrho \leq 1$ is true, From Holder's inequality, $p = \frac{2}{\varrho}, q = \frac{1}{2-\varrho}$, we have

$$\begin{aligned}|S^*_{m,q}(g;u)-g(u)| &\leq \sum_{j=0}^{\infty} S_{m,j}(q;s_m(u))\Big|g\Big(\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_q q^{j-2}}\Big)-g(u)\Big| \\ &\leq \sum_{j=0}^{\infty}\left\{S_{m,j}(q;s_m(u))\Big(\Big|g\Big(\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_q q^{j-2}}\Big)-g(u)\Big|\Big)^{\frac{2}{\varrho}}\right\}^{\frac{\varrho}{2}} \\ &\quad\times\left\{\sum_{j=0}^{\infty} S_{m,j}(q;s_m(u))\right\}^{\frac{2-\varrho}{2}}\end{aligned}$$

$$\leq \left\{\sum_{j=0}^{\infty} S_{m,j}(q;s_m(u))\Big|g\Big(\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_q q^{j-2}}\Big)-g(u)\Big|^{\frac{2}{\varrho}}\right\}^{\frac{\varrho}{2}}.$$

Since $g \in lip_D(\varrho)$, we have

$$\begin{aligned}|S^*_{m,q}(g;u)-g(u)| &\leq \frac{D}{u^{\frac{\varrho}{2}}}\left\{\sum_{j=0}^{\infty} S_{m,j}(q;s_m(u))\Big(\frac{[j]_q}{[m]_q}\frac{[m+c]_q}{[m+d]_q q^{j-2}}-u\Big)^2\right\}^{\frac{\varrho}{2}} \\ &= \frac{D}{u^{\frac{\varrho}{2}}}\left\{S^*_{m,q}((t-u))^2;u)\right\}^{\frac{\varrho}{2}} \leq D\Big(\sqrt{\frac{\zeta_{m,q}(u)}{u}}\Big)^{\varrho}.\end{aligned}$$

Therefore, the proof is completed. □

6. Conclusion

We introduced a King type modification of q-Szász-Mirakjan operators which reproduce u^2 and established different approximation results. We obtained some preliminaries such as moments, central moments and uniform convergence of these operators. Moreover, the local approximation and weighted approximation properties of these new operators in terms of modulus of continuity are studied.

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