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Compact almost Co-Kähler manifolds and Ricci-Yamabe solitons

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Abstract. In this article we establish that if the metric g of a compact almost Co-Kähler manifold $\mathcal{M}^{2n+1}$ is a Ricci-Yamabe soliton whose potential vector field is point-wise collinear with the characteristic vector field, then $\mathcal{M}^{2n+1}$ is a K-almost Co-Kähler manifold under certain condition, whereas in dimension three the restriction is not required. It is proved that if a (2n+1)-dimensional (κ, μ)-almost Co-Kähler manifold $\mathcal{M}$ with $\kappa < 0$ admits a Ricci-Yamabe soliton of gradient type, then $\mathcal{M}$ is a $N(\kappa)$-almost Co-Kähler manifold. We also show the non-existence of gradient Ricci-Yamabe structures with $D\Psi = (\zeta\Psi)\zeta$ on a compact (κ, μ)-almost Co-Kähler manifold with $\kappa < 0$. Then we establish that in a Co-Kähler 3-manifold $\mathcal{M}^3$ with gradient Ricci-Yamabe solitons, the scalar curvature of the manifold is constant and also, either $\mathcal{M}^3$ is flat, or the gradient of the potential function is collinear with the characteristic vector field ζ. Finally, we construct two non-trivial examples to ensure the existence of such solitons.

1. Introduction

According to the famous Goldberg conjecture, any compact Einstein almost Co-Kähler manifold is integrable in complex geometry and any compact Einstein almost Co-Kähler manifold is Co-Kähler, which is the analogue of this conjecture in contact geometry. Co-Kähler manifolds are actually odd-dimensional versions of Kähler manifolds and are one of the most significant research topics in contact geometry. So, one might query if the Co-Kähler geometry framework contains a Goldberg-like conjecture. For Co-Kähler manifolds, a conjecture like-wise Goldberg was recently derived in [5], stating that any compact Einstein K-almost Co-Kähler manifold is Co-Kähler. Additionally, with an η-Einstein condition, Cappelletti-Montano and Pastore [5] presented a necessary criterion for a compact K-almost Co-Kähler manifold to be Co-Kähler.

Recently, differential geometry of almost contact Riemannian manifolds has focused a lot on the Ricci-Yamabe solitons, a scalar combination of the Yamabe and Ricci soliton. Consequently, it is a fascinating subject to investigate the existence and classification of Ricci-Yamabe solitons on almost Co-Kähler manifolds. According to our awareness, the literature contains many findings regarding Riemannian manifolds [12], spacetimes [18] and $f(R)$-gravity [9] with Ricci-Yamabe solitons, but there are just a few results in

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almost Co-Kähler manifolds. We want to fill this gap in this article and focus on investigating the almost Co-Kähler manifolds that satisfy Ricci-Yamabe solitons.

Goldberg's conjecture has so far only been verified for manifolds with non-negative scalar curvatures, but its contact geometry analogue has been affirmed for manifolds with Killing characteristic vector fields in [5]. Then, Wang [21] extended this result and established that

Theorem 1.1. *A compact almost Co-Kähler manifold that admits a Ricci soliton is Ricci-flat and Co-Kähler if the potential vector field is point-wise collinear with the characteristic vector field* ζ.

In this article we generalize the above result by replacing Ricci solitons with Ricci-Yamabe solitons.

In recent years, many mathematicians have become fascinated with the theoretical rather analytical idea of geometric flows like the Yamabe flow and the Ricci flow. Under the name Ricci-Yamabe map, Guler and Crasmareanu [14] recently published the analysis of another geometric flow. This is also described as the (α, β) type Ricci-Yamabe flow. Ricci-Yamabe flow represents a development of the metrics on the Riemannian manifold proposed in [14] and defined as

$$\frac{\partial}{\partial t}g(t) = -2\alpha Ric(t) + \beta\mathcal{R}(t)g(t), \quad g_0 = g(0), \tag{1}$$

in which $\mathcal{R}$ is the scalar curvature, Ric indicates the Ricci tensor, and $\alpha, \beta \in \mathbb{R}$.

Because of the signs of the relevant scalars, α and β, Ricci-Yamabe flow can be thought of as Riemannian, semi-Riemannian, or singular Riemannian flow. This wide range of choices may be useful in a physical, or mathematical model, such as relativistic theories. Consequently, the constraint of the Ricci-Yamabe flow soliton typically causes the Ricci-Yamabe soliton to emerge. Despite the fact that Yamabe and Ricci solitons are equivalent in dimension 2, they are fundamentally different in higher dimensions, which served as even another powerful motivation for beginning the study of Ricci-Yamabe solitons.

A Ricci-Yamabe soliton in a almost Co-Kähler manifold $\mathcal{M}$ is described by

$$£_V g = -2\alpha Ric + (\beta\mathcal{R} - 2\lambda)g, \tag{2}$$

in which $\lambda \in \mathbb{R}$ and £ denotes the Lie-derivative. Here, V is called the potential vector field of the soliton.

Let V be the gradient of Ψ, a smooth function. Hence, the preceding idea is named gradient Ricci-Yamabe soliton and (2) turns into

$$\nabla^2\Psi = -\alpha Ric - (\lambda - \frac{1}{2}\beta r)g, \tag{3}$$

where $\nabla^2\Psi$ indicates the Hessian of Ψ.

The Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is termed expanding if $\lambda > 0$, whereas shrinking for $\lambda < 0$ and steady when $\lambda = 0$.

Recently, in almost Co-Kähler manifolds, several authors studied various type of solitons such as Ricci ([21], [22]) and Yamabe solitons [19], gradient Yamabe and gradient Einstein solitons[11], (m, ρ)-quasi-Einstein solitons [10], cotton solitons [7], respectively.

According to our knowledge, the study of almost Co-Kähler manifolds with Ricci-Yamabe solitons is still open. We decide to fill up this gap in this article and focus on characterizing the almost Co-Kähler manifolds that satisfy the Ricci-Yamabe soliton. Also, with and without the assumption of compactness, we study (κ, μ)-almost Co-Kähler manifolds admitting a gradient Ricci-Yamabe soliton. Next, we study the gradient Ricci-Yamabe soliton on a 3-dimensional Co-Kähler manifold. Lastly, we establish the existence of gradient Ricci-Yamabe solitons on almost Co-Kähler manifolds and Co-Kähler manifolds, respectively by creating new examples.

2. Preliminaries

Let $\mathcal{M}^{2n+1}$ be an almost contact manifold with almost contact structure (φ, ζ, η), in which ζ indicates a unit vector field, φ is a $(1, 1)$-tensor field and η denotes a 1-form satisfying:

$$\varphi^2(E) = -E + \eta(E)\zeta, \quad \eta(\zeta) = 1. \tag{4}$$

Equation (4) readily yields

$$\varphi\zeta = 0, \quad g(E,\zeta) = \eta(E), \quad \eta(\varphi X) = 0,$$
$$g(\varphi E, \varphi F) = g(E,F) - \eta(E)\eta(F), \tag{5}$$
$$g(\varphi E, F) = -g(E, \varphi F), \quad g(\varphi E, E) = 0,$$

for all smooth vector fields E, F.

Let us define the 2-form Φ as:

$$\Phi(E,F) = g(E, \varphi F),$$

for any E, F defined earlier.

An almost Co-Kähler manifold fulfills the conditions $d\Phi = 0$ and $d\eta = 0$ [13]. In particular, an almost Co-Kähler manifold is said to be a Co-Kähler manifold if it is normal, which is identical to $\nabla\Phi = 0$. An almost Co-Kähler structure is called strictly almost Co-Kähler if it is not a Co-Kähler structure. According to Blair [2], an almost cosymplectic manifold and an almost Co-Kähler manifold are the same.

Let $\mathcal{M}^{2n+1}$ be an almost Co-Kähler manifold. The operators h and ℓ in the manifold are described by $h = \frac{1}{2}\pounds_\zeta\varphi$ and $l = \mathcal{K}(.,\zeta)\zeta$ in which $\mathcal{K}$ is the curvature tensor, satisfy the subsequent relations([16], [17]):

$$h\varphi + \varphi h = 0, \quad h\zeta = 0, \quad \operatorname{tr} h = \operatorname{tr} h' = 0, \tag{6}$$
$$\nabla_\zeta\varphi = 0, \quad \nabla\zeta = h', \tag{7}$$
$$\nabla_\zeta h = -h^2\varphi - \varphi\ell,$$
$$\varphi\ell\varphi - \ell = 2h^2, \tag{8}$$
$$Ric(\zeta,\zeta) + \operatorname{tr} h^2 = 0. \tag{9}$$

Any Co-Kähler manifold is a K-almost Co-Kähler manifold, as is well known, but the converse is usually not true. Yet, in a 3-dimensional manifold, it is valid.

Lemma 2.1. *[13] In dimension three, any almost Co-Kähler manifold is Co-Kähler if and only if ζ is Killing.*

In a manifold $\mathcal{M}^3$ of dimension three, the curvature tensor $\mathcal{K}$ can be described as

$$\begin{aligned}\mathcal{K}(E,F)G &= Ric(F,G)E - Ric(E,G)F + g(F,G)QE - g(E,G)QF \\ &- \frac{\mathcal{R}}{2}\{g(F,G)E - g(E,G)F\}\end{aligned} \tag{10}$$

for any vector fields E, F, G.

In a 3-dimensional Co-Kähler manifold, we have[20]

$$QE = \frac{\mathcal{R}}{2}E - \frac{\mathcal{R}}{2}\eta(E)\zeta, \tag{11}$$

in which the Ricci operator Q is described by $g(QE,F) = Ric(E,F)$.

Proposition 2.2. *In a almost Co-Kähler 3- manifold, we have*

$$\zeta\mathcal{R} = 0. \tag{12}$$

Proof. From (11), we acquire

$$(\nabla_E Q)F = \frac{(E\mathcal{R})}{2}[F - \eta(F)\zeta]. \tag{13}$$

Contracting E from the above equation, we get (12).

Hence the result.

3. Ricci-Yamabe solitons on Almost Co-Kähler manifolds

On an almost Co-Kähler manifold $\mathcal{M}^{2n+1}(\eta,\zeta,\varphi,g)$, we acquire $(\pounds_\zeta g)(E,F) = 2g(h'E,F)$, where we have used the second equation of (7). From the previous expression we conclude that the Characteristic vector field ζ is Killing if and only if h vanishes.

Definition 3.1. *If the characteristic vector field ζ is Killing, an almost Co-Kähler manifold is referred to as a K-almost Co-Kähler manifold.*

On an almost Co-Kähler manifold, the distribution indicated by $\mathcal{D}$ is described by $\mathcal{D} = Ker\eta$. Hence, an almost Co-Kähler structure $(g_{\mathcal{D}},\varphi_{\mathcal{D}})$ on $\mathcal{D}$ can easily be obtained by using $d\Phi = 0$ and (4), (5). In [16] Olszak established that the almost Co-Kähler structure is integrable if and only if

$$(\nabla_E\varphi)F = g(hE,F)\zeta - \eta(F)hE, \tag{14}$$

for all E, F, which entails that an almost Co-Kähler manifold is Co-Kähler if and only if it is K-almost Co-Kähler, also the associated almost Co-Kähler structure is integrable. In [5], Cappelletti-Montano and Pastore have given another characterization of Co-Kähler structure and deduced that

Theorem 3.2. *Every Einstein compact K-almost Co-Kähler manifold is a Co-Kähler manifold.*

Now, we generalize the Theorem 1.1 as follows:

Theorem 3.3. *If the potential vector field is point-wise collinear with ζ, then a compact almost Co-Kähler manifold admitting a Ricci-Yamabe soliton reduces to a K-almost Co-Kähler manifold, provided the scalar curvature remains invariant under the characteristic vector field ζ.*

Proof. For a non-zero smooth function ν, we suppose that the potential function $V = \nu\zeta$.

Hence, we get

$$\nabla_E V = (E\nu)\zeta + \nu h'E, \tag{15}$$

where equation (7) is used. Now, from equation (2), we acquire

$$\begin{aligned} &\nu g(\nabla_E\zeta,F) + (E\nu)\eta(F) + \nu g(\nabla_F\zeta,E) + (F\nu)\eta(E) \\ &= -2\alpha Ric(E,F) - (2\lambda - \beta\mathcal{R})g(E,F), \end{aligned}$$

which implies

$$\begin{aligned} &2\nu g(h'E,F) + (E\nu)\eta(F) + (F\nu)\eta(E) \\ &= -2\alpha Ric(E,F) - (2\lambda - \beta\mathcal{R})g(E,F). \end{aligned} \tag{16}$$

The foregoing equation is equivalent to

$$2\nu h'E + (E\nu)\zeta + (D\nu)\eta(E) = -2\alpha QE - (2\lambda - \beta\mathcal{R})E, \tag{17}$$

in which $D\nu$ indicates the gradient of ν.

Using $\operatorname{tr} h = \operatorname{tr} h' = 0$ and putting $E = F = e_i$ in (16), where $\{e_i\}$ is an orthonormal basis and taking sum over i $(1 \le i \le 2n+1)$ gives

$$(\zeta\nu) = [\frac{2n+1}{2}\beta - \alpha]\mathcal{R} - (2n+1)\lambda. \tag{18}$$

Taking covariant Differentiation of (17) yields

$$\begin{aligned} &2\nu(\nabla_E h')F + 2(F\nu)h'E + \nabla_F(E\nu)\zeta + (E\nu)\nabla_F\zeta + \eta(E)\nabla_F D\nu + D\nu(\nabla_F\eta)E \\ &\qquad = -2\alpha(\nabla_F Q)E + \beta(F\mathcal{R})E. \end{aligned} \tag{19}$$

Contracting the preceding equation and using $div\zeta = 0$, we obtain

$$\begin{aligned} &2vdiv(h'E) + 2((h'E)v) + \zeta(Ev)\zeta + \eta(E)\triangle v \\ &= -2\alpha(divQ)E - 2\alpha\sum_{i=1}^{2n+1} g(Q\nabla_{e_i}E, e_i) + \beta(E\mathcal{R}), \end{aligned} \tag{20}$$

in which $\triangle$, the Laplacian operator is described by $\triangle = div.D$.

Putting $E = \zeta$ in the foregoing equation and using (6) and (7)

$$\zeta(\zeta v) + \triangle v = -\alpha(\zeta\mathcal{R}) - 2\alpha trQh' + \beta(\zeta\mathcal{R}). \tag{21}$$

Then, using (18) in (21) yields

$$\triangle v = 2\alpha trQh' + \frac{2n-1}{2}\beta(\zeta\mathcal{R}). \tag{22}$$

Again, substituting E by hE in the equation (17) infers that

$$2vh'^2E + (h'Ev)\zeta = -2\alpha Qh'E - (2\lambda - \beta\mathcal{R})h'E. \tag{23}$$

Since $h'^2 = h^2$, using (6) the previous equation yields

$$vtrh^2 + trQh' = 0. \tag{24}$$

Now, equations (22) and (24) together give

$$\triangle v = 2\alpha vtrh^2 + \frac{2n-1}{2}\beta(\zeta\mathcal{R}). \tag{25}$$

Let the scalar curvature remain invariant under the characteristic vector field ζ and therefore we have

$$\triangle v^2 = 2 \parallel Dv \parallel^2 + 4\alpha^2 v^2 trh^2. \tag{26}$$

By hypothesis $\mathcal{M}$ is compact, thus we acquire

$$\int_{\mathcal{M}} (2 \parallel Dv \parallel^2 + 4\alpha^2 v^2 trh^2) d\mathcal{M} = 0, \tag{27}$$

where the divergence theorem is used. Therefore, v is constant and h is vanishing, since α is a non-zero constant. Thus, $\mathcal{M}$ is a K-almost Co-Kähler manifold.

This ends the proof.

□

Now using Proposition 2.2, equation (25) reduces to

$$\triangle v^2 = 2 \parallel Dv \parallel^2 + 4\alpha^2 v^2 trh^2.$$

Hence, like-wise the last part of the above proof, we obtain that $\mathcal{M}$ is a K-almost Co-Kähler manifold.

Corollary 3.4. *If the potential vector field is point-wise collinear with ζ, then a 3-dimensional compact almost Co-Kähler manifold admitting a Ricci-Yamabe soliton reduces to a K-almost Co-Kähler manifold.*

The Ricci-Yamabe soliton turns into a Ricci soliton for $\alpha = 1$ and $\beta = 0$ [15]. Hence, equation (25) reduces to

$$\triangle v^2 = 2 \parallel Dv \parallel^2 + 4v^2 trh^2.$$

Hence, similarly to the last part of the foregoing proof, we say that$\mathcal{M}$ is a K-almost Co-Kähler manifold.

Also, from equation (17), we acquire that $\mathcal{M}$ is an Einstein manifold, since $\beta = 0$, $\alpha = 1$ and v = constant. Thus, using Theorem 3.2 we conclude that $\mathcal{M}$ is a Co-Kähler manifold.

Corollary 3.5. *If the potential vector field is point-wise collinear with ζ, then a compact almost Co-Kähler manifold admitting a Ricci soliton reduces to a Co-Kähler manifold.*

The above corollary was established by Wang [21].

4. Gradient Ricci-Yamabe solitons on (κ,μ)-almost Co-Kähler manifolds

In contact metric manifolds $\mathcal{M}$, Blair et al. [4] established the concept of (κ,μ)-nullity distribution. If

$$\mathcal{K}(E,F)\zeta = \kappa[\eta(F)E - \eta(E)F] + \mu[\eta(F)hE - \eta(E)hF], \tag{28}$$

$\kappa,\mu \in \mathbb{R}$ holds, then $\mathcal{M}$ is named (κ,μ)-contact metric manifold. In this situation we say $\zeta \in (\kappa,\mu)$-nullity distribution.

An almost Co-Kähler manifold $\mathcal{M}^{2n+1}$ is named a (κ,μ)-almost Co-Kähler manifold if ζ fulfills the equation (28).

A (κ,μ)-almost Co-Kähler manifold $\mathcal{M}^{2n+1}$ satisfies

$$h^2E = \kappa\varphi^2E, \tag{29}$$

$$Ric(E,\zeta) = 2n\kappa\eta(E) \text{ and } Q\zeta = 2n\kappa\zeta, \tag{30}$$

Q is the (1,1) Ricci tensor described by $g(QE,F) = Ric(E,F)$. Equation (29) reflects that $\kappa \le 0$. Moreover, $\kappa = 0$ if and only if $\mathcal{M}^{2n+1}$ is a K-almost Co-Kähler manifold. The manifold is named $N(\kappa)$-almost Co-Kähler manifold [8] if $\mu = 0$. Any Co-Kähler manifold obeys (28) for $k = \mu = 0$.

Lemma 4.1. *[1] In a (κ,μ)-almost Co-Kähler manifold $\mathcal{M}^{2n+1}$ with $\kappa<0$, the subsequent relations hold:*

$$QE = \mu hE + 2n\kappa\eta(E)\zeta, \tag{31}$$

$$\begin{aligned}(\nabla_E h)F - (\nabla_F h)E =& \kappa[\eta(F)\varphi E - \eta(E)\varphi F + 2g(\varphi E,F)\zeta] \\ &+ \mu[\eta(F)\varphi hE - \eta(E)\varphi hF],\end{aligned} \tag{32}$$

for any vector fields E, F.

Theorem 4.2. *Let a (κ,μ)-almost Co-Kähler manifold $\mathcal{M}^{2n+1}$ with $\kappa < 0$ admit a gradient Ricci-Yamabe soliton. Then, the manifold $\mathcal{M}$ is a $N(\kappa)$-almost Co-Kähler manifold.*

Proof. Let the (k,μ)-almost Co-Kähler manifold $\mathcal{M}$ admit a gradient Ricci-Yamabe soliton. Therefore from (3), we write

$$\nabla_E D\Psi = -\alpha QE - (\lambda - \frac{\beta}{2}\mathcal{R})E. \tag{33}$$

Covariant differentiation of (33) yields

$$\nabla_F\nabla_E D\Psi = -\alpha\nabla_F QE - (\lambda - \frac{\beta}{2}\mathcal{R})\nabla_F E + \frac{\beta}{2}(F\mathcal{R})E. \tag{34}$$

We can write from (34)

$$\nabla_E\nabla_F D\Psi = -\alpha\nabla_E QF - (\lambda - \frac{\beta}{2}\mathcal{R})\nabla_E F + \frac{\beta}{2}(E\mathcal{R})F. \tag{35}$$

Using (33), we infer

$$\nabla_{[E,F]} D\Psi = -\alpha Q(\nabla_E F - \nabla_F E) - (\lambda - \frac{\beta}{2}\mathcal{R})(\nabla_E F - \nabla_F E). \tag{36}$$

Making use of (34)-(36) and $\mathcal{R} = 2nk$ = constant, we acquire

$$\mathcal{K}(E,F)D\Psi = -\alpha[(\nabla_E Q)F - (\nabla_F Q)E]. \tag{37}$$

Using (31) in (37) yields

$$\begin{aligned}\mathcal{K}(E,F)D\Psi &= \mu[(\nabla_F h)E - (\nabla_E h)F] \\ &\quad +2nk\{\eta(E)h'F - \eta(F)h'E\} \\ &\quad +[(F\mu)hE - (E\mu)hF].\end{aligned} \tag{38}$$

Making use of (32) in the foregoing equation, we obtain

$$\begin{aligned}\mathcal{K}(E,F)D\Psi &= \alpha\{k\mu[\eta(E)\varphi F - \eta(F)\varphi E + 2g(E,\varphi F)\zeta] \\ &\quad -\mu^2[\eta(E)h'F - \eta(F)h'E] + 2nk\{\eta(E)h'F - \eta(F)h'E\} \\ &\quad +[(F\mu)hE - (E\mu)hF].\end{aligned} \tag{39}$$

Now, contracting equation (39) gives

$$S(F,D\Psi) = -\alpha(hF\mu) = 0, \quad \textit{since} \quad d\mu \wedge \eta = 0. \tag{40}$$

Equations (31) and (40) together reveal

$$\mu(hE)\Psi + 2n\kappa\eta(E)(\zeta\Psi) = 0. \tag{41}$$

Replacing E by ζ in (41) yields

$$2n\kappa(\zeta\Psi) = 0. \tag{42}$$

Since $\kappa < 0$, from the foregoing equation we acquire $(\zeta\Psi) = 0$. Hence from (41), we obtain $\mu(hE)\Psi = 0$ which implies either $\mu \neq 0$, or $\mu = 0$.

If $\mu \neq 0$, then $(hE)\Psi = 0$. Putting $E = hE$, we get

$$\begin{aligned}(h^2E)\Psi &= \kappa(\varphi^2 E)\Psi \\ &= \kappa(E + \eta(E)\zeta)\Psi \\ &= 0.\end{aligned} \tag{43}$$

Since $\kappa < 0$ and $(\zeta\Psi) = 0$, the foregoing equation gives $E\Psi = 0$, that is, $\Psi =$ constant. If we put $\Psi =$ constant in (33), we get that the manifold is an Einstein manifold which contradicts (31).

If $\mu = 0$, the (κ,μ)-almost Co-Kähler manifold reduces to a $N(\kappa)$-almost Co-Kähler manifold.

Hence the proof is finished. □

Theorem 4.3. *There does not exist gradient Ricci-Yamabe structures (g,Ψ,λ) with $D\Psi = (\zeta\Psi)\zeta$ on a compact (κ,μ)-almost Co-Kähler manifold of dimension greater than three with $\kappa < 0$.*

Proof. From (31), it follows that the scalar curvature $\mathcal{R} = 2n\kappa$. Differentiating $D\Psi = (\zeta\Psi)\zeta$ along the arbitrary vector field E, we get

$$\nabla_E D\Psi = (E(\zeta\Psi))\zeta + (\zeta\Psi)h'E. \tag{44}$$

By (31) and (56), equation (3) takes the form

$$\begin{aligned}&2n\kappa\alpha\eta(E)\eta(F) + \alpha\mu g(hE,F) + (E(\zeta\Psi))\eta(F) + (\zeta\Psi)g(h'E,F) \\ = \;& -(\kappa\beta - \lambda)g(E,F),\end{aligned} \tag{45}$$

for all vector fields E, F on $\mathcal{M}^{2n+1}$.

Replacing E by φE and F by φF, we obtain

$$\alpha\mu g(hE,F) + (\zeta\Psi)g(h'E,F) = (\kappa\beta - \lambda)g(\varphi E,\varphi F)$$

Contracting the preceding equation, we get

$$\lambda = \kappa\beta. \tag{46}$$

Setting $E = F = \zeta$ in (58) and using (46), we have

$$2n\kappa\alpha + \zeta(\zeta\Psi) = 0. \tag{47}$$

Again, contracting (56), we find

$$\Delta\Psi + \zeta(\zeta\Psi) = 0. \tag{48}$$

where $\Delta = -\mathrm{div}\, D$ is the Laplacian operator.

Using (47) in (48), we obtain

$$\Delta\Psi = 2n\kappa\alpha$$

By divergence theorem

$$2n\kappa \int_{\mathcal{M}} d\mathcal{M} = 0.$$

Given that $\mathcal{M}$ is orientable, $d\mathcal{M}$ represents the volume form of the manifold and is positive. As a result, since $\kappa < 0$, the left hand side is negative. The aforementioned relation is therefore false.

This ends the proof. □

5. Gradient Ricci-Yamabe solitons in dimension three

Let a Co-Kähler 3-manifold admit a gradient Ricci-Yamabe soliton. Therefore we write:

Theorem 5.1. *If a Co-Kähler 3-manifold admits a gradient Ricci-Yamabe soliton, then the scalar curvature of the manifold is constant. Also, the gradient of the potential function is collinear with the Reeb vector field ζ, or the manifold is flat.*

Proof. By virtue of (34)-(36) and (11), we acquire

$$\begin{aligned} \mathcal{K}(E,F)D\Psi &= -\alpha\Big[\frac{(F\mathcal{R})}{2}(E-\eta(E)\zeta) - \frac{(E\mathcal{R})}{2}(F-\eta(F)\zeta)\Big] \\ &+ \frac{\beta}{2}[(E\mathcal{R})F - (F\mathcal{R})E]. \end{aligned} \tag{49}$$

By contraction, we have

$$Ric(F, D\Psi) = -\frac{\alpha}{2}(F\mathcal{R}) + \beta(F\mathcal{R}). \tag{50}$$

From (49), we easily infer

$$g(\mathcal{K}(E,F)D\Psi, \zeta) = -\frac{\beta}{2}\{(E\mathcal{R})\eta(F) - (F\mathcal{R})\eta(E)\}.$$

In a Co-Kähler manifold we have $\mathcal{K}(E,F)\zeta = 0$ and hence we obtain

$$\frac{\beta}{2}\{(E\mathcal{R})\eta(F) - (F\mathcal{R})\eta(E)\} = 0. \tag{51}$$

As $\beta \neq 0$, the foregoing equation yields

$$(E\mathcal{R})\eta(F) - (F\mathcal{R})\eta(E) = 0. \tag{52}$$

From Proposition 2.2, we acquire $\zeta\mathcal{R} = 0$ and hence the above equation gives $E\mathcal{R} = (\zeta\mathcal{R})\eta(E) = 0$. Therefore, $\mathcal{R} =$ constant.

In view of (11) and (50) and using $\mathcal{R}$ = constant, we obtain

$$\frac{\mathcal{R}}{2}((F\Psi) - (\zeta\Psi)\eta(F)) = 0,$$

which entails either $D\Psi = (\zeta\Psi)\zeta$ or, $\mathcal{R} = 0$. If $\mathcal{R} = 0$, from equation (11) we conclude that the manifold is Ricci flat. Therefore, (10) yields that the manifold is flat.

Hence the theorem is proved. □

Since the scalar curvature $\mathcal{R}$ is constant, (11) yields that $\nabla Q = 0$ and hence the equation (10) gives $\nabla\mathcal{K} = 0$, in other words, it is a locally symmetric manifold. Therefore we get:

Corollary 5.2. *If a Co-Kähler 3-manifold admits a Ricci-Yamabe soliton of gradient type, then the manifold becomes locally symmetric.*

We are aware of the following outcome due to Perrone (Proposition 3.1 of [17]) and Wang (Corollary 4.3 of [23]):

Lemma 5.3. *Any Co-Kähler 3-manifold (locally symmetric) is locally isometric to either a Euclidean space $\mathbb{R}^3$ which is flat, or the Riemannian product of $\mathbb{R}$ and a Kähler surface with non-zero constant curvature.*

Combining Corollary 5.2 and Lemma 5.3 provides

Corollary 5.4. *Let a Co-Kähler 3-manifold admit a Ricci-Yamabe soliton of gradient type. Then the manifold is locally isometric to either a Euclidean space $\mathbb{R}^3$ which is flat, or the Riemannian product of $\mathbb{R}$ and a Kähler surface with non-zero constant curvature.*

Let in a 3-dimensional Co-Kähler manifold the scalar curvature is non-zero. Hence, we have $D\Psi = (\zeta\Psi)\zeta$. Covariant differentiation along E yields

$$\nabla_E D\Psi = (E(\zeta\Psi))\zeta. \tag{53}$$

Using (53) and (11) in the equation (3), we find

$$(E(\zeta\Psi))\zeta = -\frac{\alpha\mathcal{R}}{2}(E - \eta(E)\zeta) + (\lambda - \frac{\beta}{2}\mathcal{R})E. \tag{54}$$

Putting $E = \zeta$ in the above equation, we obtain

$$(\zeta(\zeta\Psi))\zeta = (\lambda - \frac{\beta}{2}\mathcal{R})\zeta,$$

that is,

$$\zeta(\zeta\Psi) = (\lambda - \frac{\beta}{2}\mathcal{R}).$$

Contracting (54), we see that

$$\zeta(\zeta\Psi) = 3(\lambda - \frac{\beta}{2}\mathcal{R}) - \alpha\mathcal{R}. \tag{55}$$

The last two equations yield $\lambda = \left(\frac{\alpha+\beta}{2}\right)\mathcal{R}$. Thus we write:

Corollary 5.5. *Let a Co-Kähler 3-manifold having non-zero scalar curvature admit a Ricci-Yamabe soliton of gradient type. Then the soliton is steady, expanding, or shrinking according as $\left(\frac{\alpha+\beta}{2}\right)\mathcal{R} = 0$, $\left(\frac{\alpha+\beta}{2}\right)\mathcal{R} < 0$, or $\left(\frac{\alpha+\beta}{2}\right)\mathcal{R} > 0$.*

6. Examples

Example 6.1. *Let $\mathcal{M}^3 = \{(u,v,w) \in \mathbb{R}^3 : w > 0\}$, where (u,v,w) are the standard co-ordinate of $\mathbb{R}^3$. On $\mathcal{M}^3$, the metric g is defined by*

$$g = du \otimes du + dv \otimes dv + \frac{u^2+v^2+e^{-2w}}{4w} dw \otimes dw - \frac{v}{\sqrt{w}} du \otimes dw - \frac{u}{\sqrt{w}} dv \otimes dw.$$

Let

$$\delta_1 = \frac{\partial}{\partial u}, \quad \delta_2 = \frac{\partial}{\partial v}, \quad \delta_3 = ve^w \frac{\partial}{\partial u} + ue^w \frac{\partial}{\partial v} + 2\sqrt{w}e^w \frac{\partial}{\partial w}.$$

Then, the vector fields $\delta_1, \delta_2, \delta_3$ are orthonormal with respect to the metric g.

We also acquire

$$[\delta_1, \delta_2] = 0, \quad [\delta_1, \delta_3] = e^w \delta_2, \quad [\delta_2, \delta_3] = e^w \delta_1.$$

We define the 1-form η, the vector field ζ and (1-1)-tensor field φ by

$$\eta = \frac{e^{-w}}{2\sqrt{w}} dw, \quad \zeta = \delta_3, \quad \varphi\delta_1 = \delta_2, \quad \varphi\delta_2 = -\delta_1, \quad \varphi\delta_3 = 0,$$

then it is simple to establish that the 1-form η and 2-form Φ are closed. Therefore, $\mathcal{M}^3$ is an almost Co-Kähler manifold.

Making Use of Koszul's formula, we acquire

$$\nabla_{\delta_1}\delta_1 = 0, \quad \nabla_{\delta_1}\delta_2 = -e^w\delta_3, \quad \nabla_{\delta_1}\delta_3 = e^w\delta_2,$$
$$\nabla_{\delta_2}\delta_1 = -e^w\delta_3, \quad \nabla_{\delta_2}\delta_2 = 0, \quad \nabla_{\delta_2}\delta_3 = e^w\delta_1,$$
$$\nabla_{\delta_3}\delta_1 = 0, \quad \nabla_{\delta_3}\delta_2 = 0, \quad \nabla_{\delta_3}\delta_3 = 0.$$

The components of the curvature tensor $\mathcal{K}$ are described by

$$\mathcal{K}(\delta_1,\delta_2)\delta_1 = -e^{2w}\delta_2, \quad \mathcal{K}(\delta_1,\delta_2)\delta_2 = e^{2w}\delta_1, \quad \mathcal{K}(\delta_1,\delta_2)\delta_3 = 0,$$
$$\mathcal{K}(\delta_1,\delta_3)\delta_1 = e^{2w}\delta_3, \quad \mathcal{K}(\delta_1,\delta_3)\delta_2 = 2\sqrt{w}e^{2w}\delta_3,$$
$$\mathcal{K}(\delta_1,\delta_3)\delta_3 = -e^{2w}\delta_1 - 2\sqrt{w}e^{2w}\delta_2, \quad \mathcal{K}(\delta_2,\delta_3)\delta_1 = 2\sqrt{w}e^{2w}\delta_3,$$
$$\mathcal{K}(\delta_2,\delta_3)\delta_2 = e^{2w}\delta_3, \quad \mathcal{K}(\delta_2,\delta_3)\delta_3 = -2\sqrt{w}e^{2w}\delta_1 - e^{2w}\delta_2.$$

Using the curvature tensor's expression, we calculate the Ricci operator Q by

$$Q\delta_1 = -2\sqrt{w}e^{2w}\delta_2, \quad Q\delta_2 = -2\sqrt{w}e^{2w}\delta_1, \quad Q\delta_3 = -2e^{2w}\delta_3. \tag{56}$$

The tensor field h is described by

$$h\delta_1 = -e^w\delta_1, \quad h\delta_2 = e^w\delta_2, \quad h\delta_3 = 0. \tag{57}$$

Suppose that $\Psi = w$. Then, $D\Psi = 2\sqrt{w}e^w\delta_3$. By directed computation, we have

$$\begin{cases} \nabla_{\delta_1}D\Psi = 2\sqrt{w}e^{2w}\delta_2, \\ \nabla_{\delta_2}D\Psi = 2\sqrt{w}e^{2w}\delta_1, \\ \nabla_{\delta_3}D\Psi = 2e^{2w}(1+2w)\delta_3. \end{cases} \tag{58}$$

From (3), we find that

$$\begin{cases} \nabla_{\delta_1}D\Psi + \alpha Q\delta_1 + (\lambda - \frac{\beta}{2}\mathcal{R})\delta_1 = 0, \\ \nabla_{\delta_2}D\Psi + \alpha Q\delta_2 + (\lambda - \frac{\beta}{2}\mathcal{R})\delta_2 = 0, \\ \nabla_{\delta_3}D\Psi + \alpha Q\delta_3 + (\lambda - \frac{\beta}{2}\mathcal{R})\delta_3 = 0. \end{cases}$$

Therefore, using (56) and (58), we can easily verify that the foregoing equations are satisfied for $\alpha = 1$, $\lambda = \frac{\beta}{2}\mathcal{R}$. Hence, the almost Co-Kähler manifold $\mathcal{M}^3$ admits a gradient Ricci-Yamabe soliton.

Example 6.2. *Let the Riemannian metric g of $\mathcal{M} = \mathbb{R}^3$ is defined by*

$$g = du^2 + dv^2 + e^{2w}(u^2e^{2w} + v^2e^{2w} + 1)dw^2 - 2ve^{2w}du\,dw + 2ue^{2w}dv\,dw.$$

We see that $\{\delta_1 = \frac{\partial}{\partial u}, \delta_2 = \frac{\partial}{\partial v}, \delta_3 = ve^w\frac{\partial}{\partial u} - ue^w\frac{\partial}{\partial v} + e^{-w}\frac{\partial}{\partial w}\}$ *is an orthonormal basis. Hence,* $[\delta_1, \delta_2] = 0$, $[\delta_1, \delta_3] = -e^w\delta_2$, $[\delta_2, \delta_3] = e^w\delta_1$. *We set*

$$\eta = e^w dw, \quad \zeta = \delta_3, \quad f = -\frac{\partial}{\partial u} \otimes dv + \frac{\partial}{\partial v} \otimes du - 2ue^{2w}\frac{\partial}{\partial u} \otimes dw - ve^{2w}\frac{\partial}{\partial v} \otimes dw.$$

The 2-form Φ *is described by*

$$\Phi = -2du \wedge dv - 2ve^{2w}dv \wedge dw + 2ue^{2w}dw \wedge du.$$

$\mathcal{M}$ *is an almost Co-Kähler manifold, since* η *and* Φ *are closed. As* $h = 0$, $\mathcal{M}$ *is a Co-Kähler manifold. The Riemannian connection* ∇ *is described by*

$$\nabla_{\delta_1}\delta_1 = 0, \quad \nabla_{\delta_1}\delta_2 = 0, \quad \nabla_{\delta_1}\delta_3 = 0,$$
$$\nabla_{\delta_2}\delta_1 = 0, \quad \nabla_{\delta_2}\delta_2 = 0, \quad \nabla_{\delta_2}\delta_3 = 0,$$
$$\nabla_{\delta_3}\delta_1 = e^w\delta_2, \quad \nabla_{\delta_3}\delta_2 = -e^w\delta_1, \quad \nabla_{\delta_3}\delta_3 = 0.$$

From above we see that $\mathcal{K}(E, F)G = 0$ *for all* $E, F, G \in \chi(\mathcal{M})$. *Therefore,* $Ric = 0$ *and* $\mathcal{R} = 0$.

Suppose $\Psi = e^w$. *Thus* $D\Psi = \delta_3$. *Hence,* $\nabla_E D\Psi = 0$ *for any* $E \in \chi(\mathcal{M})$. *Using this result in (3), we acquire*

$$\begin{cases} \alpha Q\delta_1 + (\lambda - \frac{\beta}{2}\mathcal{R})\delta_1 = 0, \\ \alpha Q\delta_2 + (\lambda - \frac{\beta}{2}\mathcal{R})\delta_2 = 0, \\ \alpha Q\delta_3 + (\lambda - \frac{\beta}{2}\mathcal{R})\delta_3 = 0. \end{cases}$$

Hence, the above equations are satisfied for $\lambda = -\alpha$, *since* $\mathcal{R} = 0$. *Thus, the Co-Kähler manifold* $\mathcal{M}^3$ *admits a gradient Ricci-Yamabe soliton.*

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