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Pinching results for bi-slant submanifolds in trans-Sasakian manifolds

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Abstract. In the present article, we consider bi-slant submanifolds in trans-Sasakian generalized Sasakian space forms. Specifically, we establish both the Chen first inequality and the Chen-Ricci inequality on such submanifolds. We provide an example of bi-slant submanifold.
Keywords: Chen first invariant, squared mean curvature, Ricci curvature, trans-Sasakian-manifolds, bi-slant submanifolds.

1. Introduction

The concept of slant submanifolds, which generalizes both holomorphic and totally real immersions in complex manifolds, was introduced by B.-Y. Chen [6]. In a subsequent development A. Lotta, in [11], introduced the concept of contact slant submanifolds within almost contact metric manifolds. Furthermore, the notion of bi-slant submanifolds in almost contact metric manifolds was defined by J.L. Cabrerizo and his colleagues, as discussed in [5].

B.-Y. Chen [7] proved an inequality which involves the mean curvature H, the scalar curvature τ and the sectional curvature K for n-dimensional Riemannian submanifolds M of a Riemannian space form $M^{2m+1}(c)$.

$$\tau(p) - (\inf K)(p) \leq \frac{n^2(n-2)}{2(n-1)}||H||^2 + \frac{1}{2}(n+1)(n-2)c,$$

known as Chen first inequality.
For more details, references and applications of Chen invariants, see [8].
Also, the same author [9] established another inequality relating the Ricci curvature and the mean curvature for any submanifold of dimension n in a Riemannian space form of constant sectional curvature c.

$$\mathrm{Ric}(X) \leq (n-1)c + \frac{n^2}{4}||H||^2,$$

which is known as the Chen-Ricci inequality.
The above Chen's inequalities provide relationships between the basic intrinsic invariants and the main

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extrinsic invariant (the mean curvature) of a Riemannian submanifold. Afterwards, many authors obtained some Chen inequalities for certain classes of submanifolds in different ambient spaces.
This paper is built on the above inequalities for bi-slant submanifold in trans-Sasakian generalized Sasakian space forms. These submanifolds were studied by several authors.
On Kenmotsu manifolds (a particular case of trans-Sasakian manifolds), P.K. Pandey et al. [16] proved the following inequality on an $(m+1)$-dimensional bi-slant submanifold of a $(2m+1)$-dimensional Kenmotsu space form of constant holomorphic sectional curvature c.

(i) For any plane section π tangent to D_1 and invariant by P,

$$\begin{aligned}\tau - K(\pi) \leq \frac{(m+1)^2(m-1)}{2m}\|H\|^2 + \frac{1}{2}(m+1)(m-2)\frac{c-3}{4} - m \\ + 3\frac{c+1}{4}[(n_1-1)\cos^2\theta_1 + n_2\cos^2\theta_2].\end{aligned}$$

(ii) For any plane section π tangent to D_2 and invariant by P,

$$\begin{aligned}\tau - K(\pi) \leq \frac{(m+1)^2(m-1)}{2m}\|H\|^2 + \frac{1}{2}(m+1)(m-2)\frac{c-3}{4} - m \\ + 3\frac{c+1}{4}[n_1\cos^2\theta_1 + (n_2-1)\cos^2\theta_2].\end{aligned}$$

A corresponding result for bi-slant submanifolds in cosymplectic space forms was obtained by R.S. Gupta [10].
Also D.W. Yoon [17] proved for any n-dimensional bi-slant submanifold tangent to ξ in a $(2m+1)$-dimensional cosymplectic space form $\overline{M}(c)$ the following result.

For each unit vector $X \in T_pM$ orthogonal to ξ and

(i) X tangent to D_1, we have

$$Ric(X) \leq \frac{1}{4}\left\{(n-1)c + \frac{1}{2}(3\cos^2\theta_1 - 2)c + n^2 \| H \|^2\right\}.$$

(ii) X tangent to D_2, we have

$$Ric(X) \leq \frac{1}{4}\left\{(n-1)c + \frac{1}{2}(3\cos^2\theta_2 - 2)c + n^2 \| H \|^2\right\}.$$

Corresponding inequalities for different types of submanifolds in a diversity of ambient spaces were investigated (see [1], [4], [15], [13], [14]).

2. Preliminaries

A $(2n+1)$-dimensional Riemannian manifold $(\overline{M}, g)$ is called an almost contact metric manifold if there exist a $(1,1)$-tensor field ϕ, a unit vector field ξ and a 1-form η on $\overline{M}$ satisfying

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for all tangent vector fields X, Y on $\overline{M}$. Moreover, on an almost contact metric manifold, one has

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X, \xi).$$

An almost contact metric manifold $\overline{M}$ is called a trans-Sasakian manifold if there exist two real differentiable functions α and β such that

$$(\overline{\nabla}_X\phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X),$$

which implies

$$\overline{\nabla}_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{1}$$

A Sasakian manifold is a trans-Sasakian manifold with $\alpha = 1,\ \beta = 0$, a Kenmotsu manifold is a trans-Sasakian manifold with $\alpha = 0,\ \beta = 1$, and a cosymplectic manifold is a trans-Sasakian manifold with $\alpha = 0,\ \beta = 0$.

J.C. Marrero [12] proved that a trans-Sasakian manifold of dimension greater than or equal to 5 is locally isometric to either a α-Sasakian or a β-Kenmotsu or a cosymplectic manifold. Since this result is not global it is worth to consider the general case of trans-Sasakian manifolds, not only the above mentioned particular cases.

The notion of a generalized Sasakian space from was introduced by P. Alegre, D.E. Blair and A. Carriazo [2]. It is an almost contact metric manifold $(\overline{M}, \phi, \xi, \eta, g)$ whose Riemannian curvature tensor satisfies

$$\overline{R}(X,Y)Z = f_1R_1(X,Y)Z + f_2R_2(X,Y)Z + f_3R_3(X,Y)Z, \tag{2}$$

where

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y,$$
$$R_2(X,Y)Z = g(X,\phi Z)\phi Y g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z,$$
$$R_3(X,Y)Z = \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi,$$

for any vector fields X, Y, Z, with f_1, f_2, f_3 differentiable functions on $\overline{M}$. Such a manifold is denoted by $\overline{M}(f_1, f_2, f_3)$. Particular cases of generalized Sasakian space forms are:

(i) Sasakian space forms, for $f_1 = \frac{c+3}{4}$ and $f_2 = f_3 = \frac{c-1}{4}$,

(ii) Kenmotsu space forms, for $f_1 = \frac{c-3}{4}$ and $f_2 = f_3 = \frac{c+1}{4}$,

(iii) cosymplectic space forms, for $f_1 = f_2 = f_3 = \frac{c}{4}$.

Theorem 2.1. *[3] Any contact metric generalized Sasakian-space-form $\overline{M}(f_1, f_2, f_3)$, with dimension greater than or equal to 5, is a Sasakian manifold. Therefore, if $\bar{M}$ is connected, then f_1, f_2 and f_3 must be constant functions.*

Let M be an m-dimensional Riemannian submanifold of a trans-Sasakian manifold $\overline{M}$. We denote by $\overline{\nabla}$ the Levi-Civita connection on $\overline{M}$.
Then the Gauss and Weingarten formulae are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \overline{\nabla}_X Y = -A_N X + \nabla^{\perp}_X N,$$

respectively, for all $X, Y \in TM$ and $N \in T^{\perp}M$, where ∇ is the Levi-Civita connection on M and $\nabla^{\perp}$ the normal connection, respectively. We mention the following relation between the second fundamental form h and the shape operator A.

$$g(h(X,Y),N) = g(A_N X, Y).$$

The Gauss equation is given by

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z)), \tag{3}$$

for all X, Y, Z, W tangent to M.
Let $p \in M$ and $\{e_1, ..., e_m\}$ an orthonormal basis of the tangent space T_pM. Then the scalar curvature τ at p is defined by

$$\tau(p) = \sum_{1\le i<j\le m} K(e_i \wedge e_j), \tag{4}$$

where $K(e_i \wedge e_j)$ is the sectional curvature of the 2-plane section spanned by e_i and e_j.

For $e_m = \xi$, (4) becomes

$$2\tau = \sum_{1\leq i\neq j\leq m-1} K(e_i \wedge e_j) + 2\sum_{i=1}^{m-1} K(e_i \wedge \xi). \tag{5}$$

The Chen first invariant δ_M is defined by

$$\delta_M(p) = \tau(p) - (\inf K)(p),$$

where $(\inf K)(p) = \inf\{K(\pi);\ \pi \subset T_pM,\ \dim \pi = 2\}$.

Let L be a k-plane section of T_pM and $X \in$ a unit vector; we choose an orthonormal basis $\{e_1 = X, e_2, ..., e_k\}$ of L. Then the Ricci curvature Ric_L of L at X is defined by

$$Ric_L(X) = K_{12} + K_{13} + \cdots + K_{1k}. \tag{6}$$

It is called the k-Ricci curvature.

Recall that the mean curvature vector $H(p)$ at $p \in M$ is given by

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{7}$$

Denote by $h^r_{ij} = g(h(e_i, e_j), e_r)$, $i, j = 1, ..., m, r \in \{m + 1, ..., 2n + 1\}$; then the norm of the second fundamental form h is

$$\|h\|^2 = \sum_{r=m+1}^{2n+1}\sum_{i,j=1}^{m}(h^r_{ij})^2.$$

For any vector field $X \in TM$, we decompose $\phi X = PX + FX$, where PX and FX are the tangential and normal components of ϕX, respectively. We put

$$\|P\|^2 = \sum_{i,j=1}^{m} g^2(Pe_i, e_j).$$

Lemma 2.2. *Let M be an m-dimensional submanifold tangent to ξ of a $(2n+1)$-dimensional trans-Sasakian manifold $\overline{M}$. Then, one has:*
(i) $h(\xi, \xi) = 0$;
(ii) $h(X, \xi) = -\alpha FX$, for any vector field X tangent to M orthogonal to ξ.

Proof. For any $p \in M$ and $X \in T_pM$, we have

$$\overline{\nabla}_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi).$$

By using Gauss formula we obtain

$$h(X, \xi) = -\alpha FX.$$

Taking $X = \xi$ we get (i), and taking X orthogonal to ξ we get (ii). □

Definition. A submanifold M is called
a) *totally geodesic* if the second fundamental form h vanishes identically;
b) *totally umbilical* if $h(X, Y) = g(X, Y)H$, for all $X, Y \in TM$.

Definition [5]. A differentiable distribution D on M is said to be a *slant distribution* if for any $p \in M$ and any non-zero vector $X \in D_p$, the angle $\theta_D(X)$ between ϕX and the vector subspace D_p is constant, i.e., it depends neither on the choice of p nor of X; $\theta_D(X)$ is called the slant angle.

Definition [5]. A submanifold M tangent to the structure vector field ξ is called a *bi-slant submanifold* of $\overline{M}$ if there are two orthogonal differentiable distributions D_1 and D_2 such that

(i) TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2 \oplus \{\xi\}$, where $\{\xi\}$ is the 1-dimensional distribution spanned by ξ,

(ii) D_1, D_2 are slant distributions with slant angles θ_1, θ_2.

A bi-slant submanifold of a trans-Sasakian manifold $\overline{M}$ is called proper if both the slant distributions D_1 and D_2 have the slant angles $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

Assume that M is a proper bi-slant submanifold of dimension $m+1 = 2d_1 + 2d_2 + 1$ in $\overline{M}$. We consider an orthonormal basis of T_pM

$$e_1, e_2 = \sec\theta_1 Pe_1, \cdots, e_{2d_1-1}, e_{2d_1} = \sec\theta_1 Pe_{2d_1-1}, e_{2d_1+2} = \sec\theta_2 Pe_{2d_1+1}, \cdots,$$
$$e_{2d_1+2d_2-1}, e_{2d_1+2d_2} = \sec\theta_2 e_{2d_1+2d_2-1}, e_{2d_1+2d_2+1} = \xi.$$

Then

$$g^2(\phi e_i, e_{i+1}) = \begin{cases} \cos^2\theta_1, & for \quad i \in \{1, 3, \cdots, 2d_1 - 1\}, \\ \cos^2\theta_2, & for \quad i \in \{2d_1 + 1, \cdots, 2d_1 + 2d_2 - 1\}, \end{cases}$$

$$\sum_{i,j=1}^{m} g^2(\phi e_j, e_i) = 2(d_1\cos^2\theta_1 + d_2\cos^2\theta_2). \tag{8}$$

Example. For any positive constants k_1 and k_2,
$x(u, v, s, w, z) = (u + w, 0, s, k_1\cos v + k_2\sin w, v + s, k_1\sin v + k_2\cos w, w, 0, z)$,
defines a proper bi-slant submanifold with slant angles $\theta_1 = \cos^{-1}(1/\sqrt{1+k_1^2})$ and $\theta_2 = \cos^{-1}(2/\sqrt{1+k_2^2})$ in $(\mathbb{R}^9, \phi_0, \eta, \xi, g)$.

Proof. $\mathbb{R}^9$ together with the structure

$$\eta = \frac{1}{2}\left(dz - \sum_{i=1}^{4} y^i dx^i\right), \quad \xi = 2\frac{\partial}{\partial z}.$$
$$g = -\eta\otimes\eta + \frac{1}{4}\sum_{i=1}^{4}(dx^i\otimes dx^i + dy^i\otimes dy^i),$$

$$\phi_0\left(\sum_{i=1}^{4}\left(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}\right) + Z\frac{\partial}{\partial z}\right) = \sum_{i=1}^{4}\left(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i}\right) + \sum_{i=1}^{4} Y_i y^i\frac{\partial}{\partial z},$$

is a Sasakian manifold $(\mathbb{R}^9, \phi_0, \eta, \xi, g)$.
Furthermore, it is easy to see that:

$$e_1 = (1, 0, 0, 0, 0, 0, 0, 0, 0) = \frac{\partial}{\partial x^1},$$
$$e_2 = \frac{1}{\sqrt{k_1^2+1}}(0, 0, 0, -k_1\sin v, 1, k_1\cos v, 0, 0, 0)$$
$$= \frac{1}{\sqrt{k_1^2+1}}\left(-k_1\sin v\frac{\partial}{\partial x^4} + \frac{\partial}{\partial y^1} + k_1\cos v\frac{\partial}{\partial y^2}\right),$$

$$e_3 = (0,0,1,0,1,0,0,0,0) = \frac{\partial}{\partial x^3} + \frac{\partial}{\partial y^1},$$

$$e_4 = \frac{1}{\sqrt{2+k_2^2}}(1,0,0,k_2\cos w,0,-k_2\sin w,1,0,0)$$

$$= \frac{1}{\sqrt{2+k_2^2}}\left(\frac{\partial}{\partial x^1} + k_2\cos w\frac{\partial}{\partial x^4} - k_2\sin w\frac{\partial}{\partial y^2} + \frac{\partial}{\partial y^3}\right),$$

$$e_5 = \xi$$

form a local orthonormal frame of TM. We define $D_1 = \{e_1, e_2\}$ and $D_2 = \{e_3, e_4\}$; then a standard computation yields $\cos\theta_1 = g(\phi e_1, e_2) = \frac{1}{\sqrt{1+k_1^2}}$ and $\cos\theta_2 = g(\phi e_3, e_4) = \frac{2}{\sqrt{1+k^2}}$, which prove that the distribution D_1 is θ_1-slant and the distribution D_2 is θ_2-slant. □

3. Chen first inequality

In this section, we establish a Chen first inequality for proper bi-slant submanifolds in (α,β) trans-Sasakian generalized Sasakian space forms, with respect to orthogonal subspaces to the structure vector field ξ.
We will use a well-known algebraic lemma from [7].

Lemma 3.1. *[7] Let $a_1, ...a_k, c$ be $k+1$ $(k \geq 2)$ real numbers satisfying*

$$\left(\sum_{i=1}^{k} a_i\right)^2 = (k-1)(\sum_{i=1}^{k} a_i^2 + c).$$

Then $2a_1a_2 \geq c$. The equality holds if and only if $a_1 + a_2 = a_3 = \cdots = a_k$.

Theorem 3.2. *Let $\psi : M \to \overline{M}(f_1, f_2, f_3)$ be an isometric immersion from an $(m+1)$-dimensional $(m \geq 4)$ bi-slant submanifold M in a (α,β) trans-Sasakian generalized Sasakian space form of dimension $2n+1$ $(n \geq 4)$. Then:*

(i) For any plane section π invariant by P and tangent to D_1,

$$\begin{aligned}\tau - K(\pi) \leq\ & \frac{(m+1)^2(m-1)}{2m}\|H\|^2 \\ & + f_1\frac{(m+2)(m-1)}{2} \\ & + 3f_2[(d_1-1)\cos^2\theta_1 + d_2\cos^2\theta_2] \\ & - [mf_3 + 2\alpha^2(d_1\sin^2\theta_1 + d_2\sin^2\theta_2)]. \end{aligned} \tag{9}$$

(ii) For any plane section π invariant by P and tangent to D_2,

$$\begin{aligned}\tau - K(\pi) \leq\ & \frac{(m+1)^2(m-1)}{2m}\|H\|^2 \\ & + f_1\frac{(m+2)(m-1)}{2} \\ & + 3f_2[d_1\cos^2\theta_1 + (d_2-1)\cos^2\theta_2] \\ & - [mf_3 + 2\alpha^2(d_1\sin^2\theta_1 + d_2\sin^2\theta_2)]. \end{aligned} \tag{10}$$

The equality case of the inequalities (9) *or* (10) *holds at a point $p \in M$ if and only if there exists an orthonormal basis $\{e_1, e_2, ..., e_{m+1} = \xi\}$ of T_pM and an orthonormal basis $\{e_{m+2}, ..., e_{2n}, e_{2n+1}\}$ of $T_p^{\perp}M$ such that the shape operators of M in $\overline{M}(f_1, f_2, f_3)$ at p have the following form*

$$A_r = \begin{pmatrix} h^r_{11} & h^r_{12} & 0 & \cdots & 0 & a^r_1 \\ h^r_{12} & -h^r_{11} & 0 & \cdots & 0 & a^r_2 \\ 0 & 0 & 0 & \cdots & 0 & a^r_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a^r_m \\ a^r_1 & a^r_2 & a^r_3 & \cdots & a^r_m & 0 \end{pmatrix}, \qquad r \in \{m+2, ..., 2n+1\}.$$

Proof. Let M be an $(m+1 = 2d_1 + 2d_2 + 1)$- dimensional proper bi-slant submanifold of a (α, β) trans-Sasakian generalized Sasakian space form, $p \in M$ and $\{e_1, ..., e_m, e_{m+1} = \xi\}$ an orthonormal basis of the tangent space T_pM and $\{e_{m+2}, ..., e_{2n+1}\}$ an orthonormal basis of $T^{\perp}{}_pM$, with $Fe_j = \|Fe_j\| e_{m+j+1}$, $\forall j = 1, \cdots, m$.
From equation (5) one has

$$2\tau = \sum_{i,j=1}^{m+1} R(e_i, e_j, e_j, e_i) = \sum_{1 \le i \ne j \le m} R(e_i, e_j, e_j, e_i) + 2\sum_{i=1}^{m} R(e_i, \xi, \xi, e_i). \tag{11}$$

In Gauss equation (3), we put $X = W = e_i$, $Y = Z = e_j$, $\forall i, j \in 1, 2, ..., m$, and we take the summation over $1 \le i, j \le m$. We obtain

$$\sum_{1 \le i \ne j \le m} R(e_i, e_j, e_j, e_i) = \sum_{1 \le i \ne j \le m} \overline{R}(e_i, e_j, e_j, e_i) - \sum_{1 \le i \ne j \le m} g(h(e_i, e_j), h(e_i, e_j)) + \sum_{1 \le i \ne j \le m} g(h(e_i, e_i), h(e_j, e_j)). \tag{12}$$

We calculate $\overline{R}(e_i, e_j, e_j, e_i)$ using the formula (2) of the curvature tensor and put $X = W = e_i$, $Y = Z = e_j$, for $i, j = 1, \cdots m, i \ne j$.

$$\begin{aligned} \overline{R}(e_i, e_j, e_j, e_i) = {} & f_1\{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\} \\ & + f_2\{g(e_i, \phi e_j)g(\phi e_j, e_i) - g(e_j, \phi e_j)g(\phi e_i, e_i) + 2g(e_i, \phi e_j)g(\phi e_j, e_i)\} \\ & + f_3\{\eta(e_i)\eta(e_j)g(e_j, e_i) - \eta(e_j)\eta(e_j)g(e_i, e_i) \\ & + g(e_i, e_j)\eta(e_j)\eta(e_i) - g(e_j, e_j)\eta(e_i)\eta(e_i)\}, \end{aligned} \tag{13}$$

which implies

$$\overline{R}(e_i, e_j, e_j, e_i) = f_1 + 3f_2 g^2(\phi e_i, e_j). \tag{14}$$

Then

$$\sum_{1 \le i \ne j \le m} \overline{R}(e_i, e_j, e_j, e_i) = m(m-1)f_1 + 3f_2 \sum_{i,j=1}^{m} g^2(\phi e_i, e_j). \tag{15}$$

If we introduce the equation (15) in (12), we have

$$\begin{aligned} \sum_{1 \le i \ne j \le m} R(e_i, e_j, e_j, e_i) = {} & m(m-1)f_1 + 3f_2 \sum_{i,j=1}^{m} g^2(\phi e_i, e_j) \\ & - \sum_{1 \le i \ne j \le m} g(h(e_i, e_j), h(e_i, e_j)) + \sum_{1 \le i \ne j \le m} g(h(e_i, e_i), h(e_j, e_j)). \end{aligned} \tag{16}$$

From the equation (11) one has

$$\begin{aligned}2\tau &= m(m-1)f_1 + 3f_2\sum_{i,j=1}^{m} g^2(\phi e_i, e_j)\\ &\quad - \sum_{1\le i\ne j\le m} g(h(e_i,e_j),h(e_i,e_j)) + \sum_{1\le i\ne j\le m} g(h(e_i,e_i),h(e_j,e_j))\\ &\quad + 2\sum_{j=1}^{m} R(e_i,\xi,\xi,e_i). \qquad (17)\end{aligned}$$

We calculate $\sum_{i=1}^{m} R(e_i,\xi,\xi,e_i)$.

$$\sum_{j=1}^{m} K(\xi\wedge e_j) = \sum_{j=1}^{m} R(e_i,\xi,\xi,e_i) = \sum_{j=1}^{m} \overline{R}(e_i,\xi,\xi,e_i) + \sum_{j=1}^{m} g(h(\xi,\xi),h(e_j,e_j)) - \sum_{j=1}^{m} g(h(\xi,e_j),h(\xi,e_j)). \qquad (18)$$

The equation (2) gives

$$\sum_{j=1}^{m} \overline{R}(e_i,\xi,\xi,e_i) = m(f_1 - f_3). \qquad (19)$$

If we substitute the equation (19) in (18)

$$\sum_{i=1}^{m} R(e_i,\xi,\xi,e_i) = m(f_1-f_3) + \sum_{r=m+2}^{2n+1}\sum_{j=1}^{m}[h^r_{jj}h^r_{\xi\xi} - (h_{j\xi})^2]. \qquad (20)$$

If we introduce the equation (20) in (17)

$$\begin{aligned}2\tau &= m(m-1)f_1 + 3f_2\sum_{i,j=1}^{m} g^2(\phi e_i, e_j)\\ &\quad + (m+1)^2\|H\|^2 - \|h\|^2 + 2m(f_1-f_3)\\ &\quad - 2\sum_{r=m+2}^{2n+1}\sum_{j=1}^{m}(h^r_{j\xi})^2. \qquad (21)\end{aligned}$$

By using Lemma 2.2

$$\sum_{j=1}^{m}\sum_{r=m+2}^{2n+1}(h^r_{j\xi})^2 = \sum_{j=1}^{m}\sum_{r=m+2}^{2n+1} g^2(h(e_j,\xi),e_r) = \alpha^2\sum_{j=1}^{m}\sum_{r=m+2}^{2n+1} g^2(Fe_j,e_r) = \alpha^2\sum_{j=1}^{m}\|Fe_j\|^2 = 2\alpha^2(d_1\sin^2\theta_1 + d_2\sin^2\theta_2).$$

Also, from (8) for bi-slant submanifolds we get

$$\begin{aligned}2\tau &= m(m-1)f_1 + 6f_2(d_1\cos^2\theta_1 + d_2\cos^2\theta_2)\\ &\quad + (m+1)^2\|H\|^2 - \|h\|^2 + 2m(f_1-f_3)\\ &\quad - 4\alpha^2(d_1\sin^2\theta_1 + d_2\sin^2\theta_2), \qquad (22)\end{aligned}$$

which can be rewritten as

$$\begin{aligned}2\tau &= m(m+1)f_1 + 6f_2(d_1\cos^2\theta_1 + d_2\cos^2\theta_2)\\ &\quad + (m+1)^2\|H\|^2 - \|h\|^2 - 2[mf_3 + 2\alpha^2(d_1\sin^2\theta_1 + d_2\sin^2\theta_2)]. \qquad (23)\end{aligned}$$

Now, we put

$$\begin{aligned}\epsilon = 2\tau - \frac{(m+1)^2(m-1)}{m}\|H\|^2 \\ - f_1(m+2)(m-1) \\ - 6f_2(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) \\ + 2[2\alpha^2(d_1\sin^2\theta_1 + d_2\sin^2\theta_2) + mf_3].\end{aligned} \tag{24}$$

We have

$$\begin{aligned}\|h\|^2 = (m+1)^2\|H\|^2 - 2\tau \\ + m(m+1)f_1 + 6f_2(d_1\cos^2\theta_1 + d_2\cos^2\theta_2) \\ - 2[mf_3 + 2\alpha^2(d_1\sin^2\theta_1 + d_2\sin^2\theta_2)];\end{aligned}$$

then

$$\epsilon + \|h\|^2 = \frac{(m+1)^2}{m}\|H\|^2 + 2f_1,$$

or equivalently,

$$(m+1)^2\|H\|^2 = m\|h\|^2 + m(\epsilon - 2f_1). \tag{25}$$

Let $\pi \subset T_pM$ a 2-plane section orthogonal to ξ and invariant by P.
We consider the following two cases.
Case (i). π is tangent to the differentiable distribution D_1. We may assume that π is spanned by an orthonormal basis $\{e_1, e_2\}$; if we take e_{m+2} in the direction of the mean curvature H, i.e., $H = \|H\|e_{m+2}$, then the relation (25) becomes

$$\left(\sum_{i=1}^{m+1} h_{ii}^{m+2}\right)^2 = m\left[\sum_{i=1}^{m+1}\left(h_{ii}^{m+2}\right)^2 + \sum_{1\le i\ne j\le m+1}\left(h_{ij}^{m+2}\right)^2 + \sum_{r=m+3}^{2n+1}\sum_{i,j=1}^{m+1}\left(h_{ij}^{r}\right)^2 + \epsilon - 2f_1\right]. \tag{26}$$

Using Lemma 3.1, from (26) we derive

$$2h_{11}^{m+2}.h_{22}^{m+2} \ge \sum_{1\le i\ne j\le m+1}\left(h_{ij}^{m+2}\right)^2 + \sum_{r=m+3}^{2n+1}\sum_{i,j=1}^{m+1}\left(h_{ij}^{r}\right)^2 + \epsilon - 2f_1. \tag{27}$$

From the Gauss equation

$$K(\pi) = R(e_1, e_2, e_2, e_1) = g(h(e_1, e_1), h(e_2, e_2)) - g(h(e_1, e_2), h(e_1, e_2)) + \overline{R}(e_1, e_2, e_2, e_1), \tag{28}$$

or similarly,

$$K(\pi) = R(e_1, e_2, e_2, e_1) = \sum_{r=m+2}^{2n+1}[h_{11}^r h_{22}^r - (h_{12}^r)] + \overline{R}(e_1, e_2, e_2, e_1). \tag{29}$$

For calculating $\overline{R}(e_1, e_2, e_2, e_1)$ we will use (2).

$$\overline{R}(e_1, e_2, e_2, e_1) = f_1 + 3f_2 g^2(\phi e_1, e_2). \tag{30}$$

By substituting (30) in (29), one has

$$K(\pi) = R(e_1, e_2, e_2, e_1) = \sum_{r=m+2}^{2n+1} [h_{11}^r h_{22}^r - (h_{12}^r)] + f_1 + 3f_2 \cos^2 \theta_1, \tag{31}$$

or

$$K(\pi) = h_{11}^{m+2} h_{22}^{m+2} + \sum_{r=m+3}^{2n+1} \left(h_{11}^r h_{22}^r\right) - \sum_{r=m+2}^{2n+1} \left(h_{12}^r\right)^2 + f_1 + 3f_2 \cos^2 \theta_1. \tag{32}$$

From (27) we have

$$h_{11}^{m+2}.h_{22}^{m+2} \geq \frac{1}{2} \sum_{1\leq i\neq j\leq m+1} \left(h_{ij}^{m+2}\right)^2 + \frac{1}{2} \sum_{r=m+3}^{2n+1} \sum_{i,j=1}^{m+1} \left(h_{ij}^r\right)^2 + \frac{\epsilon}{2} - f_1$$

Then equation (32) becomes

$$\begin{aligned} K(\pi) \geq\ & 3f_2 \cos^2 \theta_1 + \frac{1}{2} \sum_{1\leq i\neq j\leq m+1} \left(h_{ij}^{m+2}\right)^2 + \frac{1}{2} \sum_{r=m+3}^{2n+1} \sum_{i,j=1}^{m+1} \left(h_{ij}^r\right)^2 \\ & + \frac{\epsilon}{2} + \sum_{r=m+2}^{2n+1} h_{11}^r.h_{22}^r - \sum_{r=m+2}^{2n+1} \left(h_{12}^r\right)^2, \end{aligned}$$

or equivalently,

$$\begin{aligned} K(\pi) \geq\ & \sum_{r=m+2}^{2n+1} \sum_{j=2}^{m+1} \left[\left(h_{1j}^r\right)^2 + \left(h_{2j}^r\right)^2\right] + \frac{1}{2} \sum_{3\leq i\neq j\leq m+1} (h_{ij}^{m+2})^2 \\ & + \frac{1}{2} \sum_{r=m+3}^{2n+1} \sum_{3\leq i,j\leq m+1} \left(h_{ij}^r\right)^2 + \frac{1}{2} \sum_{r=m+2}^{2n+1} \left(h_{11}^r + h_{22}^r\right)^2 \\ & + 3f_2 \cos^2 \theta_1 + \frac{\epsilon}{2} \\ & \geq 3f_2 \cos^2 \theta_1 + \frac{\epsilon}{2}. \end{aligned} \tag{33}$$

Finally combining (24) and (33) we obtain (9).

$$\begin{aligned} \frac{\epsilon}{2} =\ & \tau - \frac{(m+1)^2(m-1)}{2m} \|H\|^2 \\ & - f_1 \frac{(m+2)(m-1)}{2} \\ & - 3f_2(d_1 \cos^2 \theta_1 + d_2 \cos^2 \theta_2) \\ & + mf_3 + 2\alpha^2(d_1 \sin^2 \theta_1 + d_2 \sin^2 \theta_2), \end{aligned}$$

It follows that

$$\begin{aligned} \tau - K(\pi) \leq\ & \frac{(m+1)^2(m-1)}{2m} \|H\|^2 \\ & + f_1 \frac{(m+2)(m-1)}{2} \\ & + 3f_2[(d_1 - 1) \cos^2 \theta_1 + d_2 \cos^2 \theta_2] \\ & - [mf_3 + 2\alpha^2(d_1 \sin^2 \theta_1 + d_2 \sin^2 \theta_2)], \end{aligned}$$

which is equivalent to the equation (9)
Case (ii) π is tangent to the differentiable distribution D_2. As in the case (i) we have

$$\begin{aligned}\tau - K(\pi) \leq& \frac{(m+1)^2(m-1)}{2m}\|H\|^2\\ &+ f_1\frac{(m+2)(m-1)}{2}\\ &+ 3f_2[d_1\cos^2\theta_1 + (d_2-1)\cos^2\theta_2]\\ &- mf_3 - 2\alpha^2(d_1\sin^2\theta_1 + d_2\sin^2\theta_2)],\end{aligned}$$

which is equivalent to the equation (10).
The case of equality for the inequality (9) holds at a point $p \in M$ if and only if it achieves the equality in the inequalities (27), (33) and Lemma (3.1), i.e.,

$$\begin{cases} h^{m+2}_{1j} = h^{m+2}_{2j} = ... = h^{m+2}_{mj} = 0, \quad \forall 3 \leq j \leq m,\\ h^r_{ij} = 0, \quad \forall i \neq j \quad 3 \leq i, j \leq m, r = m+2, \cdots, 2n+1,\\ h^r_{11} + h^r_{22} = 0, \qquad \forall r = m+3, \cdots, 2n+1.\\ h^{m+2}_{11} + h^{m+2}_{22} = h^{m+2}_{33} = \cdots = h^{m+2}_{m+1,m+1} = 0.\end{cases} \tag{34}$$

we obtain the desired form for the shape operators, where $a^r_i = h^r_{i,m+1}$. □

A bi-slant submanifold M is ideal in the sense of Chen if at any $p \in M$ there is a plane section tangent to D_1 satisfying (9) or a plane section tangent to D_2 satisfying (10).

Corollary 3.3. *Any ideal bi-slant submanifold of a trans-Sasakian generalized Sasakian space form is minimal.*

4. Chen-Ricci inequality

In this section, we establish an estimate of the mean curvature in terms of the Ricci curvature for bi-slant submanifolds in (α, β) trans-Sasakian generalized Sasakian space forms.

Theorem 4.1. *Let M be an $(m+1)$-dimensional $(m \geq 4)$ bi-slant submanifold isometrically immersed in a (α, β) trans-Sasakian generalized Sasakian space form $\overline{M}(f_1, f_2, f_3)$ of dimension $2n+1 \geq 9$. Then we have the following:*

(1) For each vector $X \in T_pM$ orthogonal to ξ and

(i) X tangent to D_1,

$$\mathrm{Ric}(X) \leq {}'\frac{1}{4}(m+1)^2\|H\|^2 + f_1 + \frac{3}{2}f_2\cos^2\theta_1 + (m-2)f_3 - \alpha^2\sin^2\theta_1; \tag{35}$$

(ii) X tangent to D_2,

$$\mathrm{Ric}(X) \leq \frac{1}{4}(m+1)^2\|H\|^2 + f_1 + \frac{3}{2}f_2\cos^2\theta_2 + (m-2)f_3 - \alpha^2\sin^2\theta_2. \tag{36}$$

(2) If $H(p) = 0$, then a unit tangent vector X at p satisfies (35) *or* (36) *if and only if $X \in N_p$, where $N_p = \{X \in T_pM |\, h(X, Y) = 0, \forall Y \in \{\xi\}^\perp\}$.*

(3) The equality case of (35) *and* (36) *holds identically for all unit tangent vector at p orthogonal to ξ if and only if $h(X, Y) = 0$, for all $X, Y \in \{\xi\}^\perp$.*

Proof. (1) (i) Let $p \in M$ and $X \in T_pM$ a unit tangent vector orthogonal to ξ. We consider an orthonormal basis $\{e_1 = X, e_2, \cdots, e_m, e_{m+1} = \xi, e_{m+2}, \cdots, e_{2n+1}\}$ in $T_p\overline{M}$ such that $e_1, \cdots, e_{m+1}$ are tangent to M at p, with $e_1, ..., e_{2d_1} \in D_1$, $e_{2d_1+1}, ..., e_m \in D_2$.
We recall the equation (23)

$$2\tau = m(m+1)f_1 + 3f_2\|P\|^2 + (m+1)^2\|H\|^2 - \|h\|^2 - 2[mf_3 + 2\alpha^2(d_1 \sin^2\theta_1 + d_2 \sin^2\theta_2)]. \tag{37}$$

Then we can write the last equation as

$$(m+1)^2\|H\|^2 = 2\tau + \|h\|^2 - m(m+1)f_1 - 3f_2\|P\|^2 + 2[mf_3 + 2\alpha^2(d_1 \sin^2\theta_1 + d_2 \sin^2\theta_2)], \tag{38}$$

or equivalently,

$$\begin{aligned}(m+1)^2\|H\|^2 = 2\tau &+ \sum_{r=m+2}^{2n+1}\left[(h_{11}^r)^2 + (h_{22}^r + ... + h_{mm}^r)^2 + 2\sum_{1\le i<j\le m+1}(h_{ij}^r)^2\right]\\ &- 2\sum_{r=m+2}^{2n+1}\sum_{2\le i<j\le m} h_{ii}^r h_{jj}^r - m(m+1)f_1\\ &- 3f_2\|P\|^2 + 2[mf_3 + 2\alpha^2(d_1\sin^2\theta_1 + d_2\sin^2\theta_2)].\end{aligned}$$

Then from the last equation we get

$$\begin{aligned}(m+1)^2\|H\|^2 = 2\tau &+ \frac{1}{2}\sum_{r=m+2}^{2n+1}[(h_{11}^r + \cdots + h_{mm}^r)^2 + (h_{11}^r - h_{22}^r - \cdots - h_{mm}^r)^2]\\ &+ 2\sum_{r=m+2}^{2n+1}\sum_{1\le i<j\le m+1}(h_{ij}^r)^2 - 2\sum_{r=m+2}^{2n+1}\sum_{2\le i<j\le m} h_{ii}^r h_{jj}^r\\ &- m(m+1)f_1 - 3f_2\|P\|^2\\ &+ 2[mf_3 + 2\alpha^2(d_1\sin^2\theta_1 + d_2\sin^2\theta_2)].\end{aligned}$$

We have

$$\begin{aligned}(m+1)^2\|H\|^2 = 2&\sum_{2\le i<j\le m+1} K_{ij} + 2\mathrm{Ric}(X) + \frac{1}{2}\sum_{r=m+2}^{2n+1}(h_{11}^r + \cdots + h_{mm}^r)^2\\ &+ \frac{1}{2}\sum_{r=m+2}^{2n+1}(h_{11}^r - h_{22}^r - \cdots - h_{mm}^r)^2\\ &+ 2\sum_{r=m+2}^{2n+1}\sum_{1\le i<j\le m+1}(h_{ij}^r)^2 - 2\sum_{r=m+2}^{2n+1}\sum_{2\le i<j\le m} h_{ii}^r h_{jj}^r\\ &- m(m+1)f_1 - 3f_2\|P\|^2\\ &+ 2[mf_3 + 2\alpha^2(d_1\sin^2\theta_1 + d_2\sin^2\theta_2)].\end{aligned}$$

We can add the term $h^r_{\xi\xi} = 0$.

$$
\begin{aligned}
(m+1)^2\|H\|^2 - \frac{1}{2}\sum_{r=m+2}^{2n+1}(h^r_{11} + \cdots + h^r_{mm} + h^r_{\xi\xi})^2 &= 2\sum_{2\le i<j\le m+1} K_{ij} + 2\mathrm{Ric}(X) \\
&+ \frac{1}{2}\sum_{r=m+2}^{2n+1}(h^r_{11} - h^r_{22} - \cdots - h^r_{mm})^2 \\
&+ 2\sum_{r=m+2}^{2n+1}\sum_{1\le i<j\le m+1}(h^r_{ij})^2 - 2\sum_{r=m+2}^{2n+1}\sum_{2\le i<j\le m} h^r_{ii}h^r_{jj} \\
&- m(m+1)f_1 - 3f_2\|P\|^2 \\
&+ 2[mf_3 + 2\alpha^2(d_1\sin^2\theta_1 + d_2\sin^2\theta_2)]. \qquad (39)
\end{aligned}
$$

We calculate

$$
\sum_{2\le i<j\le m+1} K_{ij} = \sum_{2\le i<j\le m} R(e_i, e_j, e_j, e_i) + 2\sum_{i=2}^{m} R(e_i, \xi, \xi, e_i). \qquad (40)
$$

From the Gauss equation we find

$$
\sum_{2\le i<j\le m} R(e_i, e_j, e_j, e_i) = \sum_{r=m+2}^{2n+1}\sum_{2\le i<j\le m}\left[h^r_{ii}h^r_{jj} - (h_{ij})^2\right] + \sum_{2\le i<j\le m}\overline{R}(e_i, e_j, e_j, e_i). \qquad (41)
$$

We can substitute $X = W = e_i$, $Y = Z = e_j$, $\forall i, j = 1, 2, ..., m$, in the curvature tensor formula (2); then we get

$$
\begin{aligned}
\overline{R}(e_i, e_j, e_j, e_i) &= f_1\{g(e_j, e_j)g(e_i, e_i) - g(e_i, e_j)g(e_j, e_i)\} \\
&+ f_2\{g(e_i, \phi e_j)g(\phi e_j, e_i) - g(e_j, \phi e_j)g(\phi e_i, e_i) \\
&+ 2g(e_i, \phi e_j)g(\phi e_j, e_i)\} \\
&+ f_3\{\eta(e_i)\eta(e_j)g(e_j, e_i) - \eta(e_j)\eta(e_j)g(e_i, e_i) \\
&+ g(e_i, e_j)\eta(e_j)\eta(e_i) - g(e_j, e_j)\eta(e_i)\eta(e_i)\}. \qquad (42)
\end{aligned}
$$

Because

$$
\sum_{2\le i<j\le m} g(e_j, e_j)g(e_i, e_i) = \frac{(m-1)(m-2)}{2}, \qquad (43)
$$

$$
\sum_{2\le i<j\le m}\overline{R}(e_i, e_j, e_j, e_i) = \frac{(m-1)(m-2)}{2}f_1 + 3f_2\sum_{2\le i<j\le m} g(\phi e_i, e_j). \qquad (44)
$$

If we introduce the equation (44) in (41), we find

$$
\sum_{2\le i<j\le m} R(e_i, e_j, e_j, e_i) = \sum_{r=m+2}^{2n+1}\sum_{2\le i<j\le m}\left[h^r_{ii}h^r_{jj} - (h_{ij})^2\right] + \frac{(m-1)(m-2)}{2}f_1 + 3f_2\sum_{2\le i<j\le m} g(\phi e_i, e_j). \qquad (45)
$$

If we substitute the equation (45) in (40), we get:

$$
\begin{aligned}
\sum_{2\le i<j\le m+1} K_{ij} &= \frac{(m-1)(m-2)}{2}f_1 + 3f_2\sum_{2\le i<j\le m} g(\phi e_i, e_j) \\
&+ \sum_{r=m+2}^{2n+1}\sum_{2\le i<j\le m}\left[h^r_{ii}h^r_{jj} - (h_{ij})^2\right] + 2\sum_{i=2}^{m} R(e_i, \xi, \xi, e_i). \qquad (46)
\end{aligned}
$$

We compute

$$\sum_{j=2}^{m} K(\xi \wedge e_j) = \sum_{j=2}^{m} R(e_i, \xi, \xi, e_i) = \sum_{j=2}^{m} \overline{R}(e_i, \xi, \xi, e_i) + \sum_{j=2}^{m} g(h(\xi, \xi), h(e_j, e_j)) - \sum_{j=2}^{m} g(h(\xi, e_j), h(\xi, e_j)). \tag{47}$$

From the expression of the curvature tensor (2)

$$\sum_{j=2}^{m} \overline{R}(e_i, \xi, \xi, e_i) = (m-1)(f_1 - f_3). \tag{48}$$

By substituting the equation (48) in the equation (47)

$$\sum_{i=2}^{m} R(e_i, \xi, \xi, e_i) = (m-1)(f_1 - f_3) + \sum_{j=2}^{m} \sum_{r=m+2}^{2n+1} [h_{jj}^r h_{\xi\xi}^r - (h_{j\xi}^r)^2], \tag{49}$$

which implies

$$\sum_{i=2}^{m} R(e_i, \xi, \xi, e_i) = (m-1)(f_1 - f_3) - \alpha^2[(2d_1 - 1)\sin^2\theta_1 + 2d_2 \sin^2\theta_2]. \tag{50}$$

By using the equations (50) and (46), we obtain

$$\begin{aligned}\sum_{2 \le i < j \le m+1} K_{ij} = {} & \frac{(m-1)(m-2)}{2} f_1 + \frac{3}{2} f_2[\|P\|^2 - \|Pe_1\|^2] \\ & + \sum_{r=m+2}^{2n+1} \sum_{2 \le i < j \le m} \left[h_{ii}^r h_{jj}^r - (h_{ij})^2\right] \\ & + 2(m-1)(f_1 - f_3) - 2\alpha^2[(2d_1 - 1)\sin^2\theta_1 + 2d_2 \sin^2\theta_2].\end{aligned} \tag{51}$$

If we introduce the equation (51) in the equation (39) we get:

$$\begin{aligned}\frac{1}{2}(m+1)^2\|H\|^2 = {} & 2\mathrm{Ric}(X) + \frac{1}{2}\sum_{r=m+2}^{2n+1} (h_{11}^r - h_{22}^r - \cdots - h_{mm}^r)^2 \\ & + 2\sum_{r=m+2}^{2n+1} \sum_{1 \le i < j \le m} (h_{ij}^r)^2 - 2\sum_{r=m+2}^{2n+1} \sum_{2 \le i < j \le m} h_{ii}^r h_{jj}^r \\ & - m(m+1) f_1 - 3 f_2 \|P\|^2 \\ & + 2[m f_3 + 2\alpha^2(d_1 \sin^2\theta_1 + d_2 \sin^2\theta_2)] \\ & + (m-1)(m-2) f_1 + 3 f_2[\|P\|^2 - \|Pe_1\|^2] \\ & + 2\sum_{r=m+2}^{2n+1} \sum_{2 \le i < j \le m} \left[h_{ii}^r h_{jj}^r - (h_{ij})^2\right] \\ & + 4(m-1)(f_1 - f_3) - 2\alpha^2[(2d_1 - 1)\sin^2\theta_1 + 2d_2 \sin^2\theta_2],\end{aligned} \tag{52}$$

which gives

$$\begin{aligned}\frac{1}{2}(m+1)^2\|H\|^2 = 2\mathrm{Ric}(X) &+ \frac{1}{2}\sum_{r=m+2}^{2n+1}(h^r_{11} - h^r_{22} - \cdots - h^r_{mm})^2 \\ &+ 2\sum_{r=m+2}^{2n+1}\sum_{1\le i<j\le m}(h^r_{ij})^2 - 2\sum_{r=m+2}^{2n+1}\sum_{2\le i<j\le m}h^r_{ii}h^r_{jj} \\ &- m^2 f_1 - m f_1 - 3f_2\|P\|^2 + 2m f_3 \\ &+ [m^2 f_1 - 3m f_1 + 2f_1] + 3f_2\|P\|^2 - 3f_2\|Pe_1\|^2 \\ &+ 2\sum_{r=m+2}^{2n+1}\sum_{2\le i<j\le m}\left[h^r_{ii}h^r_{jj} - (h_{ij})^2\right] \\ &+ 4(m-1)f_1 - 4(m-1)f_3 + 2\alpha^2\sin^2\theta_1. \qquad (53)\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{2}(m+1)^2\|H\|^2 = 2\mathrm{Ric}(X) &+ \frac{1}{2}\sum_{r=m+2}^{2n+1}(h^r_{11} - h^r_{22} - \cdots - h^r_{mm})^2 \\ &+ 2\sum_{r=m+2}^{2n+1}\sum_{1\le i<j\le m}(h^r_{ij})^2 - 2\sum_{r=m+2}^{2n+1}\sum_{2\le i<j\le m}h^r_{ii}h^r_{jj} \\ &- 2f_1 - 3f_2\|Pe_1\|^2 - 2(m-2)f_3 + 2\alpha^2\sin^2\theta_1 \\ &+ 2\sum_{r=m+2}^{2n+1}\sum_{2\le i<j\le m}\left[h^r_{ii}h^r_{jj} - (h_{ij})^2\right], \qquad (54)\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac{1}{2}(m+1)^2\|H\|^2 = 2\mathrm{Ric}(X) \\ &+ \frac{1}{2}\sum_{r=m+2}^{2n+1}(h^r_{11} - h^r_{22} - \cdots - h^r_{mm})^2 + 2\sum_{r=m+2}^{2n+1}\sum_{1\le i<j\le m}(h^r_{1j})^2 \\ &- 2f_1 - 3f_2\|Pe_1\|^2 - 2(m-2)f_3 - 2\alpha^2\sin^2\theta_1. \qquad (55)\end{aligned}$$

Then

$$\frac{1}{2}(m+1)^2\|H\|^2 \ge 2\mathrm{Ric}(X) - 2f_1 - 3f_2\|Pe_1\|^2 - 2(m-2)f_3 - 2\alpha^2\sin^2\theta_1, \qquad (56)$$

or equivalently,

$$\mathrm{Ric}(X) \le \frac{1}{4}(m+1)^2\|H\|^2 + f_1 + \frac{3}{2}f_2\|Pe_1\|^2 + (m-2)f_3 + \alpha^2\sin^2\theta_1, \qquad (57)$$

which represents the inequality from (i).

(ii) If X tangent to D_2, the same arguments prove the desired inequality.

(2) Assume $H(p) = 0$. Equality holds in (35) or (36) if and only if

$$\begin{cases} h^r_{12} = h^r_{13} = \cdots = h^r_{1m} = 0, \\ h^r_{11} = h^r_{22} + \cdots + h^r_{mm}, \quad r \in \{m+2, \cdots, 2n+1\}. \end{cases}$$

Then $h^r_{1j} = 0$, for all $j \in \{1, \cdots, m\}$, $r \in \{m+2, \cdots, 2n+1\}$, that is $X \in N_p$.

(3) The equality cases of (35) and (36) hold for all unit tangent vector at p orthogonal to ξ if and only if

$$\begin{cases} h^r_{ij} = 0, \quad 1 \le i \ne j \le m, & r \in \{m+2, \cdots, 2n+1\}, \\ h^r_{11} + \cdots + h^r_{mm} - 2h^r_{ii} = 0, & i \in \{1, \cdots, m\}, \quad r \in \{m+2, \cdots, 2n+1\}. \end{cases}$$

It follows that $h(X, Y) = 0, \forall X, Y \in \{\xi\}^\perp$. □

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