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PI elements and solutions of related equations in a ring with involution

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Abstract. In this paper, we study many new characterizations of *PI* elements in a rings with involution. Mainly, we firstly give some necessity and sufficient conditions for elements to be PI elements by discussing the solutions of related equations. Next, we discuss some properties of PI elements by constructing univariate and bivariate equations in a fixed set. Finally, we combine *PI* elements with *EP* elements, we study the properties of *SEP* elements with variable equations in a fixed set.

1. Introduction

Let R be a ring with an identity. If there exists $a^{\#} \in R$ such that

$$aa^{\#}a = a,\ a^{\#}aa^{\#} = a^{\#},\ aa^{\#} = a^{\#}a,$$

then a is called a group invertible element and $a^{\#}$ is called a group inverse of a [1, 4, 8], and if it exists, then it is uniquely determined by these equations. We write $R^{\#}$ to denote the set of all group invertible elements of R.

If a map $* : R \to R$ satisfies

$$(a^*)^* = a,\ (a+b)^* = a^* + b^*,\ (ab)^* = b^*a^*,$$

then R is said to be an involution ring or a $*-$ring.

Let R be a $*$-ring and $a \in R$. If there exists $a^+ \in R$ such that

$$a = aa^+a,\ a^+ = a^+aa^+,\ (aa^+)^* = aa^+,\ (a^+a)^* = a^+a,$$

then a is called a Moore-Penrose invertible element, and a^+ is called the Moore-Penrese inverse of a [6, 7]. Let R^+ denote the set of all Moore-Penrese invertible elements of R.

Let R be a $*$-ring and $a \in R$. If $a = aa^*a$, then a is called a partial isometry of R [6, 7]. Let R^{PI} denote the set of all partial isometries of R. Obviously, if $a \in R^+$, then $a \in R^{PI}$ if and only if $a^* = a^+$.

If $a \in R^{\#} \cap R^+$ and $a^{\#} = a^+$, then a is called an *EP* element. On the studies of *EP*, the readers can refer to [2, 3, 5, 6, 9–14].

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If $a \in R^{\#} \cap R^{+}$ and $a^* = a^+ = a^{\#}$, then a is called an *SEP* element of R [6, 7]. Let R^{SEP} denote the set of all *SEP* elements of R.

In [12], many characterizations of PI elements are given, we have learned some equivalent conditions for *PI* and *EP* elements.

Motivated by these results, this paper mainly study the *PI* elements by exploring the solutions of related equations in a fixed set. It plays a key role in the generalized inverse of a general ring.

2. Characterizing PI elements by the solution of univariate equations

We begin with the following lemma.

Lemma 2.1. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if $x \in R^{+}$ and $x^+ = x^*$ for some $x \in \eta_a =: \{a, a^+, a^*, (a^+)^*\}$.*

PROOF. " $\Rightarrow$ " If $a \in R^{PI}$, then $a^+ = a^*$, it follows that $x = a$ is a solution.
" $\Leftarrow$ " From the assumption, we have
(1) If $x = a$, then $a^+ = a^*$.
(2) If $x = a^+$, then $(a^+)^+ = (a^+)^*$, this implies $a^+ = a^*$.
(3) If $x = a^*$, then $(a^*)^+ = (a^*)^* = a$, this gives $a^* = a^+$.
(4) If $x = (a^+)^*$, then $((a^+)^*)^+ = ((a^+)^*)^* = a^+$, one gets $a^* = a^+$.
Hence, in any case, we have $a \in R^{PI}$. ∎

Theorem 2.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if $x \in R^{+}$ and $ax^+a = ax^*a$ for some $x \in \eta_a$.*

PROOF. " $\Rightarrow$ " If $a \in R^{PI}$, then, by Lemma 2.1, $x^+ = x^*$ for some $x \in \eta_a$. Hence, $ax^+a = ax^*a$ for some $x \in \eta_a$.
" $\Leftarrow$ " (1) If $x = a$, then

$$a = aa^+a = aa^*a.$$

Thus, $x \in R^{PI}$.
(2) If $x = a^+$, then

$$a^3 = a(a^+)^+a = a(a^+)^*a,$$

this infers

$$a = a^{\#}a^3a^{\#} = a^{\#}a(a^+)^*aa^{\#} = (a^+)^*.$$

Hence, $a \in R^{PI}$.
(3) If $x = a^*$, then

$$a(a^*)^+a = a(a^*)^*a = a^3.$$

By (2), we have $a \in R^{PI}$.
(4) If $x = (a^+)^*$, then

$$a((a^+)^*)^+a = a((a^+)^*)^*a,$$

and it follows that $aa^*a = a$.
Hence, $a \in R^{PI}$. ∎

Lemma 2.3. *Let $a \in R^{\#} \cap R^{+}$. Then*
(1) $(ax^+a)^{\#} = a^{\#}xa^{\#}$ for $x \in \chi_a =: \{a, a^{\#}, a^+, a^*, (a^+)^*, (a^{\#})^*\}$;
(2) $(ax^*a)^{\#} = a^{\#}(x^+)^*a^{\#}$ for $x \in \chi_a$.

PROOF. (1) Since

$$(ax^+a)(a^{\#}xa^{\#}) = a(x^+aa^{\#}x)a^{\#} = \begin{cases} a(a^+a)a^{\#}, x \in \tau_a =: \{a, a^{\#}, (a^+)^*\} \\ a(aa^+)a^{\#}, x \in \gamma_a =: \{a^+, a^*, (a^{\#})^*\} \end{cases} = aa^{\#},$$

$$(ax^+a)(a^{\#}xa^{\#})(ax^+a) = (aa^{\#})(ax^+a) = ax^+a,$$

$$(a^\# x a^\#)(ax^+a)(a^\# x a^\#) = a^\# x a^\# (aa^\#) = a^\# x a^\#, \text{and}$$

$$(a^\# x a^\#)(ax^+a) = a^\#(xa^\# ax^+)a = \begin{cases} a^\#(aa^+)a, x \in \tau_a \\ a^\#(a^+a)a, x \in \gamma_a \end{cases} = aa^\#.$$

Hence, $(ax^+a)^\# = a^\# x a^\#$.

(2) Noting that

$$(ax^*a)(a^\#(x^+)^*a^\#) = a(x^*aa^\#(x^+)^*)a^\# = \begin{cases} a(a^+a)a^\# = aa^\#, x \in \tau_a \\ a(aa^+)a^\# = aa^\#, x \in \gamma_a \end{cases} = aa^\#,$$

$$(ax^*a)(a^\#(x^+)^*a^\#)(ax^*a) = aa^\#(ax^*a) = ax^*a,$$

$$(a^\#(x^+)^*a^\#)(ax^*a) = a^\#((x^+)^*a^\# ax^*)a = \begin{cases} a^\#(aa^+)a, x \in \tau_a \\ a^\#(a^+a)a, x \in \gamma_a \end{cases} = aa^\#,$$

and

$$(a^\#(x^+)^*a^\#)(ax^*a)(a^\#(x^+)^*a^\#) = aa^\#(a^\#(x^+)^*a^\#) = a^\#(x^+)^*a^\#.$$

Hence, $(ax^*a)^\# = a^\#(x^+)^*a^\#$. ∎

Theorem 2.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if $x \in R^+$ and $a^\# x a^\# = a^\#(x^+)^*a^\#$ for some $x \in \eta_a$.*

PROOF. It follows from Theorem 2.2 and Lemma 2.3. ∎

Lemma 2.5. *Let $a \in R^\# \cap R^+$. Then $(a^\#)^+ = a^+a^3a^+$, $(a^+)^\# = (aa^\#)^*a(aa^\#)^*$.*

PROOF. Since

$$a^\#(a^+a^3a^+)a^\# = aa^+a^\# = a^\#; \ (a^+a^3a^+)a^\#(a^+a^3a^+) = a^+a^3a^+aa^+ = a^+a^3a^+;$$

$$a^\#(a^+a^3a^+) = aa^+ = (aa^+)^* = (a^\#(a^+a^3a^+))^*; \ (a^+a^3a^+)a^\# = a^+a = (a^+a)^* = ((a^+a^3a^+)a^\#)^*.$$

Thus, $(a^\#)^+ = a^+a^3a^+$.

Since

$$a^+((aa^\#)^*a(aa^\#)^*)a^+ = a^+aa^+ = a^+;$$

$$((aa^\#)^*a(aa^\#)^*)a^+((aa^\#)^*a(aa^\#)^*) = (aa^\#)^*aa^+(aa^\#)^*a(aa^\#)^* = (aa^\#)^*aa^+a(aa^\#)^* = (aa^\#)^*a(aa^\#)^*.$$

Meanwhile,

$$a^+((aa^\#)^*a(aa^\#)^*) = a^+a(aa^\#)^* = (aa^\#)^*; \ ((aa^\#)^*a(aa^\#)^*)a^+ = (aa^\#)^*aa^+ = (aa^\#)^*,$$

then we get $a^+((aa^\#)^*a(aa^\#)^*) = ((aa^\#)^*a(aa^\#)^*)a^+$.

Hence, $(a^+)^\# = (aa^\#)^*a(aa^\#)^*$. ∎

Lemma 2.6. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if $x \in R^+$ and $(a(x^+)^*a)^+ = a^+x^\#a^+$ for some $x \in \eta_a$.*

PROOF. " ⇒ " Since $a \in R^{PI}$, $(x^+)^* = x$ for some $x \in \eta_a$ by Lemma 2.1.
It follows that

$$(a(x^+)^*a)(a^+x^\#a^+) = a(xaa^+x^\#)a^+ = \begin{cases} a(aa^\#)a^+, x \in \{a, (a^+)^*\} \\ a(aa^\#)^*a^+, x \in \{a^+, a^*\} \end{cases} = aa^+,$$

$$(a(x^+)^*a)(a^+x^\#a^+)(a(x^+)^*a) = aa^+(a(x^+)^*a) = a(x^+)^*a,$$

$$(a^+x^\#a^+)(a(x^+)^*a) = a^+(x^\#a^+ax)a = \begin{cases} a^+(a^\#a)a, x \in \{a, (a^+)^*\} \\ a^+(aa^\#)^*a, x \in \{a^+, a^*\} \end{cases} = a^+a,$$

and

$$(a^+x^\# a^+)(a(x^+)^*a)(a^+x^\# a^+) = a^+a(a^+x^\# a^+) = a^+x^\# a^+.$$

Hence, $(a(x^+)^*a)^+ = a^+x^\# a^+$.

" $\Leftarrow$ " (1) If $x = a$, then

$$(a(a^+)^*a)^+ = a^+a^\# a^+.$$

We consider that

$$a(a^+)^*aa^+aa^\# a^*aa^\# a^+ = a(a^+)^*aa^\# a^*aa^\# a^+ = a(a^+)^*a^*aa^\# a^+ = aaa^+aa^\# a^+ = aa^+;$$

$$a^+aa^\# a^*aa^\# a^+a(a^+)^*a = a^+aa^\# a^*aa^\#(a^+)^*a = a^+aa^\# a^*(a^+)^*a = a^+aa^\# a^+aa = a^+aa^\# a = a^+a.$$

It deduces

$$(a(a^+)^*aa^+aa^\# a^*aa^\# a^+)^* = a(a^+)^*aa^+aa^\# a^*aa^\# a^+;\ (a^+aa^\# a^*aa^\# a^+a(a^+)^*a)^* = a^+aa^\# a^*aa^\# a^+a(a^+)^*a;$$

$$a(a^+)^*aa^+aa^\# a^*aa^\# a^+a(a^+)^*a = a(a^+)^*aa^+a = a(a^+)^*a;$$

$$a^+aa^\# a^*aa^\# a^+a(a^+)^*aa^+aa^\# a^*aa^\# a^+ = a^+aa^\# a^*aa^\# a^+aa^+ = a^+aa^\# a^*aa^\# a^+.$$

From above, it gives

$$(a(a^+)^*a)^+ = a^+aa^\# a^*aa^\# a^+.$$

Then

$$a^+a^\# a^+ = a^+aa^\# a^*aa^\# a^+,$$

this gives

$$a^\# = a(a^+a^\# a^+)a = a(a^+aa^\# a^*aa^\# a^+)a = aa^\# a^*aa^\#$$

and

$$a = a(a^\#)a = a(aa^\# a^*aa^\#)a = aa^*a.$$

Hence, $a \in R^{PI}$.

(2) If $x = a^+$, then

$$(a((a^+)^+)^*a)^+ = (aa^*a)^+ = a^+(a^+)^\# a^+.$$

Besides, by the following equalities:

$$(aa^*aa^+(a^\#)^*a^+)^* = (aa^*(a^\#)^*a^+)^* = (a(a^\# a)^*a^+)^* = (aa^+)^* = aa^+ = aa^*aa^+(a^\#)^*a^+;$$

$$(a^+(a^\#)^*a^+aa^*a)^* = (a^+(a^\#)^*a^*a)^* = (a^+(aa^\#)^*a)^* = (a^+a)^* = a^+(a^\#)^*a^+aa^*a;$$

$$aa^*aa^+(a^\#)^*a^+aa^*a = aa^+aa^*a = aa^*a;\ a^+(a^\#)^*a^+aa^*aa^+(a^\#)^*a^+ = a^+aa^+(a^\#)^*a^+ = a^+(a^\#)^*a^+,$$

it obviously tells $(aa^*a)^+ = a^+(a^\#)^*a^+$. Then it implies

$$a^+(a^\#)^*a^+ = a^+(aa^\#)^*a(aa^\#)^*a^+ = a^+,$$

and

$$a = aa^+(a^\#)^*a^+a = (a^+aa^\# aa^+)^* = (a^+)^*.$$

Hence, $a \in R^{PI}$.

(3) If $x = a^*$, then

$$(a((a^*)^+)^*a)^+ = (aa^+a)^+ = a^+ = a^+(a^*)^\# a^+,$$

that is,

$$a^+ = a^+(a^\#)^*a^+.$$

Hence, $a \in R^{PI}$ by (2).

(4) If $x = (a^+)^*$, then

$$(a(((a^+)^*)^+)^*a)^+ = (aaa)^+ = a^+((a^+)^*)^\# a^+.$$

We note that

$$(aaaa^+a^\#a^+)^* = (aa^+)^* = aa^+ = aaaa^+a^\#a^+;\ (a^+a^\#a^+aaa)^* = (a^+a)^* = a^+a = a^+a^\#a^+aaa;$$

$$aaaa^+a^\#a^+aaa = aa^+aaa = aaa;\ a^+a^\#a^+aaaa^+a^\#a^+ = a^+aa^+a^\#a^+ = a^+a^\#a^+.$$

So $(aaa)^+ = a^+a^\#a^+$. Thus, $a^+a^\#a^+ = a^+aa^\#a^*aa^\#a^+$. Hence, $a \in R^{PI}$ by (1). ∎

Theorem 2.7. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if $x \in R^+$ and $(ax^+a)^\# = aa^\#a^*aa^\#xa^\#$ for some $x \in \chi_a$.*

PROOF. " ⇒ "Assume that $a \in R^{PI}$. Then $a^* = a^+$.
By Lemma 2.3, one gets

$$(ax^+a)^\# = a^\#xa^\# = ((a^*)^*)^\#xa^\# = ((a^+)^*)^\#xa^\# = ((a^+)^\#)^*xa^\# = ((aa^\#)^*a(aa^\#)^*)^*xa^\# = aa^\#a^*aa^\#xa^\#$$

for some $x \in \chi_a$.

" ⇐ " From the assumption and Lemma 2.3, one obtains $a^\#xa^\# = aa^\#a^*aa^\#xa^\#$ for some $x \in \chi_a$.

If $x \in \tau_a$, then

$$xaa^\#x^+ = aa^+,$$

this gives

$$\begin{aligned} a^\#aa^+ &= a^\#(xa^\#ax^+) = (a^\#xa^\#)ax^+ = (aa^\#a^*aa^\#xa^\#)ax^+ = aa^\#a^*aa^\#(xa^\#aa^+) \\ &= aa^\#a^*aa^\#aa^+ = aa^\#a^*, \end{aligned}$$

and

$$a^+ = a^+a(a^\#aa^+) = a^+a(aa^\#a^*) = a^*.$$

Hence, $a \in R^{PI}$.

If $x \in \gamma_a$, then

$$xaa^\#x^+ = a^+a,$$

it follows that

$$\begin{aligned} a^\# = a^\#a^+a = a^\#(xa^\#ax^+) = (a^\#xa^\#)ax^+ = (aa^\#a^*aa^\#xa^\#)ax^+ = \\ aa^\#a^*aa^\#(xa^\#ax^+) = aa^\#a^*aa^\#a^+a = aa^\#a^*aa^\#, \end{aligned}$$

and

$$a = aa^\#a = a(aa^\#a^*aa^\#)a = aa^*a.$$

Thus, $a \in R^{PI}$. ∎

Theorem 2.8. *Let $a \in R^\# \cap R^+$. Then $a \in R^{PI}$ if and only if $x \in R^+$ and $(ax^+a)^\# = (a^+)^*a^\#a^\#xa^\#$ for some $x \in \chi_a$.*

PROOF. " ⇒ " Since $a \in R^{PI}$, $(a^+)^* = a$, it follows that

$$(a^+)^*a^\#a^\# = a^\#.$$

Hence, by Lemma 2.3, we get

$$(ax^+a)^\# = (a^+)^*a^\#a^\#xa^\#.$$

" ⇐ " By the hypothesis and Lemma 2.3, there exists $x_0 \in \chi_a$ such that

$$a^\#x_0a^\# = (a^+)^*a^\#a^\#x_0a^\#.$$

If $x_0 \in \tau_a$, then

$$x_0a^\#ax_0^+ = aa^+,$$

this induces

$$a^\#aa^+ = a^\#x_0a^\#ax_0^+ = (a^+)^*a^\#a^\#x_0a^\#ax_0^+ = (a^+)^*a^\#a^\#aa^+ = (a^+)^*a^\#a^+,$$

and

$$a = a^{\#}aa^{+}a^{2} = (a^{+})^{*}a^{\#}a^{+}a^{2} = (a^{+})^{*}a^{\#}a = (a^{+})^{*}.$$

Hence, $a \in R^{PI}$.

If $x_0 \in \gamma_a$, then

$$x_0a^{\#}ax_0^{+} = a^{+}a.$$

One obtains

$$a^{\#} = a^{\#}a^{+}a = a^{\#}x_0a^{\#}ax_0^{+} = (a^{+})^{*}a^{\#}a^{\#}x_0a^{\#}ax_0^{+} = (a^{+})^{*}a^{\#}a^{\#}a^{+}a = (a^{+})^{*}a^{\#}a^{\#},$$

and

$$a = a^{\#}a^{2} = (a^{+})^{*}a^{\#}a^{\#}a^{2} = (a^{+})^{*}a^{\#}a = (a^{+})^{*}.$$

Hence, $a \in R^{PI}$.

3. Construct equations to characterize PI or SEP elements in a given set

It is well known that if $a \in R^{\#}$, then $a + 1 - aa^{\#} \in R^{-1}$ with $(a + 1 - aa^{\#})^{-1} = a^{\#} + 1 - aa^{\#}$. Hence, Lemma 2.3 leads to the following lemma, which proof is routine.

Lemma 3.1. *Let $a \in R^{\#} \cap R^{+}$. Then $ax^{+}a + 1 - aa^{\#} \in R^{-1}$ with $(ax^{+}a + 1 - aa^{\#})^{-1} = a^{\#}xa^{\#} + 1 - aa^{\#}$ for each $x \in \chi_a$.*

From Lemma 3.1 and Theorem 2.7, we have

Theorem 3.2. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{PI}$ if and only if $x \in R^{+}$ and $(ax^{+}a + 1 - aa^{\#})^{-1} = (a^{+})^{*}a^{\#}a^{\#}xa^{\#} + 1 - aa^{\#}$ for some $x \in \chi_a$.*

Theorem 3.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $x \in R^{+}$ and $(ax^{+}a + 1 - aa^{\#})^{-1} = (a^{+})^{*}a^{\#}a^{\#}xa^{+} + 1 - aa^{\#}$.*

PROOF. " ⇒ " Assume that $a \in R^{SEP}$. Then $a \in R^{EP}$ and

$$(ax^{+}a + 1 - aa^{\#})^{-1} = (a^{+})^{*}a^{\#}a^{\#}xa^{\#} + 1 - aa^{\#}$$

for some $x \in \chi_a$ by Theorem 3.2.

Now $a^{\#} = a^{+}$, then

$$(ax^{+}a + 1 - aa^{\#})^{-1} = (a^{+})^{*}a^{\#}a^{\#}xa^{+} + 1 - aa^{\#}$$

for some $x \in \chi_a$.

" ⇐ " From the hypothesis, there exists $x_0 \in \chi_a$ such that

$$(ax_0^{+}a + 1 - aa^{\#})^{-1} = (a^{+})^{*}a^{\#}a^{\#}x_0a^{+} + 1 - aa^{\#},$$

this implies

$$ax_0^{+}a(a^{+})^{*}a^{\#}a^{\#}x_0a^{+} = aa^{\#}.$$

Multiplying the equality on the right by aa^{+}, one has

$$aa^{\#} = aa^{+}.$$

Hence, $a \in R^{EP}$, this induces

$$x_0a^{\#}ax_0^{+} = x_0a^{+}ax_0^{+} = aa^{\#}.$$

Thus,

$$\begin{aligned} a(a^{+})^{*}a^{\#}a^{\#} &= aa^{\#}a(a^{+})^{*}a^{\#}a^{\#}aa^{\#} = (x_0a^{\#}ax_0^{+})a(a^{+})^{*}a^{\#}a^{\#}(x_0a^{+}ax_0^{+}) \\ &= x_0a^{\#}(ax_0^{+}a(a^{+})^{*}a^{\#}a^{\#}x_0a^{+})ax_0^{+} = x_0a^{\#}aa^{\#}ax_0^{+} = x_0a^{\#}ax_0^{+} = aa^{\#}. \end{aligned}$$

It follows that

$$a^{2} = aa^{\#}a^{2} = a(a^{+})^{*}a^{\#}a^{\#}a^{2} = a(a^{+})^{*},$$

and

$$a = a^{\#}a^{2} = a^{\#}a(a^{+})^{*} = (a^{+})^{*}.$$

Hence, $a \in R^{PI}$ and so $a \in R^{SEP}$. ∎

Theorem 3.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $x \in R^{+}$ and $(ax^{+}a + 1 - aa^{\#})^{-1} = a^{*}xa^{\#} + 1 - aa^{\#}$ for some $x \in \chi_a$.*

PROOF. " $\Rightarrow$ " Since $a \in R^{SEP}$, $a \in R^{EP}$ and by Theorem 3.2, we have

$$(ax^{+}a + 1 - aa^{\#})^{-1} = (a^{+})^{*}a^{\#}a^{\#}xa^{\#} + 1 - aa^{\#}$$

for some $x \in \chi_a$.
Noting that $(a^{+})^{*}a^{\#}a^{\#} = aa^{\#}a^{\#} = a^{\#} = a^{*}$. Then $(ax^{+}a + 1 - aa^{\#})^{-1} = a^{*}xa^{\#} + 1 - aa^{\#}$.
" $\Leftarrow$ " From the assumption, one has

$$(a^{*}x_0a^{\#} + 1 - aa^{\#})(ax_0^{+}a + 1 - aa^{\#}) = 1$$

for some $x_0 \in \chi_a$, this gives

$$a^{*}x_0a^{\#}ax_0^{+}a = aa^{\#}.$$

Multiplying the equality on the left by $a^{+}a$, one yields

$$aa^{\#} = a^{+}a.$$

Hence, $a \in R^{EP}$, it follows that

$$x_0a^{\#}ax_0^{+} = aa^{\#}.$$

This induces

$$a^{*}a = a^{*}aa^{\#}a = a^{*}(x_0a^{\#}ax_0^{+})a = a^{\#}a.$$

Hence, $a \in R^{SEP}$ by [12, Theorem 1.5.3]. ∎

Lemma 3.5. *Let $a \in R^{\#} \cap R^{+}$, $x \in R^{+}$. Then $(a^{*}xa^{\#})^{\#} = a^{+}a^{3}a^{+}x^{\#}(a^{+})^{*}$ for $x \in \chi_a$.*

PROOF. Clearly, we have

$$(a^{*}xa^{\#})a^{+}a^{3}a^{+}x^{\#}(a^{+})^{*} = a^{*}(xaa^{+}x^{\#})(a^{+})^{*} = \begin{cases} a^{*}aa^{\#}(a^{+})^{*}, x \in \tau_a \\ a^{*}(aa^{\#})^{*}(a^{+})^{*}, x \in \gamma_a \end{cases} = a^{+}a,$$

$$(a^{*}xa^{\#})(a^{+}a^{3}a^{+}x^{\#}(a^{+})^{*})(a^{*}xa^{\#}) = a^{+}a(a^{*}xa^{\#}) = a^{*}xa^{\#},$$

$$(a^{+}a^{3}a^{+}x^{\#}(a^{+})^{*})(a^{*}xa^{\#}) = a^{+}a^{3}a^{+}(x^{\#}aa^{+}x)a^{\#} = \begin{cases} a^{+}a^{3}a^{+}(a^{\#}a)a^{\#}, x \in \tau_a \\ a^{+}a^{3}a^{+}(aa^{\#})^{*}a^{\#}, x \in \gamma_a \end{cases} = a^{+}a,$$

and

$$(a^{+}a^{3}a^{+}x^{\#}(a^{+})^{*})(a^{*}xa^{\#})(a^{+}a^{3}a^{+}x^{\#}(a^{+})^{*}) = a^{+}a(a^{+}a^{3}a^{+}x^{\#}(a^{+})^{*}) = a^{+}a^{3}a^{+}x^{\#}(a^{+})^{*}.$$

Hence, $(a^{*}xa^{\#})^{\#} = (a^{*}xa^{\#})^{+} = a^{+}a^{3}a^{+}x^{\#}(a^{+})^{*}$ for $x \in \chi_a$. ∎

Corollary 3.6. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $x \in R^{+}$ and $ax^{+}a = a^{+}a^{3}a^{+}x^{\#}(a^{+})^{*}$ for some $x \in \chi_a$.*

PROOF. " $\Rightarrow$ " Suppose that $a \in R^{SEP}$. Then

$$(a^{*}xa^{\#} + 1 - aa^{\#})^{-1} = ax^{+}a + 1 - aa^{\#}$$

for some $x \in \chi_a$ by Theorem 3.4.
Since $a \in R^{EP}$, $aa^{\#} = aa^{+}$.
By Lemma 3.5,

$$(a^{*}xa^{\#} + 1 - aa^{\#})^{-1} = a^{+}a^{3}a^{+}x^{\#}(a^{+})^{*} + 1 - aa^{\#}.$$

This implies

$$ax^+a = a^+a^3a^+x^\#(a^+)^*$$

for some $x \in \chi_a$.

" $\Leftarrow$ " From the hypothesis, there exists $x_0 \in \chi_a$ such that $ax_0^+a = a^+a^3a^+x_0^\#(a^+)^*$.
Then

$$ax_0^+a = a^+a(a^+a^3a^+x_0^\#(a^+)^*) = a^+a^2x_0^+a.$$

If $x_0 \in \tau_a$, then $a = aa^+a = a(x_0^+aa^\#x_0) = (a^+a^2x_0^+a)a^\#x_0 = a^+a^2(x_0^+aa^\#x_0) = a^+a^2a^+a = a^+a^2$.
Hence, $a \in R^{EP}$.

For $x_0 \in \gamma_a$, we conclusively have

$$a^2a^+ = a(x_0^+aa^\#x_0) = a^+a^2x_0^+aa^\#x_0 = a^+a^3a^+.$$

This gives

$$aa^\# = a^2a^+a^\# = a^+a^3a^+a^\# = a^+a.$$

Hence $a \in R^{EP}$, one gets

$$a^+a^3a^+ = a.$$

It follows from $ax_0^+a = a^+a^3a^+x_0^\#(a^+)^*$ that

$$ax_0^+a = ax_0^\#(a^+)^*.$$

Noting that $a \in R^{EP}$. Then

$$x_0^+ = x_0^\#$$

and

$$x_0a^\#ax_0^+ = x_0a^\#ax_0^\# = aa^\#.$$

Thus, one gets

$$a = aa^\#a = x_0a^\#ax_0^+a = x_0a^\#ax_0^\#(a^+)^* = aa^\#(a^+)^* = (a^+)^*.$$

Hence, $a \in R^{PI}$, so $a \in R^{SEP}$. ∎

4. Characterize SEP elements by the solution of equations in a fixed set

Theorem 4.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if $x \in R^+$ and $ax^+a = ax^\#(a^\#)^*$ for some $x \in \chi_a$.*

PROOF. " $\Rightarrow$ " Assume taht $a \in R^{SEP}$. Then $a \in R^{EP}$ and

$$ax^+a = a^+a^3a^+x^\#(a^+)^*$$

for some $x \in \chi_a$ by Corollary 3.6.
Noting that $a \in R^{EP}$. Then

$$a^+a^3a^+ = a^\#a^3a^\# = a$$

and

$$(a^+)^* = (a^\#)^*.$$

Hence,

$$ax^+a = ax^\#(a^\#)^*.$$

" $\Leftarrow$ " From the assumption, we have $ax^+a = ax^\#(a^\#)^*$ for some $x \in \chi_a$.

(1) If $x = a$, then $aa^+a = aa^\#(a^\#)^*$, it follows that

$$a = aa^\#(a^\#)^* = aa^\#(a^\#)^*aa^+ = a^2a^+.$$

Hence, $a \in R^{EP}$, this infers

$$a = aa^{\#}(a^{\#})^{*} = aa^{\#}(a^{+})^{*} = (a^{+})^{*}.$$

Thus, $a \in R^{SEP}$.

(2) If $x = a^{\#}$, then

$$a(a^{\#})^{+}a = a(a^{\#})^{\#}(a^{\#})^{*},$$

so

$$a^{3} = a^{2}(a^{\#})^{*},$$

and

$$a = a^{\#}a^{\#}a^{3} = a^{\#}a^{\#}a^{2}(a^{\#})^{*} = aa^{\#}(a^{\#})^{*}.$$

Thus, $a \in R^{SEP}$ by (1).

(3) If $x = a^{+}$, then

$$a(a^{+})^{+}a = a(a^{+})^{\#}(a^{\#})^{*},$$

so

$$a^{3} = a(aa^{\#})^{*}a(a^{\#})^{*} = a(aa^{\#})^{*}a(a^{\#})^{*}aa^{+} = a^{4}a^{+}.$$

Hence, $a \in R^{EP}$, this induces

$$x = a^{+} = a^{\#}.$$

By (2), $a \in R^{SEP}$.

(4) If $x = a^{*}$, then $a(a^{*})^{+}a = a(a^{*})^{\#}(a^{\#})^{*}$, so

$$a(a^{+})^{*}a = a(a^{\#})^{*}(a^{\#})^{*} = a(a^{\#})^{*}(a^{\#})^{*}aa^{+} = a(a^{+})^{*}a^{2}a^{+}.$$

Multiplying the equality on the left by $aa^{\#}a^{*}a^{\#}$, one yields

$$a = a^{2}a^{+}.$$

Hence, $a \in R^{EP}$ and we obtains

$$(a^{+})^{*}a = a^{\#}a(a^{+})^{*}a = a^{\#}a(a^{\#})^{*}(a^{\#})^{*} = a^{\#}a(a^{+})^{*}(a^{\#})^{*} = (a^{+})^{*}(a^{\#})^{*}.$$

Applying the involution on the last equality, one gets

$$a^{*}a^{+} = a^{\#}a^{+}.$$

Hence, $a \in R^{SEP}$ by [12, Theorem 1.5.3].

(5) If $x = (a^{+})^{*}$, then $a((a^{+})^{*})^{+}a = a((a^{+})^{*})^{\#}(a^{\#})^{*} = a((a^{+})^{\#})^{*}(a^{\#})^{*}$, according to Lemma 2.5, one obtains

$$aa^{*}a = aa^{*}aa^{\#}(a^{\#})^{*}.$$

Multiplying the equality on the left by $(a^{+})^{*}a^{+}$, one has

$$a = aa^{\#}(a^{\#})^{*}.$$

Hence, $a \in R^{SEP}$ by (1).

(6) If $x = (a^{\#})^{*}$, then $a((a^{\#})^{*})^{+}a = a((a^{\#})^{*})^{\#}(a^{\#})^{*}$, it gives $a((a^{\#})^{+})^{*}a = a((a^{\#})^{\#})^{*}(a^{\#})^{*}$, by Lemma 2.5, we have

$$a^{2}a^{+}a^{*}a^{+}a^{2} = aa^{*}(a^{\#})^{*} = aa^{*}(a^{\#})^{*}aa^{+} = a^{2}a^{+}a^{*}a^{+}a^{3}a^{+}.$$

Multiplying the equality on the left by $(a^{\#})^{*}a^{\#}$, one gets

$$a^{+}a^{2} = a^{+}a^{3}a^{+}.$$

Hence, $a = aa^{\#}a^{+}a^{2} = aa^{\#}a^{+}a^{3}a^{+} = a^{2}a^{+}$, one has $a \in R^{EP}$,
this infers

$$x = (a^{\#})^{*} = (a^{+})^{*}.$$

Thus, $a \in R^{SEP}$ by (5). ∎

Lemma 4.2. *Let $a \in R^{\#} \cap R^{+}$. Then $(ax^{\#}(a^{\#})^{*})^{\#} = aa^{+}a^{*}a^{+}axa^{+}$ for $x \in \chi_a$.*

PROOF. Evidently, we have

$$(ax^{\#}(a^{\#})^{*})(aa^{+}a^{*}a^{+}axa^{+}) = a(x^{\#}a^{+}ax)a^{+} = \begin{cases} a(a^{\#}a)a^{+}, x \in \tau_a \\ a(aa^{\#})^{*}a^{+}, x \in \gamma_a \end{cases} = aa^{+},$$

$$(ax^{\#}(a^{\#})^{*})(aa^{+}a^{*}a^{+}axa^{+})(aa^{\#}(a^{\#})^{*}) = aa^{+}(aa^{\#}(a^{\#})^{*}) = aa^{\#}(a^{\#})^{*},$$

$$(aa^{+}a^{*}a^{+}axa^{+})(ax^{\#}(a^{\#})^{*}) = aa^{+}a^{*}a^{+}a(xa^{+}ax^{\#})(a^{\#})^{*} = \begin{cases} aa^{+}a^{*}a^{+}a(aa^{\#})(a^{\#})^{*}, x \in \tau_a \\ aa^{+}a^{*}a^{+}a(aa^{\#})^{*}(a^{\#})^{*}, x \in \gamma_a \end{cases} = aa^{+},$$

and

$$(aa^{+}a^{*}a^{+}axa^{+})(ax^{\#}(a^{\#})^{*})(aa^{+}a^{*}a^{+}axa^{+}) = aa^{+}(aa^{+}a^{*}a^{+}axa^{+}) = aa^{+}a^{*}a^{+}axa^{+}.$$

Hence, $(ax^{\#}(a^{\#})^{*})^{\#} = (ax^{\#}(a^{\#})^{*})^{+} = aa^{+}a^{*}a^{+}axa^{+}$. ∎

From Lemma 2.3, Lemma 4.2 and Theorem 4.1, we have the following theorem.

Theorem 4.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $x \in R^{+}$ and $a^{\#}xa^{\#} = aa^{+}a^{*}a^{+}axa^{+}$ for some $x \in \chi_a$.*

Corollary 4.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $x \in R^{+}$ and $a^{\#}xa^{\#} = a^{*}xa^{+}$ for some $x \in \chi_a$.*

PROOF. " $\Rightarrow$ " Assume that $a \in R^{SEP}$, then $a \in R^{EP}$, this gives

$$a^{*} = aa^{+}a^{*}a^{+}a.$$

Hence, by Theorem 4.3, we have $a^{\#}xa^{\#} = a^{*}xa^{+}$ for some $x \in \chi_a$.

" $\Leftarrow$ " From the hypothesis, we have $a^{\#}x_0a^{\#} = a^{*}x_0a^{+}$ for some $x_0 \in \chi_a$.

Using $a^{+} = a^{+}aa^{+}$, one has

$$a^{\#}x_0a^{\#} = a^{\#}x_0a^{\#}aa^{+}.$$

If $x_0 \in \tau_a$, then

$$a^{+}aa^{\#} = (x_0^{+}aa^{\#}x_0)a^{\#} = x_0^{+}a(a^{\#}x_0a^{\#}) = x_0^{+}a(a^{\#}x_0a^{\#}aa^{+})$$
$$= (x_0^{+}aa^{\#}x_0)a^{\#}aa^{+} = a^{+}aa^{\#}aa^{+} = a^{+}.$$

Hence, $a \in R^{EP}$.

If $x_0 \in \gamma_a$, then

$$a^{\#} = aa^{+}a^{\#} = (x_0^{+}aa^{\#}x_0)a^{\#} = x_0^{+}a(a^{\#}x_0a^{\#}) = x_0^{+}a(a^{\#}x_0a^{\#}aa^{+})$$
$$= (x_0^{+}aa^{\#}x_0)a^{\#}aa^{+} = aa^{+}a^{\#}aa^{+} = a^{\#}aa^{+}.$$

Hence, $a \in R^{EP}$.

In any case, we have

$$a^{*} = aa^{+}a^{*}a^{+}a,$$

so

$$a^{\#}x_0a^{\#} = a^{*}x_0a^{+} = aa^{+}a^{*}a^{+}ax_0a^{+}.$$

By Theorem 4.3, $a \in R^{SEP}$. ∎

Theorem 4.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if $x \in R^{+}$ and $yx^{+}a = yx^{\#}(a^{\#})^{*}$ for some $(x, y) \in \chi_a^2 = \{(x, y) | x, y \in \chi_a\}$.*

PROOF. " $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then

$$ax_0^{+}a = ax_0^{\#}(a^{\#})^{*} \text{ for each } x_0 \in \chi_a.$$

Choose $(x, y) = (x_0, a)$. Then

$$yx^+a = yx^\#(a^\#)^*.$$

" $\Leftarrow$ " According to the assumption, we have

$$y_0x_0^+a = y_0x_0^\#(a^\#)^* \text{ for some } (x_0, y_0) \in \chi_a^2.$$

(1) If $y_0 = a$, then

$$ax_0^+a = ax_0^\#(a^\#)^*.$$

By Theorem 4.3, $a \in R^{SEP}$.

(2) If $y_0 = a^\#$, then

$$a^\#x_0^+a = a^\#x_0^\#(a^\#)^*,$$

it follows that

$$ax_0^+a = a^2a^\#x_0^+a = a^2a^\#x_0^\#(a^\#)^* = ax_0^\#(a^\#)^*.$$

Hence, $a \in R^{SEP}$ by (1);

(3) If $y_0 = a^+$, then

$$a^+x_0^+a = a^+x_0^\#(a^\#)^* = a^+x_0^\#(a^\#)^*(aa^+) = a^+x_0^+a^2a^+.$$

Similar to the proof of Corollary 3.6, we have $a \in R^{EP}$.
Henc, $y_0 = a^+ = a^\#$. By (2), $a \in R^{SEP}$;

(4) If $y_0 = a^*$, then

$$a^*x_0^+a = a^*x_0^\#(a^\#)^*.$$

Multiplying the equality on the left by $a^+(a^\#)^*$, one yields

$$a^+x_0^+a = a^+x_0^\#(a^\#)^*.$$

Hence, $a \in R^{SEP}$ by (3).

(5) If $y_0 = (a^\#)^*$, then

$$(a^\#)^*x_0^+a = (a^\#)^*)x_0^\#(a^\#)^*.$$

Multiplying the equality on the left by $(a^*)^2$, one gets

$$a^*x_0^+a = a^*x_0^\#(a^\#)^*.$$

Hence, $a \in R^{SEP}$ by (4).

(6) If $y_0 = (a^+)^*$, then

$$(a^+)^*x_0^+a = (a^+)^*x_0^\#(a^\#)^*.$$

Multiplying the equality on the left by aa^*, one obtains

$$ax_0^+a = ax_0^\#(a^\#)^*.$$

Hence, $a \in R^{SEP}$ by (1). ∎

5. The general solution of bivariate equations

Now we consider the following equation

$$a^\#xa^\# = a^*ya^+. \tag{5.1}$$

Lemma 5.1. *Let $a \in R^\# \cap R^+$. Then the general solution to Eq.(5.1) is given by*

$$\begin{cases} x = aa^+pa^+a + u - a^+auaa^+ \\ y = (a^+)^*a^+p + v - aa^+va^+a \end{cases}, \ p,\ u,\ v \in R \text{ with } a^+pa^+ = aa^+a^+pa^+a^+a. \tag{5.2}$$

PROOF. First, it's easy to prove that (5.2) is a solution to Eq.(5.1).
In fact,

$$a^{\#}(aa^{+}pa^{+}a+u-a^{+}auaa^{+})a^{\#}=a^{\#}aa^{+}pa^{+}aa^{\#}+a^{\#}ua^{\#}-a^{\#}a^{+}auaa^{+}a^{\#}$$
$$=a^{\#}aa^{+}pa^{+}aa^{\#}=a^{\#}a(aa^{+}a^{+}pa^{+}a^{+}a)aa^{\#}=aa^{+}a^{+}pa^{+}a^{+}a=a^{+}pa^{+},$$
$$a^{*}((a^{+})^{*}a^{+}p+v-aa^{+}va^{+}a)a^{+}=a^{*}(a^{+})^{*}a^{+}pa^{+}+a^{*}va^{+}-a^{*}va^{+}=a^{*}(a^{+})^{*}a^{+}pa^{+}=a^{+}aa^{+}pa^{+}=a^{+}pa^{+}.$$

Next, let

$$\begin{cases} x=x_0 \\ y=y_0 \end{cases} \tag{5.3}$$

be any solution to Eq.(5.1). Then

$$a^{\#}x_0a^{\#}=a^{*}y_0a^{+}.$$

Taking $p=aa^{*}y_0a^{+}a$, $u=x_0-aa^{+}pa^{+}a$, $v=y_0$.
Then

$$a^{+}pa^{+}=a^{+}(aa^{*}y_0a^{+}a)a^{+}=a^{*}y_0a^{+}=a^{\#}x_0a^{\#}=aa^{+}(a^{\#}x_0a^{\#})a^{+}a=aa^{+}a^{+}pa^{+}a^{+}a.$$
$$a^{+}auaa^{+}=a^{+}a(x_0-aa^{+}pa^{+}a)aa^{+}=a^{+}ax_0aa^{+}-a^{+}a^2a^{+}pa^{+}a^2a^{+}$$
$$=a^{+}a^2(a^{\#}x_0a^{\#})a^2a^{+}-a^{+}a^2(a^{+}pa^{+})a^2a^{+}=a^{+}a^2a^{*}y_0a^{+}a^2a^{+}-a^{+}a^2(a^{*}y_0a^{+})a^2a^{+}=0.$$

Hence,

$$x_0=aa^{+}pa^{+}a+(x_0-aa^{+}pa^{+}a)=aa^{+}pa^{+}a+u=aa^{+}pa^{+}a+u-a^{+}auaa^{+}.$$
$$aa^{+}va^{+}a=aa^{+}y_0a^{+}a=(a^{+})^{*}(a^{*}y_0a^{+}a)=(a^{+})^{*}(a^{*}y_0a^{+}a)=(a^{+})^{*}a^{+}(aa^{*}y_0a^{+}a)=(a^{+})^{*}a^{+}p.$$

Hence,

$$y_0=(a^{+})^{*}a^{+}p+y_0-(a^{+})^{*}a^{+}p=(a^{+})^{*}a^{+}p+v-aa^{+}va^{+}a.$$

■

Theorem 5.2. *Let $a\in R^{\#}\cap R^{+}$. Then $a\in R^{SEP}$ if and only if the general solution to Eq.(5.1) is given by*

$$\begin{cases} x=a^{*}apa^{+}a+u-a^{+}auaa^{+} \\ y=(a^{+})^{*}a^{+}p+v-aa^{+}va^{+}a \end{cases},\ p,\ u,\ v\in R. \tag{5.4}$$

PROOF. " ⇒ " Assume that $a\in R^{SEP}$. Then

$$aa^{+}=a^{*}a \text{ by [12, Theorem 1.5.3]}$$

and

$$a^{+}=aa^{+}a^{+}=a^{+}a^{+}a.$$

Hence, by Lemma 5.1, we are done.
" ⇐ " From the assumption, one obtains

$$a^{\#}(a^{*}apa^{+}a+u-a^{+}auaa^{+})a^{\#}=a^{*}((a^{+})^{*}a^{+}p+v-aa^{+}va^{+}a)a^{+},$$

that is,

$$a^{\#}a^{*}apa^{+}aa^{\#}=a^{+}pa^{+} \text{ for all } p\in R.$$

Especially, choose $p=a$, one yields

$$a^{\#}a^{*}a=a^{+}.$$

Hence, $a\in R^{SEP}$ by [12, Theorem 1.5.3]. ■

Similarly, we have

Theorem 5.3. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if the general solution to Eq.(5.1) is given by*

$$\begin{cases} x = aa^*pa^+a + u - a^+auaa^+ \\ y = (a^+)^*a^+p + v - aa^+va^+a \end{cases}, \ p,\ u,\ v \in R. \tag{5.5}$$

Now we construct the following equation

$$a^+axaa^{\#} = a^*a(aa^{\#})^*aa^*ya^+a. \tag{5.6}$$

Theorem 5.4. *Let $a \in R^{\#} \cap R^{+}$. Then the general solution to Eq.(5.6) is given by*

$$\begin{cases} x = a^*apa^+a + u - a^+auaa^+ \\ y = (a^+)^*a^+p + v - aa^+va^+a \end{cases}, \ p,\ u,\ v \in R \text{ with } a^+p = a^+a^+apa^+a. \tag{5.7}$$

PROOF. First, we show that the formula (5.7) is the solution to Eq.(5.6).
In fact,

$$a^+a(a^*apa^+a + u - a^+auaa^+)aa^{\#} = a^*apa^+a,$$

$$a^*a(aa^{\#})^*aa^*((a^+)^*a^+p + v - aa^+va^+a)a^+a = a^*a(aa^{\#})^*aa^*(a^+)^*a^+pa^+a$$
$$= a^*a(aa^{\#})^*aa^*(a^+)^*a^+a^+apa^+a = a^*apa^+a.$$

Next, let

$$\begin{cases} x = x_0 \\ y = y_0 \end{cases} \tag{5.8}$$

be any solution to Eq.(5.6).
Then

$$a^+ax_0aa^{\#} = a^*a(aa^{\#})^*aa^*y_0a^+a.$$

Choose $p = (aa^{\#})^*aa^*y_0a^+a$, $u = x_0 - a^*apa^+a$, $v = y_0$. Then, we have

$$a^+p = a^*y_0a^+a,$$

$$a^+a^+apa^+a = a^+a^+a(aa^{\#})^*aa^*y_0a^+a = a^*y_0a^+a.$$

Hence,

$$a^+p = a^+a^+apa^+a,$$

$$a^+auaa^+ = a^+a(x_0 - a^*apa^+a)aa^+ = (a^+ax_0aa^{\#})aa^+ - a^*apa^+a^2a^+ = a^*a(aa^{\#})^*aa^*y_0a^+a^2a^+ - a^*apa^+a^2a^+$$
$$= a^*a(aa^{\#})^*a(a^+a^+apa^+a)aa^+ - a^*apa^+a^2a^+ = a^*apa^+a^2a^+ - a^*apa^+a^2a^+ = 0.$$

It follows that

$$x_0 = a^*apa^+a + u = a^*apa^+a + u - a^+auaa^+.$$

Noting that

$$(a^+)^*a^+p = (a^+)^*a^+(aa^{\#})^*aa^*y_0a^+a = aa^+y_0a^+a = aa^+va^+a.$$

Then

$$y_0 = (a^+)^*a^+p + v - aa^+va^+a.$$

Thus, the general solution to Eq.(5.6) is given by (5.7). ∎

Theorem 5.5. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if Eq.(5.1) has the same solution as Eq.(5.6).*

PROOF. " $\Rightarrow$ " Assume that $a \in R^{SEP}$. Then $a \in R^{EP}$ and by Theorem 5.2, the general solution to Eq.(5.1) is given by (5.4). Noting that $a \in R^{EP}$. Then

$$a^+a^+a = a^+.$$

Hence, the formula (5.7) is the same as (5.4). By Theorem 5.4, the general solution to Eq.(5.6) is given by (5.4). Hence, Eq.(5.1) has the same solution as Eq.(5.6).

" $\Leftarrow$ " From the assumption, we have the general solution to Eq.(5.1) is given by (5.7).

Hence,

$$a^{\#}(a^*apa^+a + u - a^+auaa^+)a^{\#} = a^*((a^+)^*a^+p + v - aa^+va^+a)a^+.$$

That is,

$$a^{\#}a^*apa^+aa^{\#} = a^+pa^+ \text{ for } p \in R \text{ satisfying } a^+a^+ap = a^+p.$$

Especially, choose $p = a^+$, then

$$a^{\#}a^*a^+aa^{\#} = a^+a^+a^+.$$

Multiplying the equality on the left by $a^{\#}a$, one gets

$$a^+a^+a^+ = a^{\#}aa^+a^+a^+.$$

By [16, Lemma 2.11], one obtains

$$a^+ = a^{\#}aa^+.$$

Hence, $a \in R^{EP}$ by [12, Theorem 1.2.1], it follows that

$$a^{\#}a^*a^{\#} = a^{\#}a^*a^+aa^{\#} = a^+a^+a^+ = a^{\#}a^{\#}a^{\#}.$$

This induces

$$aa^*a = a^2(a^{\#}a^*a^{\#})a^2 = a^2(a^{\#}a^{\#}a^{\#})a^2 = a.$$

Thus, $a \in R^{SEP}$. ∎

6. The solution of bivariate equations in a fixed set

Now we know that if $a \in R^{SEP}$, then $a^{\#}xa^{\#}aa^+ = a^{\#}xa^{\#} = a^*xa^+$ for each $x \in \chi_a$. Hence, we can construct the following equation

$$a^{\#}xa^{\#}ay = a^*xy. \tag{6.1}$$

Theorem 6.1. *Let $a \in R^{\#} \cap R^+$. Then $a \in R^{SEP}$ if and only if Eq.(6.1) has at least one solution in $\chi_a^2 = \{(x, y) | x, y \in \chi_a\}$.*

PROOF. " $\Rightarrow$ " Suppose that $a \in R^{SEP}$. Then $a^{\#}x_0a^{\#} = a^*x_0a^+$ for some $x_0 \in \chi_a$ by Corollary 4.4. Noting that $a \in R^{EP}$. Then

$$a^{\#} = a^{\#}aa^+,$$

one obtains

$$a^{\#}x_0a^{\#}aa^+ = a^*aa^+.$$

Hence,

$$(x, y) = (x_0, a^+) \tag{6.2}$$

is a solution in χ_a^2.

" $\Leftarrow$ " From the assumption, there exists (x_0, y_0) in χ_a^2 such that

$$a^{\#}x_0a^{\#}ay_0 = a^*x_0y_0.$$

Multiplying the equatity on the left by a^+a, one yields

$$a^{\#}x_0a^{\#}ay_0 = a^+aa^{\#}x_0a^{\#}ay_0.$$

(1) If $y_0 \in \tau_a$, then

$$y_0y_0^{\#} = aa^{\#},$$

it follows that

$$a^{\#}x_0aa^{\#} = a^{\#}x_0aa^{\#}y_0y_0^{\#} = a^+aa^{\#}x_0a^{\#}ay_0y_0^{\#} = a^+aa^{\#}x_0aa^{\#}.$$

If $x_0 \in \tau_a$, then

$$x_0aa^{\#}x_0^+ = aa^+,$$

this gives

$$a^{\#}aa^+ = a^{\#}x_0aa^{\#}x_0^+ = a^+aa^{\#}x_0aa^{\#}x_0^+ = a^+aa^{\#}aa^+ = a^+.$$

Hence, $a \in R^{EP}$.
If $x_0 \in \gamma_a$, then

$$x_0aa^{\#}x_0^+ = a^+a,$$

this induces

$$a^{\#} = a^{\#}a^+a = a^{\#}x_0aa^{\#}x_0^+ = a^+aa^{\#}x_0aa^{\#}x_0^+ = a^+aa^{\#}a^+a = a^+aa^{\#}.$$

Hence, $a \in R^{EP}$.
(2) If $y_0 \in \gamma_a$, then

$$y_0y_0^{\#} = (aa^{\#})^*,$$

this leads to

$$a^{\#}x_0a^{\#}a(aa^{\#})^* = a^{\#}x_0a^{\#}ay_0y_0^{\#} = a^+aa^{\#}x_0a^{\#}ay_0y_0^{\#} = a^+aa^{\#}x_0a(aa^{\#})^*.$$

Multiplying the equality on the right by a^+a, one gets

$$a^{\#}x_0a^{\#}a = a^+aa^{\#}x_0a^{\#}a.$$

By (1), $a \in R^{EP}$.
Hence, in any case, we have $a \in R^{EP}$, and so $\chi_a = \{a, a^{\#}, a^*, (a^+)^*\}$.
① If $y_0 = a$, then

$$a^{\#}x_0a = a^{\#}x_0a^{\#}a^2 = a^*x_0a.$$

Hence, $a \in R^{SEP}$ by Corollary 4.4.
② If $y_0 = a^{\#}$, then

$$a^*x_0a^{\#} = a^*x_0a^{\#}aa^{\#} = a^*x_0a^{\#}.$$

Hence, $a \in R^{SEP}$ by Corollary 4.4.
③ If $y_0 = a^*$, then

$$a^{\#}x_0a^* = a^{\#}x_0a^{\#}aa^* = a^*x_0a^*.$$

This gives

$$a^{\#}x_0a^{\#} = a^{\#}x_0a^+ = a^{\#}x_0a^*(a^{\#})^*a^+ = a^*x_0a^*(a^{\#})^*a^+ = a^*x_0a^+ = a^*x_0a^{\#}.$$

By ②, $a \in R^{SEP}$.
④ If $y_0 = (a^+)^*$, then

$$a^{\#}x_0(a^+)^* = a^{\#}x_0a^{\#}a(a^+)^* = a^*x_0(a^+)^*$$

and

$$a^{\#}x_0a = a^{\#}x_0(a^+)^*a^*a = a^*x_0(a^+)^*a^*a = a^*x_0a.$$

By ①, $a \in R^{SEP}$. ∎

Obversing the proof of Theorem 6.1, we can obtain the following corollary.

Corollary 6.2. *Let $a \in R^{\#} \cap R^+$. Then the folllowings are equivalent:*
(1) $a \in R^{SEP}$;
(2) $a^{\#}xa^+ay = a^*xy$ has at least one solution in χ_a^2;
(3) $a^{\#}xaa^+y = a^*xy$ has at least one solution in χ_a^2;
(4) $a^{\#}x_0(aa^{\#})^*y = a^*xy$ has at least one solution in χ_a^2.

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