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Strong convergence to a solution of nonlinear equations

Shagun Sharma[a,*], Sumit Chandok[a]

[a] *Thapar Institute of Engineering and Technology, Patiala- 147004, India*

Abstract. In this paper, we propose an algorithm for approximating the common fixed points of nonexpansive mappings and strongly pseudocontractive mappings. We include a few numerical examples to support our assertions and demonstrate that our technique converges faster. Consequently, we obtain an algorithm that converges strongly to the fixed point of the mapping. Also, under some assumptions, we obtain that our iterative process converges to a solution of fractional iterative integrodifferential and ordinary differential equations.

1. Introduction and Preliminaries

In nonlinear analysis, the convergence of iterative procedures for fixed points has long been a fascinating problem. For Banach contraction maps, Picard's iterative technique converges well, but it does not converge for nonexpansive mappings even when the existence of the fixed point is guaranteed. To overcome this challenge, several researchers have been active in developing different iterative methods to approximate the fixed points of nonexpansive mappings and other classes of mappings that are more general than the class of nonexpansive mappings. Numerous scholars have developed iterative techniques for strong and weak convergence of nonlinear problems in the literature, which converge more quickly than Picard's scheme. Mann [12], Ishikawa [6], Agarwal et al. [1], Sahu [14], Sahu and Petrusel [15], Gopi. and Pragadeeswarar [5] process, and so on, used various iterative schemes on the approximation of fixed points for different classes of operators.

Let $\mathcal{X}$ be a real Banach space and J denote the normalized duality mapping from $\mathcal{X} \to 2^{\mathcal{X}^*}$ defined by

$$J(\hat{d}) = \left\{e^* \in \mathcal{X}^* : \left\langle \hat{d}, e^* \right\rangle = \|\hat{d}\|^2, \|e^*\| = \|\hat{d}\|\right\}, \text{ for all } \hat{d} \in \mathcal{X}, \tag{1}$$

where $\mathcal{X}^*$ denotes the dual space of $\mathcal{X}$ and $\langle ., . \rangle$ denotes the generalized duality pairing. We will denote the single-valued duality map by J.

Definition 1.1. *Let $\mathcal{S}$ be a non-empty subset of $\mathcal{X}$. A mapping*

(i) $T_1 : \mathcal{S} \to \mathcal{S}$ *is a nonexpansive (NE) [3] if* $\|T_1\hat{d} - T_1\hat{e}\| \le \|\hat{d} - \hat{e}\|$ *for all* $\hat{d}, \hat{e} \in \mathcal{S}$.

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* Corresponding author: Shagun Sharma
Email addresses: shagunsharmapandit8115@gmail.com (Shagun Sharma), sumit.chandok@thapar.edu (Sumit Chandok)

(ii) $T : \mathcal{S} \to \mathcal{S}$ *is a pseudocontractive [8] if there exists* $j(\hat{d} - \hat{e}) \in J(\hat{d} - \hat{e})$ *such that*

$$\left\langle T\hat{d} - T\hat{e}, j(\hat{d} - \hat{e}) \right\rangle \leq \|\hat{d} - \hat{e}\|^2,$$

for all $\hat{d}, \hat{e} \in \mathcal{S}$.

(iii) $T_2 : \mathcal{S} \to \mathcal{S}$ *is a strongly pseudocontractive (SP) [11] if there exists* $j(\hat{d} - \hat{e}) \in J(\hat{d} - \hat{e})$ *and a constant* $k \in (0, 1)$ *such that*

$$\left\langle T_2\hat{d} - T_2\hat{e}, j(\hat{d} - \hat{e}) \right\rangle \leq k\|\hat{d} - \hat{e}\|^2,$$

for all $\hat{d}, \hat{e} \in \mathcal{S}$.

Example 1.2. *Let* $\mathcal{X} = \mathbb{R}$ *with usual norm and* $\mathcal{S} = [0, 1)$. *Define* $T : \mathcal{S} \to \mathcal{S}$ *by*

$$T(\hat{d}) = \hat{d} + 1,$$

for all $\hat{d} \in \mathcal{S}$. *For all* $\hat{d}, \hat{e} \in \mathcal{S}$, *we have*

$$\left\langle T\hat{d} - T\hat{e}, j(\hat{d} - \hat{e}) \right\rangle = (\hat{d} - \hat{e})^2 = \|\hat{d} - \hat{e}\|^2.$$

This shows that T *is a pseudocontractive mapping.*

Example 1.3. *Let* $\mathcal{X} = \mathbb{R}$ *with usual norm and* $\mathcal{S} = [0, \infty)$. *Define* $T_2 : \mathcal{S} \to \mathcal{S}$ *by*

$$T_2(\hat{d}) = \frac{\hat{d}}{\sqrt{3}(1 + \hat{d})},$$

for all $\hat{d} \in \mathcal{S}$. *For all* $\hat{d}, \hat{e} \in \mathcal{S}$, *we have*

$$\left\langle T_2\hat{d} - T_2\hat{e}, j(\hat{d} - \hat{e}) \right\rangle = \frac{1}{\sqrt{3}(1 + \hat{d})(1 + \hat{e})}(\hat{d} - \hat{e})^2 = k\|\hat{d} - \hat{e}\|^2,$$

where $k = \frac{1}{\sqrt{3}(1+\hat{d})(1+\hat{e})} < 1$. *This shows that* T_2 *is a (SP) mapping.*

Later, many papers on the approximation of fixed points of (SP) and (NE) mappings appeared in the literature.

In 2011, Sahu [14], Sahu and Petrusel [15] proposed an algorithm known as the S-normal algorithm in Banach space as follows:
Suppose that T_2 is a self-mapping on a non-empty subset $\mathcal{S}$ of $\mathcal{X}$. For arbitrary $\hat{d}_0 \in \mathcal{S}$,

$$\begin{cases} \hat{e}_n = & (1 - \gamma'_n)\hat{d}_n + \gamma'_n T_2\hat{d}_n \\ \hat{d}_{n+1} = & T_2\hat{e}_n, \text{ where } \gamma'_n \in [0, 1]. \end{cases} \qquad \text{Algorithm (NS)}$$

They showed that this process converges faster than Picard, Mann [12] and Ishikawa [6] iteration processes for contraction mappings in the sense of Berinde [2] to the fixed of T_2. In 2013, Kang et al. [9] gave the following algorithm for approximate common fixed points of (NE) and (SP) mappings in real Banach space. For arbitrary $\hat{d}_0 \in \mathcal{S}$,

$$\begin{cases} \hat{e}_n = & (1 - \gamma'_n)\hat{d}_n + \gamma'_n T_2\hat{d}_n \\ \hat{d}_{n+1} = & T_1\hat{e}_n, \text{ where } \gamma'_n \in [0, 1]. \end{cases} \qquad \text{Algorithm (HNS)}$$

Dass et al. [4] proposed the following algorithm for approximate common fixed points of (NE) and (SP) mappings in uniformly smooth Banach space. For arbitrary $\hat{d}_0 \in \mathcal{S}$,

$$\begin{cases} \hat{f}_n = & (1-\gamma'_n)\hat{d}_n + \gamma'_n T_1 \hat{d}_n \\ \hat{e}_n = & (1-\delta'_n)\hat{d}_n + \delta'_n T_2 \hat{f}_n \\ \hat{d}_{n+1} = & T_1 \hat{e}_n \end{cases} \qquad \text{Algorithm (D)}$$

where $\gamma'_n, \delta'_n \in (0,1]$, $n \in \mathbb{N} \cup \{0\}$.

Recently, Okeke and Ofem [13] proposed the following algorithm for approximate common fixed points of (NE) and (SP) mappings in Banach space. For arbitrary $\hat{d}_0 \in \mathcal{S}$,

$$\begin{cases} \hat{f}_n = & (1-\gamma'_n)\hat{d}_n + \gamma'_n T_1 \hat{d}_n \\ \hat{e}_n = & (1-\eta'_n-\delta'_n)\hat{d}_n + \eta'_n T_2 \hat{f}_n + \delta'_n T_2 \hat{d}_n \\ \hat{d}_{n+1} = & T_1 \hat{e}_n \end{cases} \qquad \text{Algorithm (O)}$$

where $\gamma'_n, \eta'_n, \delta'_n \in (0,1]$, $n \in \mathbb{N} \cup \{0\}$.

The following results will be used in the sequel.

Lemma 1.4. *[9] Let $J : \mathcal{X} \to 2^{\mathcal{X}^*}$ be the normalized duality mapping. Then for all $\hat{d}, \hat{e} \in \mathcal{X}$,*

$$\|\hat{d}+\hat{e}\|^2 \leq \|\hat{d}\|^2 + 2\left\langle \hat{e}, j(\hat{d}+\hat{e})\right\rangle, \forall j(\hat{d}+\hat{e}) \in J(\hat{d}+\hat{e}).$$

Lemma 1.5. *[16] Suppose $\{l_n\}$ and $\{t_n\}$ are nonnegative sequence such that*

$$l_{n+1} \leq (1-k')t_n + c_n,$$

where $k'_n \in (0,1)$, $\sum_{n=0}^{\infty} k' = \infty$ and $\lim_{n\to\infty} \frac{c_n}{k'_n} = 0$, then $\lim_{n\to\infty} b_n = 0$.

Inspired by these interesting iterative schemes, we construct and propose a new algorithm that converges strongly to a common fixed point for a pair of mappings satisfying different classes of contractions in the context of real Banach spaces. We also include numerical examples to support our assertions and demonstrate that our technique converges faster. Consequently, we obtain an algorithm that converges strongly to the fixed point of the mapping. Also, under some assumptions, we obtain that our iterative process converges to a solution of fractional iterative integrodifferential equations (FID) and nonlinear ordinary differential equations (ODE).

2. Main Results

Throughout this section, we assume that $\mathcal{S}$ is a non-empty closed and convex subset of a real Banach space $\mathcal{X}$.

2.1. *Algorithm 1.*

Suppose that $T_1, T_2 : \mathcal{S} \to \mathcal{S}$ are self mappings. For an arbitrary element $\hat{d}_0 \in \mathcal{S}$, define a sequence $\left\{\hat{d}_n\right\}$ as follows:

$$\begin{aligned} \hat{f}_n &= (1-\gamma'_n)T_2\hat{d}_n + \gamma'_n\hat{d}_n \\ \hat{e}_n &= (1-\eta'_n-\delta'_n-\omega'_n)\hat{d}_n + \omega'_n T_1\hat{f}_n + \eta'_n T_2\hat{f}_n + \delta'_n T_2\hat{d}_n \\ \hat{d}_{n+1} &= T_1\hat{e}_n \end{aligned} \tag{2}$$

where $\omega_n, \gamma'_n, \eta'_n, \delta'_n \in [0,1]$, $n \in \mathbb{N} \cup \{0\}$.

Remark 2.1. *Algorithm 1 reduces to*

- *(NS) when* $\gamma'_n = 1$ *and* $\delta'_n, \omega'_n = 0, T_1 = T_2$,
- *(HNS) when* $\gamma'_n = 1$ *and* $\delta'_n, \omega'_n = 0, = 0$.

Now we prove our main result using Algorithm 1 as follows:

Theorem 2.2. *Let* $T_1 : \mathcal{S} \to \mathcal{S}$ *be a NE mapping, and* $T_2 : \mathcal{S} \to \mathcal{S}$ *be a uniformly continuous SP mapping with* $T_2(\mathcal{S})$ *is bounded. Further assume that* $F = \cap_{i=1}^{2} F(T_i) = \left\{\hat{d} \in \mathcal{S} : T_1\hat{d} = T_2\hat{d} = \hat{d}\right\} \neq \emptyset$ *and* $\gamma'_n, \eta'_n, \delta'_n$ *are real sequences in* $[0,1]$, *such that:*

(i) $\gamma'_n + \eta'_n + \delta'_n + \omega'_n \leq 1$;

(ii) $\lim_{n\to\infty}(\omega'_n + \eta'_n + \delta'_n) = 0 = \lim_{n\to\infty}(1-\gamma'_n)$;

(iii) $\sum_{n=1}^{\infty}(\omega'_n + \eta'_n + \delta'_n) = \infty$.

Then for any arbitrary $\hat{d}_0 \in \mathcal{S}$, *the sequence* $\left\{\hat{d}_n\right\}$ *defined by Algorithm 1 converges strongly to common fixed point* $\hat{p} \in F$.

Proof. Let $\hat{p} \in F(T_1) \cap F(T_2)$. Because T_2 has bounded range, suppose that

$$\mathcal{M}_1 = \sup\left\{\|T_2\hat{d} - T_2\hat{e}\| : \hat{d}, \hat{e} \in \mathcal{X}\right\}. \tag{3}$$

This show $\mathcal{M}_1 < \infty$. Consider

$$\begin{aligned}
\|\hat{f}_n - \hat{p}\| =& \|\gamma'_n\hat{d}_n + (1-\gamma'_n)T_2\hat{d}_n - \hat{p}\| \\
=& \|\gamma'_n(\hat{d}_n - \hat{p}) + (1-\gamma'_n)(T_2\hat{d}_n - \hat{p})\| \\
=& \|\gamma'_n(\hat{d}_n - \hat{p}) + (1-\gamma'_n)(T_2\hat{d}_n - T_2\hat{p})\| \\
\leq& \gamma'_n\|\hat{d}_n - \hat{p}\| + (1-\gamma'_n)\|T_2\hat{d}_n - T_2\hat{p}\| \\
\leq& \gamma'_n\|\hat{d}_k - \hat{p}\| + (1-\gamma'_n\|\mathcal{M}_1.
\end{aligned} \tag{4}$$

$$\begin{aligned}
\|\hat{e}_n - \hat{p}\| =& \|(1-\eta'_n-\delta'_n-\omega'_n)\hat{d}_n + \omega'_nT_1\hat{f}_n + \eta'_nT_2\hat{f}_n + \delta'_nT_2\hat{d}_n - \hat{p}\| \\
\leq& (1-\eta'_n-\delta'_n-\omega'_n)\|\hat{d}_n - \hat{p}\| + \omega'_n\|T_1\hat{f}_n - \hat{p}\| + \eta'_n\|T_2\hat{f}_n - \hat{p}\| + \delta'_n\|(T_2\hat{d}_n - \hat{p})\| \\
=& (1-\eta'_n-\delta'_n-\omega'_n)\|\hat{d}_n - \hat{p}\| + \omega'_n\|T_1\hat{f}_n - T_1\hat{p}\| + \eta'_n\|(T_2\hat{f}_n - T_2\hat{p})\| + \delta'_n\|(T_2\hat{d}_n - T_2\hat{p})\|.
\end{aligned} \tag{5}$$

We claim that $\|\hat{d}_n - \hat{p}\| \leq \mathcal{M}_2$, where

$$\mathcal{M}_2 = \mathcal{M}_1 + \|\hat{d}_0 - \hat{p}\|, n \geq 0 \text{ and } n \in \mathbb{N}. \tag{6}$$

Clearly $\mathcal{M}_1 \leq \mathcal{M}_2$. It is true for $n = 0$. Suppose that it is true for $n = k$, then we will prove for $n = k+1$. By equations (2), (4) and (5), we have

$$\begin{aligned}
\|\hat{d}_{k+1} - \hat{p}\| =& \|T_1\hat{e}_k - \hat{p}\| \\
=& \|T_1\hat{e}_k - T_1\hat{p}\| \leq \|\hat{e}_k - \hat{p}\| \\
=& (1-\eta'_k-\delta'_k-\omega'_k)\|\hat{d}_k - \hat{p}\| + \omega'_k\|T_1\hat{f}_k - T_1\hat{p}\| + \eta'_k\|(T_2\hat{f}_k - T_2\hat{p})\| + \delta'_k\|(T_2\hat{d}_k - T_2\hat{p})\| \\
\leq& (1-\eta'_k-\delta'_k-\omega'_k)\|\hat{d}_k - \hat{p}\| + \omega'_k\|\hat{f}_k - \hat{p}\| + \eta'_k\|(T_2\hat{f}_k - T_2\hat{p})\| + \delta'_k\|(T_2\hat{d}_k - T_2\hat{p})\| \\
\leq& (1-\eta'_k-\delta'_k-\omega'_k)\|\hat{d}_k - \hat{p}\| + \omega'_k(\gamma'_n\|\hat{d}_k - \hat{p}\| + (1-\gamma'_n)\mathcal{M}_1) + \eta'_k\|(T_2\hat{f}_k - T_2\hat{p})\| \\
&+ \delta'_k\|(T_2\hat{d}_k - T_2\hat{p})\| \\
\leq& (1-\eta'_k-\delta'_k-\omega'_k)\mathcal{M}_2 + \omega'_k(\gamma'_n\mathcal{M}_2 + (1-\gamma'_n)\mathcal{M}_2) + \eta'_k\mathcal{M}_2 + \delta'_k\mathcal{M}_2 \\
=& (1-\eta'_k-\delta'_k-\omega'_k)\mathcal{M}_2 + \omega'_k\mathcal{M}_2 + \eta'_k\mathcal{M}_2 + \delta'_k\mathcal{M}_2 \\
=& \mathcal{M}_2.
\end{aligned} \tag{7}$$

This shows that $\left\{\|\hat{d}_n-\hat{p}\|\right\}$ is bounded. By equations (4) and (5), $\{\|\hat{e}_n-\hat{p}\|\}$ and $\left\{\|\hat{f}_n-\hat{p}\|\right\}$ are bounded sequences.

Choose $\mathcal{D}_1=\sup\left\{\|\hat{d}_n-\hat{p}\|:n\geq 0\right\}$, $\mathcal{D}_2=\sup\{\|\hat{e}_n-\hat{p}\|:n\geq 0\}$ and $\mathcal{D}_3=\sup\left\{\|\hat{f}_n-\hat{p}\|:n\geq 0\right\}$. Denote $\mathcal{M}=\mathcal{M}_2+\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3$, then $\mathcal{M}<\infty$. Using Lemma 1.4, we have

$$\begin{aligned}
\|\hat{e}_n-\hat{p}\|^2 =&\|(1-\eta_n'-\delta_n'-\omega_n')\hat{d}_n+\omega_n'T_1\hat{f}_n+\eta_n'T_2\hat{f}_n+\delta_n'T_2\hat{d}_n-\hat{p}\|^2\\
=&\|(1-\eta_n'-\delta_n'-\omega_n')(\hat{d}_n-\hat{p})+\omega_n'(T_1\hat{f}_n-\hat{p})+\eta_n'(T_2\hat{f}_n-\hat{p})+\delta_n'(T_2\hat{d}_n-\hat{p})\|^2\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')^2\|\hat{d}_n-\hat{p}\|^2+2\left\langle\omega_n'(T_1\hat{f}_n-\hat{p})+\eta_n'(T_2\hat{f}_n-\hat{p})+\delta_n'(T_2\hat{d}_n-\hat{p}),j(\hat{e}_n-\hat{p})\right\rangle\\
=&(1-\eta_n'-\delta_n'-\omega_n')^2\|\hat{d}_n-\hat{p}\|^2+2\left\langle\omega_n'(T_1\hat{f}_n-T_1\hat{p})+\eta_n'(T_2\hat{f}_n-T_2\hat{p})+\delta_n'(T_2\hat{d}_n-T_2\hat{p}),j(\hat{e}_n-\hat{p})\right\rangle\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')^2\|\hat{d}_n-\hat{p}\|^2+2\omega_n'\left\langle T_1\hat{f}_n-T_1\hat{p},j(\hat{e}_n-\hat{p})\right\rangle+2\eta_n'\left\langle T_2\hat{f}_n-T_2\hat{p},j(\hat{e}_n-\hat{p})\right\rangle\\
&+2\delta_n'\left\langle T_2\hat{d}_n-T_2\hat{p},j(\hat{e}_n-\hat{p})\right\rangle\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')^2\|\hat{d}_n-\hat{p}\|^2+2\omega_n'\left\langle T_1\hat{f}_n-T_1\hat{e}_n+T_1\hat{e}_n-T_1\hat{p},j(\hat{e}_n-\hat{p})\right\rangle\\
&+2\eta_n'\left\langle T_2\hat{f}_n-T_2\hat{e}_n+T_2\hat{e}_n-T_2\hat{p},j(\hat{e}_n-\hat{p})\right\rangle+2\delta_n'\left\langle T_2\hat{d}_n-T_2\hat{e}_n+T_2\hat{e}_n-T_2\hat{p},j(\hat{e}_n-\hat{p})\right\rangle\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')^2\|\hat{d}_n-\hat{p}\|^2+2\omega_n'\left\langle T_1\hat{f}_n-T_1\hat{e}_n,j(\hat{e}_n-\hat{p})\right\rangle+2\omega_n'\langle T_1\hat{e}_n-T_1\hat{p},j(\hat{e}_n-\hat{p})\rangle\\
&+2\eta_n'\left\langle T_2\hat{f}_n-T_2\hat{e}_n,j(\hat{e}_n-\hat{p})\right\rangle+2\eta_n'\langle T_2\hat{e}_n-T_2\hat{p},j(\hat{e}_n-\hat{p})\rangle+2\delta_n'\left\langle T_2\hat{d}_n-T_2\hat{e}_n,j(\hat{e}_n-\hat{p})\right\rangle\\
&+2\delta_n'\langle T_2\hat{e}_n-T_2\hat{p},j(\hat{e}_n-\hat{p})\rangle\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')^2\|\hat{d}_n-\hat{p}\|^2+2\omega_n'\|T_1\hat{f}_n-T_1\hat{e}_n\|\,\|\hat{e}_n-\hat{p}\|+2\eta_n'\|T_2\hat{f}_n-T_2\hat{e}_n\|\,\|\hat{e}_n-\hat{p}\|\\
&+2\delta_n'\|T_2\hat{d}_n-T_2\hat{e}_n\|\,\|\hat{e}_n-\hat{p}\|+2(\eta_n'+\delta_n')\langle T_2\hat{e}_n-T_2\hat{p},j(\hat{e}_n-\hat{p})\rangle+2\omega_n'\|T_1\hat{e}_n-T_1\hat{p}\|\|\hat{e}_n-\hat{p}\|\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')^2\|\hat{d}_n-\hat{p}\|^2+2\omega_n'\|\hat{f}_n-\hat{e}_n\|\,\|\hat{e}_n-\hat{p}\|+2\eta_n'\|T_2\hat{f}_n-T_2\hat{e}_n\|\,\|\hat{e}_n-\hat{p}\|\\
&+2\delta_n'\|T_2\hat{d}_n-T_2\hat{e}_n\|\,\|\hat{e}_n-\hat{p}\|+2(\eta_n'+\delta_n')\langle T_2\hat{e}_n-T_2\hat{p},j(\hat{e}_n-\hat{p})\rangle+2\omega_n'\|\hat{e}_n-\hat{p}\|\|\hat{e}_n-\hat{p}\|\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')^2\|\hat{d}_n-\hat{p}\|^2+2\omega_n'\|\hat{f}_n-\hat{e}_n\|\,\|\hat{e}_n-\hat{p}\|+2\eta_n'\|T_2\hat{f}_n-T_2\hat{e}_n\|\,\|\hat{e}_n-\hat{p}\|\\
&+2\delta_n'\|T_2\hat{d}_n-T_2\hat{e}_n\|\,\|\hat{e}_n-\hat{p}\|+2(\eta_n'+\delta_n')\langle T_2\hat{e}_n-T_2\hat{p},j(\hat{e}_n-\hat{p})\rangle+2\omega_n'\|\hat{e}_n-\hat{p}\|^2\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')^2\|\hat{d}_n-\hat{p}\|^2+2\omega_n'\mathcal{M}\|\hat{f}_n-\hat{e}_n\|+2\eta_n'\mathcal{M}\|T_2\hat{f}_n-T_2\hat{e}_n\|+2\delta_n'\mathcal{M}\|T_2\hat{d}_n-T_2\hat{e}_n\|\\
&+2(\eta_n'+\delta_n')\langle T_2\hat{e}_n-T_2\hat{p},j(\hat{e}_n-\hat{p})\rangle+2\omega_n'\|\hat{e}_n-\hat{p}\|^2\\
\leq&(1-\eta_n'-\delta_n'-\omega_n')^2\|\hat{d}_n-\hat{p}\|^2+2(\eta_n'+\delta_n'+\omega_n')\max\left\{\|\hat{f}_n-\hat{e}_n\|,\|T_2\hat{f}_n-T_2\hat{e}_n\|,\|T_2\hat{d}_n-T_2\hat{e}_n\|\right\}\\
&+2(\eta_n'+\delta_n')\langle T_2\hat{e}_n-T_2\hat{p},j(\hat{y}_n-\hat{p})\rangle+2\omega_n'\|\hat{e}_n-\hat{p}\|^2.
\end{aligned}$$

From equation (6), we get

$$\begin{aligned}
\|\hat{f}_n-\hat{d}_n\| =&\|\gamma_n'\hat{d}_n+(1-\gamma_n')T_2\hat{d}_n-\hat{d}_n\|\\
=&\|\gamma_n'(\hat{d}_n-\hat{d}_n)+(1-\gamma_n')(T_2\hat{d}_n-\hat{d}_n)\|\\
=&\|\gamma_n'(\hat{d}_n-\hat{d}_n)+(1-\gamma_n')(T_2\hat{d}_n-\hat{d}_n)\|\\
\leq&(1-\gamma_n')\|T_2\hat{d}_n-\hat{d}_n\|\\
\leq&(1-\gamma_n')(\|T_2\hat{d}_n-\hat{p}\|+\|\hat{d}_n-\hat{p}\|)\\
=&(1-\gamma_n')(\|T_2\hat{d}_n-T_2\hat{p}\|+\|\hat{d}_n-\hat{p}\|)\\
\leq&(1-\gamma_n')2\mathcal{M}_2. \qquad (8)
\end{aligned}$$

By condition (ii), we obtain

$$\lim_{n\to\infty}\|\hat{f}_n-\hat{d}_n\|=0. \qquad (9)$$

Again using (6), we have

$$\begin{aligned}\|\hat{e}_n-\hat{d}_n\| =&\|(1-\eta'_n-\delta'_n-\omega'_n)\hat{d}_n+\omega'_nT_1\hat{f}_n+\eta'_nT_2\hat{f}_n+\delta'_nT_2\hat{d}_n-\hat{d}_n\|\\ =&\|(1-\eta'_n-\delta'_n-\omega'_n)(\hat{d}_n-\hat{d}_n)+\omega'_n(T_1\hat{f}_n-\hat{d}_n)+\eta'_n(T_2\hat{f}_n-\hat{d}_n)+\delta'_n(T_2\hat{d}_n-\hat{d}_n)\|\\ \leq&(1-\eta'_n-\delta'_n-\omega'_n)\|\hat{d}_n-\hat{d}_n\|+\omega'_n\|T_1\hat{f}_n-\hat{d}_n\|+\eta'_n\|T_2\hat{f}_n-\hat{d}_n\|+\delta'_n\|T_2\hat{d}_n-\hat{d}_n)\|\\ =&\omega'_n\|T_1\hat{f}_n-\hat{d}_n\|+\eta'_n\|T_2\hat{f}_n-\hat{d}_n\|+\delta'_n\|T_2\hat{d}_n-\hat{d}_n)\|\\ \leq&\omega'_n(\|T_1\hat{f}_n-\hat{p}\|+\|\hat{p}-\hat{e}_n\|)+\eta'_n(\|T_2\hat{f}_n-\hat{p}\|+\|\hat{p}-\hat{e}_n\|)+\delta'_n(\|T_2\hat{d}_n-\hat{p}\|+\|\hat{p}-\hat{e}_n\|)\\ =&\omega'_n(\|\hat{f}_n-T_1\hat{p}\|+\|\hat{p}-\hat{e}_n\|)+\eta'_n(\|T_2\hat{f}_n-T_2\hat{p}\|+\|\hat{p}-\hat{e}_n\|)+\delta'_n(\|T_2\hat{d}_n-T_2\hat{p}\|+\|\hat{p}-\hat{e}_n\|)\\ \leq&2\omega'_n\mathcal{M}_2+2\eta'_n\mathcal{M}_2+2\delta'_n\mathcal{M}_2=2(\omega'_n+\eta'_n+\delta'_n)\mathcal{M}_2.\end{aligned}$$

By condition (ii), we get

$$\lim_{n\to\infty}\|\hat{e}_n-\hat{d}_n\|=0. \tag{10}$$

Since $\|\hat{f}_n-\hat{e}_n\|\leq\|\hat{f}_n-\hat{d}_n\|+\|\hat{d}_n-\hat{e}_n\|$. By equations (9) and (10), we get

$$\lim_{n\to\infty}\|\hat{f}_n-\hat{e}_n\|=0. \tag{11}$$

The uniformly continuity of T_2 leads to thus we have

$$\lim_{n\to\infty}\|T_2\hat{f}_n-T_2\hat{e}_n\|=0 \text{ and } \lim_{n\to\infty}\|T_2\hat{d}_n-T_2\hat{e}_n\|=0. \tag{12}$$

Since T_2 is a (SP) map, it observes that

$$\begin{aligned}&\|\hat{d}_{n+1}-\hat{p}\|^2\\ &\leq(1-\eta'_n-\delta'_n-\omega'_n)^2\|\hat{e}_n-\hat{p}\|^2+2(\omega'_n+\eta'_n+\delta'_n)\max\left\{\|\hat{f}_n-\hat{e}_n\|,\|T_2\hat{f}_n-T_2\hat{e}_n\|,\|T_2\hat{d}_n-T_2\hat{e}_n\|\right\}\\ &+2(\eta'_n+\delta'_n)k\|\hat{e}_n-\hat{p}\|^2+2\omega'_n\|\hat{e}_n-\hat{p}\|^2\\ &=\frac{(1-\eta'_n-\delta'_n-\omega'_n)^2}{1-2(\omega'_n+(\eta'_n+\delta'_n)k)}\|\hat{e}_n-\hat{p}\|^2\\ &+\frac{2(\omega'_n+\eta'_n+\delta'_n)\max\left\{\|\hat{f}_n-\hat{e}_n\|,\|T_2\hat{f}_n-T_2\hat{e}_n\|,\|T_2\hat{d}_n-T_2\hat{e}_n\|\right\}}{1-2(\omega'_n+(\eta'_n+\delta'_n)k)}.\end{aligned} \tag{13}$$

Since $(\omega'_n+\eta'_n+\delta'_n)\to0$ as $n\to\infty$, for all $n\geq n_0$, there is a positive integer $n_0\in\mathbb{N}$ such that

$$(\omega'_n+\eta'_n+\delta'_n)\leq\min\left\{\frac{1}{4k},\frac{1-k}{(1-k)^2+k^2}\right\}, \tag{14}$$

where $k<\frac{1}{2}$. This implies $\frac{1-(\omega'_n+\eta'_n+\delta'_n)}{1-2(\omega'_n+(\eta'_n+\delta'_n)k)k}\leq1$ and $\frac{1}{1-2(\omega'_n+(\eta'_n+\delta'_n)k)k}\leq2$. It now follows from equation (13) that

$$\begin{aligned}&\|\hat{d}_{n+1}-\hat{p}\|^2\\ &\leq(1-\eta'_n-\delta'_n-\omega'_n)\|\hat{e}_n-\hat{p}\|^2+4(\omega'_n+\eta'_n+\delta'_n)\max\left\{\|\hat{f}_n-\hat{e}_n\|,\|T_2\hat{f}_n-T_2\hat{e}_n\|,\|T_2\hat{d}_n-T_2\hat{e}_n\|\right\}.\end{aligned}$$

Now, with the help of equations (12) , we have $\max\left\{\|\hat{f}_n-\hat{e}_n\|,\|T_2\hat{f}_n-T_2\hat{e}_n\|,\|T_2\hat{d}_n-T_2\hat{e}_n\|\right\}\to0$ as $n\to\infty$. Consider

$$\begin{aligned}\|\hat{d}_{n+1}-\hat{p}\|^2=&\|T_1\hat{e}_n-\hat{p}\|^2\\ =&\|T_1\hat{e}_n-T_1\hat{p}\|^2\\ \leq&\|\hat{e}_n-\hat{p}\|^2.\end{aligned}$$

Therefore,

$$\|\hat{d}_{n+1} - \hat{p}\|^2$$
$$\leq (1 - \eta'_n - \delta'_n - \omega'_n)\|\hat{d}_n - \hat{p}\|^2 + 4(\omega'_n + \eta'_n + \delta'_n) \max\left\{\|\hat{f}_n - \hat{e}_n\|, \|T_2\hat{f}_n - T_2\hat{e}_n\|, \|T_2\hat{d}_n - T_2\hat{e}_n\|\right\}.$$

By taking $\alpha_n = \|\hat{d}_n - \hat{p}\|, \beta_n = \omega'_n + \eta'_n + \delta'_n, \beta'_n = \max\left\{\|\hat{f}_n - \hat{e}_n\|, \|T_2\hat{f}_n - T_2\hat{e}_n\|, \|T_2\hat{d}_n - T_2\hat{e}_n\|\right\}$ and using Lemma 1.5, we obtain

$$\lim_{n\to\infty} \|\hat{d}_n - \hat{p}\| = 0.$$

□

If we take $\gamma'_n = 1, \delta'_n = 0, T_1 = T_2$ in Theorem 2.2 then Algorithm 1 reduces to Algorithm (NS) (see [15]) and we get the following fixed point result:

Corollary 2.3. *Let $T_2 : \mathcal{S} \to \mathcal{S}$ be a (SP) mapping and $F = F(T_2) = \left\{\hat{d} \in \mathcal{S} : T_2\hat{d} = \hat{d}\right\} \neq \emptyset$. Then for any arbitrary $\hat{d}_0 \in \mathcal{S}$, the sequence $\left\{\hat{d}_n\right\}$ defined by Algorithm (NS) converges strongly to fixed point $\hat{p} \in F$.*

If we take $\gamma'_n = 1, \delta'_n = 0$ in Theorem 2.2 then Algorithm 1 reduces to Algorithm (HNS) (see [9]) and we get the following fixed point result:

Corollary 2.4. *Let $T_1 : \mathcal{S} \to \mathcal{S}$ be a (NE), $T_2 : \mathcal{S} \to \mathcal{S}$ be a (SP) mappings and $F = F(T_1) \cap F(T_2) = \left\{\hat{d} \in \mathcal{S} : T_1\hat{d} = T_2\hat{d} = \hat{d}\right\} \neq \emptyset$. For any arbitrary $\hat{d}_0 \in \mathcal{S}$, the sequence iteratively defined by Algorithm (HNS) converges strongly to fixed point $\hat{p}$ in F.*

If we take $T_1 = T_2$, in Algorithm 1, we get the following algorithm:

$$\begin{cases} \hat{d}_{n+1} = & T_1\hat{e}_n \\ \hat{e}_n = & (1 - \eta'_n - \delta'_n - \omega'_n)\hat{d}_n + (\omega'_n + \eta'_n)T_1\hat{f}_n + \delta'_n T_1\hat{d}_n \\ \hat{f}_n = & (1 - \gamma'_n)T_1\hat{d}_n + \gamma'_n\hat{d}_n \end{cases} \qquad \text{Algorithm (S*)}$$

where $\gamma'_n, \omega'_n, \eta'_n, \delta'_n \in [0, 1]$, $n \in \mathbb{N} \cup \{0\}$.

In Theorem 2.2 we assume that T_2 is a (NE) mapping, we have the following result, which is very important in the application section.

Corollary 2.5. *Let $T_1 : \mathcal{S} \to \mathcal{S}$ be a (NE) mapping and $F = F(T_1) = \left\{\hat{d} \in \mathcal{S} : T_1\hat{d} = \hat{d}\right\} \neq \emptyset$. Let $\gamma'_n, \omega'_n, \eta'_n, \delta'_n \in [0, 1]$, such that:*

(i) $\omega'_n + \gamma'_n + \eta'_n + \delta'_n \leq 1$;

(ii) $\lim_{n\to\infty}(\omega'_n + \eta'_n + \delta'_n) = 0 = \lim_{n\to\infty}(1 - \gamma'_n)$;

(iii) $\sum_{n=1}^{\infty}(\omega'_n + \eta'_n + \delta'_n) = \infty$.

Then for any arbitrary $\hat{d}_0 \in \mathcal{S}$, the sequence $\{\hat{d}_n\}$ defined by Algorithm (S) converges strongly to a fixed point $\hat{p}$ in F.*

3. Numerical Examples

In this section, we give few numerical examples to back up our assertions and demonstrate that our technique converges faster than well known iterative schemes in the literature.

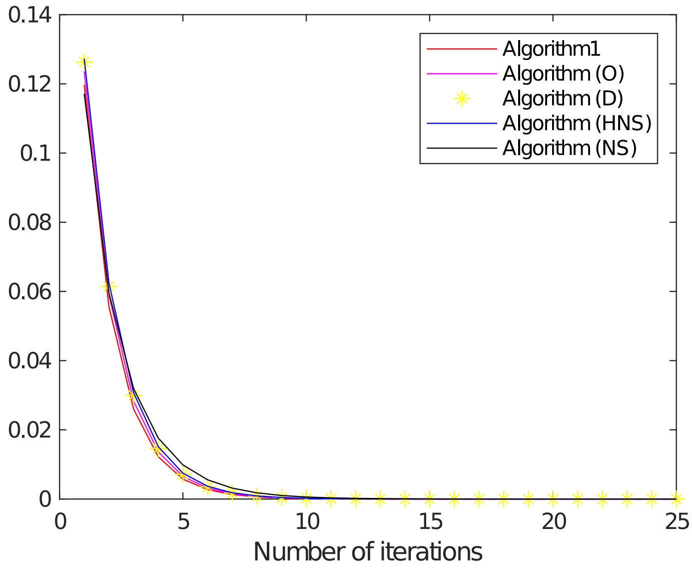

Figure 1:

Example 3.1. *Let $\mathcal{X} = \mathbb{R}$ with usual norm and $\mathcal{S} = [-1, 1]$. Define $T_1, T_2 : \mathcal{S} \to \mathcal{S}$ by*

$$T_1(\hat{d}) = \hat{d} \text{ and } T_2(\hat{d}) = \frac{\hat{d}+1}{4},$$

for all $\hat{d} \in \mathcal{S}$. Clearly T_1 is (NE) mapping. Also for all $\hat{d}, \hat{e} \in \mathcal{S}$, we get

$$\left\langle T_2\hat{d} - T_2\hat{e}, j(\hat{d}-\hat{e}) \right\rangle = \frac{1}{4}(\hat{d}-\hat{e})^2 = k\|\hat{d}-\hat{e}\|^2,$$

where $k = \frac{1}{4} < 1$. This shows that T_2 is a (SP) mapping with bounded range and both T_1, T_2 are uniformly continuous on $\mathcal{S}$. Take $\hat{d}_0 = 0.22$, $\gamma'_n = \frac{n}{n+25}$, $\omega'_n = \frac{1}{n+25}$, $\eta'_n = \frac{1}{n+25}$ and $\delta'_n = \frac{1}{n+25}$. It is easy to see that $\frac{1}{3}$ is common fixed point of T_1 and T_2. All the suppositions of Theorem 2.2, are fulfilled then the sequence defined by Algorithm 1 converges to $\frac{1}{3}$.

Example 3.2. *Let $\mathcal{X} = \mathbb{R}$ with usual norm and $\mathcal{S} = [0, \infty)$. Define $T_1, T_2 : \mathcal{S} \to \mathcal{S}$ by*

$$T_1(\hat{d}) = \frac{\hat{d}}{2} \text{ and } T_2(\hat{d}) = \frac{\hat{d}}{\sqrt{3}(1+\hat{d})},$$

for all $\hat{d} \in \mathcal{S}$. Clearly T_1 is (NE) mapping. Also for all $\hat{d}, \hat{e} \in \mathcal{S}$, we get

$$\left\langle T_2\hat{d} - T_2\hat{e}, j(\hat{d}-\hat{e}) \right\rangle = \frac{1}{\sqrt{3}(1+\hat{d})(1+\hat{e})}(\hat{d}-\hat{e})^2 = k\|\hat{d}-\hat{e}\|^2,$$

where $k = \frac{1}{\sqrt{3}(1+\hat{x})(1+\hat{y})} < 1$. This shows that T_2 is a (SP) mapping with bounded range and both T_1, T_2 are uniformly continuous on $\mathcal{S}$. It is easy to see that T_1 and T_2 has common fixed point say 0. All the suppositions of Theorem 2.2, are fulfilled then the sequence defined by Algorithm 1 converges to 0 see Table 1 and Figure 1. Take $\gamma'_n = \frac{n}{n+25}$, $\omega'_n = \frac{1}{n+25}$, $\eta'_n = \frac{1}{n+25}$ and $\delta'_n = \frac{1}{n+25}$.

$\hat{x}_{n+1}$	$Algorithm(1)$	$Algorithm(O)$	$Algorithm(D)$	$Algorithm(HNS)$	$Algorithm(NS)$
$\hat{d}_0$	0.26	0.26	0.26	0.26	0.26
$\hat{d}_1$	0.119718	0.123612	0.126231	0.0127291	0.117157
$\hat{d}_2$	0.055625	0.059069	0.061500	0.062496	0.059575
$\hat{d}_3$	0.026002	0.028327	0.029991	0.030738	0.031963
$\hat{d}_4$	0.012207	0.013619	0.014646	0.015136	0.017619
$\hat{d}_5$	0.005749	0.006560	0.007160	0.007459	0.009855
$\hat{d}_6$	0.002715	0.003166	0.003504	0.003678	0.005557
$\hat{d}_7$	0.001285	0.001530	0.001717	0.001815	0.003148
$\hat{d}_8$	0.000610	0.000740	0.000842	0.000896	0.001789
$\hat{d}_9$	0.000290	0.000359	0.000413	0.000442	0.001018
$\hat{d}_{10}$	0.000138	0.000174	0.000203	0.000218	0.000580
$\hat{d}_{11}$	0.000066	0.000084	0.000100	0.000108	0.000331
$\hat{d}_{12}$	0.000031	0.000041	0.000049	0.000053	0.000189
$\hat{d}_{13}$	0.000015	0.000020	0.000024	0.000026	0.000108
$\hat{d}_{14}$	0.000007	0.000010	0.000012	0.000013	0.000062
$\hat{d}_{15}$	0.000003	0.000005	0.000006	0.000006	0.000035
$\hat{d}_{16}$	0.000000	0.000002	0.000003	0.000003	0.000020
$\hat{d}_{17}$	0.000000	0.000001	0.000001	0.000002	0.000007
$\hat{d}_{18}$	.	0.000000	0.000000	0.000001	0.000004
$\hat{d}_{23}$	.	0.000000	0.000000	0.000000	0.000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: Camparison of convergence between different algorithms

4. Applications

4.1. Solution of Fractional Iterative Integrodifferential Equations (FIIE)

Fractional calculus differentiation and integration of arbitrary order is play an important role in the modelling of dynamical systems, mechanics, economics, control theory, signal, image processing, electrical sciences, chemical sciences, biological sciences and other allied sciences. In 2018, Kılıçman et al. [10] using the following (FIIE) including derivatives and gave a solution using (NE) mapping.

$$\begin{cases} \mathcal{D}_1^{\beta}\hat{d}(t) = \Phi\left(t, \hat{d}(\hat{d}(t)), \hat{d}(\hat{d}'(t)), \int_{t_0}^{t} \kappa(t,r).\hat{d}(\hat{d}(r))\right) dr \\ \hat{d}(t_0) = \hat{d}_0, \end{cases} \tag{FID}$$

where $t_0, \hat{d}_0$ in $I = [a,b]$, $\kappa : I \times I \to I$ and $\Phi : I \times I \times I \times I \to I$ are given continuous functions. Suppose that $\mathcal{X} = C(I,I)$ is a Banach space of all continuous functions defined on I endowed with the norm $\|.\| = \sup_{t \in I} |u(t)|$, and

$$C_{l,\beta} = \left\{ \hat{d} \in C(I,I) : |\hat{d}(t_1) - \hat{d}(t_2)| \leq l \frac{|t_1 - t_2|^{\beta}}{\Gamma(\beta+1)} \right\},$$

for all $t_1, t_2 \in I$, $l > 0$ and $C_{l,\beta}$ is a non-empty convex and compact subset of a Banach space $(\mathcal{X}, \|.\|)$.

Theorem 4.1. *Assume that the following assumptions hold :*

(i) $|\Phi(t, \hat{d}_1, \hat{e}_1, \hat{f}_1) - \Phi(t, \hat{d}_2, \hat{e}_2, \hat{f}_2)| \leq \mathcal{H}(|\hat{d}_1 - \hat{d}_2| + |\hat{e}_1 - \hat{e}_2| + |\hat{f}_1 - \hat{f}_2|)$; $\mathcal{H} > 0$, *and* $t, \hat{d}_1, \hat{d}_2, \hat{e}_1, \hat{e}_2, \hat{f}_1, \hat{f}_2 \in I$

(ii) l *is a constant such that* $|\hat{d}(t_1) - \hat{d}(t_2)| \leq l \frac{|t_1-t_2|^{\beta}}{\Gamma(\beta+1)}$, *subsequently*

$$\zeta = \max\left\{ |\Phi(t, \hat{d}_1, \hat{e}_1, \hat{f}_1)| : (t, \hat{d}_1, \hat{e}_1, \hat{f}_1) \in I \times I \times I \times I \right\} \leq \frac{l}{2};$$

(iii) One of these situations are fulfilled:

(A_1) $\frac{\zeta T^{\beta}}{\Gamma(\beta+1)} \zeta_{t_0} \leq \zeta_{\hat{d}_0}$, *where* $T = \max\{a,b\}$, *and* $\zeta_{\hat{d}_0} = \max\left\{ \hat{d}_0 - a, b - \hat{d}_0 \right\}$;

(A_2) $t_0 = a$, $\frac{\zeta T^{\beta}}{\Gamma(\beta+1)} \leq b - \hat{d}_0, \Phi(t, \hat{d}_1, \hat{e}_1, \hat{f}_1) \geq 0$ *where* $\hat{d}_1, \hat{e}_1, \hat{f}_1 \in I$;

(A_3) $t_0 = b$, $\frac{\zeta T^{\beta}}{\Gamma(\beta+1)} \leq a - \hat{d}_0, \Phi(t, \hat{d}_1, \hat{e}_1, \hat{f}_1) \geq 0$ *where* $\hat{d}_1, \hat{e}_1, \hat{f}_1 \in I$;

(iv) $\lambda = 2\mathcal{H}\zeta_{t_0}\left(1 + \frac{T^{\beta}}{\Gamma(\beta+1)}\kappa_T\right)(l+1) = 1.$

where $\kappa_T = \sup\{\kappa(r,t) : a \leq r \leq t \leq b\}$.

Further, suppose that $\left\{\hat{d}_n\right\}$ *is a sequence generated by Algorithm (S*) with real sequences* $\eta'_n, \delta'_n, \gamma'_n \in [0,1]$ *such that* $\omega'_n + \gamma'_n + \eta'_n + \delta'_n \leq 1$, $\lim_{n\to\infty}(\omega'_n + \eta'_n + \delta'_n) = 0 = \lim_{n\to\infty}(1 - \gamma'_n)$, $\sum_{n=1}^{\infty}(\omega'_n + \eta'_n + \delta'_n) = \infty$. *Then Algorithm (S*) converges strongly to a solution of (FID).*

Proof. On the lines of Theorem 3.1 of [10], define an integral operator on $C_{l,\beta}$

$$£\hat{d}(u) = \hat{d}_0 + \int_{t_0}^{u} \frac{(u-m')^{\beta}}{\Gamma(\beta+1)} \Phi\left(m', \hat{d}(\hat{d}(m')), \hat{d}(\hat{d}'(m')), \int_{t_0}^{m'} \kappa(m',r)\hat{d}(\hat{d}(r))dr\right) dm'; u \in I, u > m',$$

with $£(C_{l,\beta}) \subset C_{l,\beta}$ is (NE) mapping and the problem (FID) has at least one solution on $C_{l,\beta}$.

Therefore (FID) has a solution in $C_{l,\beta}$ that can be approximated by the Algorithm (S*) as

$$\begin{aligned}
\hat{d}_2 =&\hat{d}_0 + \int_{t_0}^{u} \frac{(u-m')^{\beta}}{\Gamma(\beta+1)}\Phi\left(m', \hat{e}_1(\hat{e}_1(m')), \hat{e}_1(\hat{e'}_1(m')), \int_{t_0}^{m'} \kappa(m',r)\hat{e}_1(\hat{e}_1(r))dr\right)dm' \\
\hat{e}_1 =&(1-\eta'_1-\delta'_1-\omega'_1)\hat{d}_1 \\
&+(\eta'_1+\omega'_1)\left(\hat{d}_0 + \int_{t_0}^{u} \frac{(u-m')^{\beta}}{\Gamma(\beta+1)}\Phi\left(m', \hat{f}_1(\hat{f}_1(m')), \hat{f}_1(\hat{f'}_1(m')), \int_{t_0}^{m'} \kappa(m',r)\hat{f}_1(\hat{f}_1(r))dr\right)dm'\right) \\
&+\delta'_1\left(\hat{d}_0 + \int_{t_0}^{u} \frac{(u-m')^{\beta}}{\Gamma(\beta+1)}\Phi\left(m', \hat{d}_1(\hat{d}_1(m')), \hat{d}_1(\hat{d'}_1(m')), \int_{t_0}^{m'} \kappa(m',r)\hat{d}_1(\hat{d}_1(r))dr\right)dm'\right) \\
\hat{f}_1 =&(1-\gamma'_1)\left(\hat{d}_0 + \int_{t_0}^{u} \frac{(u-m')^{\beta}}{\Gamma(\beta+1)}\Phi\left(m', \hat{d}_1(\hat{d}_1(m')), \hat{d}_1(\hat{d'}_1(m')), \int_{t_0}^{m'} \kappa(m',r)\hat{d}_1(\hat{d}_1(r))dr\right)dm'\right) + \gamma'_1\hat{d}_1 \\
&\vdots
\end{aligned}$$

Therefore, by Corollary 2.5, Algorithm (S*) converges strongly to a solution of (FID). □

Example 4.2. *Consider the following initial value problem linked to fractional iterative contain derivatives and integral equation :*

$$\begin{aligned}
\mathcal{D}_1^{\frac{1}{2}}\hat{d}(t) =&\frac{1}{6.3851}\left(\hat{d}(\hat{d}(t)) + \hat{d}(\hat{d}'(t))\right) + \frac{1}{6.3851}\int_0^t \frac{1}{(2+t)^2}\hat{d}(\hat{d}(r))dr \\
&\hat{d}(0) = \hat{d}'(0) = \frac{1}{4},
\end{aligned} \tag{15}$$

where $t \in [0,1], \hat{d} \in C^{l,\frac{1}{2}}([0,1]\times[0,1])$. *For all* $\hat{d}_1, \hat{d}_2, \hat{d'}_1, \hat{d'}_2 \in C^{l,\frac{1}{2}}([0,1]\times[0,1])$, *we have*

$$\begin{aligned}
&|\Phi(t, \hat{d}_1(\hat{d}_1(t)), \hat{d}_1(\hat{d'}_1(t)), \kappa^*\hat{d}_1(\hat{d}_1(t)) - \Phi(t, \hat{d}_2(\hat{d}_2(t)), \hat{d}_2(\hat{d'}_2(t)), \kappa^*\hat{d}_2(\hat{d}_2(t))| \\
\leq&\frac{1}{6.3851}\left(|\hat{d}_1(\hat{d}_1(t)) - \hat{d}_2(\hat{d}_2(t))| + |\hat{d}_1(\hat{d'}_1(t)) - \hat{d}_2(\hat{d'}_2(t))|\right) + \frac{1}{6.3851}\left(|\kappa^*\hat{d}_1(\hat{d}_1(t)) - \kappa^*\hat{d}_2(\hat{d}_2(t))|\right) \\
\leq&\frac{1}{6.3851}\left(|\hat{d}_1(\hat{d}_1(t)) - \hat{d}_2(\hat{d}_2(t))| + |\hat{d}_1(\hat{d'}_1(t)) - \hat{d}_2(\hat{d'}_2(t))| + |\kappa^*\hat{d}_1(\hat{d}_1(t)) - \kappa^*\hat{d}_2(\hat{d}_2(t))|\right).
\end{aligned}$$

where $\kappa^*\hat{d}(\hat{d}(t)) = \int_0^t \dfrac{1}{(2+t)^2}\hat{d}(\hat{d}(r))dr$.*Thus* $\mathcal{H} = \frac{1}{6.3851}, \zeta_{\hat{d}_0} = \max\left\{\hat{d}_0 - a, b - \hat{d}_0\right\} = \max\left\{\frac{1}{4}, \frac{3}{4}\right\}, \zeta_{\hat{t}_0} = \max\{t_0 - a, b - t_0\} = \max\{0, 1\}$. *Also*

$$\begin{aligned}
C_{l,\frac{1}{2}} =&\left\{\hat{d} : |\hat{d}(t_1) - \hat{d}(t_2)| \leq l\frac{|t_1 - t_2|^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)}\right\}, \text{ with } l = 1 \\
C_{1,\frac{1}{2}} =&\left\{\hat{d} : |\hat{d}(t_1) - \hat{d}(t_2)| \leq \frac{|t_1 - t_2|^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)}\right\},
\end{aligned}$$

for all $t_1, t_2 \in [0,1]$. *Presently* $\zeta \leq \frac{l}{2} = \frac{1}{2}, \frac{\zeta T^{\beta}}{\Gamma(\beta+1)}\zeta_{t_0} = (\frac{1}{2})\frac{1}{0.88622}(1) = 0.5641 < \zeta_{\hat{x}_0} = \frac{3}{4}$ *and*

$$\lambda = 2\mathcal{H}\zeta_{t_0}\left(1 + \frac{T^{\beta}}{\Gamma(\beta+1)}\kappa_T\right)(l+1) = \frac{2\times 3}{4\times 6.3851}\left(1 + \frac{1}{0.88622}\right)2 = 1.$$

All the suppositions of Theorem 4.1 are satisfied. Then the problem (15) *has at least one solution on* $C_{1,\frac{1}{2}}$ *and integral operator defined as on* $C_{1,\frac{1}{2}}$

$$\pounds\hat{d}(t) = \hat{d}_0 + \int_0^t \frac{(t-m')^{\frac{1}{2}}}{\Gamma(\frac{1}{2}+1)}\left(\frac{1}{6.3851}\left(\hat{d}(\hat{d}(t)) + \hat{d}(\hat{d}'(t))\right) + \frac{1}{6.3851}\int_0^{m'} \frac{1}{(2+t)^2}\hat{d}(\hat{d}(r))dr\right)dm'; u \in I, t > m',$$

with $\pounds(C_{1,\frac{1}{2}}) \subset C_{1,\frac{1}{2}}$ *is (NE) mapping. Take* $\gamma'_n = \frac{n}{n+2}$, $\omega'_n = \frac{1}{n+2}$, $\eta'_n = \frac{1}{n+2}$ *and* $\delta'_n = \frac{1}{n+2}$ *by Corollary 2.5, Algorithm (S*) converges strongly to a solution of (FID).*

4.2. *Solution of nonlinear ordinary differential equations*

Consider the following second order differential equation:

$$\begin{cases} \frac{-d^2\hat{d}}{dq^2} = \psi(q, \hat{d}(q)), q \in [0,1] \\ \hat{d}(1) = \hat{d}(0) = 0, \end{cases} \tag{ODE}$$

where $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.
Assume that $\mathcal{B} = C[0,1]$ is the set of all continuous function defined on $[0,1]$ with maximum norm. Green's function associated to (ODE) is defined by

$$\mathcal{G}(s,q) = \begin{cases} q(1-s), 0 \le q \le s \le 1, \\ s(1-q), 0 \le s \le q \le 1. \end{cases}$$

In theorem below, we will discuss the existence of the solution for the problem (ODE).

Theorem 4.3. *Assume that the following assumption hold:*

(i) $|\psi(q,c) - \psi(q,d)| \le \max_{c,d\in[0,1]} |c-d|$,

for all $q \in [0,1]$. *Further suppose that* $\{\hat{d}_n\}$ *be a sequence generated by Algorithm (S*) with real sequences* $\eta'_n, \delta'_n, \gamma'_n \in [0,1]$ $\omega'_n + \gamma'_n + \eta'_n + \delta'_n \le 1$, $\lim_{n\to\infty}(\omega'_n + \eta'_n + \delta'_n) = 0 = \lim_{n\to\infty}(1-\gamma'_n)$, $\sum_{n=1}^{\infty}(\omega'_n + \eta'_n + \delta'_n) = \infty$. *Then Algorithm (S*) converges strongly to a solution of (ODE).*

Proof. On the lines of Theorem 4.1 of [7], define an operator ϑ on the space $\mathcal{B}$ by

$$\vartheta\hat{d}(q) = \int_0^1 \mathcal{G}(q,s)\psi(s,\hat{d}(s))ds,$$

is a (NE) mapping and the problem (ODE) has a solution in $C[0,1]$. Thus, (ODE) has a solution in $C[0,1]$ that can be approximated by the Algorithm (S*) as

$$\begin{aligned} \hat{d}_1 &= \int_0^1 \mathcal{G}(q,s)\psi(s,\hat{e}_0(s))ds \\ \hat{e}_0 &= (1-\eta'_0-\delta'_0-\omega'_0)\hat{d}_0 + (\eta'_0+\omega'_0)\int_0^1 \mathcal{G}(q,s)\psi(s,\hat{f}_0(s))ds + \delta'_0\int_0^1 \mathcal{G}(q,s)\psi(s,\hat{d}_0(s))ds \\ \hat{f}_0 &= (1-\gamma'_0)\int_0^1 \mathcal{G}(q,s)\psi(s,\hat{d}_0(s))ds + \gamma'_0\hat{d}_0 \\ &\vdots \end{aligned}$$

Therefore, by Corollary 2.5, Algorithm (S*) converges strongly to a solution of (ODE). □

Example 4.4. *Consider the differential equation:*

$$\begin{cases} \frac{-d^2\hat{d}}{dq^2} = \cos q, q \in [0,1] \\ \hat{d}(0) = \hat{d}(1) = 0, \end{cases} \tag{16}$$

also

$$|\psi(q_1, \hat{d}(s)) - \psi(q_2, \hat{d}(s))| = |\cos q_1 - \cos q_2| \leq |q_1 - q_2|.$$

Define an operator ϑ on the space $\mathcal{B}$ by

$$\vartheta\hat{d}(q) = (1-q)\int_0^q s\cos(s)ds + q\int_q^1 (1-s)\cos(s)ds.$$

Therefore, (ODE) has a solution in $\mathcal{B}$,

$$\vartheta\hat{d}(q) = \hat{d}(q) = (1-q)\int_0^q s\cos(s)ds + q\int_q^1 (1-s)\cos(s)ds,$$

that is $\hat{d}(q) = \cos(q) - q\cos(1) + q - 1$ is solution of (16). *By Theorem 4.3, Algorithm (S^*) converges strongly to a solution of (ODE). Take $\gamma'_n = \frac{n}{n+2}$, $\omega'_n = \frac{1}{n+2}$, $\eta'_n = \frac{1}{n+2}$ and $\delta'_n = \frac{1}{n+2}$ and $\hat{d}_0(q) = 0.25$. At $q = 0.1$, approximate solution is $\hat{d}(0.1) = \cos(0.1) - 0.1\cos(1) + 0.1 - 1$.*

5. Conclusion

In this paper, we propose a new algorithm which converges strongly to a common fixed point for pair of mappings in the context of Banach spaces. We include a few numerical examples to support our assertions and demonstrate that our technique converges faster than well-known iterative schemes in the literature. Also, under some assumptions, we obtained our iterative process converges to a solution of fractional iterative integrodifferential equations (FID) and nonlinear ordinary differential equations (ODE).

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