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Iterative Hermitian R- conjugate solutions to coupled Sylvester complex matrix equations with conjugate of two unknowns

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Abstract. This paper presents an iterative algorithm to solve coupled Sylvester complex matrix equations with conjugate of two unknowns over Hermitian R - conjugate matrices. Necessary and sufficient conditions are given such that the proposed iterative algorithms converge to the exact solution for arbitrary initial Hermitian R - conjugate solution matrices V_1, W_1. A numerical example verifies the proposed method is given.

1. Introduction

In this paper, we use $A^T, A^H, \overline{A}$ and $tr(A)$ to denote the transpose, conjugate transpose, conjugate, and the trace of a matrix A, respectively. The Frobenius norm of A is denoted by$\|A\|$, that is $\|A\| = \sqrt{Re[\ tr(A^H A)]}$. For $A \in \mathbb{C}^{m\times n}$, $vec(A)$ is defined as $vec\,(A) = \left[a_1^T a_2^T \cdots\cdots a_n^T\right]^T$. For two matrices A and B, $A \otimes B$ is their Kronecker product. A well-known property of the Kronecker product is for matrices A, B and C with appropriate dimension $vec(ABC) = (C^T \otimes A)vec(B)$. Let R be an $n \times n$ symmetric orthogonal matrix, that is, $R^T = R, R^2 = I$, a matrix $A \in \mathbb{C}^{n\times n}$ is termed Hermitian R- conjugate matrix if $RAR = \overline{A}, A^H = A$. Let $\mathbb{HRC}^{n\times n}$ represent the set of all Hermitian R - conjugate matrices, that is, $\mathbb{HRC}^{n\times n} = \left\{A : RAR = \overline{A}, A^H = A\right\}$ where R be an $n \times n$ symmetric orthogonal matrix.

Consider the generalized coupled Sylvester–conjugate matrix equations

$$\begin{cases} A_{11}VB_{11} + C_{11}WD_{11} + A_{12}\overline{V}B_{12} + C_{12}\overline{W}D_{12} = E_1 \\ A_{21}VB_{21} + C_{21}WD_{21} + A_{22}\overline{V}B_{22} + C_{22}\overline{W}D_{22} = E_2 \end{cases} \tag{1}$$

where $A_{11}, A_{12}, C_{11}, C_{12}, A_{21}, A_{22}, C_{21}, C_{22} \in \mathbb{C}^{m\times n}$, $B_{11}, B_{12}, D_{11}, D_{12}, B_{21}, B_{22}, D_{21}, D_{22} \in \mathbb{C}^{n\times r}$ and $E_1, E_2 \in \mathbb{C}^{m\times r}$ are given matrices, while $V, W \in \mathbb{HRC}^{n\times n}$ are matrices to be determined. Chang et al. [10] give the expression of (R, S) - conjugate solution about $AX = C, XB = D$ by matrix decompositions. Trench investigated a system of linear equations $Az = w$ for R - conjugate matrices in Trench [12] and $min\,\|Az - w\|$ for (R, S) - conjugate matrices in Trench [13], respectively, where z, w are known column vectors. Bayoumi and Ramadan [4] provided (R, S) - conjugate solutions for solving coupled Sylvester complex matrix equations with conjugate

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of two unknowns. Dong et al. [9] presented an expression of the Hermitian R - conjugate solution to the system of complex matrix equations $AX = C, XB = D$ when the solvability conditions are satisfied and presented necessary and sufficient conditions for the existence of the Hermitian R- conjugate solution to this system. Bayoumi and Ramadan [3] presented a finite iterative Hermitian R-conjugate solution of the generalized coupled Sylvester-conjugate matrix equations. Li [11] established an iterative technique to solve the coupled Sylvester matrix equations $\sum_{j=1}^{p} A_{ij}X_jB_{ij}=C_i, i= 1, 2,\ldots, p$ over Hermitian R - conjugate matrices. Bayoumi [5] introduced a relaxed gradient iterative technique for solving coupled Sylvester conjugate transpose matrix equations with two unknowns. Bayoumi [6] introduced a method based on a shift-splitting Jacobi-gradient iterative approach for solving the matrix equation $AV - \overline{V}B = C$. Bayoumi [7] offered an accelerated Jacobi-gradient iterative technique for solving the matrix equation $AZ - \overline{Z}B = C$. Bayoumi [8] introduced two iterative methods for solving a complex matrix equation with two unknowns utilizing a real inner product in a complex matrix space.

This paper is organized as follows: In section 2, we give some definitions and lemmas that will be used in this paper. In section 3, we propose an iterative algorithm to obtain the solutions to coupled Sylvester complex matrix equation with conjugate of two unknown over Hermitian R - conjugate matrices, and we give the convergence properties of these iterative algorithms. Section 4 gives an example to demonstrate the proposed algorithms.

2. Preliminaries

Lemma 2.1 [1]

For the matrix equation $AXB = F$ where $A \in \mathbb{C}^{m\times r}$, $B \in \mathbb{C}^{s\times n}$ and $F \in \mathbb{C}^{m\times n}$ are known matrices and $X \in \mathbb{C}^{r\times s}$ is the matrix to be determined, an iterative algorithm is constructed as
$X(k + 1) = X(k) + A^H(F - AX(k)B)B^H$ with $0 < \quad < \frac{2}{\|A\|_2^2\|B\|_2^2}$

If this matrix equation has a unique solution X, then the iterative solution $X(k)$ converges to the unique solution X, that is, $lim_{k\to\infty} X(k) = X$.

Definition 2.1 Inner product [14]

A real inner product space is a vector space V over the real field $\mathbb{R}$ together with an inner product that is with a map

$$\langle .,. \rangle : V \times V \to R$$

satisfying the following three axioms for all vectors $x, y, z \in V$ and all scalars $a \in \mathbb{R}$

1. Symmetry:$\langle x, y\rangle = \langle y, x\rangle$.
2. Linearity in the first argument: $\langle ax, y\rangle = a \langle x, y\rangle$, $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$.
3. Positive definiteness: $\langle x, x\rangle > 0$ for all $x \neq 0$.

Definition 2.2 [2]

For the space $\mathbb{C}^{m\times n}$ over the field $\mathbb{R}$, an inner product can be defined as

$$\langle A, B\rangle = Re\left[\ tr\left(A^H B\ \right)\right].$$

Lemma 2.2 [11]

For any $X \in \mathbb{C}^{n\times n}$, then $X + X^H + R\overline{(X + X^H)}R \in \mathbb{HRC}^{n\times n}$ where R be an $n \times n$ symmetric orthogonal matrix.

Proof

1. $[X + X^H + R\overline{(X + X^H)}R]^H = X + X^H + R\overline{(X + X^H)}R$.
2. $R\left[X + X^H + R\overline{(X + X^H)}R\right]R = R\left(X + X^H\right)R + \overline{(X + X^H)} = \overline{X + X^H + R\overline{(X + X^H)}R}$.

Lemma 2.3

For any $X \in \mathbb{C}^{n\times n}$, $P \in \mathbb{HRC}^{n\times n}$, then $\left\langle P, \frac{X+X^H+R\overline{(X+X^H)}R}{4} \right\rangle = \langle P, X \rangle$.

Proof

$$\begin{aligned}
\left\langle P, \frac{X+X^H+R\overline{(X+X^H)}R}{4} \right\rangle &= \frac{1}{4}\langle P, X\rangle + \frac{1}{4}\left\langle P, X^H\right\rangle + \frac{1}{4}\left\langle P, R\overline{(X+X^H)}R\right\rangle \\
&= \frac{1}{4}\langle P, X\rangle + \frac{1}{4}\left\langle P^H, X^H\right\rangle + \frac{1}{4}\left\langle RPR, \overline{(X+X^H)}\right\rangle \\
&= \frac{1}{4}\langle P, X\rangle + \frac{1}{4}\langle X, P\rangle + \frac{1}{4}\left\langle \overline{P}, \overline{(X+X^H)}\right\rangle \\
&= \frac{1}{4}\langle P, X\rangle + \frac{1}{4}\langle P, X\rangle + \frac{1}{4}\left\langle P, (X+X^H)\right\rangle \\
&= \langle P, X\rangle .
\end{aligned}$$

At the end of this section, we give a very simple fact which will be utilized in the next sections.

Lemma 2.4

For any two square complex matrices A and B, if $tr(A) + tr(B)$ is real, then

$$tr(A) + tr(B) = \overline{tr(A) + tr(B)} = tr(\overline{A}) + tr\left(\overline{B}\right).$$

3. The Main Results

In this section, we propose an iterative solution to the generalized coupled Sylvester – conjugate matrix equations given in (1).

Denote $f_1(V, W) = A_{11}VB_{11} + C_{11}WD_{11} + A_{12}\overline{V}B_{12} + C_{12}\overline{W}D_{12}$,

$$f_2(V, W) = A_{21}VB_{21} + C_{21}WD_{21} + A_{22}\overline{V}B_{22} + C_{22}\overline{W}D_{22}.$$

Lemma 3.1

A necessary and sufficient condition of the consistency of the system of matrix equations (1) over Hermitian R - conjugate matrices is that the following matrix equations

$$\left.\begin{array}{r}
A_{11}VB_{11} + C_{11}WD_{11} + A_{12}\overline{V}B_{12} + C_{12}\overline{W}D_{12} = E_1 \\
A_{21}VB_{21} + C_{21}WD_{21} + A_{22}\overline{V}B_{22} + C_{22}\overline{W}D_{22} = E_2 \\
B_{11}^H VA_{11}^H + D_{11}^H WC_{11}^H + B_{12}^H \overline{V}^H A_{12}^H + D_{12}^H \overline{W}^H C_{12}^H = E_1^H \\
B_{21}^H VA_{21}^H + D_{21}^H WC_{21}^H + B_{22}^H \overline{V}^H A_{22}^H + D_{22}^H \overline{W}^H C_{22}^H = E_2^H \\
\overline{A_{11}}RVR\overline{B_{11}} + \overline{C_{11}}RWR\overline{D_{11}} + \overline{A_{12}}V\overline{B_{12}} + \overline{C_{12}}W\overline{D_{12}} = \overline{E_1} \\
\overline{A_{21}}RVR\overline{B_{21}} + \overline{C_{21}}RWR\overline{D_{21}} + \overline{A_{22}}V\overline{B_{22}} + \overline{C_{22}}W\overline{D_{22}} = \overline{E_2} \\
\overline{B_{11}}^H RVR\overline{A_{11}}^H + \overline{D_{11}}^H RWR\overline{C_{11}}^H + \overline{B_{12}}^H V\overline{A_{12}}^H + \overline{D_{12}}^H W\overline{C_{12}}^H = \overline{E_1}^H \\
\overline{B_{21}}^H RVR\overline{A_{21}}^H + \overline{D_{21}}^H RWR\overline{C_{21}}^H + \overline{B_{22}}^H V\overline{A_{22}}^H + \overline{D_{22}}^H W\overline{C_{22}}^H = \overline{E_2}^H
\end{array}\right\} \quad (2)$$

are consistent.

Proof

If the system of matrix equations (1) have solution $V^*, W^* \in \mathbb{HRC}^{n\times n}$ i.e. $V^{*H} = V^*$, $RV^*R = \overline{V^*}$ and $W^{*H} = W^*, RW^*R = \overline{W^*}$, it is easy to get that $V^*, W^*$are also solutions of Eqs. (2). Conversely, if matrix equations (2) have solutions $V_1, W_1 \in \mathbb{C}^{n\times n}$, let $V^* = \frac{1}{4}\left[V_1 + V_1^H + R\overline{\left(V_1 + V_1^H\right)}R\right]$, $W^* = \frac{1}{4}\left[W_1 + W_1^H + R\overline{\left(W_1 + W_1^H\right)}R\right]$, then $V^*, W^* \in \mathbb{HRC}^{n\times n}$ and

$$\begin{aligned}
A_{11}V^*B_{11} + C_{11}W^*D_{11} + A_{12}\overline{V^*}B_{12} + C_{12}\overline{W^*}D_{12} = \frac{1}{4}\Big[& A_{11}V_1B_{11} + A_{11}V_1^HB_{11} + A_{11}R\overline{V_1}RB_{11} + A_{11}R\overline{V_1^H}RB_{11} \\
&+ C_{11}W_1D_{11} + C_{11}W_1^HD_{11} + C_{11}R\overline{W_1}RD_{11} + C_{11}R\overline{W_1^H}RD_{11} \\
&+ A_{12}\overline{V_1}B_{12} + A_{12}\overline{V_1^H}B_{12} + A_{12}RV_1RB_{12} + A_{12}RV_1^HRB_{12} \\
&+ C_{12}\overline{W_1}D_{12} + C_{12}\overline{W_1^H}D_{12} + C_{12}RW_1RD_{12} + C_{12}RW_1^HRD_{12} \Big] \\
= \frac{1}{4}\Big[& A_{11}V_1B_{11} + C_{11}W_1D_{11} + A_{12}\overline{V_1}B_{12} + C_{12}\overline{W_1}D_{12} \\
&+ A_{11}V_1^HB_{11} + C_{11}W_1^HD_{11} + A_{12}\overline{V_1^H}B_{12} + C_{12}\overline{W_1^H}D_{12} \\
&+ A_{11}R\overline{V_1}RB_{11} + C_{11}R\overline{W_1}RD_{11} + A_{12}RV_1RB_{12} + C_{12}RW_1RD_{12} \\
&+ A_{11}R\overline{V_1^H}RB_{11} + C_{11}R\overline{W_1^H}RD_{11} + A_{12}RV_1^HRB_{12} + C_{12}RW_1^HRD_{12} \Big] \\
= \frac{1}{4}&[E_1 + E_1 + E_1 + E_1] = E_1.
\end{aligned}$$

$$\begin{aligned}
A_{21}VB_{21} + C_{21}WD_{21} + A_{22}\overline{V}B_{22} + C_{22}\overline{W}D_{22} = \frac{1}{4}\Big[& A_{21}V_1B_{21} + A_{21}V_1^HB_{21} + A_{21}R\overline{V_1}RB_{21} + A_{21}R\overline{V_1^H}RB_{21} \\
&+ C_{21}W_1D_{21} + C_{21}W_1^HD_{21} + C_{21}R\overline{W_1}RD_{21} + C_{21}R\overline{W_1^H}RD_{21} \\
&+ A_{22}\overline{V_1}B_{22} + A_{22}\overline{V_1^H}B_{22} + A_{22}RV_1RB_{22} + A_{22}RV_1^HRB_{22} \\
&+ C_{22}\overline{W_1}D_{22} + C_{22}\overline{W_1^H}D_{22} + C_{22}RW_1RD_{22} + C_{22}RW_1^HRD_{22} \Big] \\
= \frac{1}{4}\Big[& A_{21}V_1B_{21} + C_{21}W_1D_{21} + A_{22}\overline{V_1}B_{22} + C_{22}\overline{W_1}D_{22} \\
&+ A_{21}V_1^HB_{21} + C_{21}W_1^HD_{21} + A_{22}\overline{V_1^H}B_{22} + C_{22}\overline{W_1^H}D_{22} \\
&+ A_{21}R\overline{V_1}RB_{21} + C_{21}R\overline{W_1}RD_{21} + A_{22}RV_1RB_{22} + C_{22}RW_1RD_{22} \\
&+ A_{21}R\overline{V_1^H}RB_{21} + C_{21}R\overline{W_1^H}RD_{21} + A_{22}RV_1^HRB_{22} + C_{22}RW_1^HRD_{22} \Big] \\
= \frac{1}{4}&[E_2 + E_2 + E_2 + E_2] = E_2.
\end{aligned}$$

Therefore, V^*, W^* are the solutions of the system of matrix equations (1). So, the solvability of a system of matrix equations (1) is equivalent to that of matrix equations (2). ∎

By rewriting the matrix equations (2) into the equivalent system $Sz = b$, let

$$S = \begin{bmatrix}
B_{11}^H \otimes A_{11} + B_{12}^HR \otimes A_{12}R & D_{11}^H \otimes C_{11} + D_{12}^HR \otimes C_{12}R \\
B_{21}^H \otimes A_{21} + B_{22}^HR \otimes A_{22}R & D_{21}^H \otimes C_{21} + D_{22}^HR \otimes C_{22}R \\
A_{11} \otimes B_{11}^H + A_{12}R \otimes B_{12}^HR & C_{11} \otimes D_{11}^H + C_{12}R \otimes D_{12}^HR \\
A_{21} \otimes B_{21}^H + A_{22}R \otimes B_{22}^HR & C_{21} \otimes D_{21}^H + C_{22}R \otimes D_{22}^HR \\
\overline{B}_{11}^HR \otimes \overline{A_{11}}R + \overline{B}_{12}^H \otimes \overline{A_{12}} & \overline{D}_{11}^HR \otimes \overline{C_{11}}R + \overline{D}_{12}^H \otimes \overline{C_{12}} \\
\overline{B}_{21}^HR \otimes \overline{A_{21}}R + \overline{B}_{22}^H \otimes \overline{A_{22}} & \overline{D}_{21}^HR \otimes \overline{C_{21}}R + \overline{D}_{22}^H \otimes \overline{C_{22}} \\
\overline{A_{11}}R \otimes \overline{B}_{11}^HR + \overline{A_{12}} \otimes \overline{B}_{12}^H & \overline{C_{11}}R \otimes \overline{D}_{11}^HR + \overline{C_{12}} \otimes \overline{D}_{12}^H \\
\overline{A_{21}}R \otimes \overline{B}_{21}^HR + \overline{A_{22}} \otimes \overline{B}_{22}^H & \overline{C_{21}}R \otimes \overline{D}_{21}^HR + \overline{C_{22}} \otimes \overline{D}_{22}^H
\end{bmatrix}, \; b = \begin{bmatrix} vec(E_1) \\ vec(E_2) \\ vec(E_1^H) \\ vec(E_2^H) \\ vec(\overline{E_1}) \\ vec(\overline{E_2}) \\ vec(\overline{E_1}^H) \\ vec(\overline{E_2}^H) \end{bmatrix}, z = \begin{bmatrix} vec(V) \\ vec(W) \end{bmatrix}$$

We have the following well-known theorem.

Theorem 3.1 [11]

The system of matrix equations (2) has a unique Hermitian R - conjugate solutions iff $rank(S, b) = rank(S)$ and S has a full column rank.

Now, we present the iterative algorithm shown below for solving the system of matrix equations (1) over Hermitian R – conjugate matrices

Algorithm I

1. Input matrices $A_{11}, A_{12}, C_{11}, C_{12}, A_{21}, A_{22}, C_{21}, C_{22} \in \mathbb{C}^{m\times n}$, $B_{11}, B_{12}, D_{11}, D_{12}, B_{21}, B_{22}, D_{21}, D_{22} \in \mathbb{C}^{n\times r}$, *and* $E_1, E_2 \in \mathbb{C}^{m\times r}$.
2. Chosen arbitrary initial Hermitian R - conjugate matrices $V_1, W_1 \in \mathbb{HRC}^{n\times n}$ where R be an $n \times n$ symmetric orthogonal matrix.
3. Compute

$$\begin{aligned} V(k+1) = \ & V(k) + \frac{\mu}{4}\Big[A_{11}^H r_1(k) B_{11}^H + \overline{A_{12}}^H \overline{r_1(k)}\,\overline{B_{12}}^H + A_{21}^H r_2(k) B_{21}^H + \overline{A_{22}}^H \overline{r_2(k)}\,\overline{B_{22}}^H \\ & + B_{11} r_1^H(k) A_{11} + \overline{B_{12}}\,\overline{r_1(k)}^H \overline{A_{12}} + B_{21} r_2^H(k) A_{21} + \overline{B_{22}}\,\overline{r_2(k)}^H \overline{A_{22}} \\ & + R\overline{A_{11}^H r_1(k) B_{11}^H} R + R\overline{\overline{A_{12}}^H \overline{r_1(k)}\,\overline{B_{12}}^H R} + R\overline{A_{21}^H r_2(k) B_{21}^H} R + R\overline{\overline{A_{22}}^H \overline{r_2(k)}\,\overline{B_{22}}^H R} \\ & + R\overline{B_{11} r_1^H(k) A_{11}} R + R\overline{\overline{B_{12}}\,\overline{r_1(k)}^H \overline{A_{12}} R} + R\overline{B_{21} r_2^H(k) A_{21}} R + R\overline{\overline{B_{22}}\,\overline{r_2(k)}^H \overline{A_{22}} R} \Big]. \end{aligned}$$

$$\begin{aligned} W(k+1) = \ & W(k) + \frac{\mu}{4}\Big[C_{11}^H r_1(k) D_{11}^H + \overline{C_{12}}^H \overline{r_1(k)}\,\overline{D_{12}}^H + C_{21}^H r_2(k) D_{21}^H + \overline{C_{22}}^H \overline{r_2(k)}\,\overline{D_{22}}^H \\ & + D_{11} r_1^H(k) C_{11} + \overline{D_{12}}\,\overline{r_1(k)}^H \overline{C_{12}} + D_{21} r_2^H(k) C_{21} + \overline{D_{22}}\,\overline{r_2(k)}^H \overline{C_{22}} \\ & + R\overline{C_{11}^H r_1(k) D_{11}^H} R + R\overline{\overline{C_{12}}^H \overline{r_1(k)}\,\overline{D_{12}}^H R} + R\overline{C_{21}^H r_2(k) D_{21}^H} R + R\overline{\overline{C_{22}}^H \overline{r_2(k)}\,\overline{D_{22}}^H R} \\ & + R\overline{D_{11} r_1^H(k) C_{11}} R + R\overline{\overline{D_{12}}\,\overline{r_1(k)}^H \overline{C_{12}} R} + R\overline{D_{21} r_2^H(k) C_{21}} R + R\overline{\overline{D_{22}}\,\overline{r_2(k)}^H \overline{C_{22}} R} \Big]. \end{aligned}$$

where

$$r_1(k) = E_1 - f_1(V(k), W(k)) = E_1 - A_{11}V(k)B_{11} - C_{11}W(k)D_{11} - A_{12}\overline{V(k)}B_{12} - C_{12}\overline{W(k)}D_{12},$$

$$r_2(k) = E_2 - f_2(V(k), W(k)) = E_2 - A_{21}V(k)B_{21} - C_{21}W(k)D_{21} - A_{22}\overline{V(k)}B_{22} - C_{22}\overline{W(k)}D_{22}.$$

4. If $r_1(k+1) = 0, r_2(k+1) = 0$, then stop and V_k, W_k are the solution; else set $k = k+1$ go to STEP 3.

Theorem 3.2

If the system of matrix equations in (1) has a unique Hermitian R - conjugate solutions pair $[V^*, W^*]$, then the iterative solution pair $[V(k), W(k)]$ given by algorithm I, converges to $[V^*, W^*]$ for any initial Hermitian R - conjugate matrices pair $[V(1), W(1)]$ if

$$0 < \mu < \frac{2}{H} \tag{3}$$

with

$$\begin{aligned} H = \ & \|A_{11}\|^2\|B_{11}\|^2 + \|A_{12}\|^2\|B_{12}\|^2 + \|A_{21}\|^2\|B_{21}\|^2 + \|A_{22}\|^2\|B_{22}\|^2 \\ & + \|C_{11}\|^2\|D_{11}\|^2 + \|C_{12}\|^2\|D_{12}\|^2 + \|C_{21}\|^2\|D_{21}\|^2 + \|C_{22}\|^2\|D_{22}\|^2. \end{aligned}$$

Proof

First, we define the estimation error matrices as

$$\xi_1(k) = V(k) - V^* \text{ and } \xi_2(k) = W(k) - W^* \text{ for } k = 1, 2, \cdots .$$

Since $V(k), W(k), V^*, W^* \in \mathbb{HRC}^{n\times n}$, we have

$$R\xi_1(k)R = RV(k)R - RV^*R = \overline{V(k)} - \overline{V^*} = \overline{V(k) - V^*} = \overline{\xi_1(k)}$$

$$R\xi_2(k)R = RW(k)R - RW^*R = \overline{W(k)} - \overline{W^*} = \overline{W(k) - W^*} = \overline{\xi_2(k)}$$

$$\xi_1^H(k) = V^H(k) - V^{*H} = V(k) - V^* = \xi_1(k)$$

$$\xi_2^H(k) = W^H(k) - W^{*H} = W(k) - W^* = \xi_2(k)$$

These demonstrate that $\xi_1(k), \xi_2(k) \in \mathbb{HRC}^{n\times n}$

Denote

$$Z_1(k) = A_{11}\xi_1(k)B_{11} + C_{11}\xi_2(k)D_{11} + A_{12}\overline{\xi_1(k)}B_{12} + C_{12}\overline{\xi_2(k)}D_{12} \tag{4}$$

$$Z_2(k) = A_{21}\xi_1(k)B_{21} + C_{21}\xi_2(k)D_{21} + A_{22}\overline{\xi_1(k)}B_{22} + C_{22}\overline{\xi_2(k)}D_{22} \tag{5}$$

Utilizing the above error matrices and Algorithm I, we can obtain

$$\begin{aligned}
\xi_1(k+1) = \xi_1(k) - \frac{\mu}{4}\Big[& A_{11}^H Z_1(k) B_{11}^H + \overline{A_{12}}^H\,\overline{Z_1(k)}\,\overline{B_{12}}^H + A_{21}^H Z_2(k) B_{21}^H + \overline{A_{22}}^H\,\overline{Z_2(k)}\,\overline{B_{22}}^H \\
& + B_{11} Z_1^H(k) A_{11} + \overline{B_{12}}\,\overline{Z_1(k)}^H\,\overline{A_{12}} + B_{21} Z_2^H(k) A_{21} + \overline{B_{22}}\,\overline{Z_2(k)}^H\,\overline{A_{22}} \\
& + R\overline{A_{11}^H Z_1(k) B_{11}^H}R + R\overline{\overline{A_{12}}^H\,\overline{Z_1(k)}\,\overline{B_{12}}^H}R + R\overline{A_{21}^H Z_2(k) B_{21}^H}R + R\overline{\overline{A_{22}}^H\,\overline{Z_2(k)}\,\overline{B_{22}}^H}R \\
& + R\overline{B_{11} Z_1^H(k) A_{11}}R + R\overline{\overline{B_{12}}\,\overline{Z_1(k)}^H\,\overline{A_{12}}}R + R\overline{B_{21} Z_2^H(k) A_{21}}R + R\overline{\overline{B_{22}}\,\overline{Z_2(k)}^H\,\overline{A_{22}}}R \Big].
\end{aligned} \tag{6}$$

$$\begin{aligned}
\xi_2(k+1) = \xi_2(k) - \frac{\mu}{4}\Big[& C_{11}^H Z_1(k) D_{11}^H + \overline{C_{12}}^H\,\overline{Z_1(k)}\,\overline{D_{12}}^H + C_{21}^H Z_2(k) D_{21}^H + \overline{C_{22}}^H\,\overline{Z_2(k)}\,\overline{D_{22}}^H \\
& + D_{11} Z_1^H(k) C_{11} + \overline{D_{12}}\,\overline{Z_1(k)}^H\,\overline{C_{12}} + D_{21} Z_2^H(k) C_{21} + \overline{D_{22}}\,\overline{Z_2(k)}^H\,\overline{C_{22}} \\
& + R\overline{C_{11}^H Z_1(k) D_{11}^H}R + R\overline{\overline{C_{12}}^H\,\overline{Z_1(k)}\,\overline{D_{12}}^H}R + R\overline{C_{21}^H Z_2(k) D_{21}^H}R + R\overline{\overline{C_{22}}^H\,\overline{Z_2(k)}\,\overline{D_{22}}^H}R \\
& + R\overline{D_{11} Z_1^H(k) C_{11}}R + R\overline{\overline{D_{12}}\,\overline{Z_1(k)}^H\,\overline{C_{12}}}R + R\overline{D_{21} Z_2^H(k) C_{21}}R + R\overline{\overline{D_{22}}\,\overline{Z_2(k)}^H\,\overline{C_{22}}}R \Big].
\end{aligned} \tag{7}$$

Now, by taking the norm of both sides of (6) and (7) and utilizing the following facts for two square complex matrices $tr(AB) = tr(BA)$, $\|A + B\| \leq \|A\| + \|B\|$, $\|RAR\| = \|A\| = \left\|\overline{A}\right\|$, we have

$$
\begin{aligned}
\|\xi_1(k+1)\|^2 &= \mathrm{Re}\left(tr\left(\xi_1^H(k+1)\,\xi_1(k+1)\right)\right) \\
&= \mathrm{Re}\left(tr\left(\xi_1^H(k)\,\xi_1(k)\right)\right) - \frac{\mu}{2}\mathrm{Re}\Big(tr\Big(\Big[A_{11}^H Z_1(k) B_{11}^H + \overline{A}_{12}^H\overline{Z_1(k)}\,\overline{B}_{12}^H + A_{21}^H Z_2(k) B_{21}^H \\
&+ \overline{A}_{22}^H\overline{Z_2(k)}\,\overline{B}_{22}^H + B_{11}Z_1^H(k)A_{11} + \overline{B_{12}}\,\overline{Z_1(k)}^H\overline{A_{12}} + B_{21}Z_2^H(k)A_{21} + \overline{B_{22}}\,\overline{Z_2(k)}^H\overline{A_{22}} \\
&+ R\overline{A_{11}^H Z_1(k) B_{11}^H}R + R\overline{\overline{A}_{12}^H\overline{Z_1(k)}\,\overline{B}_{12}^H}R + R\overline{A_{21}^H Z_2(k) B_{21}^H}R + R\overline{\overline{A}_{22}^H\overline{Z_2(k)}\,\overline{B}_{22}^H}R \\
&+ R\overline{B_{11}Z_1^H(k)A_{11}}R + R\overline{\overline{B_{12}}\,\overline{Z_1(k)}^H\overline{A_{12}}}R + R\overline{B_{21}Z_2^H(k)A_{21}}R + R\overline{\overline{B_{22}}\,\overline{Z_2(k)}^H\overline{A_{22}}}R\Big]^H\xi_1(k)\Big)\Big) \\
&+ \frac{\mu^2}{16}\Big[\Big\|A_{11}^H Z_1(k) B_{11}^H + \overline{A}_{12}^H\overline{Z_1(k)}\,\overline{B}_{12}^H + A_{21}^H Z_2(k) B_{21}^H + \overline{A}_{22}^H\overline{Z_2(k)}\,\overline{B}_{22}^H + B_{11}Z_1^H(k)A_{11} \\
&+ \overline{B_{12}}\,\overline{Z_1(k)}^H\overline{A_{12}} + B_{21}Z_2^H(k)A_{21} + \overline{B_{22}}\,\overline{Z_2(k)}^H\overline{A_{22}} + R\overline{A_{11}^H Z_1(k) B_{11}^H}R \\
&+ R\,\overline{\overline{A}_{12}^H\overline{Z_1(k)}\,\overline{B}_{12}^H}\,R + R\overline{A_{21}^H Z_2(k) B_{21}^H}R + R\overline{\overline{A}_{22}^H\overline{Z_2(k)}\,\overline{B}_{22}^H}R + R\overline{B_{11}Z_1^H(k)A_{11}}R \\
&+ R\overline{\overline{B_{12}}\,\overline{Z_1(k)}^H\overline{A_{12}}}R + R\overline{B_{21}Z_2^H(k)A_{21}}R + R\,\overline{\overline{B_{22}}\,\overline{Z_2(k)}^H\,\overline{A_{22}}}\,R\Big\|^2\Big]
\end{aligned} \tag{8}
$$

Applying properties of the trace of a matrix, one has

$$
\begin{aligned}
&\mathrm{Re}\Big(tr\Big(\Big[A_{11}^H Z_1(k) B_{11}^H + \overline{A}_{12}^H\overline{Z_1(k)}\,\overline{B}_{12}^H + A_{21}^H Z_2(k) B_{21}^H + \overline{A}_{22}^H\overline{Z_2(k)}\,\overline{B}_{22}^H + B_{11}Z_1^H(k)A_{11} + \overline{B_{12}}\,\overline{Z_1(k)}^H\overline{A_{12}} \\
&+ B_{21}Z_2^H(k)A_{21} + \overline{B_{22}}\,\overline{Z_2(k)}^H\overline{A_{22}} + R\overline{A_{11}^H Z_1(k) B_{11}^H}R + R\overline{\overline{A}_{12}^H\overline{Z_1(k)}\,\overline{B}_{12}^H}R + R\overline{A_{21}^H Z_2(k) B_{21}^H}R + R\overline{\overline{A}_{22}^H\overline{Z_2(k)}\,\overline{B}_{22}^H}R \\
&+ R\overline{B_{11}Z_1^H(k)A_{11}}R + R\overline{\overline{B_{12}}\,\overline{Z_1(k)}^H\overline{A_{12}}}R + R\overline{B_{21}Z_2^H(k)A_{21}}R + R\overline{\overline{B_{22}}\,\overline{Z_2(k)}^H\overline{A_{22}}}R\Big]^H\xi_1(k)\Big)\Big) \\
&\quad = \mathrm{Re}\Big(tr\Big(Z_1^H(k)A_{11}\xi_1(k)B_{11} + \overline{Z_1(k)}^H\overline{A}_{12}\xi_1(k)\overline{B}_{12} + Z_2^H(k)A_{21}\xi_1(k)B_{21} + \overline{Z_2(k)}^H\overline{A}_{22}\xi_1(k)\overline{B}_{22} \\
&\quad + Z_1(k)B_{11}^H\xi_1(k)A_{11}^H + \overline{Z_1(k)}\,\overline{B}_{12}^H\xi_1(k)\overline{A}_{12}^H + Z_2(k)B_{21}^H\xi_1(k)A_{21}^H + \overline{Z_2(k)}\,\overline{B}_{22}^H\xi_1(k)\overline{A}_{22}^H \\
&\quad + \overline{Z_1^H(k)}\,\overline{A_{11}}R\xi_1(k)R\overline{B_{11}} + Z_1^H(k)A_{12}R\xi_1(k)RB_{12} + \overline{Z_2^H(k)}\,\overline{A_{21}}R\xi_1(k)R\overline{B_{21}} \\
&\quad + Z_2^H(k)A_{22}R\xi_1(k)RB_{22} + \overline{Z_1(k)}\,\overline{B}_{11}^H R\xi_1(k)R\overline{A}_{11}^H + Z_1(k)B_{12}^H R\xi_1(k)RA_{12}^H \\
&\quad + \overline{Z_2(k)}\,\overline{B}_{21}^H R\xi_1(k)R\overline{A}_{21}^H + Z_2(k)B_{22}^H R\xi_1(k)RA_{22}^H\Big)\Big) \\
&\quad = 4\,\mathrm{Re}\Big(tr\Big(Z_1^H(k)\Big(A_{11}\xi_1(k)B_{11} + A_{12}\overline{\xi_1(k)}B_{12}\Big) + Z_2^H(k)\Big(A_{21}\xi_1(k)B_{21} + A_{22}\overline{\xi_1(k)}B_{22}\Big)\Big)\Big)
\end{aligned}
$$

Substituting from the preceding relation into (8), gives

$$
\begin{aligned}
\|\xi_2(k+1)\|^2 = {} & \|\xi_2(k)\|^2 - 2\mu \mathrm{Re}\left(tr\left(Z_1^H(k)\left(A_{11}\xi_1(k)B_{11} + A_{12}\overline{\xi_1(k)}B_{12}\right) + Z_2^H(k)\left(A_{21}\xi_1(k)B_{21} + A_{22}\overline{\xi_1(k)}B_{22}\right)\right)\right) \\
& + \frac{\mu^2}{16}\Bigg[\Big\| A_{11}^H Z_1(k) B_{11}^H + \overline{A}_{12}^H \overline{Z_1(k)}\overline{B}_{12}^H + A_{21}^H Z_2(k) B_{21}^H + \overline{A}_{22}^H \overline{Z_2(k)}\overline{B}_{22}^H + B_{11} Z_1^H(k) A_{11} + \overline{B_{12}}\,\overline{Z_1(k)}^H \overline{A_{12}} \\
& + B_{21} Z_2^H(k) A_{21} + \overline{B_{22}}\,\overline{Z_2(k)}^H \overline{A_{22}} + R\overline{A_{11}^H Z_1(k) B_{11}^H}R + R\,\overline{\overline{A}_{12}^H \overline{Z_1(k)}\overline{B}_{12}^H}\,R + R\overline{A_{21}^H Z_2(k) B_{21}^H}R \\
& + R\overline{\overline{A}_{22}^H \overline{Z_2(k)}\overline{B}_{22}^H}R + R\overline{B_{11} Z_1^H(k) A_{11}}R + R\overline{\overline{B_{12}}\,\overline{Z_1(k)}^H \overline{A_{12}}}R + R\overline{B_{21} Z_2^H(k) A_{21}}R + R\,\overline{\overline{B_{22}}\,\overline{Z_2(k)}^H\,\overline{A_{22}}}\,R\Big\|^2\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\|\xi_1(k+1)\|^2 \le {} & \|\xi_1(k)\|^2 - 2\mu \mathrm{Re}\left(tr\left(Z_1^H(k)\left(A_{11}\xi_1(k)B_{11} + A_{12}\overline{\xi_1(k)}B_{12}\right) + Z_2^H(k)\left(A_{21}\xi_1(k)B_{21} + A_{22}\overline{\xi_1(k)}B_{22}\right)\right)\right) \\
& + \mu^2\left(\|A_{11}\|^2\|B_{11}\|^2 + \|A_{12}\|^2\|B_{12}\|^2 + \|A_{21}\|^2\|B_{21}\|^2 + \|A_{22}\|^2\|B_{22}\|^2\right)\left(\|Z_1(k)\|^2 + \|Z_2(k)\|^2\right)
\end{aligned}
\tag{9}
$$

Similarly to the above, we can write

$$
\begin{aligned}
\|\xi_2(k+1)\|^2 \le {} & \|\xi_2(k)\|^2 - 2Re\left(tr\left(Z_1^H(k)\left(C_{11}\xi_2(k)D_{11} + C_{12}\overline{\xi_2(k)}D_{12}\right) + Z_2^H(k)\left(C_{21}\xi_2(k)D_{21} + C_{22}\overline{\xi_2(k)}D_{22}\right)\right)\right) \\
& + \mu^2\left(\|C_{11}\|^2\|D_{11}\|^2 + \|C_{12}\|^2\|D_{12}\|^2 + \|C_{21}\|^2\|D_{21}\|^2 + \|C_{22}\|^2\|D_{22}\|^2\right)\left(\|Z_1(k)\|^2 + \|Z_2(k)\|^2\right)
\end{aligned}
\tag{10}
$$

From (9) and (10)

$$
\begin{aligned}
\|\xi_1(k+1)\|^2 + \|\xi_2(k+1)\|^2 \le {} & \|\xi_1(k)\|^2 + \|\xi_2(k)\|^2 - 2Re\Big(tr\Big(Z_1^H(k)\Big(A_{11}\xi_1(k)B_{11} + A_{12}\overline{\xi_1(k)}B_{12} + C_{11}\xi_2(k)D_{11} \\
& + C_{12}\overline{\xi_2(k)}D_{12}\Big) + Z_2^H(k)\Big(A_{21}\xi_1(k)B_{21} + A_{22}\overline{\xi_1(k)}B_{22} + C_{21}\xi_2(k)D_{21} + C_{22}\overline{\xi_2(k)}D_{22}\Big)\Big)\Big) \\
& + \mu^2\Big(\|A_{11}\|^2\|B_{11}\|^2 + \|A_{12}\|^2\|B_{12}\|^2 + \|A_{21}\|^2\|B_{21}\|^2 + \|A_{22}\|^2\|B_{22}\|^2 + \|C_{11}\|^2\|D_{11}\|^2 \\
& + \|C_{12}\|^2\|D_{12}\|^2 + \|C_{21}\|^2\|D_{21}\|^2 + \|C_{22}\|^2\|D_{22}\|^2\Big)\left(\|Z_1(k)\|^2 + \|Z_2(k)\|^2\right)
\end{aligned}
$$

Define the non negative definite function $\eta(k)$ by:

$$\eta(k) = \|\xi_1(k)\|^2 + \|\xi_2(k)\|^2$$

From the previous results, this function can be computed as

$$\eta(k+1) = \|\xi_1(k+1)\|^2 + \|\xi_2(k+1)\|^2$$

$$\eta(k+1) \le \eta(k) - 2\mu Re\left(tr\left(Z_1^H(k)Z_1(k) + Z_2^H(k)Z_2(k)\right)\right) + \mu^2(H)(\|Z_1(k)\|^2 + \|Z_2(k)\|^2) \tag{11}$$

where

$$
\begin{aligned}
H = {} & \|A_{11}\|^2\|B_{11}\|^2 + \|A_{12}\|^2\|B_{12}\|^2 + \|A_{21}\|^2\|B_{21}\|^2 + \|A_{22}\|^2\|B_{22}\|^2 \\
& + \|C_{11}\|^2\|D_{11}\|^2 + \|C_{12}\|^2\|D_{12}\|^2 + \|C_{21}\|^2\|D_{21}\|^2 + \|C_{22}\|^2\|D_{22}\|^2.
\end{aligned}
$$

$$\eta(k+1) \le \eta(k) - 2\mu\left(\|Z_1(k)\|^2 + \|Z_2(k)\|^2\right) + \mu^2(H)\left(\|Z_1(k)\|^2 + \|Z_2(k)\|^2\right)$$

$$\eta(k+1) \le \eta(k) - 2\mu\left(1 - \frac{\mu}{2}H\right)\left(\|Z_1(k)\|^2 + \|Z_2(k)\|^2\right)$$

$$\eta(k+1) \le \eta(k) - 2\mu\left(1 - \frac{\mu}{2}H\right)\left(\sum_{m=1}^{k} \|Z_1(m)\|^2 + \sum_{m=1}^{k} \|Z_2(m)\|^2\right)$$

If the convergence factor $\mu$is chosen to satisfy (3), then one has

$$\sum_{m=1}^{\infty} \|Z_1(m)\|^2 + \sum_{m=1}^{\infty} \|Z_2(m)\|^2 < \infty$$

Since the matrix equation (1) has a unique solution pair. It follows from the definition (4) and (5) of $Z_i(k)$ that

$lim_{i\to\infty}\xi_1(i) = 0$ and $lim_{i\to\infty}\xi_2(i) = 0$

Or

$lim_{i\to\infty}V(i) = V^*$ and $lim_{i\to\infty}W(i) = W^*$

This completes the proof of the theorem.

4. Numerical examples

In this section, we will give two examples to illustrate the effectiveness of our Algorithm I to solve generalized Sylvester matrix equations (1).

Example 4.1

Given $A_{11} = \begin{bmatrix} -i & 2 & 1+i \\ 0 & 2i & -2-i \\ 3-i & 1 & -1+i \end{bmatrix}$, $A_{21} = \begin{bmatrix} i & 0 & -2+i \\ -1 & -3i & 1+i \\ 2i & 1+2i & i \end{bmatrix}$, $A_{12} = \begin{bmatrix} -1-i & 2i & -i \\ 1+2i & -1 & -i \\ -3i & 0 & 2 \end{bmatrix}$

, $B_{11} = \begin{bmatrix} i & 2+i \\ -3 & 1+i \\ 0 & 1 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 0 & -i & -1-i \\ 2-2i & 1 & 2+i \\ 1+i & 1-i & 3i \end{bmatrix}$, $C_{11} = \begin{bmatrix} i & 3 & -2i \\ 1-i & 2+i & -1 \\ -i & 2-i & 1+i \end{bmatrix}$

, $C_{21} = \begin{bmatrix} 0 & i & 1+i \\ -i & 1 & 2i \\ 1-i & -3i & 1+i \end{bmatrix}$, $B_{21} = \begin{bmatrix} 1-i & 2+i \\ 1-3i & -2i \\ 0 & 0 \end{bmatrix}$, $D_{11} = \begin{bmatrix} 1-i & 2+2i \\ -i & 0 \\ 0 & -1+3i \end{bmatrix}$

, $D_{21} = \begin{bmatrix} -1+i & 2+2i \\ -i & 0 \\ -3 & 1+3i \end{bmatrix}$, $B_{12} = \begin{bmatrix} -1+i & -2 \\ -i & 1-i \\ 2+2i & 0 \end{bmatrix}$, $B_{22} = \begin{bmatrix} 0 & -3i \\ 1+i & 0 \\ 0 & 2-i \end{bmatrix}$

, $C_{12} = \begin{bmatrix} 0 & 2-i & -3i \\ 1+i & 2+2i & 0 \\ -1 & 0 & -2+i \end{bmatrix}$, $D_{12} = \begin{bmatrix} 2+i & -3i \\ 0 & 0 \\ 1+i & 3 \end{bmatrix}$, $C_{22} = \begin{bmatrix} 0 & 0 & -i \\ 2+i & 1-i & 5 \\ 2+2i & 0 & -3 \end{bmatrix}$,

$D_{22} = \begin{bmatrix} 0 & -i \\ 1+i & 2-2i \\ -1+3i & -i \end{bmatrix}$, $E_1 = \begin{bmatrix} 20+18i & -187-125i \\ 114+38i & -179+13i \\ -46-86i & -75+63i \end{bmatrix}$, $E_2 = \begin{bmatrix} 48+14i & 11-29i \\ -46+16i & 155+217i \\ -112+58i & -18+136i \end{bmatrix}$.

This system of matrix equations (1) has a unique Hermitian R - conjugate solution of the following form

$$V = \begin{bmatrix} -8 & -3-3i & 3+3i \\ 3+3i & -2 & 4i \\ 3-3i & -4i & -2 \end{bmatrix}, W = \begin{bmatrix} -12 & 3-7i & 3+7i \\ 3+7i & -2 & -2+4i \\ 3-7i & -2-4i & -2 \end{bmatrix}.$$

We apply Algorithm I to solve generalized Sylvester matrix equations (1). When the initial Hermitian R - conjugate solution matrices are chosen as

$$V_1 = \begin{bmatrix} 4 & 2+4i & 2-4i \\ 2-4i & 2 & 6i \\ 2+4i & -6i & 2 \end{bmatrix}, W_1 = \begin{bmatrix} 4 & -1-3i & -1+3i \\ -1+3i & 0 & 4+4i \\ -1-3i & 4-4i & 0 \end{bmatrix}, let\ R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

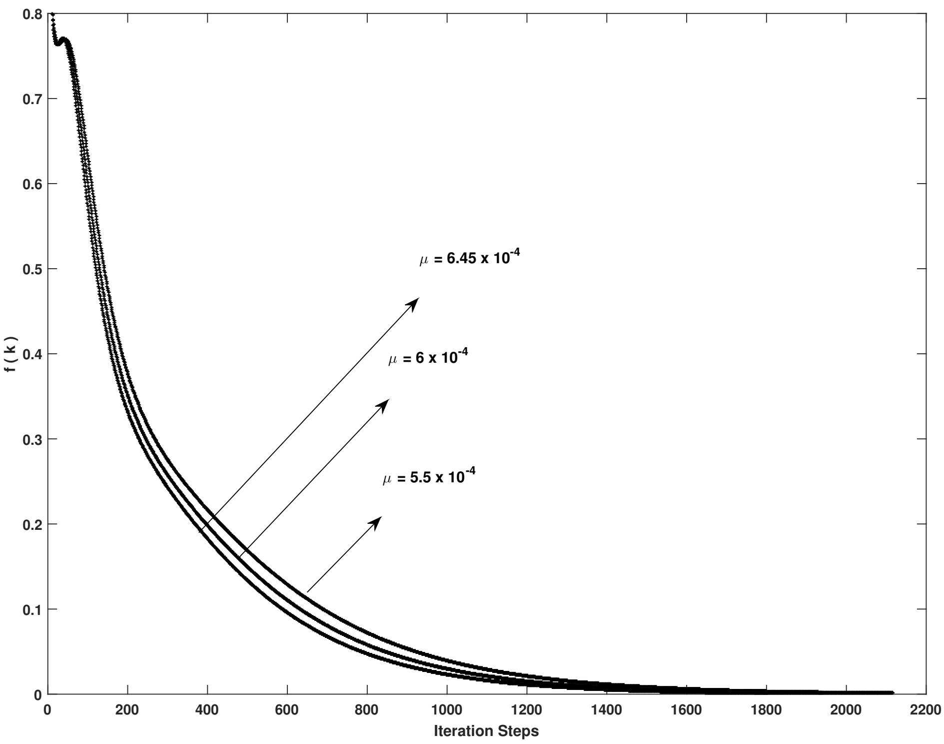

Figure 1: the convergence performance of Algorithm I

According to theorem 3.2, the Algorithm I is convergent for $0 < \mu < 6.45 \times 10^{-4}$. We can see in fig.1 that for $\mu = 6.45 \times 10^{-4}$, $\mu = 6 \times 10^{-4}$ and $\mu = 5.5 \times 10^{-4}$, then the iteration stops at $k = 1805$, $k = 1941$, and $k = 2117$, respectively. Define the relative iterative error as

$$f(k) = \sqrt{\frac{||V(k) - V||^2 + ||W(k) - W||^2}{||V||^2 + ||W||^2}}$$

From fig. 1, it is clear that the error f becomes smaller and goes to zero as k increases. The effect of changing the convergence factor μ is illustrated in fig. 1. As we can see, the larger the convergence factor μ, the faster the rate of convergence.

Example 4.2

This example illustrates the theoretical findings of algorithm I for solving the matrix equation (1) with the following randomly produced matrices

Given A1=randi([−5, 4], 10, 10)+$i{*}randi([-3,5],10,10)$, $A2 = randi([-6,3],10,10)+i{*}randi([-4,5],10,10)$,
$E1 = randi([-3,1],10,10) + i * randi([-6,1],10,10)$, $E2 = randi([-4,2],10,10) + i * randi([-5,4],10,10)$,
$C1 = randi([-1,5],10,10) + i * randi([-2,6],10,10)$, $C2 = randi([-2,6],10,10) + i * randi([-1,7],10,10)$,
$B1 = randi([-7,3],10,10) + i * randi([-4,8],10,10)$, $B2 = randi([-6,7],10,10) + i * randi([-5,4],10,10)$,
$D1 = randi([-3,1],10,10) + i * randi([-8,4],10,10)$, $D2 = randi([-2,5],10,10) + i * randi([-7,0],10,10)$,
$F1 = randi([-8,1],10,10) + i * randi([-3,7],10,10)$, $F2 = randi([-6,5],10,10) + i * randi([-7,3],10,10)$,
$C12 = randi([-4,6],10,10) + i * randi([-6,2],10,10)$, $D12 = randi([-5,4],10,10) + i * randi([-4,5],10,10)$,
$C22 = randi([-3,6],10,10) + i * randi([-3,7],10,10)$, $D22 = randi([-2,4],10,10) + i * randi([-8,5],10,10)$,

$R(k,k) = (-1)^k, \quad k = 1 : n.$

If we choose the Hermitian R - conjugate solutions V, W of the matrix equations (1) as follows v=randi($[-2,5]\,,10,10) + i * randi\,([-4,4]\,,10,10)\,; w = randi\,([-3,2]\,,10,10) + i * randi\,([-5,3]\,,10,10)\,;$
$V = v + v^H + R\overline{(v + v^H)}R, W = w + w^H + R\overline{(w + w^H)}R.$

It follows that from the complex matrix equations (1), we can calculate E_1, E_2. When the initial Hermitian R - conjugate matrices are selected as $V1 = eye\,(10,10)\,,\ W1 = eye\,(10,10)$. We can see in fig.2 that the iteration stops at $k = 5903$.

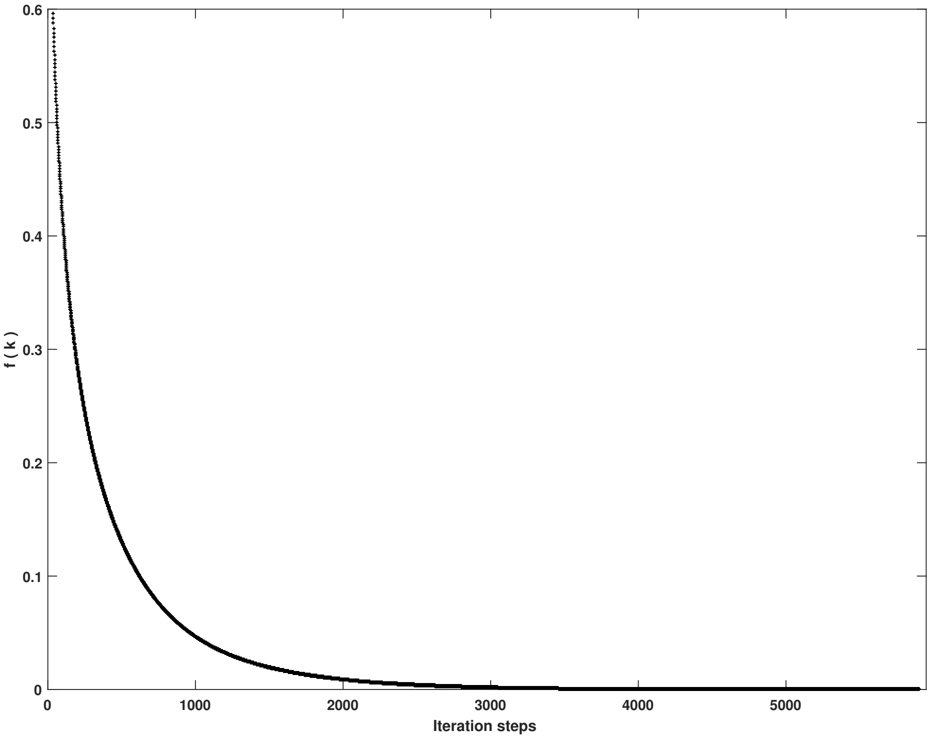

Figure 2: the convergence performance of Algorithm I

5. Conclusions

An iterative algorithm is constructed to give Hermitian R - conjugate solutions to coupled Sylvester complex matrix equations with conjugate of two unknowns. We have established the necessary and sufficient conditions for the existence of Hermitian R - conjugate solution to system (1). our future work to determine the optimal value of the convergence factor μ. We test the proposed algorithm using MATLAB and the results verify our theoretical findings.

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Conflict of interest

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