Filomat 38:24 (2024), 8481–8493
https://doi.org/10.2298/FIL2424481C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some new characterizations of normal and SEP elements

Liufeng Cao[a,b], Mengge Guan[b], Junchao Wei[b]

[a]*Department of Mathematics, Yancheng Institute of Technology*
[b]*School of Mathematical Sciences, Yangzhou University*

Abstract. We give some new characterizations of normal and SEP elements in a ring with an involution by the solutions of equations in a certain set. Furthermore, we first use w-core inverses to describe SEP elements in this paper.

1. Introduction

Let R be an associative ring with 1. An element $a \in R$ is called group invertible if there exists $a^\# \in R$ such that

$$a = aa^\# a,\ \ a^\# = a^\# aa^\#,\ \ aa^\# = a^\# a.$$

We call the element $a^\#$ the group inverse of a [3], which is unique if it exists. The set of all group invertible elements of R is denoted by $R^\#$. In particular, if $a = a^2 b = ca^2$ for some $b, c \in R$, then $a^\# = cab = c^2 a = ab^2$.

A map $* : R \to R, a \mapsto a^*$ is said to be an involution of R if:

$$(a^*)^* = a, (ab)^* = b^* a^*, (a + b)^* = a^* + b^*.$$

An element $a^+ \in R$ is called the Moore-Penrose inverse (or MP-inverse) of a, if

$$aa^+ a = a,\ \ a^+ aa^+ = a^+,\ \ (aa^+)^* = aa^+,\ \ (a^+ a)^* = a^+ a.$$

If a^+ exists, then it is unique [9]. The set of all MP-invertible elements in R is denoted by R^+. Let $a \in R$, a is said to be normal if $aa^* = a^* a$. Mosić and Djordjević proved that $a \in R^+$ is normal if and only if $aa^+ = a^+ a$ and $a^* a^+ = a^+ a^*$ [4, Lemma 1.2]. For other studies of normal elements, one can refer to [6, 10, 11].

We say that an element $a \in R^\# \cap R^+$ is EP if $a^\# = a^+$. The set of all EP elements of R is denoted by R^{EP}. For the works of EP elements, one can see [1, 4–8]. An element $a \in R^{EP}$ is called strongly EP (or SEP) if $a^* = a^+$. The third author of this paper and his cooperators have done some works concerning SEP elements [12, 13].

Let $a, w \in R$, a is called the w-core invertible element of R if there exists $a_w^{\textcircled{\#}} \in R$ such that

---

2020 *Mathematics Subject Classification*. Primary 16U90.
Keywords. normal element, SEP element, w-core inverse
Received: 21 August 2023; Revised: 02 April 2024; Accepted: 03 May 2024
Communicated by Dijana Mosić
L. Cao was supported by National Natural Science Foundation of China (Grant No. 12371041). J. Wei was supported by Jiangsu Province University Brand Specialty Construction Support Project (Mathematics and Applied Mathematics) (Grant No. PPZY2025B109) and Yangzhou University Science and Innovation Fund (Grant No. XCX20240259, XCX20240272).
Email addresses: 1204719495@qq.com (Liufeng Cao), 2530374647@qq.com (Mengge Guan), jcweiyz@126.com (Junchao Wei)

$$a_w^{\textcircled{\#}} = aw(a_w^{\textcircled{\#}})^2, \ a = a_w^{\textcircled{\#}}awa, \ (awa_w^{\textcircled{\#}})^* = awa_w^{\textcircled{\#}},$$

where $a_w^{\textcircled{\#}}$ is called the w-core inverse of a [14, 15]. If $a_w^{\textcircled{\#}}$ exists, then it is unique. We write $R_w^{\textcircled{\#}}$ for the set of all w-core invertible elements of R. In particular, a is called the core invertible element of R if

$$a^{\textcircled{\#}} = a(a^{\textcircled{\#}})^2, \ a = a^{\textcircled{\#}}a^2, \ (aa^{\textcircled{\#}})^* = aa^{\textcircled{\#}},$$

where $a^{\textcircled{\#}}$ is said to be the core inverse of a. We denote the set of all core invertible elements of R by $R^{\textcircled{\#}}$. Dually, we call a dual core invertible if there exists $a_{\textcircled{\#}} \in R$ such that

$$a_{\textcircled{\#}} = (a_{\textcircled{\#}})^2a, \ a = a^2a_{\textcircled{\#}}, \ (a_{\textcircled{\#}}a)^* = a_{\textcircled{\#}}a,$$

where $a_{\textcircled{\#}}$ is called the dual core inverse of a. The set of all dual core invertible elements of R is denoted by $R_{\textcircled{\#}}$.

The studies of generalized inverses in rings with involution are popular recently [1–3, 7, 8, 10–15]. Such as Mosić and Djordjević considered Moore-Penrose invertible elements, EP elements and partial isometries in rings [4, 6–8]. In [1], Cao et al explicitly determined all EP and SEP elements in $\mathbb{Z}[x]/(x^2+x)$. Among these works, the study of characterizations of normal elements, EP elements and SEP elements is an important ingredient in rings. For this reason, the third author Wei and his cooperators have done many works in recent years [2, 10–13]. For example, Zhao and Wei proved that an element $a \in R^\# \cap R^+$ is SEP if and only if the equation $a^*xa = aa^+x$ has at least one solution in the given set $\chi_a := \{a, a^\#, a^+, a^*, (a^\#)^*, (a^+)^*\}$. Then Li and Wei showed that $a \in R^\# \cap R^+$ is SEP if and only if $xx^\# = aa^*$ for some $x \in \chi_a$.

Motivated by the existing results, in this paper we give new characterizations of normal and SEP elements by the solutions of some new equations in the given set $\rho_a := \{a, a^\#, a^+, a^*, (a^\#)^*, (a^+)^*, a^{\textcircled{\#}}, a_{\textcircled{\#}}, (a^\#)^+, (a^+)^\#\}$. Moreover, we first give some sufficient and necessary conditions for an element in an involution ring to be SEP by the solutions of some equations related to w-core inverses in the given set $\phi_a := \{a, a^\#, a^+, a^*, (a^\#)^*, (a^+)^*, a^{\textcircled{\#}}, (a^\#)^+, (a^+)^\#\}$. It is obvious that $\chi_a \subseteq \phi_a \subseteq \rho_a$.

2. Characterizations of normal and SEP elements related to the solutions of equations in a given set

In this section, we will give some new characterizations of normal and SEP elements by the solutions of some equations in a certain given set $\rho_a = \{a, a^\#, a^+, a^*, (a^\#)^*, (a^+)^*, a^{\textcircled{\#}}, a_{\textcircled{\#}}, (a^\#)^+, (a^+)^\#\}$. It is proved that

$$a^{\textcircled{\#}} = a^\#aa^+, \ a_{\textcircled{\#}} = a^+aa^\#, \ (a^\#)^+ = a^+a^3a^+,$$
$$(a^+)^\# = (aa^\#)^*a(aa^\#)^*, \ (a^{\textcircled{\#}})^+ = (a^{\textcircled{\#}})^\# = a^2a^+, \ (a_{\textcircled{\#}})^+ = (a_{\textcircled{\#}})^\# = a^+a^2,$$
$$((a^+)^\#)^+ = aa^+a^+a^+a, \ ((a^\#)^+)^\# = (aa^\#)^*a^\#(aa^\#)^*.$$

Moreover in case $a \in R^{EP}$, we have

$$a^+ = a^\#, \ (a^+)^* = (a^\#)^*, \ a^{\textcircled{\#}} = a^\#aa^+ = a^\#,$$
$$a_{\textcircled{\#}} = a^+aa^\# = a^\#, \ (a^{\textcircled{\#}})^+ = (a^{\textcircled{\#}})^\# = a, \ (a_{\textcircled{\#}})^+ = (a_{\textcircled{\#}})^\# = a,$$
$$(a^\#)^+ = a^+a^3a^+ = a, \ (a^+)^\# = (aa^\#)^*a(aa^\#)^* = a, \ ((a^\#)^+)^* = a^*,$$
$$(a^+)^\#)^* = a^*, \ ((a^\#)^+)^\# = a^\#, \ ((a^+)^\#)^+ = a^\#.$$

In this case $\rho_a = \{a, a^\#, a^*, (a^\#)^*\}$.

Theorem 2.1. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Nor}$ if and only if for some $x \in \rho_a$,*

$$a^*x(a^+)^*a^\#x^+ = a^\#. \quad (1)$$

Proof. $\Rightarrow$ Since $a \in R^{Nor}$, $a^*a = aa^*$ and $a \in R^{EP}$, i.e., $a^+ = a^\#$. Choosing $x = a$, then

$$a^*a(a^+)^*a^\#a^+ = (aa^*(a^+)^*)a^\#a^+ = aa^\#a^+ = aa^+a^\# = a^\#.$$

$\Leftarrow$ Multiplying the equation (1) on the left by a^+a, then

$$a^+aa^*x(a^+)^*a^\#x^+ = a^+aa^\#.$$

Since $a^+aa^* = a^*$, the above equality induces $a^\# = a^+aa^\#$, and hence $a \in R^{EP}$ by [3, Theorem 1.2.1]. By above, in this condition, $\rho_a = \{a, a^\#, a^*, (a^\#)^*\}$.

(1) If $x = a$, then

$$a^*a(a^+)^*a^\#a^+ = a^\#.$$

Multiplying the equality $a^*a(a^+)^*a^\#a^+ = a^\#$ on the right by a^2, then one gets

$$a^*a(a^+)^*a^\#a^+a^2 = a^\#a^2,$$

i.e.,

$$a = a^\#a^2 = a^*a(a^+)^*a^\#(a^+a^2) = a^*a((a^+)^*a^\#a) = a^*a(a^+)^*.$$

This induces

$$aa^* = a^*a((a^+)^*a^*) = a^*(a^2a^+) = a^*a.$$

Hence, $a \in R^{Nor}$.

(2) If $x = a^\#$, then

$$a^*a^\#(a^+)^*a^\#(a^\#)^+ = a^\#.$$

By $a \in R^{EP}$, then $a^+ = a^\#$ and $(a^\#)^+ = (a^+)^+ = a$, it follows that

$$a^+ = a^\# = a^*a^\#(a^+)^*a^\#(a^\#)^+ = a^*a^\#(a^+)^*a^\#a = a^*a^\#(a^+)^* = a^*a^+(a^+)^*.$$

Multiplying the equality $a^*a^+(a^+)^* = a^+$ on the right by a^* and remind $a^+(a^+)^*a^* = a^+$, one gets

$$a^*a^+ = a^*(a^+(a^+)^*a^*) = a^+a^*,$$

which implies that $a \in R^{Nor}$.

(3) If $x = a^*$, then

$$a^*a^*(a^+)^*a^\#(a^*)^+ = a^\#.$$

Since $a \in R^{EP}$, $a^+ = a^\# = (a^*a^*(a^+)^*)a^\#(a^*)^+ = a^*a^+(a^+)^*$. As the same as the case of $x = a^+$, one can prove that $a \in R^{Nor}$.

(4) If $x = (a^\#)^*$, then

$$a^*(a^\#)^*(a^+)^*a^\#((a^\#)^*)^+ = a^\#.$$

According to $a \in R^{EP}$, one has $a^+ = a^\# = (a^*(a^\#)^*(a^+)^*)a^\#((a^\#)^*)^+ = (a^+)^*a^+a^*$. Multiplying the equality $(a^+)^*a^+a^* = a^+$ on the left by a^*, then

$$a^+a^* = (a^*(a^+)^*a^+)a^* = a^*a^+.$$

Hence, $a \in R^{Nor}$. $\square$

Noting that when $a \in R^{EP}$, $\rho_a = \{a, a^\#, a^*, (a^\#)^*\}$ and $x^\# = x^+$ for any $x \in \rho_a$. Hence, by Theorem 2.1, we can obtain the following conclusion.

Corollary 2.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Nor}$ if and only if for some $x \in \rho_a$,*

$$a^*x(a^+)^*a^+x^\# = a^\#.$$

Theorem 2.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{Nor}$ if and only if for some $x \in \rho_a$,*

$$a^\#xaa^*x^\# = a^*. \tag{2}$$

Proof. $\Rightarrow$ By $aa^* = a^*a$ and $a^+ = a^\#$, one has $a^*aa^\# = a^*aa^+ = a^*$, and hence

$$(a^\#aa)a^*a^\# = aa^*a^\# = a^*aa^\# = a^*.$$

$\Leftarrow$ Multiplying the equation (2) on the left by aa^+, then

$$aa^+a^\#xaa^*x^\# = aa^+a^*.$$

Noting that $aa^+a^\# = a^\#$, one gets $a^* = (a^2a^+)^*$, i.e., $a = a^2a^+$, and hence $a \in R^{EP}$ by [3, Theorem 1.2.1].
(1) If $x = a$, then

$$a^\#a^2a^*a^\# = a^*.$$

Thus by $a \in R^{EP}$, $a^* = (a^\#a^2)a^*a^\# = aa^*a^+$. Hence, $a \in R^{Nor}$ by [3, Theorem 1.3.2].
(2) If $x = a^\#$, then

$$(a^\#a^\#a)a^*(a^\#)^\# = a^\#a^*a = a^*.$$

So $a \in R^{Nor}$ by [3, Theorem 1.3.2].
(3) If $x = a^*$, then

$$a^\#a^*aa^*(a^*)^\# = a^*.$$

Since $a \in R^{EP}$, $a^* = a^\#a^*aa^*(a^*)^\# = a^\#a^*(aa^*(a^+)^*) = a^\#a^*a$. As the same the case of $x = a^\#$, one gets $a \in R^{Nor}$.
(4) If $x = (a^\#)^*$, then

$$a^\#(a^\#)^*aa^*((a^\#)^*)^\# = a^*.$$

This induces $a^* = a^\#(a^\#)^*aa^*((a^\#)^*)^\# = a^+(a^+)^*aa^*a^*$. Multiplying the equality $a^+(a^+)^*aa^*a^* = a^*$ on the left by a^*a and on the right by $(a^+)^*$ respectively, and according to $a^*a^*(a^\#)^* = a^*$ and $a^*(a^+)^*a = (a^+)^*a^*a = a$, one gets

$$a^*a = a^*(aa^*(a^+)^*) = a^*(aa^+(a^+)^*)a(a^*a^*(a^+)^*) = (a^*(a^+)^*a)a^* = aa^*.$$

It follows that $a \in R^{Nor}$. □

Proposition 2.4. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if for some $x \in \rho_a$,*

$$a^\#xaa^+x^\# = a^*. \tag{3}$$

Proof. $\Rightarrow$ Choosing $x = a$, then by $a \in R^{SEP}$, one has $a^\# = a^*$, and hence

$$a^* = a^\#a(aa^+a^\#) = aa^\#a^\# = a^\#.$$

$\Leftarrow$ Similar to Theorem 2.3, multiplying the equation (3) on the left by aa^+, one gets

$$aa^+a^\#xaa^+x^\# = aa^+a^*.$$

Since $aa^+a^\# = a^\#$, one has $aa^+a^* = (aa^+a^\#)xaa^+x^\# = a^*$, which implies that $a \in R^{EP}$ by [3, Theorem 1.2.1].
(1) If $x = a$, then

$$a^* = (a^\#aa)a^+a^\# = aa^+a^\# = a^\#,$$

which shows that $a \in R^{SEP}$.

(2) If $x = a^{\#}$, then

$$a^* = a^{\#}a^{\#}(aa^+(a^{\#})^{\#}) = a^{\#}a^+a = a^{\#}.$$

Hence, $a \in R^{SEP}$.

(3) If $x = a^*$, then

$$a^* = a^{\#}a^*(aa^+(a^*)^{\#}) = a^{\#}a^*(a^+)^* = a^{\#}.$$

It follows that $a \in R^{SEP}$.

(4) If $x = (a^{\#})^*$, then

$$a^* = a^{\#}((a^{\#})^*aa^+)((a^{\#})^*)^{\#} = a^{\#}(a^{\#})^*a^* = a^+(a^+)^*a^* = a^+.$$

So $a \in R^{SEP}$. □

3. Characterizations of SEP elements via w-core inverses

The aim of this section is to give some new characterizations of SEP elements via w-core inverses in the certain set $\phi_a = \{a, a^{\#}, a^+, a^*, (a^{\#})^*, (a^+)^*, a^{\textcircled{\#}}, (a^{\#})^+, (a^+)^{\#}\}$. It is the first time to use w-core inverses to describe SEP elements.

Theorem 3.1. *Let $a \in R^{\#} \cap R^+$. Then $({a^*}_{x^{\#}(a^+)^*})^{\textcircled{\#}} = a^*x(a^+)^*$ for $x \in \phi_a$.*

Proof. (1) If $x = a$, then

$$\begin{aligned} a^*a^{\#}((a^+)^*a^*a)((a^+)^*a^*a)(a^+)^* &= a^*(a^{\#}a^2)(a^+)^* \\ &= a^*a(a^+)^*, \end{aligned}$$

$$\begin{aligned} a^*a((a^+)^*a^*a^{\#})(a^+)^*a^* &= a^*(aa^{\#}(a^+)^*)a^* \\ &= a^*(a^+)^*a^* \\ &= a^*, \end{aligned}$$

and

$$\begin{aligned} a^*a^{\#}((a^+)^*a^*a)(a^+)^* &= a^*(a^{\#}a(a^+)^*) \\ &= a^*(a^+)^* \\ &= (a^+a)^* \\ &= a^+a \\ &= (a^*a^{\#}(a^+)^*a^*a(a^+)^*)^*. \end{aligned}$$

Hence, $({a^*}_{a^{\#}(a^+)^*})^{\textcircled{\#}} = a^*a(a^+)^*$.

(2) If $x = a^{\#}$, then

$$\begin{aligned} a^*(a^{\#})^{\#}((a^+)^*a^*a^{\#})((a^+)^*a^*a^{\#})(a^+)^* &= a^*(aa^{\#}a^{\#})(a^+)^* \\ &= a^*a^{\#}(a^+)^*, \end{aligned}$$

$$\begin{aligned} a^*a^\#((a^+)^*a^*(a^\#)^\#)(a^+)^*a^* &= a^*(a^\#a(a^+)^*)a^* \\ &= a^*(a^+)^*a^* \\ &= a^*, \end{aligned}$$

and

$$\begin{aligned} a^*(a^\#)^\#((a^+)^*a^*a^\#)(a^+)^* &= a^*(aa^\#(a^+)^*) \\ &= a^*(a^+)^* \\ &= (a^+a)^* \\ &= a^+a \\ &= (a^*(a^\#)^\#(a^+)^*a^*a^\#(a^+)^*)^*, \end{aligned}$$

which implies that $({a^*}_{a(a^+)^*})^{\textcircled{\#}} = a^*a^\#(a^+)^*$.

(3) If $x = a^+$, then

$$\begin{aligned} a^*(a^+)^\#(a^+)^*a^*(a^+(a^+)^*a^*)a^+(a^+)^* &= (a^*(aa^\#)^*)a(aa^\#)^*(a^+)^*a^*a^+a^+(a^+)^* \\ &= a^*a((aa^\#)^*(a^+)^*a^*)a^+a^+(a^+)^* \\ &= a^*a(aa^+aa^\#)^*a^+a^+(a^+)^* \\ &= a^*a((aa^\#)^*a^+)a^+(a^+)^* \\ &= (a^*aa^+)a^+(a^+)^* \\ &= a^*a^+(a^+)^*, \end{aligned}$$

$$\begin{aligned} a^*(a^+(a^+)^*a^*)(a^+)^\#(a^+)^*a^* &= a^*(a^+(aa^\#)^*)a((aa^\#)^*(a^+)^*a^*) \\ &= a^*a^+a(aa^+aa^\#)^* \\ &= a^*(a^+a)^*(aa^\#)^* \\ &= (a^\#aa^+aa)^* \\ &= (a^\#a^2)^* \\ &= a^*, \end{aligned}$$

and

$$\begin{aligned} a^*(a^+)^\#(a^+)^*a^*a^+(a^+)^* &= (a^*(aa^\#)^*)a((aa^\#)^*(a^+)^*a^*)a^+(a^+)^* \\ &= a^*a((aa^\#)^*a^+)(a^+)^* \\ &= a^*(aa^+(a^+)^*) \\ &= a^*(a^+)^* \\ &= (a^+a)^* \\ &= a^+a \\ &= (a^*(a^+)^\#(a^+)^*a^*a^+(a^+)^*)^*. \end{aligned}$$

It follows that $({a^*}_{(a^+)^\#(a^+)^*})^{\textcircled{\#}} = a^*a^+(a^+)^*$.

(4) If $x = a^*$, then

$$\begin{aligned} a^*(a^*)^\#(a^+)^*a^*a^*(a^+)^*a^*a^*(a^+)^* &= (a^+a(aa^+a)(aa^+a^\#)a)^* \\ &= (a^+(aaa^\#)a)^* \\ &= (a^+aa)^* \\ &= a^*a^*(a^+)^*, \end{aligned}$$

$$\begin{aligned} a^*a^*(a^+)^*a^*(a^*)^\#(a^+)^*a^* &= ((aa^+a^\#)(aa^+a)a)^* \\ &= (a^\#aa)^* \\ &= a^*, \end{aligned}$$

and

$$\begin{aligned} a^*(a^*)^\#(a^+)^*a^*a^*(a^+)^* &= (a^+a(aa^+a^\#)a)^* \\ &= (a^+aa^\#a)^* \\ &= (a^+a)^* \\ &= a^+a \\ &= (a^*(a^*)^\#(a^+)^*a^*a^*(a^+)^*)^*. \end{aligned}$$

Hence, $(a^*_{(a^\#)^*(a^+)^*})^{\textcircled{\#}} = a^*a^*(a^+)^*$.

(5) If $x = (a^\#)^*$, then

$$\begin{aligned} a^*((a^\#)^*)^\#(a^+)^*a^*(a^\#)^*(a^+)^*a^*(a^\#)^*(a^+)^* &= (a^+a^\#(aa^+a^\#)(aa^+a)a)^* \\ &= (a^+(a^\#a^\#a)a)^* \\ &= (a^+a^\#a)^* \\ &= a^*(a^\#)^*(a^+)^*, \end{aligned}$$

$$\begin{aligned} a^*(a^\#)^*(a^+)^*a^*((a^\#)^*)^\#(a^+)^*a^* &= ((aa^+a)(aa^+a^\#)a)^* \\ &= (aa^\#a)^* \\ &= a^*, \end{aligned}$$

and

$$\begin{aligned} a^*((a^\#)^*)^\#(a^+)^*a^*(a^\#)^*(a^+)^* &= (a^+a^\#(aa^+a)a)^* \\ &= (a^+(a^\#a^2))^* \\ &= (a^+a)^* \\ &= a^+a \\ &= (a^*((a^\#)^*)^\#(a^+)^*a^*(a^\#)^*(a^+)^*)^*. \end{aligned}$$

It follows that $(a^*_{a^*(a^+)^*})^{\textcircled{\#}} = a^*(a^\#)^*(a^+)^*$.

(6) If $x = (a^+)^*$, then

$$\begin{aligned}
a^*((a^+)^*)^\#((a^+)^*a^*(a^+)^*)((a^+)^*a^*(a^+)^*)(a^+)^* &= a^*aa^\#a^*(aa^\#(a^+)^*)(a^+)^*(a^+)^* \\
&= a^*a(a^\#a^*(a^+)^*)(a^+)^*(a^+)^* \\
&= a^*(aa^\#(a^+)^*)(a^+)^* \\
&= a^*(a^+)^*(a^+)^*,
\end{aligned}$$

$$\begin{aligned}
a^*(a^+)^*(a^+)^*a^*((a^+)^*)^\#(a^+)^*a^* &= a^*(a^+)^*((a^+)^*a^*a)a^\#a^*(aa^\#(a^+)^*)a^* \\
&= a^*((a^+)^*aa^\#)(a^*(a^+)^*a^*) \\
&= a^*(a^+)^*a^* \\
&= a^*,
\end{aligned}$$

and

$$\begin{aligned}
a^*((a^+)^*)^\#((a^+)^*a^*(a^+)^*)(a^+)^* &= a^*aa^\#a^*(aa^\#(a^+)^*)(a^+)^* \\
&= a^*a(a^\#a^*(a^+)^*)(a^+)^* \\
&= a^*(aa^\#(a^+)^*) \\
&= a^*(a^+)^* \\
&= (a^+a)^* \\
&= a^+a \\
&= (a^*((a^+)^*)^\#(a^+)^*a^*(a^+)^*(a^+)^*)^*,
\end{aligned}$$

which implies that $(a^*_{((a^+)^*)^\#(a^+)^*})^{\textcircled{\#}} = a^*(a^+)^*(a^+)^*$.

(7) If $x = a^{\textcircled{\#}}$, then note that $a^{\textcircled{\#}} = a^\#aa^+$, we have $a^*a^{\textcircled{\#}}(a^+)^* = a^*a^\#(aa^+(a^+)^*) = a^*a^\#(a^+)^*$. Then

$$\begin{aligned}
a^*(a^{\textcircled{\#}})^\#((a^+)^*a^*a^\#)((a^+)^*a^*a^\#)(a^+)^* &= a^*a(aa^+a^\#)a^\#(a^+)^* \\
&= a^*(aa^\#a^\#)(a^+)^* \\
&= a^*a^\#(a^+)^*,
\end{aligned}$$

$$\begin{aligned}
a^*a^\#(a^+)^*a^*(a^{\textcircled{\#}})^\#(a^+)^*a^* &= a^*a^\#((a^+)^*a^*a)(aa^+(a^+)^*)a^* \\
&= a^*(a^\#a(a^+)^*)a^* \\
&= a^*(a^+)^*a^* \\
&= a^*,
\end{aligned}$$

and

$$\begin{aligned}
a^*(a^{\textcircled{\#}})^\#((a^+)^*a^*a^\#)(a^+)^* &= a^*a(aa^+a^\#)(a^+)^* \\
&= a^*(aa^\#(a^+)^*) \\
&= a^*(a^+)^* \\
&= (a^+a)^* \\
&= a^+a \\
&= (a^*(a^{\textcircled{\#}})^\#(a^+)^*a^*a^\#(a^+)^*)^*,
\end{aligned}$$

which shows that $(a^*_{(a^{\textcircled{\#}})^\#(a^+)^*})^{\textcircled{\#}} = a^*a^{\textcircled{\#}}(a^+)^*$.

(8) If $x = (a^\#)^+ = a^+a^3a^+$, then $a^*(a^\#)^+(a^+)^* = a^*a^+a^2(aa^+(a^+)^*) = a^*a^+a^2(a^+)^*$. Thus, we have

$$\begin{aligned}
a^*((a^\#)^+)^\#(a^+)^*a^*a^+a^2(a^+)^*a^*a^+a^2(a^+)^* &= (a^*(aa^\#)^*)a^\#((aa^\#)^*(a^+)^*a^*)a^+a^2((a^+)^*a^*)a^+a^2(a^+)^* \\
&= a^*a^\#(aa^+aa^\#)^*a^+a^3a^+a^+a^2(a^+)^* \\
&= a^*a^\#((aa^\#)^*a^+)a^3a^+a^+a^2(a^+)^* \\
&= a^*(a^\#a^+a)a^2a^+a^+a^2(a^+)^* \\
&= a^*(a^\#a^2)a^+a^+a^2(a^+)^* \\
&= (a^*aa^+)a^+a^2(a^+)^* \\
&= a^*a^+a^2(a^+)^*,
\end{aligned}$$

$$\begin{aligned}
a^*a^+a^2(a^+)^*a^*((a^\#)^+)^\#(a^+)^*a^* &= a^*a^+a^2(a^+)^*(a^*(aa^\#)^*)a^\#((aa^\#)^*(a^+)^*a^*) \\
&= a^*a^+a^2((a^+)^*a^*a^\#)(aa^+aa^\#)^* \\
&= a^*a^+a(aa^\#)^* \\
&= a^*(a^+a)^*(aa^\#)^* \\
&= a^*(a(a^\#a^+a))^* \\
&= a^*(aa^\#)^* \\
&= a^*,
\end{aligned}$$

and

$$\begin{aligned}
a^*((a^\#)^+)^\#)(a^+)^*a^*a^+a^2(a^+)^* &= (a^*(aa^\#)^*)a^\#((aa^\#)^*(a^+)^*a^*)a^+a^2(a^+)^* \\
&= a^*a^\#((aa^\#)^*a^+)a^2(a^+)^* \\
&= a^*(a^\#a^+a)a(a^+)^* \\
&= a^*(a^\#a(a^+)^*) \\
&= a^*(a^+)^* \\
&= (a^+a)^* \\
&= a^+a \\
&= (a^*((a^\#)^+)^\#(a^+)^*a^*a^+a^2(a^+)^*)^*.
\end{aligned}$$

So $(a^*_{((a^\#)^+)^\#(a^+)^*})^{\textcircled{\#}} = a^*(a^\#)^+(a^+)^*$.

(9) If $x = (a^+)^\# = (aa^\#)^*a(aa^\#)^*$, then $a^*(a^+)^\#(a^+)^* = (a^*(aa^\#)^*)a^*aa^\#)^*(a^+)^* = a^*a(aa^\#)^*(a^+)^*$. Thus, we have

$$\begin{aligned}
a^*((a^+)^\#)^\#((a^+)^*a^*a)(aa^\#)^*((a^+)^*a^*a)(aa^\#)^*(a^+)^* &= a^*a^+a(aa^\#)^*a(aa^\#)^*(a^+)^* \\
&= a^*(a^+a(a^\#)^*)a^*a(aa^\#)^*(a^+)^* \\
&= (a^*(a^\#)^*a^*)a(aa^\#)^*(a^+)^* \\
&= a^*a(aa^\#)^*(a^+)^*,
\end{aligned}$$

$$\begin{aligned}
a^*a((aa^\#)^*(a^+)^*a^*)((a^+)^\#)^\#(a^+)^*a^* &= a^*a(aa^\#)^*(a^+(a^+)^*a^*) \\
&= a^*a((aa^\#)^*a^+) \\
&= a^*aa^+ \\
&= a^*,
\end{aligned}$$

and

$$\begin{aligned} a^*((a^+)^\#)^\#((a^+)^*a^*a)(aa^\#)^*(a^+)^* &= a^*a^+a(aa^\#)^*(a^+)^* \\ &= a^*(a^+a(a^\#)^*)a^*(a^+)^* \\ &= (a^*(a^\#)^*a^*)(a^+)^* \\ &= a^*(a^+)^* \\ &= (a^+a)^* \\ &= a^+a \\ &= (a^*((a^+)^\#)^\#(a^+)^*a^*a(aa^\#)^*(a^+)^*)^*. \end{aligned}$$

It follows that $(a^*{}_{a^+(a^+)^*})^{\textcircled{\#}} = a^*(a^+)^\#(a^+)^*$. □

Theorem 3.2. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if for some $x \in \phi_a$, $x^\#$ and $(a^\#{}_{x^\#(a^+)^*})^{\textcircled{\#}}$ exist and*

$$(a^\#{}_{x^\#(a^+)^*})^{\textcircled{\#}} = a^*x(a^+)^*.$$

Proof. ⇒ Since $a \in R^{SEP}$, $a^* = a^\# = a^+$. Then by Theorem 3.1, this assertion holds.

⇐ By $(a^\#{}_{x^\#(a^+)^*})^{\textcircled{\#}} = a^*x(a^+)^*$, we have

$$a^*x(a^+)^*a^\#x^\#(a^+)^*a^\# = a^\#. \tag{4}$$

Multiplying the equation (4) on the left by a^+a, then

$$a^+aa^*x(a^+)^*a^\#x^\#(a^+)^*a^\# = a^+aa^\#.$$

By $a^+aa^* = a^*$, one can get $a^\# = a^+aa^\#$, it follows that $a \in R^{EP}$ by [3, Theorem 1.2.1].

(1) If $x = a$, then by $(a^\#{}_{a^\#(a^+)^*})^{\textcircled{\#}} = a^*a(a^+)^*$,

$$a^\#a^\#(a^+)^*a^*a(a^+)^*a^*a(a^+)^* = a^*a(a^+)^*.$$

Since $a \in R^{EP}$, it is easy to compute

$$a^\#(a^\#(a^+)^*a^*)a((a^+)^*a^*a)(a^+)^* = a^\#(a^\#a^2)(a^+)^* = a^\#a(a^+)^* = (a^+)^*.$$

Thus we have $(a^+)^* = a^*a(a^+)^*$. Multiplying the equality $(a^+)^* = a^*a(a^+)^*$ on the right by a^*a^+, and according to $(a^+)^*a^*a^+ = (a^\#)^*a^*a^+ = a^+$, one gets

$$a^+ = (a^+)^*a^*a^+ = a^*a((a^+)^*a^*a^+) = a^*aa^+ = a^*.$$

It follows that $a \in R^{SEP}$.

(2) If $x = a^\#$, then

$$a^*a^\#(a^+)^*a^\#(a^\#)^\#(a^+)^*a^\# = a^\#,$$

i.e.,

$$a^\# = a^*a^\#(a^+)^*a^\#(a^\#)^\#(a^+)^*a^\# = a^*a^+(a^+)^*(a^+)^*a^+.$$

Multiplying the equality $a^*a^+(a^+)^*(a^+)^*a^+ = a^\#$ on the left by $a(a^+)^*$ and on the right by aa^*, then

$$a(a^+)^*a^*a^+(a^+)^*(a^+)^*a^+a(a^+)^* = a(a^+)^*a^\#a(a^+)^*.$$

By $a \in R^{EP}$, one has that $a((a^+)^* a^* a^+)(a^+)^*(a^+)^*(a^+ a a^*) = aa^+((a^+)^*(a^+)^* a^*) = aa^+(a^+)^* = (a^+)^*$ and $a(a^+)^*(a^\# a a^*) = a(a^+)^* a^* = a$, which implies that $a = (a^+)^*$, i.e., $a^* = a^+$, and hence $a \in R^{SEP}$.

(3) If $x = a^*$, then

$$a^* a^*(a^+)^* a^\#(a^*)^\#(a^+)^* a^\# = a^\#.$$

This induces

$$a^\# = a^*(a^*(a^+)^* a^\#)(a^*)^\#(a^+)^* a^\# = a^* a^+(a^+)^*(a^+)^* a^+.$$

As the same as the case of $x = a^\#$, one can prove that $a \in R^{SEP}$.

(4) If $x = (a^\#)^*$, then by $a^*(a^\#)^*(a^+)^* = a^*(a^\#)^*(a^\#)^* = (a^\#)^* = (a^+)^*$, one has

$$a^\# = (a^*(a^\#)^*(a^+)^*)a^\#((a^\#)^*)^\#(a^+)^* a^\# = (a^+)^* a^+(a^*(a^+)^* a^+) = (a^+)^* a^+ a^+.$$

Multiplying the above equality $(a^+)^* a^+ a^+ = a^\#$ on the left by a^* and on the right by a, and according to $a^+ a^+ a = a^\# a^\# a = a^\# = a^+$ and $a^* a^\# a = a^* a^+ a = a^*$, one gets

$$a^* = a^* a^\# a = a^*(a^+)^*(a^+ a^+ a) = a^*(a^+)^* a^+ = a^+.$$

It follows that $a \in R^{SEP}$. □

Theorem 3.3. *Let $a \in R^\# \cap R^+$. Then $a \in R^{SEP}$ if and only if for some $x \in \phi_a$, $x^\#$ and $({a^*}_{x^\#(a^+)^*})^{\textcircled{\#}}$ exist and*

$$({a^*}_{x^\#(a^+)^*})^{\textcircled{\#}} = a^\# x (a^\#)^*.$$

Proof. ⇒ By $a \in R^{SEP}$ and Theorem 3.1, this conclusion holds.

⇐ By $({a^*}_{x^\#(a^+)^*})^{\textcircled{\#}} = a^\# x (a^\#)^*$, we have

$$a^\# x (a^\#)^* a^* x^\#(a^+)^* a^* = a^*. \tag{5}$$

Multiplying (5) on the left by aa^+, then

$$aa^+ a^\# x (a^\#)^* a^* x^\#(a^+)^* a^* = aa^+ a^*.$$

By $aa^+ a^\# = a^\#$, one has $a^* = aa^+ a^*$, and hence $a \in R^{EP}$ by [3, Theorem 1.2.1].

(1) If $x = a$, then by $(a^\#)^* a^* a^\#(a^+)^* a^* = (a^+)^* a^* a^\# a^*(a^+)^* = a^\#$, one has

$$a^* = a^\# a((a^\#)^* a^* a^\#(a^+)^* a^*) = a^\# a a^\# = a^\#.$$

So $a \in R^{SEP}$.

(2) If $x = a^\#$, then by $a^\#(a^\#)^* a^* = a^+(a^+)^* a^* = a^+$ and $(a^\#)^\#(a^+)^* a^* = aa^*(a^+)^* = a$, one gets

$$a^* = a^\#(a^\#(a^\#)^* a^*)((a^\#)^\#(a^+)^* a^*) = a^+ a^+ a = a^+.$$

It follows that $a \in R^{SEP}$.

(3) If $x = a^*$, then according to $(a^+)^* a^*(a^*)^\#(a^+)^* a^* = (aa^\# a^\# aa^\#)^* = (a^\#)^* = (a^+)^*$ and $a^+ a^*(a^+)^* = a^+ a^*(a^\#)^* = a^+$, one obtains

$$a^* = a^+ a^*((a^+)^* a^*(a^*)^\#(a^+)^* a^*) = a^+ a^*(a^+)^* = a^+,$$

which implies that $a \in R^{SEP}$.

(4) If $x = (a^\#)^*$, then according to $(a^+)^* a^* a^*(a^+)^* a^* = (a^\#)^* a^* a^*(a^\#)^* a^* = (a^\#)^* = (a^+)^*$, one obtains

$$a^* = a^+(a^\#)^*((a^+)^* a^* a^*(a^+)^* a^*) = a^+(a^+)^* a^* = a^+.$$

Hence, $a \in R^{SEP}$. □

Theorem 3.4. *Let $a \in R^{\#} \cap R^{+}$. Then $a \in R^{SEP}$ if and only if for some $x \in \phi_a$, $x^{\#}$ and $(a^*{}_{x^{\#}(a^+)^*})^{\textcircled{\#}}$ exist and*

$$(a^*{}_{x^{\#}(a^+)^*})^{\textcircled{\#}} = a^{\#}x(a^+)^*.$$

Proof. $\Rightarrow$ By $a \in R^{SEP}$ and Theorem 3.1, the statement holds.

$\Leftarrow$ Since $(a^*{}_{x^{\#}(a^+)^*})^{\textcircled{\#}} = a^{\#}x(a^+)^*$,

$$a^* = a^{\#}x(a^+)^*a^*x^{\#}(a^+)^*a^*. \tag{6}$$

Multiplying (6) on the left by aa^+, then

$$aa^+a^{\#}x(a^+)^*a^*x^{\#}(a^+)^*a^* = aa^+a^*.$$

Since $aa^+a^{\#} = a^{\#}$, one has $a^2a^+ = a$, and hence $a \in R^{EP}$ by [3, Theorem 1.2.1].

(1) If $x = a$, then

$$a^* = a^{\#}a(a^+)^*a^*a^{\#}(a^+)^*a^*.$$

Since $a \in R^{EP}$, one has $a^* = a^{\#}(a(a^+)^*a^*)(a^{\#}(a^+)^*a^*) = a^{\#}aa^{\#} = a^{\#}$, and hence $a \in R^{SEP}$.

(2) If $x = a^{\#}$, then

$$a^* = a^{\#}a^{\#}(a^+)^*a^*(a^{\#})^{\#}(a^+)^*a^*.$$

By $a \in R^{EP}$, $a^* = a^{\#}a^{\#}((a^+)^*a^*(a^{\#})^{\#}(a^+)^*a^*) = a^{\#}a^{\#}a = a^{\#}$, which implies that $a \in R^{SEP}$.

(3) If $x = a^*$, then

$$a^* = a^{\#}a^*(a^+)^*a^*(a^*)^{\#}(a^+)^*a^*.$$

Noting that $a \in R^{EP}$, one gets $a^* = a^{\#}(a^*(a^+)^*a^*)((a^*)^{\#}(a^+)^*a^*) = a^{\#}a^*(a^+)^* = a^{\#}$. Hence, $a \in R^{SEP}$.

(4) If $x = (a^{\#})^*$, then

$$a^* = a^{\#}(a^{\#})^*(a^+)^*a^*((a^{\#})^*)^{\#}(a^+)^*a^*.$$

Because $a \in R^{EP}$, one can have that $a^* = a^{\#}((a^{\#})^*(a^+)^*a^*)(((a^{\#})^*)^{\#}(a^+)^*a^*) = a^{\#}(a^+)^*a^* = a^{\#}$. It follows that $a \in R^{SEP}$. □

In [2], Li and Wei proved that for an element $x \in \chi_a := \{a, a^{\#}, a^+, a^*, (a^{\#})^*, (a^+)^*\}$, then $xx^* \in R^{SEP}$ if and only if $(xx^*)^3 = xx^*$ ([2, theorem 4.7]). In fact, this statement is true for any $x \in R^{\#} \cap R^{+}$. Thus, we have the following conclusion.

Theorem 3.5. *Let $a \in R^{\#} \cap R^{+}$. Then $xx^* \in R^{SEP}$ if and only if for some $x \in \phi_a$, $x_{x^*}^{\textcircled{\#}}$ exists and*

$$x_{x^*}^{\textcircled{\#}} = xx^*.$$

Proof. $\Rightarrow$ Choosing $x = a$, then by $a \in R^{SEP}$, $a^* = a^{\#} = a^+$, we have $aa^*a = a$. Thus, following equalities hold:

$$aa^*(aa^*)^2 = (aa^*a)a^*aa^* = (aa^*a)a^* = aa^*,$$

$$(aa^*a)a^*a = aa^*a = a,$$

and

$$((aa^*a)a^*)^* = (aa^*)^* = aa^*.$$

It follows that $a_{a^*}^{\#} = aa^*$.

$\Leftarrow$ By $x_{x^*}^{\#} = xx^*$, we have

$$xx^*(xx^*)^2 = (xx^*)^3 = xx^*.$$

Thus $xx^* \in R^{SEP}$ by [2, theorem 4.7]. □

Acknowledgement

The authors thank the anonymous referee for numerous suggestions that helped improve our paper substantially.

Conflict of Interest

The authors declared that they have no conflict of interest.

References

[1] L. F. Cao, L. You, J. C. Wei, *EP elements of* $\mathbb{Z}[x]/(x^2+x)$, Filomat **37(22)** (2023) 7467-7478.
[2] A. Q. Li, J. C. Wei, *The influence of the expression form of solutions to related equations on SEP elements in a ring with involution*. J. Algebra Appl. DOI: 10.1142/S0219498824501135.
[3] D. Mosić, *Generalized inverses*, Faculty of Sciences and Mathematics, University of Niš, Niš 2018.
[4] D. Mosić, D. S. Djordjević, *Moore-Penrose invertible normal and Hermitian elements in rings*, Linear Algebra Appl. **431** (2009), 732-745.
[5] D. Mosić, D. S. Djordjević, J. J. Koliha, *EP elements in rings*, Linear Algebra Appl. **431** (2009), 527-535.
[6] D. Mosić, D. S. Djordjević, *New characterizations of EP, generalized normal and generalized Hermitian elements in rings*, Appl. Math. Comput. **218** (2012), 6702-6710.
[7] D. Mosić, D. S. Djordjević, *Further results on partial isometries and EP elements in rings with involution*, Math. Comput. Model. **54** (2011), 460-465.
[8] D. Mosić, D. S. Djordjević, *Partial isometries and EP elements in rings with involution*, Electron. J. Linear Algebra **18** (2009) 761-722.
[9] R. Penrose, *A generalized inverse for matrices*. Proc. Cambridge Philos. Soc. **51** (1955), 406-413.
[10] Y. C. Qu, J. C. Wei, H. Yao, *Characterizations of normal elements in rings with involution*, Acta. Math. Hungar. **156(2)** (2018), 459-464.
[11] L. Y. Shi, J. C. Wei, *Some new characterizations of normal elements*, Filomat **33(13)** (2019), 4115-4120.
[12] Z. C. Xu, R. J. Chen, J. C. Wei, *Strongly EP elements in a ring with involution*, Filomat, **34(6)** (2020), 2101-2107.
[13] D. D. Zhao, J. C. Wei, *Strongly EP elements in rings with involution*, J. Algebra Appl. **21(5)** (2022), 2250088, 10pp.
[14] H. H. Zhu, L. Y. Wu, J. L. Chen, *A new class of generalized inverses in semigroups and rings with involution*. Comm. Algebra. DOI: 10.1080/00927872.2022.2150771
[15] H. H. Zhu, L. Y. Wu, D. Mosić, *One-sided w-core inverses in rings with an involution*. Linear and Multilinear Algebra. DOI: 10.1080/03081087.2022.2025208