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On generalized ψ-conformable calculus: Properties and inequalities

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Abstract. In this paper, we first introduce a new fractional derivatives and integrals called generalized ψ-conformable derivative and generalized ψ-conformable integral operators, respectively. We also show that these operators generalize various well-known fractional integral operators. Then, we present several properties of these operators including semi-group property. Moreover, we apply these operators to obtain a new Hermite-Hadamard-type inequality for convex functions. Furthermore, we obtain corresponding midpoint and trapezoid type inequalities for functions whose derivatives in absolute value are convex.

1. Introduction

Throughout history, various forms of fractional derivative definitions have been introduced, including Riemann-Liouville, Caputo, Grunwald-Letnikov, Riesz, and Weyl. The majority of them are defined using fractional integrals, with the loss of some basic properties that ordinary derivatives have, such as the product rule and the chain rule, among others.
Essential tools that allowed mathematicians to define fractional integrals were Cauchy's iterations of n integrations

$$(I^n f)(x) = \int_a^x d\sigma_1 \int_a^{\sigma_1} d\sigma_2 \cdots \int_a^{\sigma_{n-1}} f(\sigma_n)\, d\sigma_n = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)\, dt,$$

and the Gamma function defined for complex values α with $\mathfrak{Re}(\alpha) > 0$ as follows

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt,$$

which satisfies the following properties

$$\forall n \in \mathbb{N}^* : \ \Gamma(n) = (n-1)! \quad \text{and} \quad \alpha\,\Gamma(\alpha) = \Gamma(\alpha+1),$$

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and the Beta function defined by

$$B(z,s) = \int_0^1 t^{z-1} (1-t)^{s-1}\, dt \qquad \mathcal{R}e(z) > 0,\ \mathcal{R}e(s) > 0.$$

It verifies

$$B(z,s) = \frac{\Gamma(z)\,\Gamma(s)}{\Gamma(z+s)}.$$

In 2014, Khalil et al. [3] proposed well-behaved type of fractional derivative named conformable fractional derivative whose most properties coincide with Newton derivative.

Definition 1.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function. The conformable fractional derivative of order $\alpha \in (0,1]$ is defined for all $t > 0$ by*

$$f^{\alpha}(t) = \lim_{\epsilon \to 0} \frac{f\left(t + \epsilon t^{1-\alpha}\right) - f(t)}{\epsilon}$$

If f is α-differentiable in some $(0,a), a > 0$, and $\lim_{t\to 0^+} f^{\alpha}(t)$ exists, then define

$$f^{\alpha}(0) = \lim_{t \to 0^+} f^{\alpha}(t)$$

If f is differentiable at t; then the α-derivative is $f^{\alpha}(t) = t^{1-\alpha} f'(t)$.

The α-fractional integral of a function f starting from $a \geq 0$ is defined as follows

Definition 1.2. *: Let $f : [0,\infty) \to \mathbb{R}$ be a continuous function. The conformable fractional integral starting from $a \geq 0$ of order $\alpha \in (0,1]$ of the function f is defined by*

$$(I_{\alpha}^{a} f)(x) = \int_a^x f(t)\, \frac{dt}{t^{1-\alpha}} = \int_a^x f(t)\, d_{\alpha} t \tag{1}$$

where the integral is the usual Riemann improper integral and $d_{\alpha} t = \dfrac{dt}{t^{1-\alpha}}$.

Iterating n-times (1) leads to (taking $s = \alpha - 1$)

$$\int_a^x t_1^s\, dt_1 \int_a^{t_1} t_2^s\, dt_2 \ldots \int_a^{t_{n-1}} t_n^s f(t_n)\, dt_n = \frac{(s+1)^{1-n}}{\Gamma(n)} \int_a^x \left(x^{s+1} - t^{s+1}\right)^{n-1} f(t)\, t^s\, dt.$$

The generalized fractional integral of order $\beta > 0$, with parameter $\alpha \in (0,1]$ [2] is

$${}^{\beta}\mathcal{I}_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \left(\frac{x^{\alpha} - t^{\alpha}}{\alpha}\right)^{\beta-1} f(t)\, \frac{dt}{t^{1-\alpha}}. \tag{2}$$

The major focus of this work is to introduce a novel class of fractional conformable integration and differentiation operators of order $\alpha \in (0,1]$ with respect to a differentiable monotone function ψ. Then, using Cauchy's iteration, generalized fractional conformable integral operators of order $\beta > 0$ and parameter $\alpha \in (0,1]$ are defined, and their boundedness and semi-properties are given. Next, we derive the generalized fractional conformable derivatives. Finally, for convex functions, we prove new Hermite-Hadamard inequalities.

2. New Conformable Fractional Integrals

In this section, we introduced new conformable derivatives and integrals. In this section, we assume that $\psi(.)$ is a positive continuously differentiable function such that $\psi' > 0$ and $\alpha \in (0,1]$.

2.1. ψ-conformable fractional derivative and integral

The definition of ψ- conformable fractional derivative is now given.

Definition 2.1. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function.*

- *For all $t > 0$, the ψ-conformable fractional derivative of order α is defined by*

$$\mathcal{D}^{\psi} f(t) = \lim_{\epsilon \to 0} \frac{f\left(t + \epsilon\, \psi'^{1-\alpha}(t)\right) - f(t)}{\epsilon}.$$

- *If f is differentiable at t, then the ψ-derivative is*

$$\mathcal{D}^{\psi} f(t) = \psi'^{1-\alpha}(t)\, f'(t).$$

- *The corresponding ψ-conformable fractional integral of a function f starting from $a \geq 0$ is*

$$(\mathcal{CF}_a^{\psi} f)(x) := \int_a^x \psi'(t)\, f(t)\, d_{\psi}\, t, \tag{3}$$

where $d_{\psi}\, t = \dfrac{dt}{\psi^{1-\alpha}(t)}$.

2.2. Generalized ψ-conformable fractional integral

Iterating n times the ψ-conformable fractional integral (3), gives

$$\int_a^x \psi'(t_1)\, d_{\psi} t_1 \int_a^{t_1} \psi'(t_2)\, d_{\psi} t_2 \ldots \int_a^{t_{n-1}} \psi'(t_n)\, f(t_n)\, d_{\psi} t_n$$

$$= \frac{1}{\Gamma(n)} \int_a^x \left(\frac{\psi^{\alpha}(x) - \psi^{\alpha}(t)}{\alpha} \right)^{n-1} \psi'(t)\, f(t)\, d_{\psi} t.$$

According to this result, the generalized ψ-conformable fractional integral operators of order $\beta > 0$ are defined as follow.

Definition 2.2. *Let $0 \leq a < b < \infty$, $f : [a, b] \subset [0, +\infty) \to \mathbb{R}$ be a continuous function. The generalized left and right ψ-conformable fractional integral operators of the function f, with order $\beta > 0$, are defined as*

$${}^{\psi}_{\mathcal{CF}} I^{\beta}_{a^+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \psi'(t) \Big[\frac{\psi^{\alpha}(x) - \psi^{\alpha}(t)}{\alpha} \Big]^{\beta - 1} f(t)\, d_{\psi}\, t, \tag{4}$$

and

$${}^{\psi}_{\mathcal{CF}} I^{\beta}_{b^-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \psi'(t) \Big[\frac{\psi^{\alpha}(t) - \psi^{\alpha}(x)}{\alpha} \Big]^{\beta - 1} f(t)\, d_{\psi}\, t. \tag{5}$$

For special choices of ψ , α and β, we get already known results.

1. Taking $\alpha = 1$, the operators reduce to the ψ-Hilfer integral operators of order $\beta > 0$.
2. For $\psi(t) = t$, we get the Katugompola operators of order $\beta > 0$ and parameter $\alpha > 0$.
3. For $\psi(t) = t$, $\alpha = 1$, the operators are simplified to Riemann-Liouville integral operators.
4. Taking $\psi(t) = t$, $\alpha = 1$ and $\beta = 1$, the operators reduce to classical Riemann integrals.

5. Setting $\psi(t) = \ln(t)$ and $a > 1$, We get the α-Hadamard operators of order $\beta > 0$ given in (6) and (7).
6. Setting $\psi(t) = \ln(t)$, $\alpha = 1$ and $a > 1$, We get Hadamard operators with order $\beta > 0$.

Definition 2.3. *Let $1 \leq a < b < \infty$, $f : [a,b] \subset [0,+\infty) \to \mathbb{R}$ be a continuous function. The left and right α-Hadamard integral operators with order $\beta > 0$, are defined as*

$$ {}^{\alpha}\mathcal{H}^{\beta}_{a^+} f(x) = \frac{1}{\Gamma(\beta)} \int_a^x \Big[\frac{\ln^{\alpha}(x) - \ln^{\alpha}(t)}{\alpha}\Big]^{\beta-1} \frac{f(t)}{t} \frac{dt}{\ln^{1-\alpha}(t)}, \tag{6} $$

and

$$ {}^{\alpha}\mathcal{H}^{\beta}_{b^-} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b \Big[\frac{\ln^{\alpha}(t) - \ln^{\alpha}(x)}{\alpha}\Big]^{\beta-1} \frac{f(t)}{t} \frac{dt}{\ln^{1-\alpha}(t)}. \tag{7} $$

Next, some examples are given.

Example 2.4. *For $\delta > 0$, we have*

$$ {}^{\psi}_{CF}I^{\beta}_{a^+}\left(\Big[\frac{\psi^{\alpha}(x) - \psi^{\alpha}(a)}{\alpha}\Big]^{\delta}\right) = \frac{\Gamma(\delta+1)}{\Gamma(\beta+\delta+1)}\Big[\frac{\psi^{\alpha}(x) - \psi^{\alpha}(a)}{\alpha}\Big]^{\delta+\beta} \tag{8} $$

$$ {}^{\psi}_{CF}I^{\beta}_{b^-}\left(\Big[\frac{\psi^{\alpha}(b) - \psi^{\alpha}(x)}{\alpha}\Big]^{\delta}\right) = \frac{\Gamma(\delta+1)}{\Gamma(\beta+\delta+1)}\Big[\frac{\psi^{\alpha}(b) - \psi^{\alpha}(x)}{\alpha}\Big]^{\beta+\delta} \tag{9} $$

Proof. Using the definition (8), we get

$$ {}^{\psi}_{CF}I^{\beta}_{a^+}\left(\Big[\frac{\psi^{\alpha}(x) - \psi^{\alpha}(a)}{\alpha}\Big]^{\delta}\right) = \frac{1}{\Gamma(\beta)} \int_a^x \Big[\frac{\psi^{\alpha}(x) - \psi^{\alpha}(t)}{\alpha}\Big]^{\beta-1} \Big[\frac{\psi^{\alpha}(t) - \psi^{\alpha}(a)}{\alpha}\Big]^{\delta} \psi'(t) \frac{dt}{\psi^{1-\alpha}(t)}. $$

Posing $u = \dfrac{\psi^{\alpha}(t) - \psi^{\alpha}(a)}{\psi^{\alpha}(x) - \psi^{\alpha}(a)}$ then $1 - u = \dfrac{\psi^{\alpha}(x) - \psi^{\alpha}(t)}{\psi^{\alpha}(x) - \psi^{\alpha}(a)}$ and

$$ \begin{aligned} {}^{\psi}_{CF}I^{\beta}_{a^+}\left(\Big[\frac{\psi^{\alpha}(x) - \psi^{\alpha}(a)}{\alpha}\Big]^{\delta}\right) &= \frac{1}{\Gamma(\beta)}\Big[\frac{\psi^{\alpha}(x) - \psi^{\alpha}(a)}{\alpha}\Big]^{\delta+\beta} \int_0^1 \Big[1-u\Big]^{\beta-1} u^{\delta}\, du \\ &= \frac{\Gamma(\delta+1)}{\Gamma(\beta+\delta+1)}\Big[\frac{\psi^{\alpha}(x) - \psi^{\alpha}(a)}{\alpha}\Big]^{\delta+\beta}. \end{aligned} $$

The proof of (9) is similar. □

Example 2.5. *Consider the Mittag-Leffler function defined for $\beta > 0$ by*

$$ E_{\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)}, $$

then

$$ {}^{\psi}_{CF}I^{\beta}_{a^+} E_{\beta}\left(\Big[\frac{\psi^{\alpha}(x) - \psi^{\alpha}(a)}{\alpha}\Big]^{\beta}\right) = E_{\beta}\left(\Big[\frac{\psi^{\alpha}(x) - \psi^{\alpha}(a)}{\alpha}\Big]^{\beta}\right) - 1, \tag{10} $$

$$ {}^{\psi}_{CF}I^{\beta}_{b^-} E_{\beta}\left(\Big[\frac{\psi^{\alpha}(b) - \psi^{\alpha}(x)}{\alpha}\Big]^{\beta}\right) = E_{\beta}\left(\Big[\frac{\psi^{\alpha}(b) - \psi^{\alpha}(x)}{\alpha}\Big]^{\beta}\right) - 1. \tag{11} $$

Proof. Using (8), we have

$$
\begin{aligned}
{}^{\psi}_{\mathcal{CF}}I^{\beta}_{a^+}\, E_\beta\left(\left[\frac{\psi^\alpha(x)-\psi^\alpha(a)}{\alpha}\right]^\beta\right) &= \sum_{k=0}^{\infty}\frac{1}{\Gamma(\beta k+1)}\, {}^{\psi}_{\mathcal{CF}}I^{\beta}_{a^+}\left[\frac{\psi^\alpha(x)-\psi^\alpha(a)}{\alpha}\right]^{k\beta} \\
&= \sum_{k=0}^{\infty}\frac{1}{\Gamma(k\beta+\beta+1)}\left[\frac{\psi^\alpha(x)-\psi^\alpha(a)}{\alpha}\right]^{k\beta+\beta} \\
&= \sum_{k=0}^{\infty}\frac{1}{\Gamma(\beta(k+1)+1)}\left[\frac{\psi^\alpha(x)-\psi^\alpha(a)}{\alpha}\right]^{\beta(k+1)} \\
&= E_\beta\left(\left[\frac{\psi^\alpha(x)-\psi^\alpha(a)}{\alpha}\right]^\beta\right)-1.
\end{aligned}
$$

The proof of (11) is similar. □

Now, we state some theorems related to the generalized ψ-conformable integral operators.

Theorem 2.6. *Let be a function $f \in C^1(a,b)$, then*

$$
\lim_{\beta\to 0^+} {}^{\psi}_{\mathcal{CF}}I^{\beta}_{a^+}f(x) = f(x), \qquad \lim_{\beta\to 0^+} {}^{\psi}_{\mathcal{CF}}I^{\beta}_{b^-}f(x) = f(x). \tag{12}
$$

Proof. Integrating by parts, we get

$$
\begin{aligned}
{}^{\psi}_{\mathcal{CF}}I^{\beta}_{a^+}f(x) &= \frac{1}{\Gamma(\beta)}\int_a^x \psi'(t)\left[\frac{\psi^\alpha(x)-\psi^\alpha(t)}{\alpha}\right]^{\beta-1} f(t)\, d_\psi t \\
&= \frac{1}{\Gamma(\beta+1)}\left(\left[\frac{\psi^\alpha(x)-\psi^\alpha(a)}{\alpha}\right]^\beta f(a) + \int_a^x \left[\frac{\psi^\alpha(x)-\psi^\alpha(t)}{\alpha}\right]^\beta f'(t)\, dt\right),
\end{aligned}
$$

consequently $\lim_{\beta\to 0^+} {}^{\psi}_{\mathcal{CF}}D^{\beta}_{a^+}f(x) = f(x)$. □

Now, we show that these operators are well defined on a specified space of functions. Define

$$
X_\psi\left([a,b]\right) = \left\{ f : \| f \|_{X_\psi} = \int_a^b | f(t) |\, \psi'(t)\, d_\psi t < \infty \right\}, \tag{13}
$$

where $d_\psi t = \dfrac{dt}{\psi^{1-\alpha}(t)}$.

Theorem 2.7. *(Boundedness property) Let $f \in X_\psi([a,b])$, then for $\beta > 0$*

$$
{}^{\psi}_{\mathcal{CF}}I^{\beta}_{a^+} f \in X_\psi([a,b]), \text{ and } {}^{\psi}_{\mathcal{CF}}I^{\beta}_{b^-} f \in X_\psi([a,b]).
$$

Moreover the fractional integral operators ${}^{\psi}_{\mathcal{CF}}I^{\beta}_{a^+}$ and ${}^{\psi}_{\mathcal{CF}}I^{\beta}_{b^-}$ are bounded on X_ψ, explicitly

$$
\left\| {}^{\psi}_{\mathcal{CF}}I^{\beta}_{a^+} f \right\|_{X_\psi} \leq \frac{(\psi^\alpha(b)-\psi^\alpha(a))^\beta}{\alpha^\beta\, \Gamma(\beta+1)} \left\|f\right\|_{X_\psi}, \tag{14}
$$

and

$$
\left\| {}^{\psi}_{\mathcal{CF}}I^{\beta}_{b^-} f \right\|_{X_\psi} \leq \frac{(\psi^\alpha(b)-\psi^\alpha(a))^\beta}{\alpha^\beta\, \Gamma(\beta+1)} \left\|f\right\|_{X_\psi}. \tag{15}
$$

Proof. Let $f \in X_\psi([a,b])$, by Fubini's Theorem, we get

$$\begin{aligned}
\left\| {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+} f \right\|_{X_\psi} &= \int_a^b \mid {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+} f(x) \mid \psi'(x)\, d_\psi x \\
&\le \frac{1}{\Gamma(\beta)} \int_a^b \int_a^x \left| f(t) \right| \psi'(t) \Big[\frac{\psi^\alpha(x) - \psi^\alpha(t)}{\alpha} \Big]^{\beta-1} \psi'(x)\, d_\psi t\, d_\psi x \\
&= \frac{1}{\Gamma(\beta)} \int_a^b \left| f(t) \right| \left(\int_t^b \Big[\frac{\psi^\alpha(x) - \psi^\alpha(t)}{\alpha} \Big]^{\beta-1} \psi'(x)\, d_\psi x \right) \psi'(t)\, d_\psi t \\
&= \frac{1}{\Gamma(\beta+1)} \int_a^b \left| f(t) \right| \Big[\frac{\psi^\alpha(b) - \psi^\alpha(t)}{\alpha} \Big]^{\beta} \psi'(t)\, d_\psi t \\
&\le \frac{(\psi^\alpha(b) - \psi^\alpha(a))^\beta}{\alpha^\beta\, \Gamma(\beta+1)} \int_a^b \left| f(t) \right| \psi'(s)\, d_\psi t.
\end{aligned}$$

Similarly, we establish (15). □

Remark 2.8. *Denote by $C([a,b])$ the space of continuous functions on $[a,b]$.*

1. *Let $f \in C([a,b])$, then*

$$\begin{aligned}
\int_a^b \mid f(t) \mid \psi'(t)\, d_\psi t &\le \max_{a\le x\le b}(\mid f(t) \mid) \int_a^b \psi'(t)\, d_\psi t \\
&= \max_{a\le x\le b}(\mid f(t) \mid) \Big[\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha} \Big] < \infty,
\end{aligned}$$

so that $C([a,b]) \subset X_\psi([a,b])$.

2. *Let $f \in C([a,b])$ then*

$$\begin{aligned}
\mid {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+} f(x) \mid &\le \frac{1}{\Gamma(\beta)} \int_a^x \psi'(t) \Big[\frac{\psi^\alpha(x) - \psi^\alpha(t)}{\alpha} \Big]^{\beta-1} \left| f(t) \right|\, d_\psi t \\
&\le \frac{1}{\Gamma(\beta)} \max_{a\le x\le b}(\mid f(t) \mid) \int_a^x \psi'(t) \Big[\frac{\psi^\alpha(x) - \psi^\alpha(t)}{\alpha} \Big]^{\beta-1} d_\psi t \\
&\le \frac{1}{\Gamma(\beta+1)} \Big[\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha} \Big]^{\beta} \max_{a\le x\le b}(\mid f(t) \mid).
\end{aligned}$$

Hence, ${}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+} f \in C([a,b])$ and the operator ${}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}$ is bounded on $(C([a,b]), max)$.

Theorem 2.9. *If $f \in C([a,b])$, then*

$$\lim_{x\to a^+} {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+} f(x) = 0, \qquad \lim_{x\to b^-} {}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-} f(x) = 0.$$

Proof. Let $f \in C([a,b])$, then

$$\begin{aligned}
\mid {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+} f(x) \mid &\le \frac{1}{\Gamma(\beta)} \max_{a\le x\le b}(\mid f(t) \mid) \int_a^x \psi'(t) \Big[\frac{\psi^\alpha(x) - \psi^\alpha(t)}{\alpha} \Big]^{\beta-1} d_\psi t \\
&= \frac{1}{\Gamma(\beta+1)} \Big[\frac{\psi^\alpha(x) - \psi^\alpha(a)}{\alpha} \Big]^{\beta} \max_{a\le x\le b}(\mid f(t) \mid),
\end{aligned}$$

consequently $\lim_{x\to a^+} {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+} f(x) = 0.$ □

Theorem 2.10. *(Semi group property) Let $f \in X_\psi([a,b])$ and $\beta_1, \beta_2 > 0$, then*

$$ {}^{\psi}_{\mathcal{CF}}I^{\beta_1}_{a^+}\left({}^{\psi}_{\mathcal{CF}}I^{\beta_2}_{a^+}f(x)\right) = {}^{\psi}_{\mathcal{CF}}I^{\beta_1+\beta_2}_{a^+}f(x), \tag{16}$$

and

$$ {}^{\psi}_{\mathcal{CF}}I^{\beta_1}_{b^-}\left({}^{\psi}_{\mathcal{CF}}I^{\beta_2}_{b^-}f(x)\right) = {}^{\psi}_{\mathcal{CF}}I^{\beta_1+\beta_2}_{b^-}f(x). \tag{17}$$

Proof. Using Fubini's Theorem, we get

$$ {}^{\psi}_{\mathcal{CF}}I^{\beta_1}_{a^+}\left({}^{\psi}_{\mathcal{CF}}I^{\beta_2}_{a^+}f(x)\right) = \frac{1}{\Gamma(\beta_1)}\int_a^x \psi'(t)\Big[\frac{\psi^\alpha(x)-\psi^\alpha(t)}{\alpha}\Big]^{\beta_1-1}\left({}^{\beta_2}_{\mathcal{CF}}I^{\psi}_{a^+}f(t)\right)\,d_\psi t$$

$$= \frac{1}{\Gamma(\beta_1)\,\Gamma(\beta_2)}\int_a^x\int_a^t \psi'(t)\,\psi'(s)\Big[\frac{\psi^\alpha(x)-\psi^\alpha(t)}{\alpha}\Big]^{\beta_1-1}\Big[\frac{\psi^\alpha(t)-\psi^\alpha(s)}{\alpha}\Big]^{\beta_2-1}f(s)\,d_\psi s\,d_\psi t$$

$$= \frac{1}{\Gamma(\beta_1)\,\Gamma(\beta_2)}\int_a^x\int_s^x \psi'(t)\,\psi'(s)\Big[\frac{\psi^\alpha(x)-\psi^\alpha(t)}{\alpha}\Big]^{\beta_1-1}\Big[\frac{\psi^\alpha(t)-\psi^\alpha(s)}{\alpha}\Big]^{\beta_2-1}f(s)\,d_\psi t\,d_\psi s.$$

By changing the variable $\tau = \dfrac{\psi^\alpha(t)-\psi^\alpha(s)}{\psi^\alpha(x)-\psi^\alpha(s)}$, we obtain

$$ {}^{\beta_1}_{\mathcal{CF}}I^{\psi}_{a^+}\left({}^{\beta_2}_{\mathcal{CF}}I^{\psi}_{a^+}f(x)\right)$$

$$= \frac{1}{\Gamma(\beta_1)\,\Gamma(\beta_2)}\int_a^x\left(\int_0^1 \tau^{\beta_2-1}(1-\tau)^{\beta_1-1}\,d\tau\right)\psi'(s)\Big[\frac{(\psi^\alpha(x)-\psi^\alpha(s))}{\alpha}\Big]^{\beta_1+\beta_2-1}f(s)\,d_\psi s$$

$$= \frac{B(\beta_1,\beta_1)}{\Gamma(\beta 1)\,\Gamma(\beta_2)}\int_a^x \psi'(s)\Big[\frac{(\psi^\alpha(x)-\psi^\alpha(s))}{\alpha}\Big]^{\beta_1+\beta_2-1}f(s)\,d_\psi s$$

$$= \frac{1}{\Gamma(\beta 1+\beta 2)}\int_a^x \psi'(s)\Big[\frac{(\psi^\alpha(x)-\psi^\alpha(s))}{\alpha}\Big]^{\beta 1+\beta 2-1}f(s)\,d_\psi s$$

$$= {}^{\psi}_{\mathcal{CF}}I^{\beta 1+\beta 2}_{a^+}f(x).$$

The result (17) can be proved similarly. □

2.3. Generalized ψ-conformable fractional derivative

Definition 2.11. *(ψ-conformable fractional derivative) The generalized left and right ψ-conformable fractional derivative operators of order $\beta > 0$, of a function f are defined by*

- *If $\beta = n \in \mathbb{N}^*$ then*

$$ {}^{\psi}_{\mathcal{CF}}D^{n}_{a^+}f(x) = \left(\frac{\psi^{1-\alpha}(x)}{\psi'(x)}\frac{d}{dx}\right)^n f(x) \qquad \text{and} \qquad {}^{\psi}_{\mathcal{CF}}D^{n}_{b^-}f(x) = \left(-\frac{\psi^{1-\alpha}(x)}{\psi'(x)}\frac{d}{dx}\right)^n f(x).$$

- *If $n-1 < \beta < n$, $n \in \mathbb{N}^*$ then*

$$\begin{aligned} {}^{\psi}_{\mathcal{CF}}D^{\beta}_{a^+}f(x) &= \left(\tfrac{\psi^{1-\alpha}(x)}{\psi'(x)}\tfrac{d}{dx}\right)^n\left({}^{\psi}_{\mathcal{CF}}I^{n-\beta}_{a^+}f\right)(x) \\ &= \frac{1}{\Gamma(n-\beta)}\left(\frac{\psi^{1-\alpha}(x)}{\psi'(x)}\frac{d}{dx}\right)^n\int_a^x \psi'(t)\Big[\frac{\psi^\alpha(x)-\psi^\alpha(t)}{\alpha}\Big]^{n-\beta-1}f(t)\,d_\psi t, \end{aligned} \tag{18}$$

and

$$\begin{aligned} {}^{\psi}_{C\mathcal{F}}D^{\beta}_{b^-}f(x) &= \left(-\tfrac{\psi^{1-\alpha}(x)}{\psi'(x)}\ \tfrac{d}{dx}\right)^n \left({}^{\psi}_{C\mathcal{F}}I^{n-\beta}_{b^-}f\right)(x) \\ &= \frac{1}{\Gamma(n-\beta)}\left(-\frac{\psi^{1-\alpha}(x)}{\psi'(x)}\ \frac{d}{dx}\right)^n \int_x^b \psi'(t)\left[\frac{\psi^{\alpha}(t)-\psi^{\alpha}(x)}{\alpha}\right]^{n-\beta-1} f(t)\, d_{\psi}\, t. \end{aligned} \tag{19}$$

3. Hermite-Hadamard inequality

Now, we give a version of Hermite-Hadamard inequality involving the generalized ψ-conformable fractional integral.

Theorem 3.1. *Let $f \in X_{\psi}([a,b])$ be a convex function and $\beta > 0$, then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)}{4\,\Omega(\alpha,\psi)}\left[{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a) \ + \ {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right] \leq \frac{f(a)+f(b)}{2}, \tag{20}$$

where

$$F(s) = f(s) + f(a+b-s), \tag{21}$$

and $\Omega(\alpha,\psi) = \left(\dfrac{\psi^{\alpha}(b)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta}.$

Proof. For any $s \in [a,b]$, we have

$$2\,f\left(\frac{a+b}{2}\right) = 2\,f\left(\frac{a+b-s}{2}+\frac{s}{2}\right) \leq f(a+b-s) + f(s),$$

then

$$2\,f\left(\frac{a+b}{2}\right) \leq \tilde{f}(s) + f(s), \tag{22}$$

where

$$\tilde{f}(s) = f(a+b-s).$$

Multiplying (22) by $\left(\dfrac{\psi^{\alpha}(b)-\psi^{\alpha}(s)}{\alpha}\right)^{\beta-1}\psi'(s)\,\psi^{\alpha-1}(s)$ and integrating over $s \in [a,b]$, we result

$$f\left(\frac{a+b}{2}\right)\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta} \leq \frac{\Gamma(\beta+1)}{2}\ {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b). \tag{23}$$

Multiplying (22) by $\left(\dfrac{\psi^{\alpha}(s)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta-1}\psi'(s)\psi^{\alpha-1}(s)$ and integrating over $s \in [a,b]$, we get

$$f\left(\frac{a+b}{2}\right)\left(\frac{\psi^{\alpha}(b)-\psi^{\alpha}(a)}{\alpha}\right)^{\beta} \leq \frac{\Gamma(\beta+1)}{2}\ {}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a). \tag{24}$$

By adding the inequalities (23) and (24), we deduce

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)}{4\,\Omega(\alpha,\psi)}\left[{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a) \ + \ {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right]. \tag{25}$$

To prove the second part of inequality (20), putting $s = (1-t)\,a + t\,b$ and by using the convexity of f, we get

$$\tilde{f}\,(s) + f\,(s) \leq \; f\,(a) + f\,(b)\,. \tag{26}$$

Doing the same calculus as above with (26) gives us

$$\frac{\Gamma(\beta+1)}{4\,\Omega(\alpha,\psi)}\left[{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a) \; + \; {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right] \leq \; \frac{f\,(a)+f\,(b)}{2}. \tag{27}$$

By combining the inequality (25) and the inequality (27), we obtain the desired result. □

We suggest some new results for special choices of α and ψ.

Remark 3.2. *Under the hypothesis of Theorem (3.1), taking $\alpha = 1$ we deduce the following Hermite-Hadamard inequality related to the ψ-Hilfer fractional operators ${}^{\psi}\mathcal{I}^{\beta}_{a^+}f(x)$ and ${}^{\psi}\mathcal{I}^{\beta}_{b^-}f(x)$*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\beta+1)}{4\,(\psi(b)-\psi(a))^{\beta}}\left[{}^{\psi}\mathcal{I}^{\beta}_{b^-}F(a) + {}^{\psi}\mathcal{I}^{\beta}_{a^+}F(b)\right] \leq \frac{f(a)+f(b)}{2},$$

which proved by Jleli and Samet in [1].

The following results are obtained under the hypothesis of Theorem (3.1) and dependent on the function ψ given.

Corollary 3.3. *Under the assumptions of Theorem (3.1);*

1. *Putting $\psi(s) = s$, we result Hermite-Hadamard inequality related to Katugompola operators ${}^{\alpha}\mathcal{I}^{\beta}_{a^+}f(x)$ and ${}^{\alpha}\mathcal{I}^{\beta,\alpha}_{b^-}f(x)$*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\alpha^{\beta}\Gamma(\beta+1)}{4(b^{\alpha}-a^{\alpha})^{\beta}}\left[{}^{\alpha}\mathcal{I}^{\beta}_{b^-}F(a) + {}^{\alpha}\mathcal{I}^{\beta}_{a^+}F(b)\right] \leq \frac{f(a)+f(b)}{2}.$$

   Particularly, by $\alpha = 1$, we obtain Hermite-Hadamard inequality via the fractional Riemann-Liouville operators proved by Sarikaya et al. in [5].

2. *Setting $\psi(t) = \ln t$, we get Hermite-Hadamard inequality related to α-Hadamard operators*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\alpha^{\beta}\Gamma(\beta+1)}{4\;(\ln^{\alpha} b - \ln^{\alpha} a)^{\beta}}\left[{}^{\alpha}\mathcal{H}^{\beta}_{b^-}F(a) + {}^{\alpha}\mathcal{H}^{\beta}_{a^+}F(b)\right] \leq \frac{f(a)+f(b)}{2}.$$

   If $\alpha = 1$, we obtain Hermite-Hadamard inequality involving Hadamard operators.

3.1. Trapezoid Type Inequalities

In this section, we describe certain trapezoid type inequalities using ψ-conformable fractional integral operators, as well as their distinctive results. To do this, we first provide an equality in the following Lemma.

Lemma 3.4. *Assume that α, β, ψ are defined as in Theorem 3.1 and let $f : [a,b] \to \mathbb{R}$ is a differentiable mapping on (a,b), then the following identity holds*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a) + {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right] \\ &= \frac{b-a}{4\,\Omega(\psi,\alpha)}\int_0^1 \left(2\,\Omega(\psi,\alpha) - A_{\psi,\alpha}(s)\right)\left[f'\,(sa+(1-s)b) - f'\,((1-s)a+sb)\right]ds, \end{aligned} \tag{28}$$

where

$$A_{\psi,\alpha}(s) = \left(\frac{\psi^{\alpha}(b) - \psi^{\alpha}\,(sa+(1-s)b)}{\alpha}\right)^{\beta} + \left(\frac{\psi^{\alpha}\,((1-s)a+sb) - \psi^{\alpha}(a)}{\alpha}\right)^{\beta}. \tag{29}$$

Proof. Let

$$J_1 = \int_a^b \left[\left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha} \right)^\beta - \left(\frac{\psi^\alpha(b) - \psi^\alpha(t)}{\alpha} \right)^\beta \right] F'(t) dt. \tag{30}$$

Integrating by parts (30) and using (21), we get

$$J_1 = \left[\left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha} \right)^\beta - \left(\frac{\psi^\alpha(b) - \psi^\alpha(t)}{\alpha} \right)^\beta \right] F(t) \Big|_a^b - \beta \int_a^b \left(\frac{\psi^\alpha(b) - \psi^\alpha(t)}{\alpha} \right)^{\beta-1} \psi'(t) F(t) d_\psi t,$$

therefore

$$J_1 = \left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha} \right)^\beta F(b) - \Gamma(\beta+1) \ {}^{\psi}_{C\mathcal{F}}I^\beta_{a^+} F(b). \tag{31}$$

Similarly, let

$$J_2 = \int_a^b \left[\left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha} \right)^\beta - \left(\frac{\psi^\alpha(t) - \psi^\alpha(a)}{\alpha} \right)^\beta \right] F'(t) dt. \tag{32}$$

Integrating by parts (32) , we deduce

$$J_2 = -\left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha} \right)^\beta F(a) + \Gamma(\beta+1) \ {}^{\psi}_{C\mathcal{F}}I^\beta_{b^-} F(a). \tag{33}$$

Since $F(a) = F(b) = f(a) + f(b)$, from (31) and (33), we obtain

$$J_1 - J_2 = 2\,\Omega(\psi,\alpha)\,(f(a) + f(b)) - \Gamma(\beta+1) \left[{}^{\psi}_{C\mathcal{F}}I^\beta_{b^-} F(a) + {}^{\psi}_{C\mathcal{F}}I^\beta_{a^+} F(b) \right],$$

thus

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)} \left[{}^{\psi}_{C\mathcal{F}}I^\beta_{b^-} F(a) + {}^{\psi}_{C\mathcal{F}}I^\beta_{a^+} F(b) \right] = \frac{1}{4\,\Omega(\psi,\alpha)} \left(J_1 - J_2 \right). \tag{34}$$

On the other hand, given that $F'(t) = f'(t) - f'(a+b-t)$, from (30), we get

$$J_1 = \int_a^b \left[\left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha} \right)^\beta - \left(\frac{\psi^\alpha(b) - \psi^\alpha(t)}{\alpha} \right)^\beta \right] \left(f'(t) - f'(a+b-t) \right) dt.$$

By changing the variable $t = sa + (1-s)b$ for $s \in [0,1]$, we obtain

$$\begin{aligned} J_1 = (b-a) \int_0^1 & \left[\left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha} \right)^\beta - \left(\frac{\psi^\alpha(b) - \psi^\alpha(sa+(1-s)b)}{\alpha} \right)^\beta \right] \\ & \times \left[f'\left(sa+(1-s)b\right) - f'\left((1-s)a+sb\right) \right] ds. \end{aligned}$$

Similarly, from (32), we get

$$\begin{aligned} J_2 &= \int_a^b \left[\left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha} \right)^\beta - \left(\frac{\psi^\alpha(t) - \psi^\alpha(a)}{\alpha} \right)^\beta \right] \left(f'(t) - f'(a+b-t) \right) dt \\ &= (b-a) \int_0^1 \left[\left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha} \right)^\beta - \left(\frac{\psi^\alpha((1-s)a+sb) - \psi^\alpha(a)}{\alpha} \right)^\beta \right] \\ &\qquad \times \left[f'\left((1-s)a+sb\right) - f'\left(sa+(1-s)b\right) \right] ds. \end{aligned}$$

As a result,

$$J_1 - J_2 = (b-a)\int_0^1 \Big(2\,\Omega(\psi,\alpha) - A_{\psi,\alpha}(s)\Big)\big[f'\left(sa+(1-s)b\right) - f'\left((1-s)a+sb\right)\big]\,ds. \tag{35}$$

The desired equality (28) is attained by replacing (35) in (34). □

Theorem 3.5. *Assume that α,β,ψ are defined as in Lemma 3.4. If $|f'|$ is a convex mapping on $[a,b]$, then the trapezoid type inequality is obtained as*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[\,{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a) + {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right]\right|\\ &\leq \frac{b-a}{4\,\Omega(\psi,\alpha)}\Big[|f'(a)|+|f'(b)|\Big]\int_0^1 \Big|2\,\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\Big|ds.\end{aligned} \tag{36}$$

Proof. Using the absolute value of identity (28) and the convexity of the function $|f'|$, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[\,{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a) + {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right]\right|$$

$$\leq \frac{b-a}{4\,\Omega(\psi,\alpha)}\int_0^1 \Big|2\,\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\Big|\Big|f'\left(sa+(1-s)b\right) - f'\left((1-s)a+sb\right)\Big|ds$$

$$\leq \frac{b-a}{4\,\Omega(\psi,\alpha)}\int_0^1 \Big|2\,\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\Big|\Big[s\Big|f'(a)\Big| + (1-s)\Big|f'(b)\Big| + (1-s)\Big|f'(a)\Big| + s\Big|f'(b)\Big|\Big]\,ds$$

$$= \frac{b-a}{4\,\Omega(\psi,\alpha)}\Big[|f'(a)|+|f'(b)|\Big]\int_0^1 \Big|2\,\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\Big|ds.$$

This end the proof. □

For $\alpha = 1$, we get the following Corollary.

Corollary 3.6. *With the hypothesis of Theorem (3.5), we possess*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)}{4\,\Omega(\psi)}\left[\,{}^{\psi}\mathcal{I}^{\beta}_{b^-}F(a) + {}^{\psi}\mathcal{I}^{\beta}_{a^+}F(b)\right]\right|$$

$$\leq \frac{b-a}{4\,\Omega(\psi)}\Big[|f'(a)|+|f'(b)|\Big]\int_0^1 \Big|2\,\Omega(\psi)-A_{\psi}(s)\Big|ds,$$

where

$$A_{\psi}(s) = \left(\psi(b) - \psi\left(sa+(1-s)b\right)\right)^{\beta} + \left(\psi\left((1-s)a+sb\right) - \psi(a)\right)^{\beta}, \tag{37}$$

and

$$\Omega(\psi) = \left(\psi(b)-\psi(a)\right)^{\beta}. \tag{38}$$

It is proved in Theorem 2.5 of [1].

Depending on the choice of the function ψ we obtain the following Corollary.

Corollary 3.7. 1. *Taking $\psi(s) = s$, we get*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)}{4\,\Omega_1(\alpha)}\left[\,{}^{\alpha}\mathcal{I}^{\beta}_{b^-}F(a) + {}^{\alpha}\mathcal{I}^{\beta}_{b^+}F(b)\right]\right| \leq \frac{b-a}{4\,\Omega_1(\alpha)}\Big[|f'(a)|+|f'(b)|\Big]\int_0^1 \Big|2\,\Omega_1(\alpha) - A_{1,\alpha}(s)\Big|ds, \tag{39}$$

where

$$A_{1,\alpha}(s) = \left(\frac{b^{\alpha}-(sa+(1-s)b)^{\alpha}}{\alpha}\right)^{\beta} + \left(\frac{((1-s)a+sb)^{\alpha}-a^{\alpha}}{\alpha}\right)^{\beta}, \tag{40}$$

and

$$\Omega_1(\alpha) = \left(\frac{b^{\alpha}-a^{\alpha}}{\alpha}\right)^{\beta}. \tag{41}$$

If $\alpha = 1$ in (39)*, we result Corollary 2.6 of [1].*

2. *Choose $\psi(s) = \ln s$, we obtain*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)}{4\,\Omega_2(\alpha)}\left[\,{}^{\alpha}\mathcal{H}^{\beta}_{b^-}F(a) + {}^{\alpha}\mathcal{H}^{\beta}_{a^+}F(b)\right]\right| \leq \frac{b-a}{4\,\Omega_2(\alpha)}\Big[|f'(a)|+|f'(b)|\Big]\int_0^1 \Big|2\,\Omega_2(\alpha) - A_{2,\alpha}(s)\Big|ds, \tag{42}$$

where

$$A_{2,\alpha}(s) = \left(\frac{\ln^{\alpha}(b)-\ln^{\alpha}(sa+(1-s)b)}{\alpha}\right)^{\beta} + \left(\frac{\ln^{\alpha}((1-s)a+sb)-\ln^{\alpha}(a)}{\alpha}\right)^{\beta}, \tag{43}$$

and

$$\Omega_2(\alpha) = \left(\frac{\ln^{\alpha}(b)-\ln^{\alpha}(a)}{\alpha}\right)^{\beta}. \tag{44}$$

If $\alpha = 1$ in (42)*, we result Corollary 2.7 of [1].*

Theorem 3.8. *Let $p > 1$ and $\frac{1}{p'} + \frac{1}{p} = 1$. Assume that α, β, ψ are defined as in Lemma 3.4. If $\left|f'\right|^{p}$, is a convex mapping on $[a,b]$, then the following trapezoid type inequality is obtained*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[\,{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a) + {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right]\right| \\
&\leq \frac{b-a}{4\,\Omega(\psi,\alpha)}\left(2\int_0^1 \left|2\,\Omega(\psi,\alpha) - A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|^{p} + \left|f'(b)\right|^{p}\right)^{\frac{1}{p}} \\
&\leq \frac{b-a}{4\,\Omega(\psi,\alpha)}\left(2\int_0^1 \left|2\,\Omega(\psi,\alpha) - A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right| + \left|f'(b)\right|\right).
\end{aligned} \tag{45}$$

Proof. By Lemma (3.4), using the absolute value

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a)+{}^{\alpha\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right]\right|$$

$$\leq\frac{b-a}{4\,\Omega(\psi,\alpha)}\int_0^1\left|2\,\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|\left|f'\left(sa+(1-s)b\right)\right|ds$$

$$+\frac{b-a}{4\,\Omega(\psi,\alpha)}\int_0^1\left|2\,\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|\left|f'\left((1-s)a+sb\right)\right|ds,$$

Hölder's inequality yields

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a)+{}^{\alpha\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right]\right|$$

$$\leq\frac{b-a}{4\,\Omega(\psi,\alpha)}\left(\int_0^1\left|2\,\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\times\left(\int_0^1\left|f'\left(sa+(1-s)b\right)\right|^{p}ds\right)^{\frac{1}{p}}$$

$$+\frac{b-a}{4\,\Omega(\psi,\alpha)}\left(\int_0^1\left|2\,\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\times\left(\int_0^1\left|f'\left((1-s)a+sb\right)\right|^{p}ds\right)^{\frac{1}{p}}.$$

Since $\left|f'\right|^p$ is a convex function, we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a)+{}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right]\right|$$

$$\leq\frac{b-a}{4\,\Omega(\psi,\alpha)}\left(\int_0^1\left|2\,\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}2^{1-\frac{1}{p}}\left(\left|f'(a)\right|^p+\left|f'(b)\right|^p\right)^{\frac{1}{p}}$$

$$=\frac{b-a}{4\,\Omega(\psi,\alpha)}\left(2\int_0^1\left|2\,\Omega(\psi,\alpha)-A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|^p+\left|f'(b)\right|^p\right)^{\frac{1}{p}}.$$

This accomplishes the first inequality in (45). Notice that $A^p+B^p\leq(A+B)^p$ for $p>1$, it results the second inequality in (45). □

3.2. Midpoint Type Inequalities

In this section, a midpoint type inequality involving ψ-conformable fractional integral operators is established using the identity stated in the following Lemma.

Lemma 3.9. *Under the hypothesis of Lemma 3.4, the following identity holds*

$$\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a)+{}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)$$

$$=\frac{b-a}{4\,\Omega(\psi,\alpha)}\int_0^1A_{\psi,\alpha}(s)\left[f'\Big(sa+(1-s)b\Big)-f'\Big((1-s)a+sb\Big)\right]ds \tag{46}$$

$$-\frac{b-a}{2}\int_{\frac{1}{2}}^1\left[f'\Big(sa+(1-s)b\Big)-f'\Big((1-s)a+sb\Big)\right]ds.$$

Proof. Let

$$K_1 = -\int_a^b \left(\frac{\psi^\alpha(b) - \psi^\alpha(t)}{\alpha}\right)^\beta F'(t)dt + \left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha}\right)^\beta \int_a^{\frac{a+b}{2}} F'(t)dt. \tag{47}$$

Integrating by parts (47), we get

$$\begin{aligned} K_1 &= -\left(\frac{\psi^\alpha(b) - \psi^\alpha(t)}{\alpha}\right)^\beta F(t)\Big|_a^b - \beta \int_a^b \left(\frac{\psi^\alpha(b) - \psi^\alpha(t)}{\alpha}\right)^{\beta-1} \psi'^{\alpha-1}(t)\, F(t)dt \\ &+\left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha}\right)^\beta F(t)\Big|_a^{\frac{a+b}{2}}, \end{aligned}$$

which gives

$$K_1 = \left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha}\right)^\beta F\left(\frac{a+b}{2}\right) - \Gamma(\beta+1)\ {}^{\psi}_{C\mathcal{F}}I^\beta_{a^+}F(b). \tag{48}$$

Similarly, consider

$$K_2 = \int_a^b \left(\frac{\psi^\alpha(t) - \psi^\alpha(a)}{\alpha}\right)^\beta F'(t)dt - \left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha}\right)^\beta \int_{\frac{a+b}{2}}^b F'(t)dt. \tag{49}$$

Integrating by parts yields

$$\begin{aligned} K_2 &= \left(\frac{\psi^\alpha(t) - \psi^\alpha(a)}{\alpha}\right)^\beta F(t)\Big|_a^b - \beta \int_a^b \left(\frac{\psi^\alpha(t) - \psi^\alpha(a)}{\alpha}\right)^{\beta-1} \psi'^{\alpha-1}(t)\, F(t)dt \\ &-\left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha}\right)^\beta F(t)\Big|_{\frac{a+b}{2}}^b, \end{aligned}$$

which gives

$$K_2 = \left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha}\right)^\beta F\left(\frac{a+b}{2}\right) - \Gamma(\beta+1)\ {}^{\psi}_{C\mathcal{F}}I^\beta_{b^-}F(a). \tag{50}$$

Adding (48) and (50), then using (21) we result

$$K_1 + K_2 = 4\,\Omega(\psi,\alpha)\, f(\tfrac{a+b}{2}) - \Gamma(\beta+1)\left[\ {}^{\psi}_{C\mathcal{F}}I^\beta_{b^-}F(a) + \ {}^{\psi}_{C\mathcal{F}}I^\beta_{a^+}F(b)\right]. \tag{51}$$

Moreover, by changing the variable $t = sa + (1-s)b$ in (47) and using (21), we get

$$\begin{aligned} K_1 &= -(b-a)\int_0^1 \left(\frac{\psi^\alpha(b) - \psi^\alpha(sa+(1-s)b)}{\alpha}\right)^\beta \left[f'\left(sa+(1-s)b\right) - f'\left((1-s)a+sb\right)\right] ds \\ &+(b-a)\left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha}\right)^\beta \int_{\frac{1}{2}}^1 \left[f'\left(sa+(1-s)b\right) - f'\left((1-s)a+sb\right)\right] ds. \end{aligned}$$

Changing the variable $t = (1-s)a + sb$ in (49) and using (21), we obtain

$$\begin{aligned} K_2 &= (b-a)\int_0^1 \left(\frac{\psi^\alpha((1-s)a+sb) - \psi^\alpha(a)}{\alpha}\right)^\beta \left[f'\left((1-s)a+sb\right) - f'\left(sa+(1-s)b\right)\right] ds \\ &-(b-a)\left(\frac{\psi^\alpha(b) - \psi^\alpha(a)}{\alpha}\right)^\beta \int_{\frac{1}{2}}^1 \left[f'\left((1-s)a+sb\right) - f'\left(sa+(1-s)b\right)\right] ds. \end{aligned}$$

Thus

$$K_1 + K_2 = -(b-a)\int_0^1 A_{\psi,\alpha}(s)\left[f'(sa+(1-s)b) - f'((1-s)a+sb)\right]ds$$
$$+2(b-a)\,\Omega(\psi,\alpha)\int_{\frac{1}{2}}^1 \left[f'(sa+(1-s)b) - f'((1-s)a+sb)\right]ds, \tag{52}$$

replacing (52) in (51) we get the desired equality (46). □

Theorem 3.10. *Assume that α, β, ψ are defined as in Lemma 3.4. If $|f'|$ is a convex mapping on $[a,b]$, then the midpoint type inequality is obtained as*

$$\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a) + {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right] - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(1+\frac{1}{\Omega(\psi,\alpha)}\int_0^1\left|A_{\psi,\alpha}(s)\right|ds\right)\left[|f'(a)|+|f'(b)|\right]. \tag{53}$$

Proof. Using the absolute value of identity (46) and the convexity of $|f'|$ function, we obtain

$$\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[{}^{\psi}_{C\mathcal{F}}I^{\beta}_{b^-}F(a) + {}^{\psi}_{C\mathcal{F}}I^{\beta}_{a^+}F(b)\right] - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4\,\Omega(\psi,\alpha)}\int_0^1 \left|A_{\psi,\alpha}(s)\right|\left[\left|f'\big(sa+(1-s)b\big)\right| + \left|f'\big((1-s)a+sb\big)\right|\right]ds$$
$$+\frac{b-a}{2}\int_{\frac{1}{2}}^1\left[\left|f'\big(sa+(1-s)b\big)\right| + \left|f'\big((1-s)a+sb\big)\right|\right]ds$$
$$\leq \frac{b-a}{4}\left(1+\frac{1}{\Omega(\psi,\alpha)}\int_0^1\left|A_{\psi,\alpha}(s)\right|ds\right)\left[|f'(a)|+|f'(b)|\right].$$

This end the proof. □

For $\alpha = 1$, we get the below new result.

Corollary 3.11. *With the hypothesis of Theorem (3.10), we possess*

$$\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi)}\left[{}^{\psi}\mathcal{I}^{\beta}_{b^-}F(a) + {}^{\psi}\mathcal{I}^{\beta}_{a^+}F(b)\right] - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(1+\frac{1}{\Omega(\psi)}\int_0^1\left|A_{\psi}(s)\right|ds\right)\left[|f'(a)|+|f'(b)|\right].$$

where $A_\psi(s)$ and $\Omega(\psi)$ are defined by (37) *and* (38)§.

The following new results are obtained according to the function ψ used.

Corollary 3.12. 1. *Taking $\psi(s) = s$, we get*

$$\left|\frac{\Gamma(\beta+1)}{4\,\Omega_1(\alpha)}\left[{}^{\alpha}\mathcal{I}^{\beta}_{b^-}F(a) + {}^{\alpha}\mathcal{I}^{\beta}_{a^+}F(b)\right] - f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(1+\frac{1}{\Omega_1(\alpha)}\int_0^1\left|A_{1,\alpha}(s)\right|ds\right)\left[|f'(a)|+|f'(b)|\right]. \tag{54}$$

where $A_{1,\alpha}(s)$ *and* $\Omega_1(\alpha)$ *are defined by* (40) *and* (41)§.

If $\alpha = 1$ *in* (54), *we get*

$$\left|\frac{\Gamma(\beta+1)}{4\,(b-a)^{\beta}}\left[\mathcal{I}^{\beta}_{b^-}F(a)+\mathcal{I}^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\right|\le\frac{(b-a)}{4}\left(1+\frac{2}{\beta+1}\right)\left[|f'(a)|+|f'(b)|\right].$$

2. *Choose* $\psi(s)=\ln s$, *we obtain*

$$\begin{aligned}&\left|\frac{\Gamma(\beta+1)}{4\,\Omega_2(\alpha)}\left[\,{}^{\alpha}\mathcal{H}^{\beta}_{b^-}F(a)+{}^{\alpha}\mathcal{H}^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{b-a}{4}\left(1+\frac{1}{\Omega_2(\alpha)}\int_0^1\left|A_{2,\alpha}(s)\right|ds\right)\left[|f'(a)|+|f'(b)|\right].\end{aligned}\tag{55}$$

where $A_{2,\alpha}(s)$ *and* $\Omega_2(\alpha)$ *are defined by* (43) *and* (44)§.

If $\alpha = 1$ *in* (55), *we get*

$$\begin{aligned}&\left|\frac{\Gamma(\beta+1)}{4\,\Omega_2(1)}\left[\mathcal{H}^{\beta}_{b^-}F(a)+\mathcal{H}^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{b-a}{4}\left(1+\frac{1}{\Omega_2(1)}\int_0^1\left|A_{2,1}(s)\right|ds\right)\left[|f'(a)|+|f'(b)|\right].\end{aligned}$$

Theorem 3.13. *Let* $p>1$ *and* $\frac{1}{p'}+\frac{1}{p}=1$. *Assume that* α,β,ψ *are defined as in Lemma 3.9. If* $\left|f'\right|^{p}$, *is a convex mapping on* $[a,b]$, *then the following trapezoid type inequality is obtained*

$$\begin{aligned}&\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[\,{}^{\psi}_{\mathcal{CF}}I^{\beta}_{b^-}F(a)+{}^{\psi}_{\mathcal{CF}}I^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{b-a}{4\,\Omega(\psi,\alpha)}\left(2\int_0^1\left|A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|^{p}+\left|f'(b)\right|^{p}\right)^{\frac{1}{p}}\\&+\frac{b-a}{4}\left[\left(\frac{\left|f'(a)\right|^{p}+3\left|f'(b)\right|^{p}}{4}\right)^{\frac{1}{p}}+\left(\frac{3\left|f'(a)\right|^{p}+\left|f'(b)\right|^{p}}{4}\right)^{\frac{1}{p}}\right]\\&\le\frac{b-a}{2}\left(2+\frac{1}{\Omega^{p'}(\psi,\alpha)}\int_0^1\left|A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right).\end{aligned}\tag{56}$$

Proof. Applying Lemma (3.4) and the absolute value, we get

$$\begin{aligned}&\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[\,{}^{\psi}_{\mathcal{CF}}I^{\beta}_{b^-}F(a)+{}^{\psi}_{\mathcal{CF}}I^{\beta}_{a^+}F(b)\right]-f\left(\frac{a+b}{2}\right)\right|\\&\le\frac{b-a}{4\,\Omega(\psi,\alpha)}\int_0^1\left|A_{\psi,\alpha}(s)\right|\left[\left|f'\big(sa+(1-s)b\big)\right|+\left|f'\big((1-s)a+sb\big)\right|\right]ds\\&+\frac{b-a}{2}\int_{\frac{1}{2}}^1\left[\left|f'\big(sa+(1-s)b\big)\right|+\left|f'\big((1-s)a+sb\big)\right|\right]ds,\end{aligned}$$

Hölder's inequality gives us

$$\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[{}_{C\mathcal{F}}^{\;\psi}I_{b^-}^{\beta}F(a)+{}_{C\mathcal{F}}^{\;\psi}I_{a^+}^{\beta}F(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{4\,\Omega(\psi,\alpha)}\left(\int_0^1\left|A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left[\left(\int_0^1\left|f'\left(sa+(1-s)b\right)\right|^p ds\right)^{\frac{1}{p}}+\left(\int_0^1\left|f'\left((1-s)a+sb\right)\right|^p ds\right)^{\frac{1}{p}}\right]$$

$$+\frac{b-a}{2}\left(\int_{\frac{1}{2}}^1 ds\right)^{\frac{1}{p'}}\left[\left(\int_{\frac{1}{2}}^1\left|f'\left(sa+(1-s)b\right)\right|^p ds\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^1\left|f'\left((1-s)a+sb\right)\right|^p ds\right)^{\frac{1}{p}}\right].$$

Given that $\left|f'\right|^p$ is a convex function, we obtain

$$\left|\frac{\Gamma(\beta+1)}{4\,\Omega(\psi,\alpha)}\left[{}_{C\mathcal{F}}^{\;\psi}I_{b^-}^{\beta}F(a)+{}_{C\mathcal{F}}^{\;\psi}I_{a^+}^{\beta}F(b)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{4\,\Omega(\psi,\alpha)}\left(\int_0^1\left|A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}2^{1-\frac{1}{p}}\left(\left|f'(a)\right|^p+\left|f'(b)\right|^p\right)^{\frac{1}{p}}$$

$$+\frac{b-a}{2}\left(\int_{\frac{1}{2}}^1 ds\right)^{\frac{1}{p'}}\left(\frac{1}{2}\right)^{\frac{1}{p}}\left[\left(\frac{\left|f'(a)\right|^p+3\left|f'(b)\right|^p}{4}\right)^{\frac{1}{p}}+\left(\frac{3\left|f'(a)\right|^p+\left|f'(b)\right|^p}{4}\right)^{\frac{1}{p}}\right]$$

$$=\frac{b-a}{4\,\Omega(\psi,\alpha)}\left(2\int_0^1\left|A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|^p+\left|f'(b)\right|^p\right)^{\frac{1}{p}}$$

$$+\frac{b-a}{4}\left[\left(\frac{\left|f'(a)\right|^p+3\left|f'(b)\right|^p}{4}\right)^{\frac{1}{p}}+\left(\frac{3\left|f'(a)\right|^p+\left|f'(b)\right|^p}{4}\right)^{\frac{1}{p}}\right].$$

This accomplishes the first inequality in (45). For the second inequality, notice that for p, $p'>1$

$$A^p+B^p\leq(A+B)^p,\ 1+3^{\frac{1}{p}}\leq 4 \text{ and } A^{\frac{1}{p'}}+B^{\frac{1}{p'}}\leq 2^{1-\frac{1}{p'}}(A+B)^{\frac{1}{p'}},$$

then

$$\frac{b-a}{4\,\Omega(\psi,\alpha)}\left(2\int_0^1 \left|A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|^p+\left|f'(b)\right|^p\right)^{\frac{1}{p}}$$

$$+\frac{b-a}{4}\left[\left(\frac{\left|f'(a)\right|^p+3\left|f'(b)\right|^p}{4}\right)^{\frac{1}{p}}+\left(\frac{3\left|f'(a)\right|^p+\left|f'(b)\right|^p}{4}\right)^{\frac{1}{p}}\right]$$

$$\leq \frac{b-a}{4\,\Omega(\psi,\alpha)}\left(2\int_0^1 \left|A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)+\frac{b-a}{4}\left[4^{1-\frac{1}{p}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)\right]$$

$$=\frac{b-a}{4}\left[4^{\frac{1}{p'}}+\left(\frac{2}{\Omega^{p'}(\psi,\alpha)}\int_0^1 \left|A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\right]\left(\left|f'(a)\right|+\left|f'(b)\right|\right)$$

$$\leq \frac{b-a}{4}\,2^{1-\frac{1}{p'}}\left(4+\frac{2}{\Omega^{p'}(\psi,\alpha)}\int_0^1 \left|A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)$$

$$=\frac{b-a}{2}\left(2+\frac{1}{\Omega^{p'}(\psi,\alpha)}\int_0^1 \left|A_{\psi,\alpha}(s)\right|^{p'}ds\right)^{\frac{1}{p'}}\left(\left|f'(a)\right|+\left|f'(b)\right|\right),$$

which yields to the second inequality in (45). □

4. Conclusion

In this paper, we introduced a novel set of fractional derivatives and integrals, namely the generalized ψ-conformable derivative and integral operators. We demonstrated that these operators extend various well-known fractional integral operators and explored their key properties, highlighting their semi-group property. Moreover, the application of these operators led to the derivation of a new Hermite-Hadamard-type inequality for convex functions. We also present some midpoint and trapezoid type inequalities

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