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A new two-step method with identical coefficient matrices for complex symmetric linear systems

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Abstract. A new two-step iteration method is constructed to solve complex symmetric linear systems inspired by the CRI method, and we call it as the ICCRI method by using two linear subsystems with identical coefficient matrices in each iteration. We present the elaborate discussion of the spectral radius of the iteration matrix for the ICCRI method, and obtain the quasi-optimal parameter. Particularly, the spectral radius of the iteration matrix is no more than 0.5 when the quasi-optimal parameter is used for the ICCRI method. Moreover, to make comparison with the CRI method, we also give detailed analyses on the quasi-optimal parameter and the corresponding convergence factor of the CRI method, which are not proved in the original article. Some numerical experiments are implemented and the results show that the new ICCRI method is more efficient than the PMHSS and the CRI methods.

1. Motivation and construction of the new ICCRI method

In this paper, we consider a class of complex symmetric linear systems which is given by

$$Ax := (W + iT)x = b, \tag{1}$$

with the imaginary unit $i = \sqrt{-1}$, the constant vector $b \in \mathbb{C}^n$ and the unknown $x \in \mathbb{C}^n$. Throughout the paper, assume that $W, T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices, and $W \in \mathbb{R}^{n\times n}$ is positive definite. Here we only study the case $T \neq 0$, otherwise the complex system (1) becomes a real symmetric linear system $Wx = b$. This kinds of systems are widely used in applications, see [1–5].

For solving the system (1) numerically, a well-known Hermitian and skew-Hermitian splitting (HSS) method was introduced in [6], and a modified HSS (MHSS) iteration algorithm was proposed in[2]. Moreover, a preconditioned MHSS (PMHSS) method was proposed in [7], which can be given as

$$\begin{cases} (\alpha V + W)x^{(k+\frac{1}{2})} = (\alpha V - iT)x^{(k)} + b, \\ (\alpha V + T)x^{(k+1)} = (\alpha V + iW)x^{(k+\frac{1}{2})} - ib, \end{cases} \tag{2}$$

where a constant $\alpha > 0$ is given and a prescribed matrix $V \in \mathbb{R}^{n\times n}$ is symmetric positive definite.

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Later, a mass of HSS-type methods were constructed one after another, which can be referred as lopsided version of PMHSS iteration algorithm [8], new HSS iteration algorithm [9], single-step HSS (SHSS) iteration algorithm [10], parameterized SHSS (PSHSS) iteration algorithm [11], the PSS and the BTSS methods in [12], and the QHSS method [13]. For more methods, the readers can refer to [14–20].

In addition, by letting $x = u + iv$ and $b = p + iq$ with the vectors $u, v, p, q \in \mathbb{R}^n$, then the complex linear system (1) can be transformed to the following form

$$\begin{bmatrix} W & -T \\ T & W \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}, \tag{3}$$

where $u, v \in \mathbb{R}^n$ are unknown real vectors. To get the numerical solution of (3), Salkuyeh et al. [21] used the efficient generalized successive overrelaxation (GSOR) method first constructed in [22]. Later, the symmetric SOR (SSOR) method [23], the preconditioned GSOR (PGSOR) method [24], the accelerated GSOR (AGSOR) method [25], and a preconditioned AGSOR (PAGSOR) method [26], were proposed one after another.

Recently, for solving the complex symmetric linear system (1), a scale-splitting (SCSP) method was proposed in [27], and it was extended to a two-step version in [28] named TSCSP, which is also called as the double-step scale splitting (DSS) method [29]. Later, the DSS method was generalized to a real-valued method [30], and the TSCSP method was extended to a two-parameter version (TTSCSP) in [31], which is also named the accelerated DSS (ADSS) method [32]. Furthermore, a modified TSCSP method was introduced in [33]. For more latest research results, please see [34–38].

Additionally, Wang et al.[39] proposed a combination method by using the real part and imaginary part of the coefficient matrix A, which is called CRI method and can be given by

$$\begin{cases} (\alpha T + W)x^{k+\frac{1}{2}} = (\alpha - i)Tx^k + b, \\ (\alpha W + T)x^{k+1} = (\alpha + i)Wx^{k+\frac{1}{2}} - ib, \end{cases} \tag{4}$$

where the parameter $\alpha > 0$. Wang et al.[39] proved that the CRI method is more efficient than the PMHSS method. Later, a generalised CRI (GCRI) method with two parameters

$$\begin{cases} (\alpha T + W)x^{k+\frac{1}{2}} = (\alpha - i)Tx^k + b, \\ (\beta W + T)x^{k+1} = (\beta + i)Wx^{k+\frac{1}{2}} - ib, \end{cases} \tag{5}$$

was introduced in [40], where α and β are given parameters. A modified CRI (MCRI) method with two parameters was proposed in [41], and the MCRI method is given by

$$\begin{cases} (\alpha T + W)x^{k+1} = (1 - \omega)(\alpha T + W)x^k + \omega(\alpha - i)Ty^k + \omega b, \\ (\alpha W + T)y^{k+1} = (1 - \omega)(\alpha W + T)y^k + \omega(\alpha + i)Wx^{k+1} - i\omega b, \end{cases} \tag{6}$$

where α and $\omega \in (0, 2)$ are two positive constants. A single step iterative scheme using the real part and imaginary part of the coefficient matrix A was proposed in [42], which is named SSRI method and can be given by

$$(\alpha T + W)x^{k+1} = (i\alpha + 1)Wx^k - i\alpha b, \tag{7}$$

with the parameter $\alpha > 0$.

Now, let us go back to the CRI method (4). Notice that when $\alpha \neq 1$, we must solve two linear subsystems with two different coefficient matrices $\alpha T + W$ and $\alpha W + T$ in each iteration by using the CRI method.

However, notice the fact that

$$(\alpha W + T)x = \alpha(-iTx + b) + Tx = (1 - \alpha i)Tx + \alpha b.$$

Then, we can establish the following so-called ICCRI iteration scheme, by solving two linear subsystems with identical coefficient matrices which are the combination of the real part and imaginary part of the coefficient matrix A.

The ICCRI Iteration method: Choose a starting value $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \ldots$ until the sequence of iterations $\{x^{(k)}\}$ converges, compute $x^{(k+1)}$ with the scheme:

$$\begin{cases}(\alpha W + T)x^{k+\frac{1}{2}} = (1-\alpha i)Tx^{(k)} + \alpha b,\\ (\alpha W + T)x^{k+1} = (\alpha + i)Wx^{k+\frac{1}{2}} - ib,\end{cases} \tag{8}$$

where the parameter $\alpha > 0$.

In the following Section 2, we are going to analyze the convergence properties of the new ICCRI method and discuss the quasi-optimal parameter α_*. In Section 3, we also explore the quasi-optimal parameter of the CRI method and then make a simple comparison between the CICRI and the CRI methods. In Section 4, several numerical tests are carried out to show the effectiveness of the ICCRI method. Finally, we make a brief conclusion of this work in Section 5.

2. Convergence properties and quasi-optimal parameter of the ICCRI iteration scheme

It is obvious that the ICCRI iteration scheme (8) can be rewritten as

$$x^{(k+1)} = M(\alpha)x^{(k)} + G^{-1}(\alpha)b, \quad k = 0, 1, 2, \ldots,$$

with

$$\begin{aligned} M(\alpha) &= [2\alpha + (1-\alpha^2)i](\alpha W + T)^{-1}W(\alpha W + T)^{-1}T \\ &= [2\alpha + (1-\alpha^2)i]W^{-\frac{1}{2}}\tilde{M}(\alpha)W^{\frac{1}{2}}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} G^{-1}(\alpha) &= (\alpha W + T)^{-1}[(\alpha^2 + \alpha i)W(\alpha W + T)^{-1} - iI] \\ &= (\alpha W + T)^{-1}(\alpha^2 W - iT)(\alpha W + T)^{-1}, \end{aligned}$$

where

$$\begin{aligned} \tilde{M}(\alpha) &= [W^{\frac{1}{2}}(\alpha W + T)^{-1}W^{\frac{1}{2}}]^2 W^{-\frac{1}{2}}TW^{-\frac{1}{2}} \\ &= (\alpha I + W^{-\frac{1}{2}}TW^{-\frac{1}{2}})^{-2}W^{-\frac{1}{2}}TW^{-\frac{1}{2}}. \end{aligned} \tag{10}$$

Let $H(\alpha) = G(\alpha) - A$, i.e.,

$$H(\alpha) = [2\alpha + (1-\alpha^2)i](\alpha W + T)(\alpha^2 W - iT)^{-1}W(\alpha W + T)^{-1}T,$$

then it is clear that the iteration matrix $M(\alpha) = G^{-1}(\alpha)H(\alpha)$ and

$$A = G(\alpha) - H(\alpha)$$

defines a splitting way of the coefficient matrix A. Therefore, we can use the splitting matrix

$$G(\alpha) = (\alpha W + T)(\alpha^2 W - iT)^{-1}(\alpha W + T)$$

as a preconditioner for the coefficient matrix $A \in \mathbb{C}^{n\times n}$.

It is clear that all the eigenvalues of the symmetric positive semi-definite matrix $W^{-\frac{1}{2}}TW^{-\frac{1}{2}}$ are nonnegative real numbers. For convenience of later studies, we use the following notations

$$T_W = W^{-\frac{1}{2}}TW^{-\frac{1}{2}} \quad \text{and} \quad \lambda_{max} = \max_{\lambda_j \in sp(T_W)} \{\lambda_j\}, \tag{11}$$

where $sp(B)$ represents the spectral set of matrix B. For the ICCRI iteration method, we obtain the following results.

Theorem 2.1. *Assume that $W \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $T \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite.*
(i) If $\lambda_{max} \geq \alpha$, then we have

$$\rho(M(\alpha)) \leq \delta_1(\alpha) := \frac{\alpha^2+1}{4\alpha}.$$

Particularly, if $2-\sqrt{3} < \alpha < 2+\sqrt{3}$ then $\delta_1(\alpha) < 1$, i.e., the ICCRI scheme is convergent.
(ii) If $\lambda_{max} \leq \alpha$, then we have

$$\rho(M(\alpha)) = \delta_2(\alpha) := \frac{(\alpha^2+1)\lambda_{max}}{(\alpha+\lambda_{max})^2}.$$

Particularly, if $2-\sqrt{3} < \alpha < 2+\sqrt{3}$ then $\delta_2(\alpha) < 1$; if $0 < \alpha \leq 2-\sqrt{3}$ or $\alpha \geq 2+\sqrt{3}$ then $\delta_2(\alpha) < 1$ provided that

$$\lambda_{max} < \frac{(\alpha-1)^2 - \sqrt{(\alpha-1)^4 - 4\alpha^2}}{2} =: h(\alpha). \tag{12}$$

Proof. From the relational expression (9), we have

$$\rho(M(\alpha)) = (\alpha^2+1)\cdot\rho(\tilde{M}(\alpha)) = (\alpha^2+1)\cdot \max_{\lambda_j\in sp(T_W)} \frac{\lambda_j}{(\alpha+\lambda_j)^2}.$$

Let

$$g(x) = \frac{x}{(\alpha+x)^2} \quad \text{for} \quad x \geq 0. \tag{13}$$

By direct calculation, we obtain the derivation

$$g'(x) = \frac{\alpha - x}{(\alpha+x)^3},$$

which implies that $g'(\alpha) = 0$, and the function $g(x)$ is monotonically increasing in $[0,\alpha]$ and decreasing in $[\alpha,+\infty)$.

(i) If $\lambda_{max} \geq \alpha$, then we have

$$\rho(M(\alpha)) \leq (\alpha^2+1)\cdot g(\alpha) = \frac{\alpha^2+1}{4\alpha}.$$

Moreover, it is easy to see that if $2-\sqrt{3} < \alpha < 2+\sqrt{3}$ then $\delta_1(\alpha) < 1$.

(ii) If $\lambda_{max} \leq \alpha$, then we have

$$\rho(M(\alpha)) = (\alpha^2+1)\cdot g(\lambda_{max}) = \frac{(\alpha^2+1)\lambda_{max}}{(\alpha+\lambda_{max})^2}.$$

Moreover, $\delta_2(\alpha) < 1$ is equivalent to

$$\lambda_{max}^2 - (\alpha-1)^2\lambda_{max} + \alpha^2 > 0. \tag{14}$$

If $2-\sqrt{3}<\alpha<2+\sqrt{3}$, we can see that the discriminant

$$\Delta=(\alpha-1)^4-4\alpha^2=(\alpha^2+1)(\alpha^2-4\alpha+1)<0$$

of the quadratic function defined on the left hand side of (14), and the inequality (14) is always true.

On the other hand, if $0<\alpha\le 2-\sqrt{3}$ or $\alpha\ge 2+\sqrt{3}$, which is equivalent to

$$\alpha^2-4\alpha+1\ge 0, \tag{15}$$

and thus the discriminant

$$\Delta=(\alpha^2+1)(\alpha^2-4\alpha+1)\ge 0.$$

Then by solving the inequality (14), we obtain

$$\lambda_{max}>\frac{(\alpha-1)^2+\sqrt{(\alpha-1)^4-4\alpha^2}}{2}$$

or

$$\lambda_{max}<\frac{(\alpha-1)^2-\sqrt{(\alpha-1)^4-4\alpha^2}}{2}.$$

However, we can show that

$$\frac{(\alpha-1)^2+\sqrt{(\alpha-1)^4-4\alpha^2}}{2}\ge\alpha \quad\text{and}\quad \frac{(\alpha-1)^2-\sqrt{(\alpha-1)^4-4\alpha^2}}{2}\le\alpha.$$

In fact,

$$(\alpha-1)^2+\sqrt{(\alpha-1)^4-4\alpha^2}-2\alpha\ge\alpha^2-4\alpha+1\ge 0,$$

and

$$\begin{aligned}&(\alpha-1)^2-\sqrt{(\alpha-1)^4-4\alpha^2}-2\alpha\\ &\quad=\sqrt{\alpha^2-4\alpha+1}(\sqrt{\alpha^2-4\alpha+1}-\sqrt{\alpha^2+1})\le 0,\end{aligned}$$

where we have used (15). In a word, if $0<\alpha\le 2-\sqrt{3}$ or $\alpha\ge 2+\sqrt{3}$, the inequality (14) is equivalent to

$$\lambda_{max}<\frac{(\alpha-1)^2-\sqrt{(\alpha-1)^4-4\alpha^2}}{2}.$$

Thus, we have completed the proof of this theorem. □

Proposition 2.2. *Assume that $W\in\mathbb{R}^{n\times n}$ is symmetric positive definite and $T\in\mathbb{R}^{n\times n}$ is symmetric positive semi-definite. Then for any $\alpha\in(2-\sqrt{3},2+\sqrt{3})$, we have $\rho(M(\alpha))<1$, i.e., the ICCRI scheme is unconditionally convergent.*

Moreover, if $\lambda_{max}\ge 1$, then the quasi-optimal parameter is $\alpha_*=1$, and the corresponding spectral radius satisfies

$$\rho(M(\alpha_*))\le\delta_1(\alpha_*)=\delta_1(1)=\frac{1}{2}. \tag{16}$$

If $\lambda_{max}<1$, then the quasi-optimal parameter is $\alpha_*=1/\lambda_{max}$, and the corresponding spectral radius satisfies

$$\rho(M(\alpha_*))=\delta_2\Big(\frac{1}{\lambda_{max}}\Big)=\frac{\lambda_{max}}{1+\lambda_{max}^2}<\delta_2(1)=\frac{2\lambda_{max}}{(1+\lambda_{max})^2}<\frac{1}{2}. \tag{17}$$

Proof. According to the results of Theorem 2.1, it is clear that for any $\alpha \in (2-\sqrt{3}, 2+\sqrt{3})$, we have $\rho(M(\alpha)) < 1$. Moreover, since

$$\delta_1(\alpha) - \delta_2(\alpha) = \frac{\alpha^2+1}{4\alpha} - \frac{(\alpha^2+1)\lambda_{max}}{(\alpha+\lambda_{max})^2}$$
$$= (\alpha^2+1)\frac{(\alpha+\lambda_{max})^2 - 4\alpha\lambda_{max}}{4\alpha(\alpha+\lambda_{max})^2}$$
$$= (\alpha^2+1)\frac{(\alpha-\lambda_{max})^2}{4\alpha(\alpha+\lambda_{max})^2} \geq 0,$$

we know that $\delta_1(\alpha) \geq \delta_2(\alpha)$ holds true for any $\alpha > 0$ and $\delta_1(\alpha) = \delta_2(\alpha)$ if and only if $\alpha = \lambda_{max}$.

By taking differentiations for $\delta_1(\alpha)$ and $\delta_2(\alpha)$, we get

$$\delta_1'(\alpha) = \frac{\alpha^2-1}{4\alpha^2} \quad (\alpha \leq \lambda_{max}),$$
$$\delta_2'(\alpha) = \frac{2\lambda_{max}(\alpha\lambda_{max}-1)}{(\alpha+\lambda_{max})^3} \quad (\alpha \geq \lambda_{max}).$$

Then it is apparent that the function $\delta_1(\alpha)$ achieves the minimum at the point $\alpha = 1$, and $\delta_2(\alpha)$ achieves the minimum at the point $\alpha = 1/\lambda_{max}$.

Therefore, if $\lambda_{max} \geq 1$, according to the left hand side plots in Figure 1, we know that the quasi-optimal parameter is $\alpha_* = 1$, and (16) holds true. If $\lambda_{max} < 1$, according to the right hand side plots in Figure 1, we know that the quasi-optimal parameter $\alpha_* = 1/\lambda_{max}$ provided that $\rho(M(1/\lambda_{max})) < 1$, which is true according to the following analysis.

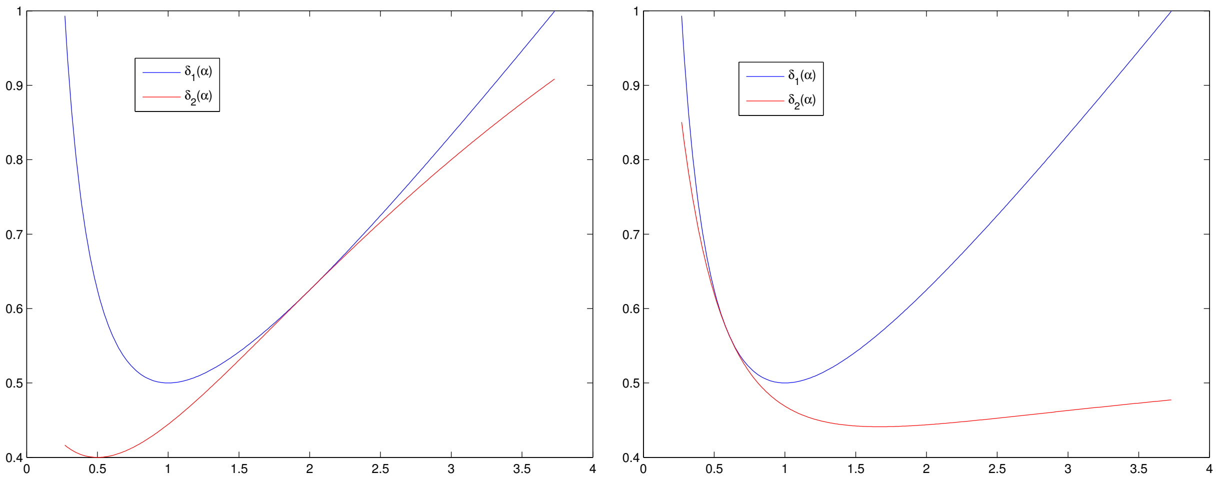

Figure 1: The plots of $\delta_1(\alpha)$ and $\delta_2(\alpha)$ for the ICCRI method varying with $\alpha \in (0.27, 3.73)$ for $\lambda_{max} \geq 1$ on the left and $\lambda_{max} < 1$ on the right, respectively.

Next, we are going to confirm that $\rho(M(1/\lambda_{max})) < 1$ when $\lambda_{max} < 1$. If $2-\sqrt{3} < \lambda_{max} < 1$, then the quasi-optimal parameter $\alpha_* = 1/\lambda_{max} \in (1, 2+\sqrt{3})$ and then $\rho(M(1/\lambda_{max})) < 1$ holds true according to Theorem 2.1.

On the other hand, if $\lambda_{max} \leq 2-\sqrt{3}$, i.e., $\alpha_* = 1/\lambda_{max} \geq 2+\sqrt{3}$, in this case,

$$\lambda_{max} < h\left(\frac{1}{\lambda_{max}}\right) = \frac{(1-\lambda_{max})^2 - \sqrt{(1-\lambda_{max})^4 - 4\lambda_{max}^2}}{2\lambda_{max}^2}$$

always holds true, which implies $\rho(M(1/\lambda_{max})) < 1$ in terms of Theorem 2.1, here the function $h(x)$ is defined in (12). In fact, for $0 < \lambda_{max} \leq 2-\sqrt{3} \approx 0.268$, we have

$$(1-\lambda_{max})^2 - 2\lambda_{max}^3 > 0,$$

since

$$(1-\lambda_{max})^2 > 0.7^2 > 2\times 0.3^3 > 2\lambda_{max}^3.$$

In addition, the inequality $\lambda_{max} < h\Big(\frac{1}{\lambda_{max}}\Big)$ is true because

$$\begin{aligned}&2\lambda_{max}^2\Big[h\Big(\frac{1}{\lambda_{max}}\Big)-\lambda_{max}\Big]\\&\quad=(1-\lambda_{max})^2-2\lambda_{max}^3-\sqrt{(1-\lambda_{max})^4-4\lambda_{max}^2}>0,\end{aligned}$$

where we have used the fact that

$$\begin{aligned}&[(1-\lambda_{max})^2-2\lambda_{max}^3]^2-[(1-\lambda_{max})^4-4\lambda_{max}^2]\\&=4\lambda_{max}^2(\lambda_{max}^4+2\lambda_{max}^2+1-\lambda_{max}^3-\lambda_{max})\\&=4\lambda_{max}^2(\lambda_{max}^2+1)(\lambda_{max}^2-\lambda_{max}+1)>0,\end{aligned}$$

since

$$\lambda_{max}^2-\lambda_{max}+1>0$$

is always true. In a word, if $\lambda_{max} < 1$, then the quasi-optimal parameter $\alpha_* = 1/\lambda_{max}$, and it is easy to show that (17) holds true.

Now we have completed the proof. □

Remark 2.3. *According to the result given in Proposition 2.2, the best choice of the parameter for the ICCRI method is $\alpha \in [1, 2+\sqrt{3})$. In addition, there is a symmetric version of the ICCRI iteration method in (8) which is given by*

$$\begin{cases}(\omega T+W)x^{(k+\frac{1}{2})}=(\omega-i)Tx^{(k)}+b,\\(\omega T+W)x^{(k+1)}=(1+\omega i)Wx^{(k+\frac{1}{2})}-i\omega b,\end{cases}\tag{18}$$

with the parameter $\omega > 0$.

However, by letting $\omega=\frac{1}{\alpha}$ and multiplying the both sides of the two equations in (18) by α, then the symmetric version in (18) is exactly the same as the ICCRI iteration scheme in (8). Therefore, we do not need to analyze the iteration scheme (18) here.

3. Comparison with the CRI method

Next we shall make a simple comparison between the ICCRI and the CRI methods. Actually, the spectral radius $\rho(\mathcal{T}(\alpha))$ for the CRI method [39] is bounded by

$$\hat{\delta}(\alpha) := \frac{\alpha^2+1}{(\alpha+1)^2},\tag{19}$$

where

$$\mathcal{T}(\alpha)=(\alpha^2+1)(\alpha W+T)^{-1}W(\alpha T+W)^{-1}T$$

is the iteration matrix of the CRI method.

It is pointed out in [39] that when $\alpha = 1$, the upper bound $\hat{\delta}(\alpha)$ in (19) of the spectral radius $\rho(\mathcal{T}(\alpha))$ reaches its minimum value, however the detailed theoretical analyses on the quasi-optimal parameter α_* and the corresponding spectral radius $\rho(\mathcal{T}(\alpha_*))$ are not given. Next, we will complete this task.

In our case, let $W \in \mathbb{R}^{n\times n}$ be symmetric positive definite and $T \in \mathbb{R}^{n\times n}$ be symmetric positive semi-definite, we have

$$\mathcal{T}(\alpha)=(\alpha^2+1)W^{-\frac{1}{2}}\hat{\mathcal{T}}(\alpha)W^{\frac{1}{2}},\tag{20}$$

where

$$\hat{\mathcal{T}}(\alpha) = (\alpha I + W^{-\frac{1}{2}}TW^{-\frac{1}{2}})^{-1}(\alpha W^{-\frac{1}{2}}TW^{-\frac{1}{2}} + I)^{-1}W^{-\frac{1}{2}}TW^{-\frac{1}{2}}. \tag{21}$$

With the same definition in (11), we have the following results.

Theorem 3.1. *Assume that $W \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $T \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite. Then the CRI scheme is convergent for any $\alpha > 0$.*
(i) If $\lambda_{max} \geq 1$, then we have

$$\rho(\mathcal{T}(\alpha)) \leq \hat{\delta}_1(\alpha) := \frac{\alpha^2+1}{(\alpha+1)^2} < 1.$$

(ii) If $\lambda_{max} \leq 1$, then we have

$$\rho(\mathcal{T}(\alpha)) = \hat{\delta}_2(\alpha) := \frac{(\alpha^2+1)\lambda_{max}}{(\alpha+\lambda_{max})(\alpha\lambda_{max}+1)} < 1.$$

Proof. From the relational expression (20), we obtain

$$\rho(\mathcal{T}(\alpha)) = (\alpha^2+1)\cdot\rho(\hat{\mathcal{T}}(\alpha)) = (\alpha^2+1)\cdot \max_{\lambda_j \in sp(T_W)} \frac{\lambda_j}{(\alpha+\lambda_j)(\alpha\lambda_j+1)}.$$

Let

$$\hat{g}(x) = \frac{x}{(\alpha+x)(\alpha x+1)}, \quad \text{for} \quad x \geq 0. \tag{22}$$

By direct calculation, we have

$$\hat{g}'(x) = \frac{\alpha(1-x^2)}{(\alpha+x)^2(\alpha x+1)^2},$$

which implies that $\hat{g}'(1) = 0$, and the function $\hat{g}(x)$ is monotonically increasing in $[0,1]$ and decreasing in $[1,+\infty)$.

(i) If $\lambda_{max} \geq 1$, then we have

$$\rho(\mathcal{T}(\alpha)) \leq (\alpha^2+1)\cdot\hat{g}(1) = \frac{\alpha^2+1}{(\alpha+1)^2} < 1,$$

for any $\alpha > 0$.

(ii) If $\lambda_{max} \leq 1$, then we have

$$\rho(\mathcal{T}(\alpha)) = (\alpha^2+1)\cdot\hat{g}(\lambda_{max}) = \frac{(\alpha^2+1)\lambda_{max}}{(\alpha+\lambda_{max})(\alpha\lambda_{max}+1)} < 1.$$

Thus, we have completed the proof. □

Proposition 3.2. *Assume that $W \in \mathbb{R}^{n\times n}$ is symmetric positive definite and $T \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite. If $\lambda_{max} \geq 1$, then the quasi-optimal parameter for the CRI method is $\alpha_* = 1$, and the corresponding spectral radius*

$$\rho(\mathcal{T}(\alpha_*)) \leq \hat{\delta}_1(\alpha_*) = \hat{\delta}_1(1) = \frac{1}{2}. \tag{23}$$

If $\lambda_{max} \leq 1$, then the quasi-optimal parameter for the CRI method is also $\alpha_* = 1$, and the corresponding spectral radius

$$\rho(\mathcal{T}(\alpha_*)) = \hat{\delta}_2(1) = \frac{2\lambda_{max}}{(1+\lambda_{max})^2} \leq \frac{1}{2}. \tag{24}$$

Proof. According to the results of Theorem 3.1, since

$$\begin{aligned}\hat{\delta}_1(\alpha)-\hat{\delta}_2(\alpha)&=\frac{\alpha^2+1}{(\alpha+1)^2}-\frac{(\alpha^2+1)\lambda_{max}}{(\alpha+\lambda_{max})(\alpha\lambda_{max}+1)}\\&=(\alpha^2+1)\frac{(\alpha+\lambda_{max})(\alpha\lambda_{max}+1)-(\alpha+1)^2\lambda_{max}}{(\alpha+1)^2(\alpha+\lambda_{max})(\alpha\lambda_{max}+1)}\\&=(\alpha^2+1)\frac{\alpha(\lambda_{max}-1)^2}{(\alpha+1)^2(\alpha+\lambda_{max})(\alpha\lambda_{max}+1)}\geq 0,\end{aligned}$$

we know that $\hat{\delta}_1(\alpha)\geq\hat{\delta}_2(\alpha)$ holds true for any $\alpha>0$ and $\hat{\delta}_1(\alpha)=\hat{\delta}_2(\alpha)$ if and only if $\lambda_{max}=1$.

By taking differentiations for $\hat{\delta}_1(\alpha)$ and $\hat{\delta}_2(\alpha)$, we get

$$\begin{aligned}\hat{\delta}_1'(\alpha)&=\frac{2(\alpha-1)}{(\alpha+1)^3}\quad(\lambda_{max}\geq 1),\\\hat{\delta}_2'(\alpha)&=\frac{(\alpha^2-1)\lambda_{max}(\lambda_{max}^2+1)}{(\alpha+\lambda_{max})^2(\alpha\lambda_{max}+1)^2}\quad(\lambda_{max}\leq 1).\end{aligned}$$

Then it is apparent that both the functions $\hat{\delta}_1(\alpha)$ and $\hat{\delta}_2(\alpha)$ achieve the minimum at the point $\alpha=1$.

Therefore, it is easy to show that (23) and (24) holds true. Now we have completed the proof. □

Corollary 3.3. *Assume that $W\in\mathbb{R}^{n\times n}$ is symmetric positive definite and $T\in\mathbb{R}^{n\times n}$ is symmetric positive semi-definite. If $\lambda_{max}\geq 1$, then the quasi-optimal parameters for the CRI and ICCRI methods are the same, i.e., $\alpha_*=1$, and the corresponding spectral radius is no more than $\frac{1}{2}$. If $\lambda_{max}<1$, then for the CRI and ICCRI methods with different quasi-optimal parameters, we have*

$$\rho(M(\alpha_*))=\delta_2\Big(\frac{1}{\lambda_{max}}\Big)=\frac{\lambda_{max}}{1+\lambda_{max}^2}<\rho(\mathcal{T}(\alpha_*))=\hat{\delta}_2(1)=\frac{2\lambda_{max}}{(1+\lambda_{max})^2}. \tag{25}$$

Proof. From the results given in Propositions 2.2 and 3.2, we can easily draw the conclusion. □

Remark 3.4. *It is obvious that the ICCRI method is the same as the CRI method when $\alpha=1$, and the quasi-optimal parameter for the CRI method is always $\alpha_*=1$. However, if $\lambda_{max}<1$, the optimal parameter for the ICCRI method is $\alpha_*=1/\lambda_{max}>1$, and the corresponding sequence of iterations converges faster.*

4. Numerical results

In this section, we shall test the efficiency of the ICCRI method by solving some complex linear systems, and compare it with the PMHSS [7] and the CRI [39] iteration methods, based on the iteration steps (denoted as IT) and CPU times (denoted as CPU) in seconds. The programs are performed in a personal computer with the processor, Intel(R) Core(TM) i5-10210U CPU @ 1.60GHz 2.11 GHz Microsoft Windows 10 Professional with the 64-bit Operating System and RAM 8.00 GB. The programming package is Matlab R2014a (8.3.0.532) with 4096 digits.

In our experiments, similar as in [39], we choose $V=W$ for the PMHSS method, then the PMHSS method can be transformed to the following single-step iteration method

$$(\alpha W+T)x^{k+1}=\frac{\alpha+i}{1+\alpha}(\alpha W-iT)x^k+\frac{\alpha(1-i)}{1+\alpha}b.$$

In addition, we compute the matrix inversions in the considered iteration methods with the Cholesky decomposition, and the initial value $x^{(0)}=0$ is always fixed for all the methods with the stopping criteria

$$ERR:=\frac{\|b-Ax^{(k)}\|_2}{\|b\|_2}\leq 10^{-6},$$

where $x^{(k)}$ is the approximation after k-th iterations.

Example 4.1. *Consider the following complex Helmholtz equation [5, 8, 11, 24]*

$$-\Delta u + \sigma_1 u + i\sigma_2 u = f,$$

where the function u satisfies the Dirichlet boundary conditions in the domain $D = [0,1] \times [0,1]$*. By discretizing this complex Helmholtz equation with finite differences on an* $m \times m$ *grid with mesh size* $h = 1/(m+1)$*, a complex linear system*

$$[(K + \sigma_1 I) + i\sigma_2 I]x = b,$$

is obtained, where the matrix $K \in \mathbb{R}^{n\times n}$ *is the sum of tensor-product*

$$K = I \otimes B_m + B_m \otimes I \quad with \quad B_m = \frac{1}{h^2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}.$$

In our tests, we set $\sigma_1 = 100$, $\sigma_2 = 10$ *and the constant vector* $b = (1 + i)A\mathbf{1}$, *where all the entries of the vector* $\mathbf{1}$ *are equal to* 1*. By multiplying both sides by* h^2*, we obtain a normalization of the complex linear system.*

Example 4.2. *Consider the complex linear system [2]*

$$[(K - \varpi^2 M) + i(C_H + \varpi C_V)]x = b,$$

which arises from direct frequency domain analysis. For more details, please see [4, 43].

Here, we consider the linear system

$$\left[\left(K - \varpi^2 I\right) + i\left(\beta K + 10\varpi I\right)\right]x = b,$$

where the two parameters $\varpi, \beta \in \mathbb{R}$ *are known, and the matrix*

$$K = I \otimes B_m + B_m \otimes I \quad with \quad B_m = \frac{1}{h^2} tridiag(-1, 2, -1) \in \mathbb{R}^{m\times m}.$$

In our tests, we choose $\varpi = 0.5, \beta = 0.2$, *and the constant vector b with its j-th entry*

$$b_j = \frac{(1+i)j}{h^2(j+1)^2}, \quad j = 1, 2, \ldots, n.$$

Furthermore, by multiplying both sides by h^2*, we obtain a normalization of the complex linear system.*

Example 4.3. *Consider the complex linear system* $Ax = b$*, where A has quasi-tridiagonal form [42]*

$$A = \begin{bmatrix} 1+\omega i & \frac{1}{8} & 0 & \cdots & 0 & \frac{1}{2} \\ \frac{1}{8} & 1+\omega i & \frac{1}{8} & \ddots & \vdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & 0 & \ddots & \ddots & \frac{1}{8} & 0 \\ 0 & \vdots & \ddots & \frac{1}{8} & 1+\omega i & 0 \\ \frac{1}{2} & 0 & \cdots & 0 & \frac{1}{8} & 1+\omega i \end{bmatrix}_{n\times n}$$

and the right hand side vector b can be determined when the exact solution is

$$x^* = \left[1, \frac{1}{2}, \cdots, \frac{1}{n-1}, \frac{1}{n}\right].$$

Here, we let $n = m^2$ *and choose* $\omega = 0.2$.

Table 1: Numerical results for the considered schemes by solving Example 4.1 when $\sigma_1 = 100$, $\sigma_2 = 10$.

Method		Grid				
		32×32	64×64	128×128	192×192	256×256
PMHSS	α_*	1.0	1.0	1.0	1.0	1.0
	IT	40	40	40	40	40
	CPU(s)	0.013085	0.179380	1.670092	4.057580	12.261971
CRI	α_*	1.0	1.0	1.0	1.0	1.0
	IT	7	6	6	6	5
	CPU(s)	0.007257	0.053230	0.510335	1.222703	3.431233
ICCRI	α_*	2.0	2.0	2.0	2.0	3.0
	IT	6	5	5	5	4
	CPU(s)	0.007153	0.048121	0.428986	1.087861	2.892065

Table 2: Numerical results for the considered schemes by solving Example 4.2 when $\varpi = 0.5$, $\beta = 0.2$.

Method		Grid				
		32×32	64×64	128×128	192×192	256×256
PMHSS	α_*	0.5	0.5	0.5	0.5	0.5
	IT	25	25	25	25	25
	CPU(s)	0.010314	0.102970	0.935825	2.865143	9.106758
CRI	α_*	1.0	1.0	1.0	1.0	1.0
	IT	15	14	13	12	12
	CPU(s)	0.010991	0.111618	0.940385	2.685790	8.539651
ICCRI	α_*	2.0	2.0	2.0	2.0	2.0
	IT	13	12	11	11	11
	CPU(s)	0.010064	0.098414	0.796742	2.429729	7.862425

Table 3: Numerical results for the considered schemes by solving Example 4.3 when $\omega = 0.2$.

Method		Grid				
		32×32	64×64	128×128	192×192	256×256
PMHSS	α_*	0.5	0.5	0.5	0.5	0.5
	IT	28	28	28	28	28
	CPU(s)	0.005313	0.016506	0.057196	0.106444	0.242996
CRI	α_*	1.0	1.0	1.0	1.0	1.0
	IT	15	15	15	15	15
	CPU(s)	0.007123	0.016936	0.043028	0.082240	0.194431
ICCRI	α_*	2.5	2.5	2.5	2.5	2.5
	IT	12	12	12	12	12
	CPU(s)	0.004909	0.014700	0.034760	0.067385	0.180144

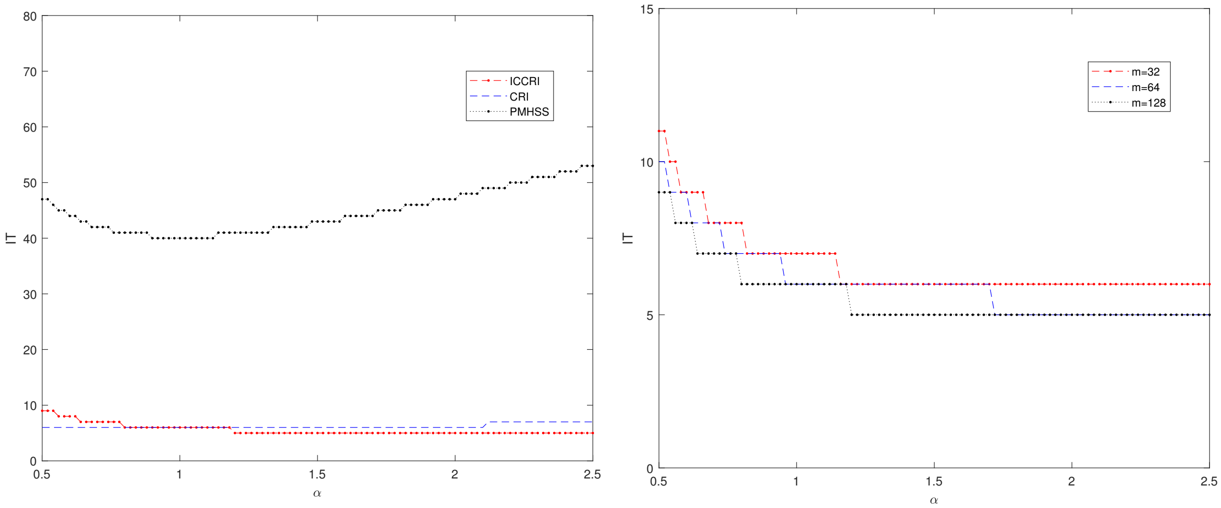

Figure 2: Iteration steps varying with the parameter α for Example 4.1, the left figure is for the ICCRI, the CRI and the PMHSS methods when $m = 128$, and the right one is for the ICCRI method when $m = 32, 64, 128$, respectively.

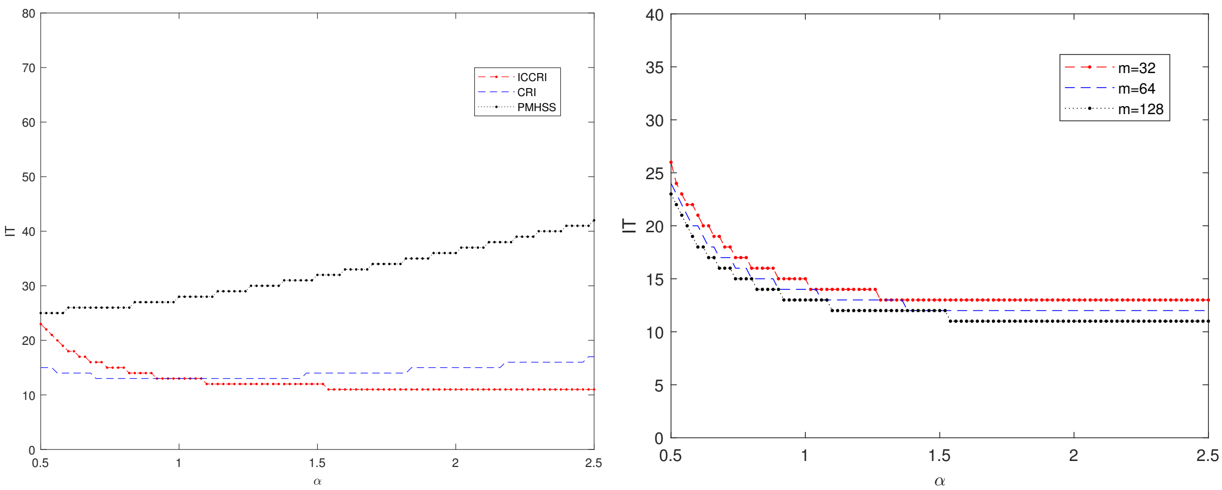

Figure 3: Iteration steps varying with the parameter α for Example 4.2, the left figure is for the ICCRI, the CRI and the PMHSS methods when $m = 128$, and the right one is for the ICCRI method when $m = 32, 64, 128$, respectively.

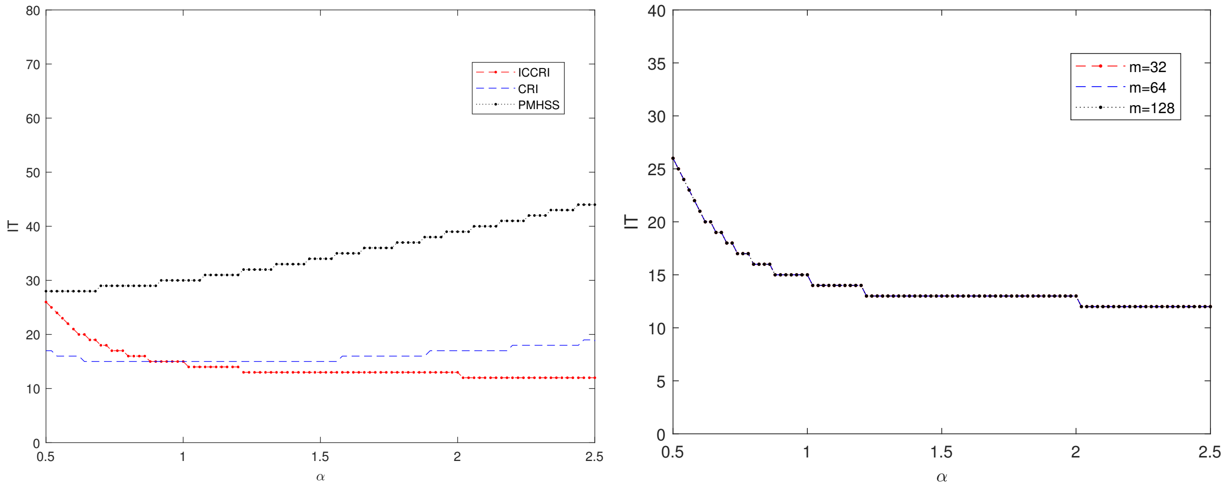

Figure 4: Iteration steps varying with the parameter α for Example 4.3, the left figure is for the ICCRI, the CRI and the PMHSS methods when $m = 128$, and the right one is for the ICCRI method when $m = 32, 64, 128$, respectively.

Tables 1-3 show the numerical results of experiments by applying the PMHSS, the CRI and the ICCRI iteration schemes when the optimal experimental parameters α_* are used, with various problem sizes of Examples 4.1-4.3, respectively. We can see that the considered PMHSS, CRI and ICCRI iteration schemes are convergent. Moreover, it can be seen that the CRI and the ICCRI iteration schemes have significant advantages over the PMHSS method, since they converge faster with less iteration steps and CPU times, meanwhile the ICCRI method is superior to the CRI method by comparison of iteration steps and CPU times. Notice also that the optimal experimental parameter of the CRI method is always $\alpha_* = 1$, which coincides with the results given in Section 3. In addition, we also find an interesting phenomenon that for the CRI and the ICCRI iteration schemes, the iteration steps decrease with the growth of the problem size $n = m^2$, particularly in the Examples 4.1-4.2.

On the other hand, Figures 2-4 show the plots of iteration steps varying with the parameter $\alpha \in (0.5, 2.5)$ for the considered three methods for Examples 4.1-4.3, respectively. The left figures are for the ICCRI, the CRI and the PMHSS methods when $m = 128$, and the right ones are only for the ICCRI method when $m = 32, 64, 128$, respectively. It is clear that the optimal experimental parameters given in the Tables 1-3 can be verified according to these figures. Moreover, we can see that the ICCRI method needs more iteration steps than the CRI method when $0 < \alpha < 1$, while needs less iteration steps when $\alpha > 1$. However, the ICCRI method requires only one matrix inversion for any α, the CRI method requires two matrix inversions as long as $\alpha \neq 1$, and these two methods are the same when $\alpha = 1$.

5. Conclusions

In this work, we construct a so-called ICCRI method for solving complex symmetric linear systems, based on the CRI method. The spectral radius of the iteration matrix and the quasi-optimal parameter of the ICCRI scheme are discussed elaborately. Furthermore, the quasi-optimal parameter and the corresponding convergence factor of the CRI method are also given detailed analysis, for the purpose of comprison. Numerical results confirm that the ICCRI scheme is superior to the PMHSS and CRI methods.

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