Filomat 38:26 (2024), 9223–9236
https://doi.org/10.2298/FIL2426223B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On the existence of χ-differentiable solutions for sequential differential system involving a mixed derivative in Banach space

Moustafa Beddani[a,*], Hamid Beddani[b], Belqassim Azzouz[c]

[a] *Department of Mathematics, E. N. S. of Mostaganem, 27000 Mostaganem, Algeria*
[b] *Laboratory of Complex Systems, Higher School in Electrical and Energy Engineering, 31000 Oran, Algeria*
[c] *Laboratoire de Mathématiques Fondamentales et Appliquées d'Oran, Relizane University, Relizane 48000, Algeria*

Abstract. In this article, we aim to study the existence of χ-continuously differentiable solutions for an integral boundary value problem to sequential differential system involving a mixed derivative in Banach space. We apply the Mönch fixed point theorem combined with the Kuratowski measure of noncompactness to obtain this result. We also obtain the compactness of the solution set for the given problem, an example is given to illustrate the results obtained.

1. Introduction

Fractional differential problems with different conditions have been investigated by many Authors, for example [2, 10, 14, 17], those differential problems still used to understand various phenomena in physical sciences, engineering, electrochemistry, fluid flow, economic and biomedical sciences, for more details, see the following references [14, 16, 18] and the references therein.

Fractional differential equations with nonlocal conditions have been discussed in [5, 6, 20]. In [11], Byszewaski showed that nonlocal conditions can be more effective than others to describe some physical phenomena. The topological properties of the solution set for differential problems are studied by many mathematicians, in the references [2–4, 8, 9], Authors studied the compactness and the stability of some fractional differential problems.

In 1930, Kuratowski introduced the notion of measure of noncompactness. This notion is very useful in the functional analysis, many researchers have used such notion to study the existence of solutions for sereval ordinary and fractional differential problems, see [12, 13]. In view of the above considerations, we consider the following system

$$(\mathbf{S}) \quad \begin{cases} {}^{RL}\mathcal{D}_{\underline{\xi}^+}^{\rho_1,\chi} D^\chi y_1(\xi) = \hbar_1\Big(\xi, y_1(\xi), y_2(\xi), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\sigma_1,\chi} y_1(\xi), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\sigma_2,\chi} y_2(\xi)\Big), \\ {}^{RL}\mathcal{D}_{\underline{\xi}^+}^{\rho_2,\chi} D^\chi y_2(\xi) = \hbar_2\Big(\xi, y_1(\xi), y_2(\xi), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\sigma_1,\chi} y_1(\xi), {}^{C}\mathcal{D}_{\underline{\xi}^+}^{\sigma_2,\chi} y_2(\xi)\Big), \quad \xi \in (\underline{\xi}, \overline{\xi}], \end{cases}$$

2020 *Mathematics Subject Classification*. Primary 26A33, 34A60; Secondary 34A08, 34A37, 47H08.
Keywords. χ-continuously differentiable solutions, measure of non-compactness, Banach space, fixed point theorem, χ-Riemann-Liouville fractional derivative.
Received: 02 April 2024; Accepted: 02 April 2024
Communicated by Maria Alessandra Ragusa
* Corresponding author: Moustafa Beddani
Email addresses: beddani2004@yahoo.fr (Moustafa Beddani), beddanihamid@gmail.com (Hamid Beddani), azzouzbel27@gmail.com (Belqassim Azzouz)

associated with the following nonlocal integral boundary conditions

$$(\mathbf{NIB})\quad \begin{cases} \mathcal{I}_{\underline{\xi}^+}^{1-\rho_1,\chi} D^\chi y_1(\underline{\xi}^+) = \sum_{j=1}^{n_1} \zeta_{1j} D^\chi y_1(\xi_{1j}), \quad y_1(\underline{\xi}) = \Lambda_1, \\ \mathcal{I}_{\underline{\xi}^+}^{1-\rho_2,\chi} D^\chi y_2(\underline{\xi}^+) = \sum_{k=1}^{n_2} \zeta_{2k} D^\chi y_2(\xi_{2k}), \quad y_2(\underline{\xi}) = \Lambda_2, \end{cases}$$

where ${}^{RL}\mathcal{D}_{\underline{\xi}^+}^{\rho_i,\chi}$ denotes the χ-Riemann-Liouville fractional derivative of order $0 < \rho_i < 1,\ i = 1,2,$ ${}^{C}\mathcal{D}_{\underline{\xi}^+}^{\rho_i,\chi}$ is the χ-Caputo fractional derivative of order $\rho_i,\ i = 1,2,$ E is a Banach space, $\hbar_i : (\underline{\xi}, \overline{\xi}] \times E^2 \to E,\ i = 1,2$ are a functions satisfying some specified conditions (see, section 3), $\chi \in C^1([\underline{\xi}, \overline{\xi}], \mathbb{R}^+)$ satisfied $\chi'(\xi) > 0$, for all $\xi \in [\underline{\xi}, \overline{\xi}]$, $D^\chi = \frac{1}{\chi'(\xi)} \frac{d}{d\xi}$, $\underline{\xi},\ \overline{\xi} \in \mathbb{R}_*^+$ with $\underline{\xi} < \overline{\xi}$ and $\xi_{1j},\ \xi_{2k}, \in (\underline{\xi}, \overline{\xi})$, $j = 1, \cdots, n_1, k = 1, \cdots, n_2$ with $\Gamma(\rho_i) \neq \sum_{k=1}^{n_i} \zeta_{ik}(\chi(\xi) - \chi(\underline{\xi}))^{\rho_i - 1}, i = 1,2.$

Note that in our problem, the derivative we took is a composition of fractional and ordinary derivatives, it is clear that the relation ${}^{RL}\mathcal{D}_{\underline{\xi}^+}^{\rho_1,\chi} D^\chi y_i = {}^{RL}\ \mathcal{D}_{\underline{\xi}^+}^{\rho_i+1,\chi} y_i$ is not correct except in the case $y_i(\underline{\xi}) = 0$. This is the second motivation to consider the (**S**) − (**NIB**) problem involving a mixed derivatives,

The present work is organized as follows: In Section 2, we give some general results and preliminaries and in Section 4, we show the existence solution for the problem (**S**) − (**NIB**) by applying the fixed point theorem, also the compactness of the solution set. Finally an example to reinforce our work in Section 5

2. Basic results and Background

In this section, we will give some concepts and notations about the functional spaces, fractional calculus, noncompactness measure which are used throughout this paper. we denote by $C([\underline{\xi}, \overline{\xi}])$ (resp. by $L^1([\underline{\xi}, \overline{\xi}])$) the space of E-valued continuous functions (resp. the space of E-Bochner's integrable functions) with the following norm

$$\|u\|_\infty = \sup\left\{\|u(\xi)\|,\ \xi \in [\underline{\xi}, \overline{\xi}]\right\} \ \left(\text{resp. } \|u\|_{L^1} = \int_{\underline{\xi}}^{\overline{\xi}} \|u(\xi)\| d\xi\right).$$

Let $C_{1-\rho_i}([\underline{\xi}, \overline{\xi}])$ be the Banach spaces of functions from $(\underline{\xi}, \overline{\xi}]$ into E which is defined as:

$$C_{1-\rho_i,\chi}([\underline{\xi}, \overline{\xi}]) = \left\{u \in C((\underline{\xi}, \overline{\xi}]) :\ \ (\chi(.) - \chi(\underline{\xi}))^{1-\rho_i} u(.) \in C([\underline{\xi}, \overline{\xi}], E)\right\},\ i = 1,2.$$

with his norm $\|u\|_{\rho_i,\chi}$, that is given by

$$\|u\|_{\rho_i,\chi} = \sup_{\xi \in (\underline{\xi}, \overline{\xi}]} (\chi(\xi) - \chi(\underline{\xi}))^{1-\rho_i} \|u(\xi)\|,\ i = 1,2.$$

Next, we denote by $C^1_{1-\rho_i,\chi}((\underline{\xi}, \overline{\xi}])$ the space of functions χ-continuously differentiable defined as folows

$$C^1_{1-\rho_i,\chi}([\underline{\xi}, \overline{\xi}]) = \left\{u : (\underline{\xi}, \overline{\xi}] \to E :\ \ u(.) \in C([\underline{\xi}, \overline{\xi}]) \text{ and } D^\chi u(.) \in C_{1-\rho_i,\chi}([\underline{\xi}, \overline{\xi}])\right\},\ i = 1,2.$$

with the norm

$$\|u\|^1_{\rho_i,\chi} = \|u\|_\infty + \|D^\chi u\|_{\rho_i,\chi},\ i = 1,2.$$

Let $\prod_{i=1}^2 C^1_{1-\rho_i,\chi}([\underline{\xi}, \overline{\xi}])$ be the product space (will be denoted in all that follows by $\Pi^1_\chi([\underline{\xi}, \overline{\xi}])$), which is a Banach space with the following norm

$$\|(u_1, u_2)\|^1_\chi = \max\left\{\|u_1\|^1_{\rho_1,\chi}, \|u_2\|^1_{\rho_2,\chi}\right\}.$$

In the following, for all $\eta > -1$, we pose $\Psi^{\eta}(r,s) = (\chi(r) - \chi(s))^{\eta}$, for all $s, r \in [\underline{\xi}, \overline{\xi}]$ with $r > s$ and for all $\eta > 0$, we put $\Psi_*^{\eta} = (\chi(\overline{\xi}) - \chi(\underline{\xi}))^{\eta}$.

First, we introduce the concepts of χ- fractional derivative in the Riemann-Liouville and Hilfer sense and their properties.

Definition 2.1. *[14, 19] Let $\ell \in L^1([\underline{\xi}, \overline{\xi}])$ and $\chi \in C^1([\underline{\xi}, \overline{\xi}])$ such that $\chi'(\xi) > 0$, for all $\xi \in [\underline{\xi}, \overline{\xi}]$,*

(i) the χ-Riemann- Liouville fractional integral of order $\rho > 0$ of the function ℓ is defined by

$$\mathcal{I}_{\underline{\xi}^+}^{\rho,\chi} \ell(\xi) = \frac{1}{\Gamma(\rho)} \int_{\underline{\xi}}^{\xi} \chi'(s) \Psi^{\rho-1}(\xi, s) \ell(s) ds,$$

(ii) the χ-Riemann- Liouville fractional derivative of order $\rho > 0$ of the function ℓ is defined by

$${}^{RL}\mathcal{D}_{\underline{\xi}^+}^{\rho,\chi} \ell(\xi) = \frac{1}{\Gamma(n-\rho)} \left(\frac{1}{\chi'(\xi)} \frac{d}{d\xi} \right)^n \left(\int_{\underline{\xi}}^{\xi} \chi'(s) \Psi^{n-\rho-1}(\xi, s) \ell(s) ds \right),$$

where Γ is the gamma function and $n = [\rho] + 1$ ($[\rho]$ represents the integer part of the real number ρ).

Lemma 2.2. *[14, 15] Let $\rho, \mu \in \mathbb{R}_*^+$, $\xi > \underline{\xi}$. We have then*

(i_1) $\mathcal{I}_{\underline{\xi}^+}^{\rho,\chi} \Psi^{\mu-1}(\xi, \underline{\xi}) = \frac{\Gamma(\mu)}{\Gamma(\rho+\mu)} \Psi^{\rho+\mu-1}(\xi, \underline{\xi})$.

(i_2) ${}^{RL}\mathcal{D}_{\underline{\xi}^+}^{\rho,\chi} \Psi^{\mu-1}(\xi, \underline{\xi}) = \frac{\Gamma(\mu)}{\Gamma(\mu-\rho)} \Psi^{\mu-\rho-1}(\xi, \underline{\xi}), 0 < \rho < 1,\ \mu > 1$, in the case when $\rho = \mu$, we find ${}^{RL}\mathcal{D}_{\underline{\xi}^+}^{\rho,\chi} \Psi^{\mu-1}(\xi, \underline{\xi}) = 0$.

Lemma 2.3 ([14, 15]). *Let $\rho > 0$ and $0 \le \gamma < 1$. If $\gamma \le \rho$. Then $\mathcal{I}_{\underline{\xi}^+}^{\rho,\chi}$ is bounded from $C_{\gamma,\chi}\left([\underline{\xi}, \overline{\xi}]\right)$ into $C\left([\underline{\xi}, \overline{\xi}]\right)$.*

Lemma 2.4 ([14, 15]). *Let $0 < \rho < 1,\ 0 \le \gamma < 1$. If $y \in C_{\gamma,\chi}\left(\underline{\xi}, \overline{\xi}\right)$ and $\mathcal{I}_{\underline{\xi}^+}^{1-\rho,\chi} y \in C_{\gamma,\chi}^1\left([\underline{\xi}, \overline{\xi}]\right)$, then*

$$\mathcal{I}_{\underline{\xi}^+}^{\rho,\chi} {}^{RL}\mathcal{D}_{\underline{\xi}^+}^{\rho,\chi} y(\xi) = y(\xi) - \frac{\mathcal{I}_{\underline{\xi}^+}^{1-\rho,\chi} y(\underline{\xi}^+)}{\Gamma(\rho)} \Psi^{\rho-1}(\xi, \underline{\xi}),\ \textit{for all}\ \xi \in (\underline{\xi}, \overline{\xi}].$$

Next, in this part we begin to give the notion of the Kuratowski measure of noncompactness and its properties which will be used in the next section, for this purpose, we denote by $\mathrm{Set}_b(E)$ the set of all bounded subsets of the Banach space E.

Definition 2.5. *[7, 12] Let $D \in \mathrm{Set}_b(E)$. The Kuratowski measure of noncompactness ϑ of the subset D is defined as follows:*

$\vartheta(\Omega) = \inf\{e > 0 : \Omega$ *admits a finite cover by sets of diameter* $\le e\}$.

Lemma 2.6. *[7, 12] Let $A, B \in \mathrm{Set}_b(E)$, we have the following properties*

(i_1) $\vartheta(A) = 0$ if and only if A is relatively compact,

(i_2) $\vartheta(A) = \vartheta(\overline{A})$, where $\overline{A}$ denotes the closure of A,

(i_3) $\vartheta(A + B) \le \vartheta(A) + \vartheta(B)$,

(i_4) $A \subset B$ implies $\vartheta(A) \le \vartheta(B)$,

(i_5) $\vartheta(a.A) = |a|.\vartheta(A)$ for all $a \in \mathbb{R}$,

(i_6) $\vartheta(\{a\} \cup A) = \vartheta(A)$ for all $a \in E$,

(i_7) $\vartheta(A) = \vartheta(Conv(A))$, where $Conv(A)$ is the smallest convex that contains A.

Lemma 2.7. *[12] If D is a equicontinuous and bounded subset of $C([\underline{\xi}, \overline{\xi}])$, then $\vartheta(D(.)) \in C([\underline{\xi}, \overline{\xi}], \mathbb{R}_+)$*

$$\vartheta_C(D) = \max_{\xi \in [\underline{\xi}, \overline{\xi}]} \vartheta(D(\xi)), \ \vartheta\left(\left\{ \int_{\underline{\xi}}^{\overline{\xi}} w(\xi) d\xi : w \in D \right\}\right) \leq \int_{\underline{\xi}}^{\overline{\xi}} \vartheta(D(\xi)) dr,$$

where $D(\xi) = \{w(\xi) : w \in D\}$ and ϑ_C is the Kuratowski measure of noncompactness on the space $C([\underline{\xi}, \overline{\xi}])$.

Theorem 2.8. *[1] Let E be a Banach space and D a closed and convex subset of E such that D is bounded and contains 0, and let $N : D \longrightarrow D$ be a continuous mapping. If the following implication:*

$$V = N(V) \cup \{0\} \ \ or \ \ V = \overline{conv}\, N(V) \implies \rho_i(V) = 0,$$

is satisfied for every subset V of D, then N has at least one fixed point.

3. Integral equation

In the content of Lemma below, we will illustrate the equivalence between the problem at hand (**S**)−(**NIB**) and the following system of integral equations

$$(\mathbf{S}^*) \quad \begin{cases} y_1(\xi) = \Lambda_1 + \frac{\sum_{j=1}^{n_1} \zeta_{1j} \mathcal{I}_{\underline{\xi}^+}^{\rho_1} \hbar_1\left(\xi_{1j}, y_1(\xi_{1j}), y_2(\xi_{1j}), {}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_1,\chi} y_1(\xi_{1j}), {}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_2,\chi} y_2(\xi_{1j})\right)}{\Gamma(\rho_1+1) - \rho_1 \sum_{j=1}^{n_1} \zeta_{1j} \Psi^{\rho_1 - 1}(\xi_{1j}, \underline{\xi})} \Psi^{\rho_1}(\xi, \underline{\xi}) \\ + \frac{1}{\Gamma(\rho_1+1)} \int_{\underline{\xi}}^{\xi} \chi'(s) \Psi^{\rho_1}(\xi, s) \hbar_1\left(s, y_1(s), y_2(s), {}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_1,\chi} y_1(s), {}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_2,\chi} y_2(s)\right) ds, \\ y_2(\xi) = \Lambda_2 + \frac{\sum_{k=1}^{n_2} \zeta_{2k} \mathcal{I}_{\underline{\xi}^+}^{\rho_2} \hbar_2\left(\xi_{2k}, y_1(\xi_{2k}), y_2(\xi_{2k}), {}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_1,\chi} y_1(\xi_{2k}), {}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_2,\chi} y_2(\xi_{2k})\right)}{\Gamma(\rho_2+1) - \rho_2 \sum_{k=1}^{n_2} \zeta_{2k} \Psi^{\rho_1 - 1}(\xi_{2k}, \underline{\xi})} \Psi^{\rho_1}(\xi, \underline{\xi}) \\ + \frac{1}{\Gamma(\rho_2+1)} \int_{\underline{\xi}}^{\xi} \chi'(s) \Psi^{\rho_2}(\xi, s) \hbar_2\left(s, y_1(s), y_2(s), {}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_1,\chi} y_1(s), {}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_2,\chi} y_2(s)\right) ds. \end{cases}$$

In all that follows, we put

$$\aleph_i(\xi, Y(\xi)) = \hbar_i\left(\xi, y_1(\xi), y_2(\xi), {}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_1,\chi} y_1(\xi), {}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_2,\chi} y_2(\xi)\right), \ i = 1, 2,$$

where $Y(.) = (y_1(.), y_2(.))$.

Lemma 3.1. *Let $0 \leq \rho_1, \rho_2 \leq 1$, we assume that $\aleph_i(., Y(.)) \in C([\underline{\xi}, \overline{\xi}]), i = 1, \ 2$, for all $\left(y_1, y_2\right) \in \Pi_{\chi}^1([\underline{\xi}, \overline{\xi}])$. Then, $\left(y_1, y_2\right)$ is a solution of the system* (**S**) − (**NIB**) *if and only if $\left(y_1, y_2\right)$ satisfies the system of integral equations* ($\mathbf{S}^*$).

Proof. Let $\left(y_1, y_2\right) \in \Pi_{\chi}^1([\underline{\xi}, \overline{\xi}])$ be a solution of the system (**S**) − (**NIB**), We want to prove that $\left(y_1, y_2\right)$ is a solution of ($\mathbf{S}^*$). From Definition of $\prod_{i=1}^2 C_{1-\rho_i,\chi}^1([\underline{\xi}, \overline{\xi}])$, Lemma 2.3 and Definition of $\mathcal{I}_{\underline{\xi}^+}^{1-\rho_i,\chi}$, we have $\mathcal{I}_{\underline{\xi}^+}^{1-\rho_i,\chi} D^{\chi} y(.) \in C([\underline{\xi}, \overline{\xi}]), i = 1, \ 2$ and $\frac{d}{\chi'(\xi) d\xi}\left(\mathcal{I}_{\underline{\xi}^+}^{1-\rho_i,\chi} D^{\chi} y(\xi)\right) = {}^{RL}\mathcal{D}_{\underline{\xi}^+}^{\rho_i} D^{\chi} y(\xi) = \aleph_i(\xi, Y(\xi)) \in C([\underline{\xi}, \overline{\xi}]) \subset C_{1-\rho_i,\chi}([\underline{\xi}, \overline{\xi}]), \ i = 1, \ 2$.
From Lemma (2.4) we have

$$D^{\chi} y_i(\xi) = \frac{\mathcal{I}_{\underline{\xi}^+}^{1-\rho_i,\chi} D^{\chi} y(\underline{\xi}^+)}{\Gamma(\rho_i)} \Psi^{\rho_i - 1}(\xi, \underline{\xi}) + \kappa y(\xi) + \mathcal{I}_{\underline{\xi}^+}^{\rho,\chi} \aleph_i(\xi, Y(\xi)), \ i = 1, \ 2. \tag{3.1}$$

Next, we substitute ξ by ξ_{1j}, ξ_{2k} into (3.1), we get

$$\begin{cases} D^{\chi}y(\xi_{1j}) = \frac{\mathcal{I}_{\underline{\xi}^+}^{1-\rho_1,\chi}D^{\chi}y(\underline{\xi}^+)}{\Gamma(\rho_1)}\Psi^{\rho_1-1}(\xi_{2j},\underline{\xi}) + \mathcal{I}_{\underline{\xi}^+}^{\rho_1,\chi}\aleph_1(\xi_{1j}, Y(\xi_{1j})),\ j = 1,\cdots,n_1, \\ D^{\chi}y(\xi_{2k}) = \frac{\mathcal{I}_{\underline{\xi}^+}^{1-\rho_2,\chi}D^{\chi}y(\underline{\xi}^+)}{\Gamma(\rho_2)}\Psi^{\rho_2-1}(\xi_{2k},\underline{\xi}) + \mathcal{I}_{\underline{\xi}^+}^{\rho_1,\chi}\aleph_2(\xi_{2k}, Y(\xi_{2k})),\ k = 1,\cdots,n_2. \end{cases}$$

By utilizing the second condition (**NIB**), we obtain

$$\begin{cases} \mathcal{I}_{\underline{\xi}^+}^{1-\rho_1,\chi}D^{\chi}y(\underline{\xi}^+) = \frac{\Gamma(\rho_1)\sum_{j=1}^{n_1}\zeta_{1j}\mathcal{I}_{\underline{\xi}^+}^{\rho_1,\chi}\aleph_1(\xi_{1j},Y(\xi_{1j}))}{\Gamma(\rho_1)-\sum_{j=1}^{n_1}\zeta_{1j}\Psi^{\rho_1-1}(\xi_{1j},\underline{\xi})}, \\ \mathcal{I}_{\underline{\xi}^+}^{1-\rho_2,\chi}D^{\chi}y(\underline{\xi}^+) = \frac{\Gamma(\rho_2)\sum_{k=1}^{n_2}\zeta_{2k}\mathcal{I}_{\underline{\xi}^+}^{\rho_2,\chi}\aleph_2(\xi_{2k},Y(\xi_{2k}))}{\Gamma(\rho_2)-\sum_{k=1}^{n_2}\zeta_{2k}\Psi^{\rho_2-1}(\xi_{2k},\underline{\xi})}. \end{cases}$$

By substituting, we deduce that

$$\begin{cases} D^{\chi}y(\xi) = \frac{\sum_{j=1}^{n_1}\zeta_{1j}\mathcal{I}_{\underline{\xi}^+}^{\rho_1,\chi}\aleph_1(\xi_{1j},Y(\xi_{1j}))}{\Gamma(\rho_1)-\sum_{j=1}^{n_1}\zeta_{1j}\Psi^{\rho_1-1}(\xi_{1j},\underline{\xi})}\Psi^{\rho_1-1}(\xi,\underline{\xi}) + \mathcal{I}_{\underline{\xi}^+}^{\rho_1,\chi}\aleph_1(\xi, Y(\xi)), \\ D^{\chi}y(\xi) = \frac{\sum_{k=1}^{n_2}\zeta_{2k}\mathcal{I}_{\underline{\xi}^+}^{\rho_2,\chi}\aleph_2(\xi_{2k},Y(\xi_{2k}))}{\Gamma(\rho_2)-\sum_{k=1}^{n_2}\zeta_{2k}\Psi^{\rho_2-1}(\xi_{2k},\underline{\xi})}\Psi^{\rho_2-1}(\xi,\underline{\xi}) + \mathcal{I}_{\underline{\xi}^+}^{\rho_1,\chi}\aleph_2(\xi, Y(\xi)). \end{cases} \tag{3.2}$$

Next, applying $I_{\underline{\xi}^+}^{\chi}$ to both sides of each equation of (3.2), we obtain

$$\begin{cases} y_1(\xi) = \Lambda_1 + \frac{\left(\sum_{j=1}^{n_1}\zeta_{1j}\mathcal{I}_{\underline{\xi}^+}^{\rho_1}\aleph_1(\xi_{1j},Y(\xi_{1j}))\right)\Psi^{\rho_1}(\xi_{1j},\underline{\xi})}{\Gamma(\rho_1+1)-\rho_1\sum_{j=1}^{n_1}\zeta_{1j}\Psi^{\rho_1-1}(\xi_{1j},\underline{\xi})} + \frac{1}{\Gamma(\rho_1+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_1}(\xi,s)\aleph_1(s,Y(s))ds, \\ y_2(\xi) = \Lambda_2 + \frac{\left(\sum_{k=1}^{n_2}\zeta_{2k}\mathcal{I}_{\underline{\xi}^+}^{\rho_2}\aleph_2(\xi_{2k},Y(\xi_{2k}))\right)\Psi^{\rho_2}(\xi_{2k},\underline{\xi})}{\Gamma(\rho_2+1)-\rho_2\sum_{k=1}^{n_2}\zeta_{2k}\Psi^{\rho_2-1}(\xi_{2k},\underline{\xi})} + \frac{1}{\Gamma(\rho_2+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_2}(\xi,s)\aleph_2(s,Y(s))ds. \end{cases}$$

Conversely, let $\left(y_1, y_2\right) \in \Pi_{\chi}^1([\underline{\xi},\overline{\xi}])$ be a solution of the system ($\mathbf{S}^*$), it is clear that $y_1(\underline{\xi}) = \Lambda_1$ and $y_1(\underline{\xi}) = \Lambda_1$. By applying $D_{\underline{\xi}^+}^{\chi}$ to both sides of each equation of ($\mathbf{S}^*$), we obtain the system of equations (3.2), Applying $\mathcal{I}_{\underline{\xi}^+}^{1-\rho_i,\chi}$ to both sides of the equation

$$D^{\chi}y(\xi) = \frac{\sum_{j=1}^{n_i}\zeta_{ij}\mathcal{I}_{\underline{\xi}^+}^{\rho_i,\chi}\aleph_i(\xi_{ij}, Y(\xi_{ij}))}{\Gamma(\rho_i) - \sum_{j=1}^{n_i}\zeta_{ij}\Psi^{\rho_i-1}(\xi_{ij},\underline{\xi})}\Psi^{\rho_1-1}(\xi,\underline{\xi}) + \mathcal{I}_{\underline{\xi}^+}^{\rho_1,\chi}\aleph_i(\xi, Y(\xi)),\ i = 1,\ 2, \tag{3.3}$$

and utilizing Lemma 2.2, we get

$$\mathcal{I}_{\underline{\xi}^+}^{1-\rho_i,\chi}D^{\chi}y(\xi) = \frac{\Gamma(\rho_i)\sum_{k=1}^{n_i}\zeta_{ij}\mathcal{I}_{\underline{\xi}^+}^{\rho_i,\chi}\aleph_i(\xi_{ij}, Y(\xi_{ij}))}{\Gamma(\rho_i) - \sum_{k=1}^{n_i}\zeta_{ij}\Psi^{\rho_i-1}(\xi_{ij},\underline{\xi})} + \mathcal{I}_{\underline{\xi}^+}^{1,\chi}\aleph_i(\xi, Y(\xi)),\ i = 1,\ 2$$

Taking $\xi \longrightarrow 0$, we get

$$\mathcal{I}_{\underline{\xi}^+}^{1-\rho_i,\chi}D^{\chi}y(\underline{\xi}^+) = \frac{\Gamma(\rho_i)\sum_{k=1}^{n_i}\zeta_{ij}\mathcal{I}_{\underline{\xi}^+}^{\rho_i,\chi}\aleph_i(\xi_{ij}, Y(\xi_{ij}))}{\Gamma(\rho_i) - \sum_{k=1}^{n_i}\zeta_{ij}\Psi^{\rho_i-1}(\xi_{ij},\underline{\xi})}\ i = 1,\ 2. \tag{3.4}$$

Substituting and adding side to side in the equation (3.3), we find,

$$\sum_{j=1}^{n_i}\zeta_{ij}D^{\chi}y(\xi_j) = \frac{\sum_{j=1}^{n_i}\zeta_{ij}\mathcal{I}_{\underline{\xi}^+}^{\rho_i,\chi}\aleph_i(\xi_{ij}, Y(\xi_{ij}))}{\Gamma(\rho_i) - \sum_{j=1}^{n_i}\zeta_{ij}\aleph_i(\xi_{ij}, Y(\xi_{ij}))}\sum_{j=1}^{n_i}\zeta_{ij}\Psi^{\rho_i-1}(\xi_{ij},\underline{\xi}) + \sum_{j=1}^{n_i}\zeta_{ij}\mathcal{I}_{\underline{\xi}^+}^{\rho_i,\chi}\aleph_i(\xi_{ij}, Y(\xi_{ij})),\ i = 1,\ 2. \tag{3.5}$$

From (3.4) and (3.5), we have

$$\mathcal{I}_{\underline{\xi}^+}^{1-\rho_i,\chi} D^{\chi} y(\underline{\xi}^+) = \sum_{j=1}^{n_i} \zeta_{ij} D^{\chi} y(\xi_j),\ i = 1,\ 2.$$

□

4. Main Results

In this section, we will prove the existence of χ-differential solutions of the system (**S**) − (**NIB**), also the compactness of its solution set. We necessarily assume the following hypotheses

(H$_1$) Suppose that $\aleph_i(., Y(.)) \in \mathcal{C}([\underline{\xi}, \overline{\xi}])$, for all $\left(y_1, y_2\right) \in \Pi_\chi^1([\underline{\xi}, \overline{\xi}])$, and there exists $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}_+,\ i = 1,\ 2,$ such that

(H$_{1-1}$) For all $\left(y_1, y_2\right), \left(\overline{y_1}, \overline{y_2}\right) \in \Pi_\chi^1([\underline{\xi}, \overline{\xi}]), \xi \in [\underline{\xi}, \overline{\xi}]$:

$$\begin{aligned} \|\aleph_i(\xi, Y(\xi)) - \aleph_i(\xi, \overline{Y}(\xi))\| &\le \alpha_i \|y_1(\xi) - \overline{y_1}(\xi)\| + \beta_i \|y_2(\xi) - \overline{y_2}(\xi)\| \\ &+ \gamma_i \|{}^C\mathcal{D}^{\rho_1,\chi}(y_1(\xi) - \overline{y_1}(\xi))\| + \delta_i \|{}^C\mathcal{D}^{\rho_2,\chi}(y_2(\xi) - \overline{y_2}(\xi))\|, i = 1,\ 2. \end{aligned}$$

(H$_{1-2}$) For each nonempty, bounded set $\Omega_i \subset C^1_{1-\rho_i,\chi}([\underline{\xi}, \overline{\xi}])$, for all $\xi \in (\underline{\xi}, \overline{\xi}]$, we have

$$\begin{aligned} \vartheta\Big(\aleph_i(\xi, Y(\xi))\Big) &\le \alpha \vartheta\Big(\Omega_1(\xi)\Big)\Big) + \beta_i \vartheta\Big(\Omega_2(\xi)\Big) \\ &+ \gamma_i \vartheta\Big({}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_1,\chi} \Omega_1(\xi)\Big) + \delta_i \vartheta\Big({}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_2,\chi} \Omega_2(\xi)\Big),\ i = 1,\ 2. \end{aligned}$$

where

$$\Omega_i(\xi) = \Big\{y_i(\xi),\ y_i \in C^1_{1-\rho_i,\chi}([\underline{\xi}, \overline{\xi}])\Big\} \text{ and}$$
$${}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_i,\chi} \Omega_i(\xi) = \Big\{{}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_i,\chi} y_i(\xi),\ y_i \in C^1_{1-\rho_i,\chi}([\underline{\xi}, \overline{\xi}])\Big\},\ i = 1,\ 2.$$

(H$_2$)

$$\Big[1 + (\rho_i + 1)\Big(\mathcal{T} \zeta_i^* n_i (\Psi_*^{\rho_i} + \rho_i) + \Psi_*^{1-\rho_i}\Big)\Big] A_0 < \frac{\Gamma(\rho_i + 2)}{2},\ i = 1,\ 2.$$

where

$$\mathcal{T} = \max\Big\{\frac{1}{|\Gamma(\rho_i + 1) - \rho_i \sum_{j=1}^{n_i} \zeta_j \Psi^{\rho_i - 1}(\xi_{ij}, \underline{\xi})|}, i = 1,\ 2\Big\},$$
$$A_0 = \max\Big\{\Big(\alpha_i + \beta_i + \gamma_i \Gamma(\rho_1) + \delta_i \Gamma(\rho_2)\Big)\psi_*^{\rho_i},\ i = 1,\ 2\Big\} \text{ and } \zeta_i^* = \max_{j=1,\ldots,n_i} \{\zeta_{ij}\}.$$

Define the operator $\Xi : \Pi_\chi^1([\underline{\xi}, \overline{\xi}]) \to \Pi_\chi^1([\underline{\xi}, \overline{\xi}])$ by

$$\Xi(y_1, y_2) = \begin{cases} \Xi_1(y_1, y_2), \\ \Xi_2(y_1, y_2), \end{cases}$$

where, for $i = 1,\ 2$, we have

$$\Xi_i(y_1, y_2)(\xi) = \Lambda_i + \frac{\Psi^{\rho_i}(\xi, \underline{\xi}) \sum_{j=1}^{n_i} \zeta_{ij} \mathcal{I}_{\underline{\xi}^+}^{\rho_i} \aleph_i(\xi_{ij}, Y(\xi_{ij}))}{\Gamma(\rho_i + 1) - \rho_i \sum_{j=1}^{n_i} \zeta_{ij} \Psi^{\rho_i - 1}(\xi_{ij}, \underline{\xi})} + \frac{1}{\Gamma(\rho_i + 1)} \int_{\underline{\xi}}^{\xi} \chi'(s) \Psi^{\rho_i}(\xi, s) \aleph_i(s, Y(s)) ds.$$

We note that, for all $\xi \in ([\underline{\xi}, \overline{\xi}]$ we have

$$D^{\chi}\Xi(y_1, y_2)(\xi) = \begin{cases} D^{\chi}\Xi_1(y_1, y_2)(\xi), \\ D^{\chi}\Xi_2(y_1, y_2)(\xi), \end{cases}$$

where, for $i = 1, 2$, we have

$$D^{\chi}\Xi_i(y_1, y_2)(\xi) = \frac{\Psi^{\rho_i-1}(\xi, \underline{\xi})\sum_{j=1}^{n_i}\zeta_{ij}\mathcal{I}_{\underline{\xi}^+}^{\rho_i}\aleph_i(\xi_{ij}, Y(\xi_{ij}))}{\Gamma(\rho_i) - \sum_{j=1}^{n_i}\zeta_{ij}\Psi^{\rho_i-1}(\xi_{ij}, \underline{\xi})} + \frac{1}{\Gamma(\rho_i)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_i-1}(\xi, s)\aleph_i(s, Y(s))ds.$$

4.1. Existence Results

In this part, we will present the result concerning the existence of χ - continuously differentiable solutions of the problem (**S**) − (**NIB**). First, we will give some useful lemmas to demonstrate this result.

Lemma 4.1. *We assume the hypotheses* ($\mathbf{H_1}$) *and* ($\mathbf{H_{1-1}}$) *are hold. Then*

(1) Ξ is bounded and continuous.
(2) $\Xi(B)$ is equicontinuous for all bounded subset B of $\Pi_{\chi}^1([\underline{\xi}, \overline{\xi}])$.

Proof. Let us show the axiom (1), we begin to prove that Ξ is bounded operator. Let $(y_1, y_2) \in \Pi_{\chi}^1([\underline{\xi}, \overline{\xi}]), \xi \in [\underline{\xi}, \overline{\xi}]$, it is clear to see that $\Xi(y_1, y_2) \in \Pi_{\chi}^1([\underline{\xi}, \overline{\xi}]), \xi \in [\underline{\xi}, \overline{\xi}]$. Using ($\mathbf{H_1}$) and ($\mathbf{H_{1-1}}$), for all $y \in B_r = \{(y_1, y_2) \in \Pi_{\chi}^1([\underline{\xi}, \overline{\xi}]) : \|(y_1, y_2)\|_{\chi}^1 < r\}$, $i = 1, 2$ and $\xi \in (\underline{\xi}, \overline{\xi}]$, we have

$$\begin{aligned}\|\Xi_i(y_1, y_2)(\xi)\| \le& \|\Lambda_i\| + \mathcal{T}\Psi^{\rho_i}(\xi, \underline{\xi})\sum_{j=1}^{n_i}|\zeta_{ij}|\mathcal{I}_{\underline{\xi}^+}^{\rho_i}\|\aleph_i(\xi_{ij}, Y(\xi_{ij}))\| + \frac{1}{\Gamma(\rho_i+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_i}(\xi, s)\|\aleph_i(s, Y(s))\|ds \\ \le& \|\Lambda_i\| + \mathcal{T}\zeta_i^* n_i\Psi_*^{\rho_i}\left[\frac{\hbar_i^*\Psi_*^{\rho_i}}{\Gamma(\rho_i+1)} + \frac{rA_0}{\Gamma(\rho_i+1)}\right] + \frac{\hbar_i^*\Psi_*^{\rho_i+1}}{\Gamma(\rho_i+2)} + \frac{rA_0}{\Gamma(\rho_i+2)},\end{aligned}$$

where $\hbar_i^* = \sup_{t\in[\underline{\xi}, \overline{\xi}]}\hbar_i(\xi, 0, 0, 0, 0)$, $i = 1, 2$, we also have, for all $\xi \in (\underline{\xi}, \overline{\xi}]$

$$\begin{aligned}\|\Psi^{1-\rho_i}(\xi, \underline{\xi})D^{\chi}\Xi_i(y_1, y_2)(\xi)\| \le& \rho_i\mathcal{T}\sum_{j=1}^{n_i}|\zeta_{ij}|\mathcal{I}_{\underline{\xi}^+}^{\rho_i}\|\aleph_i(\xi_{ij}, Y(\xi_{ij}))\| + \frac{1}{\Gamma(\rho_i)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_i-1}(\xi, s)\|\aleph_i(s, Y(s))\|ds \\ \le& \left(\rho_i\mathcal{T}\zeta_i^* n_i + \Psi_*^{1-\rho_i}\right)\left[\frac{\hbar_i^*\Psi_*^{\rho_i}}{\Gamma(\rho_i+1)} + \frac{rA_0}{\Gamma(\rho_i+1)}\right].\end{aligned}$$

So,

$$\begin{aligned}\|\Xi_i(y_1, y_2)\|_{\infty} + \|D^{\chi}\Xi_i(y_1, y_2)\|_{\rho_i,\chi} \le& \left(\mathcal{T}\zeta_i^* n_i(\rho_i + \Psi_*^{\rho_i}) + \Psi_*^{1-\rho_i}\right)\left[\frac{\hbar_i^*\Psi_*^{\rho_i}}{\Gamma(\rho_i+1)} + \frac{rA_0}{\Gamma(\rho_i+2)}\right] \\ &+ \frac{\hbar_i^*\Psi_*^{\rho_i+1}}{\Gamma(\rho_i+2)} + \frac{rA_0}{\Gamma(\rho_i+2)} = \mathbf{M_i}.\end{aligned}$$

Thus, $\|\Xi(y_1, y_2)\|_{\chi}^1 \le \max\{\mathbf{M_1}, \mathbf{M_2}\}$.
Now we will show that Ξ is continuous. Let $\{(y_{1n}(.), y_{2n}(.))\}_{n\in\mathbb{N}}$ be a sequence converges to $(y_1^*(.), y_2^*(.))$ in $\Pi_{\chi}^1([\underline{\xi}, \overline{\xi}])$, it enough to prove $\Xi_i(y_{1n}, y_{2n})(.) \to \Xi_i(y_1^*, y_2^*)(.)$ as $n \to \infty$ in $C_{1-\rho_i.\chi}^1([\underline{\xi}, \overline{\xi}])$, $i = 1, 2$, from ($\mathbf{H_{1-1}}$) and Lemma 2.2 we can easily prove that $\Xi_i(y_{1n}, y_{2n})(.) \to \Xi_i(y_1^*, y_2^*)(.)$ in $C([\underline{\xi}, \overline{\xi}])$ and $D^{\chi}\Xi_i(y_{1n}, y_{2n})(.) \to$

$D^{\chi}\Xi_i(y_1^*, y_2^*)(.)$ in $C_{1-\rho_i,\chi}([\underline{\xi}, \overline{\xi}])$, $i = 1, 2$, that implies $\Xi(y_{1n}, y_{2n})(.) \to \Xi(y_1^*, y_2^*)(.)$ in $C^1_{1-\rho_i\cdot\chi}([\underline{\xi}, \overline{\xi}])$, then Ξ is continuous.

Let us show the second axiom (2), it is enough to show that $\Xi_i(B_r)$ (resp. $D^{\chi}\Xi_i(B_r)$) is equicontinuous on $C([\underline{\xi}, \overline{\xi}])$ (resp. on $C_{1-\rho_i,\chi}([\underline{\xi}, \overline{\xi}])$), $i = 1, 2$. Let $(y_1, y_2) \in B_r$ and $\xi_1, \xi_2 \in (\underline{\xi}, \overline{\xi}]$ with $\xi_1 < \xi_2$, from $(\mathbf{H_{1-1}})$, we have

$$\begin{aligned}
\|\Xi_i(y_1, y_2)(\xi_2) - \Xi_i(y_1, y_2)(\xi_1)\| \leq{}& \mathcal{T}\zeta_i^* n_i \Psi_*^{\rho_i}\Big[\frac{rA_0}{\Gamma(\rho_i+1)} + \frac{\hbar_i^* \Psi_*^{\rho_i}}{\Gamma(\rho_i+1)}\Big]\Big(\Psi^{\rho_i}(\xi_2, \underline{\xi}) - \Psi^{\rho_i}(\xi_1, \underline{\xi})\Big) \\
&+ \frac{1}{\Gamma(\rho_i+1)}\int_{\underline{\xi}}^{\xi_1} \chi'(s)[\Psi^{\rho_i}(\xi_2, s) - \Psi^{\rho_i}(\xi_1, s)]\|\aleph_i(s, Y(s))\|ds \\
&+ \frac{1}{\Gamma(\rho_i+1)}\int_{\xi_1}^{\xi_2} \chi'(s)\Psi^{\rho_i}(\xi_2, s)\|\aleph_i(s, Y(s))\|ds \\
\leq{}& \mathcal{T}\zeta_i^* n_i \Psi_*^{\rho_i}\Big[\frac{rA_0}{\Gamma(\rho_i+1)} + \frac{\hbar_i^* \Psi_*^{\rho_i}}{\Gamma(\rho_i+1)}\Big]\Big(\Psi^{\rho_i}(\xi_2, \underline{\xi}) - \Psi^{\rho_i}(\xi_1, \underline{\xi})\Big) \\
&+ \frac{\hbar_i^* + r\Psi_*^{-\rho_i}A_0}{\Gamma(\rho_i+2)}\Big[\Psi^{\rho_i+1}(\xi_2, \underline{\xi}) - \Psi^{\rho_i+1}(\xi_1, \underline{\xi}) + 2\Psi^{\rho_i+1}(\xi_2, \xi_1)\Big].
\end{aligned}$$

As ξ_2 tends to ξ_1, the right-hand side of the last inequality tends to 0. Therefore $\Xi_i(B_r)$, $i = 1, 2$ is equicontinuous on $C([\underline{\xi}, \overline{\xi}])$. And, we also have

$$\begin{aligned}
&\|\Psi^{1-\rho_i}(\xi_2, \underline{\xi})D^{\chi}\Xi(y_1, y_2)(\xi_2) - \Psi^{1-\rho_i}(\xi_1, \underline{\xi})D^{\chi}\Xi(y_1, y_2)(\xi_1)\| \leq \\
&\quad \Bigg\| \frac{\Psi^{1-\rho_i}(\xi_2, \underline{\xi})}{\Gamma(\rho_i)}\int_{\underline{\xi}}^{\xi_2} \chi'(s)\Psi^{\rho_i-1}(\xi_2, s)\aleph_i(s, Y(s))ds \\
&\quad - \frac{\Psi^{1-\rho_i}(\xi_1, \underline{\xi})}{\Gamma(\rho_i)}\int_{\underline{\xi}}^{\xi_1} \chi'(s)\Psi^{\rho_i}(\xi_1, s)\aleph_i(s, Y(s)) \Bigg\| \\
&\leq \frac{\Psi^{1-\rho_i}(\xi_1, \underline{\xi})}{\Gamma(\rho_i)}\int_{\underline{\xi}}^{\xi_1} \chi'(s)\Big[\Psi^{\rho_i-1}(\xi_1, s) - \Psi^{\rho_i-1}(\xi_2, s)\Big]\|\aleph_i(s, Y(s))\|ds \\
&\quad + \frac{\Psi^{1-\rho_i}(\xi_2, \underline{\xi}) - \Psi^{1-\rho_i}(\xi_1, \underline{\xi})}{\Gamma(\rho_i)}\int_{\underline{\xi}}^{\xi_1} \chi'(s)\Psi^{\rho_i-1}(\xi_2, s)\|\aleph_i(s, Y(s))\|ds \\
&\quad + \frac{\Psi^{1-\rho_i}(\xi_2, \underline{\xi})}{\Gamma(\rho_i)}\int_{\xi_1}^{\xi_2} \chi'(s)\Psi^{\rho_i-1}(\xi_2, s)\|\aleph_i(s, Y(s))\|ds \\
&\leq \frac{\Big(\hbar_i^* + r\Psi_*^{-\rho_i}A_0\Big)\Psi^{1-\rho_i}(\xi_1, \underline{\xi})}{\Gamma(\rho_i)}\int_{\underline{\xi}}^{\xi_1} \chi'(s)\Big[\Psi^{\rho_i-1}(\xi_1, s) - \Psi^{\rho_i-1}(\xi_2, s)\Big]ds \\
&\quad + \frac{\Big(\hbar_i^* + r\Psi_*^{-\rho_i}A_0\Big)\Big(\Psi^{1-\rho_i}(\xi_2, \underline{\xi}) - \Psi^{1-\rho_i}(\xi_1, \underline{\xi})\Big)}{\Gamma(\rho_i)}\int_{\underline{\xi}}^{\xi_1} \chi'(s)\Psi^{\rho_i-1}(\xi_2, s)ds \\
&\quad + \frac{\Big(\hbar_i^* + r\Psi_*^{-\rho_i}A_0\Big)\Psi^{1-\rho_i}(\xi_2, \underline{\xi})}{\Gamma(\rho_i)}\int_{\xi_1}^{\xi_2} \chi'(s)\Psi^{\rho_i-1}(\xi_2, s)ds \\
&\leq \frac{\Big(\hbar_i^* + r\Psi_*^{-\rho_i}A_0\Big)\Psi_*^{1-\rho_i}}{\Gamma(\rho_i+1)}\Big[\Psi^{\rho_i}(\xi_2, \underline{\xi}) - \Psi^{\rho_i}(\xi_1, \underline{\xi}) + 2\Psi^{\rho_i}(\xi_2, \xi_1)\Big]
\end{aligned}$$

$$+\frac{\left(\hbar_i^* + r\Psi_*^{-\rho_i}A_0\right)\Psi_*^{\rho_i}}{\Gamma(\rho_i)}\left[\Psi^{1-\rho_i}(\xi_2,\underline{\xi}) - \Psi^{1-\rho_i}(\xi_1,\underline{\xi})\right].$$

By taking ξ_2 tends to ξ_1, the right-hand side of the last inequality tends to 0, and hence $D^\chi\Xi_i(B_r),\ i=1,\ 2$ is equicontinuous on $C_{1-\rho_i,\chi}([\underline{\xi},\overline{\xi}])$, thus, $\Xi(B_r)$ is equicontinuous on $\Pi^1_\chi([\underline{\xi},\overline{\xi}])$. □

We denote by $\vartheta_C,\ \vartheta_{\rho_i},\ \vartheta^1_{\rho_i}$ and ϑ^1_χ the Kuratowski measure of noncompactness defined respectively on $C([\underline{\xi},\overline{\xi}]))$, $C_{1-\rho_i,\chi}([\underline{\xi},\overline{\xi}]))$, $C^1_{1-\rho_i,\chi}([\underline{\xi},\overline{\xi}]))$ and $\Pi^1_\chi([\underline{\xi},\overline{\xi}])$.

Lemma 4.2. *Let $B = B_1\times B_2$ be a bounded subset of $\Pi^1_\chi([\underline{\xi},\overline{\xi}])$, we have*

(i) $\vartheta^1_{\rho_i}(B_i) \le \vartheta(B_i) + \vartheta_{\rho_i}(D^\chi B_i) \le 2\vartheta^1_{\rho_i}(B_i),\ i=1,\ 2.$

(ii) $\vartheta^1_\chi(B) = \max\left\{\vartheta^1_{\rho_1}(B_1),\ \vartheta^1_{\rho_2}(B_2)\right\}.$

Proof. Let $B = B_1\times B_2$ be a bounded subset of $\Pi^1_\chi([\underline{\xi},\overline{\xi}])$. Let us show the axiom (*i*), we have $B_i, i=1,\ 2$ is a bounded subset of $C^1_{1-\rho_i,\chi}([\underline{\xi},\overline{\xi}]))$, let ϵ be a strictly positive real number. So, for $i=1,\ 2.$ there exists a finite partition $B_i^j,\ j=1,\cdots m$, such that

$$\mathrm{Diam}(B_i^j) \le \epsilon + \vartheta^1_{\rho_i}(B_i),\quad j=1,\cdots m.$$

Then for all y_1, y_2 in $B_i^j,\ j=1,\cdots m$ and $\xi\in(\underline{\xi},\overline{\xi}]$, we have

$$\|y_2(\xi)-y_1(\xi)\| \le \epsilon + \vartheta^1_{\rho_i}(B_i) \text{ and } \|D^\chi y_2(\xi) - D^\chi y_1(\xi)\| \le \epsilon + \vartheta^1_{\rho_i}(B_i).$$

So,

$$\mathrm{Diam}(B_i^j) \le \epsilon + \vartheta^1_{\rho_i}(B_i) \text{ and } \mathrm{Diam}(D^\chi B_i^j) \le \epsilon + \vartheta^1_{\rho_i}(B_i),\ j=1,\cdots m.$$

Thus,

$$\vartheta(B_i) + \vartheta_{\rho_i}(D^\chi B_i) \le 2\epsilon + 2\vartheta^1_{\rho_i}(B_i).$$

Since ϵ is arbitrary, this means that we arrive at

$$\vartheta(B_i) + \vartheta_{\rho_i}(D^\chi B_i) \le 2\vartheta^1_{\rho_i}(B_i),\ i=1,\ 2. \tag{4.1}$$

Conversely, we want to prove that $\vartheta^1_{\rho_i}(B) \le \vartheta(B) + \vartheta_{\rho_i}(D^\chi B)$, from the definition of Kuratowski measure of noncompactness, we have, for each $\epsilon > 0$, there are a finite partitions $\{B_i^j\}_{j=1,\cdots,m_1}$ of B_i and $\{D_i^k\}_{k=1,\cdots,m_2}$ of $D^\chi B_i$ such that

$$\mathrm{Diam}(B_i^j) \le \epsilon + \vartheta(B_i),\ \text{ and }\ \mathrm{Diam}(D_i^k) \le \epsilon + \vartheta_{\rho_i}(D^\chi B_i),$$

it is clear that the partition $\{B_i^j \cap I^\chi_{\underline{\xi}^+} D_i^k\}_{j,k}$ belongs to $C^1_{1-\rho_i,\chi}([\underline{\xi},\overline{\xi}]))$ and verifies the following inequality

$$\mathrm{Diam}(B_i^j \cap I^\chi_{\underline{\xi}^+} D_i^k) + \mathrm{Diam}(D^\chi(B_i^j \cap I^\chi_{\underline{\xi}^+} D_i^k)) \le 2\epsilon + \vartheta(B_i) + \vartheta_{\rho_i}(D^\chi B_i).$$

As ϵ is arbitrary, we obtain

$$\vartheta^1_{\rho_i}(B_i) \le \vartheta(B_i) + \vartheta_{\rho_i}(D^\chi B_i),\ i=1,\ 2. \tag{4.2}$$

From (4.1)-(4.2), we get

$$\vartheta^1_{\rho_i}(B) \le \vartheta(B) + \vartheta_{\rho_i}(D^\chi B) \le 2\vartheta^1_{\rho_i}(B).$$

Let us prove the second axiom (*ii*), Let $B = B_1\times B_2$ be a bounded subset of $\Pi^1_\chi([\underline{\xi},\overline{\xi}])$, for all $\epsilon > 0$, there exists a finite partition $B^j,\ j=1,\cdots m$, such that,

$$\mathrm{Diam}(B^j) \le \epsilon + \vartheta^1_\chi(B),\quad j=1,\cdots m.$$

Then for all $(x_1, y_1), (x_2, y_2)$ in $B^j = B_1^j \times B_2^j,\ j = 1, \cdots m$ and $\xi \in (\underline{\xi}, \overline{\xi}]$, we have

$$\max\left\{\|x_2(\xi) - x_1(\xi)\|, \|y_2(\xi) - y_1(\xi)\|\right\} \leq \epsilon + \vartheta_\chi^1(B),$$

that implies

$$\|x_2(\xi) - x_1(\xi)\| \leq \epsilon + \vartheta_\chi^1(B) \text{ and } \|x_2(\xi) - x_1(\xi)\| \leq \epsilon + \vartheta_\chi^1(B).$$

So,

$$\mathrm{Diam}(B_1^j) \leq \epsilon + \vartheta_\chi^1(B) \text{ and } \mathrm{Diam}(B_2^j) \leq \epsilon + \vartheta_\chi^1(B),\ j = 1, \cdots m.$$

As $B_i \subset \cup_j B_i^j,\ i = 1,\ 2$, we have

$$\max\left\{\vartheta_{\rho_i}^1(B_i),\ i = 1,\ 2\right\} \leq \epsilon + \vartheta_\chi^1(B).$$

As ϵ is arbitrary, we obtain

$$\max\left\{\vartheta_{\rho_i}^1(B_i),\ i = 1,\ 2\right\} \leq \vartheta_\chi^1(B). \tag{4.3}$$

Conversely, for all $\epsilon > 0$, there are a finite partitions $\{B_1^j\}_{j=1,\cdots,m_1}$ of B_1 and $\{B_2^k\}_{k=1,\cdots,m_2}$ of B_2 such that

$$\mathrm{Diam}(B_1^j) \leq \epsilon + \vartheta_{\rho_1}^1(B_1), \text{ and } \mathrm{Diam}(B_2^k) \leq \epsilon + \vartheta_{\rho_2}^1(B_2),$$

it is clear that the partition $\bigcup_{j,k} B_1^j \times B_2^k$ belongs to $\Pi_\chi^1([\underline{\xi}, \overline{\xi}])$ and verifies the following inequality

$$\mathrm{Diam}(B_1^j \times B_2^k) \leq \epsilon + \max\left\{\vartheta_{\rho_i}^1(B_i),\ i = 1,\ 2\right\}.$$

Since ϵ is arbitrary, we get

$$\vartheta_\chi^1(B) \leq \max\left\{\vartheta_{\rho_i}^1(B_i),\ i = 1,\ 2\right\}. \tag{4.4}$$

From (4.3)-(4.4), we have

$$\vartheta_\chi^1(B) = \max\left\{\vartheta_{\rho_i}^1(B_i),\ i = 1,\ 2\right\}.$$

□

From Lemma 2.7 and Lemma 4.2, we easily show the following inequality

$$\vartheta_{\rho_i}^1(D) \leq \sup_{\xi \in [\underline{\xi}, \overline{\xi}]} \vartheta(D(\xi)) + \sup_{\xi \in [\underline{\xi}, \overline{\xi}]} \vartheta(\Psi^{1-\rho_i}(\xi, \underline{\xi}) D^\chi D(\xi)) \leq 2\vartheta_{\rho_i}^1(D),\ i = 1,\ 2, \tag{4.5}$$

where D is a bounded and equicontinuous subset of $C_{1-\rho_i,\chi}^1([\underline{\xi}, \overline{\xi}]))$, $D(\xi) = \{y(\xi) : y \in D\}$ and $D^\chi D(\xi) = \{D^\chi y(\xi) : y \in D\}$.
Let

$$B_R = \left\{(y_1, y_2) \in \Pi_\chi^1([\underline{\xi}, \overline{\xi}]) : \|(y_1, y_2)\|_\chi^1 \leq R\right\}.$$

We are about to present our main result which is as follows.

Theorem 4.3. *Assume that the hypotheses* $(\mathbf{H_1}) - (\mathbf{H_2})$ *are satisfied and that* R *verifies the following inequality*

$$R > \max\left\{\frac{(\rho_i + 1)\Big(\mathcal{T}\zeta_i^* n_i(\Psi_*^{\rho_i} + \rho_i) + \Psi_*^{1-\rho_i}\Big)\Psi_*^{\rho_i}\hbar_i^* + \Psi_*^{\rho_i+1}\hbar_i^* + \|\Lambda_i\|\Gamma(\rho_i + 2)}{\Gamma(\rho_i + 2) - \Big[1 + (\rho_i + 1)\Big(\mathcal{T}\zeta_i^* n_i(\Psi_*^{\rho_i} + \rho_i) + \Psi_*^{1-\rho_i}\Big)\Big]A_0},\ i = 1,\ 2\right\}. \tag{4.6}$$

Then, Problem $(\mathbf{S}) - (\mathbf{NIB})$ *has at least one solution in* $\Pi_\chi^1([\underline{\xi}, \overline{\xi}])$*. In addition, the soluion set* $\mathbf{SS}$ *of the problem* $(\mathbf{S}) - (\mathbf{NIB})$ *is compact.*

Proof. By using Lemma 3.1, it is clear that the fixed points of the operator Ξ are solutions of the problem $(\mathbf{S})-(\mathbf{NIB})$. We want to verify that Ξ satisfies the assumptions of Mönch fixed point theorem. First, we will prove that Ξ is well defined from B_R to B_R, indeed, let $(y_1, y_2) \in B_R$. By using the condition $(\mathbf{H_{1-1}})$ and after some calculations, for each $\xi \in (\underline{\xi}, \overline{\xi}]$, $i = 1, 2$ and $(y_1, y_2) \in B_R$, we get

$$\begin{aligned}
&\|\Xi_i(y_1,y_2)(\xi)\| + \|\Psi^{1-\rho_i}(\xi,\underline{\xi})D_i^{\chi}\Xi(y_1,y_2)(\xi)\| \le \|\Lambda_i\| + \mathcal{T}\Psi^{\rho_i}(\xi,\underline{\xi})\sum_{j=1}^{n_i}|\zeta_{ij}|\mathcal{I}_{\underline{\xi}^+}^{\rho_i}\|\aleph_i(\xi_{ij},Y(\xi_{ij}))\| \\
&\quad + \frac{1}{\Gamma(\rho_i+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_i}(\xi,s)\|\aleph_i(s,Y(s))\|ds + \rho_i\mathcal{T}\sum_{j=1}^{n_i}|\zeta_{ij}|\mathcal{I}_{\underline{\xi}^+}^{\rho_i}\|\aleph_i(\xi_{ij},Y(\xi_{ij}))\| \\
&\quad + \frac{1}{\Gamma(\rho_i)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_i-1}(\xi,s)\|\aleph_i(s,Y(s))\|ds \\
&\quad \le \frac{(\rho_i+1)\Big(\mathcal{T}\zeta_i^* n_i(\Psi_*^{\rho_i}+\rho_i)+\Psi_*^{1-\rho_i}\Big)\Psi_*^{\rho_i}\hbar_i^* + \Psi_*^{\rho_i+1}\hbar_i^* + \|\Lambda_i\|\Gamma(\rho_i+2)}{\Gamma(\rho_i+2)} \\
&\quad + \frac{\Big[1+(\rho_i+1)\Big(\mathcal{T}\zeta_i^* n_i(\Psi_*^{\rho_i}+\rho_i)+\Psi_*^{1-\rho_i}\Big)\Big]A_0}{\Gamma(\rho_i+2)}R.
\end{aligned}$$

From $(\mathbf{H_2})$ and the inequality (4.6), we obtain

$$\forall (y_1,y_2)\in B_R : \|\Xi(y_1,y_2)\|_{\chi}^{1} < R.$$

Note that B_R is bounded, convex and closed subset of $\Pi_{\chi}^1([\underline{\xi},\overline{\xi}])$ and Ξ is continuous on B_R. Next, it is enough to show the following implication

$$V \subset \overline{conv}\{\Xi V \cup \{0\}\} \Longrightarrow \vartheta_{\chi}^1(V) = 0, \text{ for any } V \subset B_R.$$

Let $V = V_1 \times V_2$ be a subset of B_R such that $V \subset \overline{conv}\{\Xi V \cup \{0\}\}$. By using Lemmas 2.6 and 2.7, for all $\xi \in (\underline{\xi}, \overline{\xi}]$, $i = 1, 2$, we obtain

$$\begin{aligned}
&\vartheta(\Xi_i V(\xi)) + \vartheta\Big(\psi_{1-\rho_i}(\xi,\underline{\xi})D^{\chi}\Xi_i(V(\xi))\Big) \le \\
&\Psi_*^{\rho_i}\mathcal{T}\Bigg[\sum_{j=1}^{n_i}\zeta_{ij}\vartheta\mathcal{I}_{\underline{\xi}^+}^{\rho_i}\vartheta\Big(\hbar_i\Big(\xi_{ij},V_1(\xi_{ij}),V_2(\xi_{ij}),{}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_1,\chi}V_1(\xi_{ij}),{}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_2,\chi}V_2(\xi_{ij})\Big)\Big)\Bigg] \\
&+ \frac{1}{\Gamma(\rho_i+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_i}(\xi,s)\vartheta\Big(\hbar_i\Big(s,V_1(s),V_2(s),{}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_1,\chi}V_1(s),{}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_2,\chi}V_2(s)\Big)\Big)ds \\
&+ \rho_i\mathcal{T}\sum_{j=1}^{n_i}|\zeta_{ij}|\mathcal{I}_{\underline{\xi}^+}^{\rho_i}\vartheta\Big(\hbar_i\Big(\xi_{ij},V_1(\xi_{ij}),V_2(\xi_{ij}),{}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_1,\chi}V_1(\xi_{ij}),{}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_2,\chi}V_2(\xi_{ij})\Big)\Big) \\
&+ \frac{1}{\Gamma(\rho_i)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_i-1}(\xi,s)\vartheta\Big(\hbar_i\Big(s,V_1(s),V_2(s),{}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_1,\chi}V_1(s),{}^C\mathcal{D}_{\underline{\xi}^+}^{\rho_2,\chi}V_2(s)\Big)\Big)ds.
\end{aligned}$$

From Lemmas 2.7, 4.1 and 4.2 and the hypotheses $(\mathbf{H_{1-2}})-(\mathbf{H_2})$ and inequality (4.5), we arrive at

$$\begin{aligned}
\vartheta_{\rho_i}^1(\Xi_i V) &\le \sup_{\xi\in[\underline{\xi},\overline{\xi}]}\vartheta(\Xi_i V(\xi)) + \sup_{\xi\in[\underline{\xi},\overline{\xi}]}\vartheta\Big(\psi_{1_{\rho_i}(\xi,\underline{\xi})}D^{\chi}\Xi_i(V(\xi))\Big) \\
&\le \frac{2\Big[1+(\rho_i+1)\Big(\mathcal{T}\zeta_i^* n_i(\Psi_*^{\rho_i}+\rho_i)+\Psi_*^{1-\rho_i}\Big)\Big]A_0}{\Gamma(\rho_i+2)}\vartheta_{\rho_i}^1(\Xi_i V).
\end{aligned}$$

From Lemma 4.2, we have

$$\vartheta^1_\chi(\Xi V) \le \max\Big\{\frac{2\Big[1+(\rho_i+1)\Big(\mathcal{T}\zeta^*_i n_i(\Psi^{\rho_i}_* + \rho_i) + \Psi^{1-\rho_i}_*\Big)\Big]A_0}{\Gamma(\rho_i+2)},\ i=1,\ 2\Big\}\vartheta^1_\chi(\Xi V).$$

By the condition ($\mathbf{H_2}$), we get $\vartheta^1_\chi(\Xi V)=0$, that means $\vartheta^1_\chi(V)=0$. From the theorem 2.8, the operator Ξ has at least one fixed point $(y_1,y_2)\in B_R$. By using Lemma 3.1, we conclude that the problem (**S**) − (**NIB**) has at least one solution.

4.2. Compactness of Solution Set

In this part, we will show that the soluion set of the problem (**S**) − (**NIB**) is compact subset of $\Pi^1_\chi([\underline{\xi},\overline{\xi}])$. Let $\{(y^n_1,y^n_2)\}_{n\in\mathbb{N}}$ be a sequence of the solution set, as $\Pi^1_\chi([\underline{\xi},\overline{\xi}])$ is compact space, there exists a subsequence of $\{(y^n_1,y^n_2)\}_{n\in\mathbb{N}}$ (still denoted $\{(y^n_1,y^n_2)\}_{n\in\mathbb{N}}$) converges to (y^*_1,y^*_2), it enough to demonstrate that (y^*_1,y^*_2) is a solution of (**S**) − (**NIB**), for each $\xi\in(\underline{\xi},\overline{\xi}]$, $i=1,\ 2$, we have

$$y^n_i(\xi) = \Lambda_i + \mathcal{T}\Psi^{\rho_i}(\xi,\underline{\xi})\sum_{j=1}^{n_i}|\zeta_{ij}|\mathcal{I}^{\rho_i}_{\underline{\xi}^+}\aleph_i(\xi_{ij},Y^n(\xi_{ij})) + \frac{1}{\Gamma(\rho_i+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_i}(\xi,s)\aleph_i(s,Y^n(s))ds,$$

and

$$D^\chi y^n_i(\xi) = \rho_i\mathcal{T}\sum_{j=1}^{n_i}|\zeta_{ij}|\mathcal{I}^{\rho_i}_{\underline{\xi}^+}\aleph_i(\xi_{ij},Y^n((\xi_{ij})) + \frac{1}{\Gamma(\rho_i)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_i-1}(\xi,s)\aleph_i(s,Y^n(s))ds.$$

From ($\mathbf{H_1}$), we have $\aleph_i(.,Y^n(.)) = \hbar_i\Big(.,y^n_1(.),y^n_2(.),^C\mathcal{D}^{\rho_1,\chi}_{\underline{\xi}^+}y^n_1(.),^C\mathcal{D}^{\rho_2,\chi}_{\underline{\xi}^+}y^n_2(.)\Big)$ simply converges to $\aleph_i(.,Y^*(.)) = \hbar_i\Big(.,y^*_1(.),y^*_2(.),^C\mathcal{D}^{\rho_1,\chi}_{\underline{\xi}^+}y^*_1(.),^C\mathcal{D}^{\rho_2,\chi}_{\underline{\xi}^+}y^*_2(.)\Big)$ as $n\to+\infty$, let $\xi\in(\underline{\xi},\overline{\xi}]$, form ($\mathbf{H_{1-1}}$), for all $n\in\mathbb{N}$, $i=1,\ 2$, we have

$$\chi'(s)\Psi^{\rho_i}(\xi,s)\|\aleph_i(s,Y^n(s))\| \le \Big(\hbar^*_i + \big(\Big(\alpha_i+\beta_i+\gamma_i\Gamma(\rho_1)+\delta_i\Gamma(\rho_2)\Big)M\Big)\chi'(s)\Psi^{\rho}(\xi,s) \text{ and}$$
$$\chi'(s)\Psi^{\rho_i-1}(\xi,s)\|\aleph_i(s,Y^n(s))\| \le \Big(\hbar^*_i + \Big(\alpha_i+\beta_i+\gamma_i\Gamma(\rho_1)+\delta_i\Gamma(\rho_2)\Big)M\Big)\chi'(s)\Psi^{\rho-1}(\xi,s).$$

Using Lebesgue's dominated convergence theorem, for each $\xi\in(\underline{\xi},\overline{\xi}]$, $i=1,\ 2$, we obtain

$$y^*_i(\xi) = \Lambda_i + \mathcal{T}\Psi^{\rho_i}(\xi,\underline{\xi})\sum_{j=1}^{n_i}|\zeta_{ij}|\mathcal{I}^{\rho_i}_{\underline{\xi}^+}\aleph_i(\xi_{ij},Y^*(\xi_{ij})) + \frac{1}{\Gamma(\rho_i+1)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_i}(\xi,s)\aleph_i(s,Y^*(s))ds,$$

and

$$D^\chi y^*_i(\xi) = \rho_i\mathcal{T}\sum_{j=1}^{n_i}|\zeta_{ij}|\mathcal{I}^{\rho_i}_{\underline{\xi}^+}\aleph_i(\xi_{ij},Y^*(\xi_{ij})) + \frac{1}{\Gamma(\rho_i)}\int_{\underline{\xi}}^{\xi}\chi'(s)\Psi^{\rho_i-1}(\xi,s)\aleph^*_i(s,Y^*(s))ds.$$

Thus, the soluion set of the problem (**S**) − (**NIB**) is compact subset of $\Pi^1_\chi([\underline{\xi},\overline{\xi}])$. □

5. Example

We take $\chi(t)=\frac{4\arctan t}{\pi}$, $\underline{\xi}=0,\overline{\xi}=1$, $\rho_1=\rho_2=0.25$, $n_1=n_2=1$, $\xi_{11}=\xi_{21}=1$ and E the Banach space defined by

$$E=\Big\{\Big(y^n\Big)_{n\in\mathbb{N}} : \sup_n|y^n|<\infty\Big\},$$

with the norm $\|y\| = \sup_n |y_n|$, we define the function $\aleph_i(., Y(.)),\ i = 1,\ 2$ by

$$\aleph_i(., Y(.)) = \Big(\aleph_i(., Y^1(.)), \ldots, \aleph_i(.Y^n(.)), \ldots\Big),,\ i = 1,\ 2$$

where

$$\aleph_1(\xi, Y^n(\xi)) = \frac{y_1^n(\xi)}{40+\xi^n} + \frac{y_2^n(\xi)}{39+e^{n\xi}} + \frac{{}^C\mathcal{D}^{\rho_1,\chi}_{\underline{\xi}^+} y_1^n(\xi)}{40+n\xi^2} + \frac{{}^C\mathcal{D}^{\rho_2,\chi}_{\underline{\xi}^+} y_2^n(\xi)}{39+e^{n\xi}} \text{ and}$$
$$\aleph_2(\xi, Y^n(\xi)) = \frac{y_1^n(\xi) + y_2^n(\xi) + {}^C\mathcal{D}^{\rho_1,\chi}_{\underline{\xi}^+} y_1^n(\xi) + {}^C\mathcal{D}^{\rho_2,\chi}_{\underline{\xi}^+} y_2^n(\xi)}{20+30e^{n\xi}},\ \ \xi \in (0,1].$$

We easily see that for all $\big(y_1, y_2\big), \big(\overline{y_1}, \overline{y_2}\big) \in \Pi^1_\chi([\underline{\xi}, \overline{\xi}]), \xi \in [\underline{\xi}, \overline{\xi}]$:

$$\begin{aligned}\|\aleph_1(\xi, Y(\xi)) - \aleph_1(\xi, \overline{Y}(\xi))\| \leq \frac{1}{40}\Big[&\|y_1(\xi) - \overline{y_1}(\xi)\| + \|y_2(\xi) - \overline{y_2}(\xi)\| \\ &+ \|{}^C\mathcal{D}^{\rho_1,\chi}(y_1(\xi) - \overline{y_1}(\xi))\| + \|{}^C\mathcal{D}^{\rho_2,\chi}(y_2(\xi) - \overline{y_2}(\xi))\|\Big] \text{ and} \\ \|\aleph_2(\xi, Y(\xi)) - \aleph_2(\xi, \overline{Y}(\xi))\| \leq \frac{1}{50}\Big[&\|y_1(\xi) - \overline{y_1}(\xi)\| + \|y_2(\xi) - \overline{y_2}(\xi)\| \\ &+ \|{}^C\mathcal{D}^{\rho_1,\chi}(y_1(\xi) - \overline{y_1}(\xi))\| + \|{}^C\mathcal{D}^{\rho_2,\chi}(y_2(\xi) - \overline{y_2}(\xi))\|\Big].\end{aligned}$$

Next, For all Ω_i a bounded subset of $C^1_{1-\rho_i,\chi}([0,1]$, we have

$$\vartheta\Big(\aleph_i(\xi, Y(\xi))\Big) \leq \frac{\vartheta\Big(\Omega_1(\xi))\Big) + \vartheta\Big(\Omega_2(\xi)\Big) + \vartheta\Big({}^C\mathcal{D}^{\rho_1,\chi}_{\underline{\xi}^+}\Omega_1(\xi)\Big) + \vartheta\Big({}^C\mathcal{D}^{\rho_2,\chi}_{\underline{\xi}^+}\Omega_2(\xi)\Big)}{40}, \text{ and}$$

$$\vartheta\Big(\aleph_2(\xi, Y(\xi))\Big) \leq \frac{\vartheta\Big(\Omega_1(\xi))\Big) + \vartheta\Big(\Omega_2(\xi)\Big) + \vartheta\Big({}^C\mathcal{D}^{\rho_1,\chi}_{\underline{\xi}^+}\Omega_1(\xi)\Big) + \vartheta\Big({}^C\mathcal{D}^{\rho_2,\chi}_{\underline{\xi}^+}\Omega_2(\xi)\Big)}{50},\ \xi \in (0,1]$$

So, ($\mathbf{H_1}$), ($\mathbf{H_{1-1}}$) and ($\mathbf{H_{1-2}}$) are satisfied. A quick calculation gives us, for $i = 1,\ 2$, we have

$$\Big[1 + (\rho_i + 1)\Big(\mathcal{T}\zeta_i^* n_i(\Psi_*^{\rho_i} + \rho_i) + \Psi_*^{1-\rho_i}\Big)\Big]A_0 < \frac{\Gamma(\rho_i+2)}{2}.$$

So, ($\mathbf{H_2}$) holds. Therefore, Theorem 4.3 ensures that the solution set of Problem (**S**) − (**NIB**) is nonempty and compact.

Acknowledgments

The authors would like to express their deep gratitude to the referee for his/her meticulous reading and valuable suggestions which have definitely improved the paper.

References

[1] R. P. Agarwal, M. Meehan, D. O'Regan, *Fixed Point Theory and Applications*, Cambridge Tracts in Mathematics, Cambridge University Press, Cambridge, 2001.

[2] H. M. Ahmed, M. A. Ragusa, *Nonlocal controllability of Sobolev-type conformable fractional stochastic evolution inclusions with Clarke subdifferential*, Bull. Malays. Math. Sci. Soc., **45** (6) (2022), 3239–3253.

[3] J. Andres, L. Górniewicz, *Topological Principles for Boundary Value Problems*. Kluwer, Dordrecht, 2003.

[4] J. Andres, M. Pavlackova, *Topological structure of solution sets to asymptotic boundary value problems*, J. Differ. Equ. **248** (2010), 127–150.

[5] K. Balachandran, J. Y. Park, Nonlocal Cauchy problem for abstract fractional semilinear evolution equations, *Nonlinear Anal.* **71** (2009), 4471–4475.

[6] K. Balachandran, J. J. Trujillo, *The nonlocal Cauchy problem for nonlinear fractional integrodifferential equations in Banach spaces*, Nonlinear Anal.**72** (2010), 4587–4593.

[7] J. Banaŝ, K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Note in Pure App. Math. Marcel Dekker, New York, 1980.

[8] M. Beddani, H. Beddani, *Compactness of boundary value problems for impulsive integro-differential equation*, Filomat, **37** (20)(2023), 6855–6866.

[9] M. Beddani, H. Beddani, *Qualitative study for impulsive pantograph fractional integro-differential equation via ψ-Hilfer derivative*, Miskolc Math. Notes., **24**(2) (2023), 635–651.

[10] M. Beddani and B. Hedia, *Existence result for fractional differential equations on unbounded domain*, Kragujevac J. Math. **48** (5) (2024), 755–766.

[11] L. Byszewski, *Existence and uniqueness of mild and classical solutions of semilinear functional differential evolution nonlocal Cauchy problem*, Selected problems of mathematics, 50th Anniv. Cracow Univ. Technol. Anniv. Issue 6, Cracow Univ. Technol. Krakow, (1995), 25–33.

[12] D. J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht, 1996.

[13] K. Benia, A. D. El Hadi, M. Beddani, B. Hedia, *Nonlocal Fractional Differential Equation On The Half Line in Banach Space*, Adv. Theory Nonlinear Anal. Appl., **6** (2022), 118–134.

[14] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier B.V. Amsterdam, 2006.

[15] K. D. Kucche, A. D. Mali, *On the Nonlinear (k, χ)-Hilfer Hybrid Fractional Differential Equations*, Chaos, Solitons and Fractals, **152** (2021), 111335.

[16] I. Podlubny, *Fractional Differential Equations, in: Mathematics in Science and Engineering*, Academic Press, New York, 1999.

[17] M. A. Ragusa, A. Razani, *Weak solutions for a system of quasilinear elliptic equations*, Contrib. Math. (Shahin Digital Publisher), **1** (2020), 11–16.

[18] S. G. Samko, A. A. Kilbas, O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[19] J. Vanterler da C. Sousa, E. Capelas de Oliveira, *On the χ-Hilfer fractional derivative*, Commun. Nonlinear Sci. Numer. Simul. **60** (2018), 72–91.

[20] J. Wang, Y. Zhang, *Nonlocal initial value problems for differential equations with Hilfer fractional derivative*, Appl. Math. Comput. **266** (2015), 850—859.