Filomat 38:26 (2024), 9237–9247
https://doi.org/10.2298/FIL2426237A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Blow-up of solutions for a viscoelastic Kirchhoff equation with a logarithmic nonlinearity, delay and Balakrishnan-Taylor damping terms

Asma Alharbi[a,*], Abdelbaki Choucha[b,c], Salah Boulaaras[a]

[a]*Department of Mathematics, College of Science, Qassim University, 51452, Buraydah, Saudi Arabia*
[b]*Department of Material Sciences, Faculty of Sciences, Amar Teleji Laghouat University, Algeria*
[c]*Laboratory of Mathematics and Applied Sciences, Ghardaia University, Algeria*

Abstract. We examine a Kirchhoff-type equation with nonlinear viscoelasticity, incorporating logarithmic nonlinearity, delay, and Balakrishnan-Taylor damping terms. The blow-up of solutions is established under appropriate assumptions.

1. Introduction

In this study, we examine the Kirchhoff equation described as follows:

$$\begin{cases} \varsigma_{tt} - M(t)\Delta\varsigma(t) + \int_0^t \mathfrak{R}(t-\varphi)\Delta\varsigma(\varphi)d\varphi + g(\varsigma_t) = \varsigma|\varsigma|^{\aleph-2}\ln|\varsigma|^{\jmath}, \quad x \in \Xi, t > 0, \\ \varsigma(x,0) = \varsigma_0(x), \quad \varsigma_t(x,0) = \varsigma_1(x), \quad x \in \Xi, \\ \varsigma_t(x,t-\omega) = q_0(x,t-\omega), \quad x \in \Xi, \quad t \in (0,\omega), \\ \varsigma(x,t) = 0, \quad x \in \partial\Xi, \end{cases} \tag{1}$$

where

$$\begin{aligned} M(t) &:= \left(\chi_0 + \chi_1\|\nabla\varsigma\|_2^2 + \sigma(\nabla\varsigma(t), \nabla\varsigma_t(t))_{\mathscr{P}^2(\Xi)}\right), \\ g(\varsigma_t) &:= \vartheta_1\varsigma_t + \vartheta_2\varsigma_t(x,t-\omega). \end{aligned}$$

Here, ϑ_2 represents a real number, $\Xi \in \mathbb{R}^N$ represents a bounded domain with sufficiently smooth boundary $\partial\Xi$, $\aleph \geq 2$, χ_0, χ_1, σ, ϑ_1, $\jmath$ are positive constants, $\mathfrak{R}$ is a positive function and $\omega > 0$ represents the time delay.

The connection between stress and strain history in the beam, influenced by Boltzmann theory, is termed viscoelastic damping. The memory term within this damping is governed by the function $\mathfrak{R}$.

2020 *Mathematics Subject Classification*. Primary 35B40; Secondary 35L70, 76Exx, 93D20.
Keywords. Kirchhoff equation; Blow-up; Delay term; Viscoelastic term; Logarithmic nonlinearity, partial differential equations.
Received: 16 April 2024; Revised: 13 May 2024; Accepted: 14 May 2024
Communicated by Maria Alessandra Ragusa
* Corresponding author: Asma Alharbi
Email addresses: ao.alharbi@qu.edu.sa (Asma Alharbi), abdel.choucha@lagh-univ.dz (Abdelbaki Choucha), s.boularas@qu.edu.sa (Salah Boulaaras)

See ([8], [13], [14],[18],[19],[21],[23],[24],[31]).
In [3], Taylor and Balakrishnan introduced a novel damping model termed Balakrishnan-Taylor damping, which pertains to the span problem and the plate equation. For further exploration, several papers have delved into the investigation of this damping, including references such as ([3], [6], [10], [18], [23], [24], [30], [32]).
The presence of delay frequently manifests in numerous applications and practical scenarios, transforming many systems into distinct problems worthy of investigation. Recently, numerous authors have explored the stability and asymptotic behavior of evolution systems incorporating time delays. Relevant studies include references such as ([9], [13]-[19], [22], [24], [33–36]).
The significant relevance of logarithmic nonlinearity in physics lies in its occurrence across various theories and concepts, encompassing cosmology, symmetry, nuclear physics, and quantum mechanics. Furthermore, it finds utility in numerous applications, spanning nuclear, optical, and even subterranean physics. Several researchers have also explored this particular type of problem across various contexts, investigating aspects such as the global existence of solutions, stability, blow-up, and growth of solutions. For further details, readers are directed to references such as ([4], [5], [7], [10], [18], [20], [25]-[27]).
Considering all the previously mentioned factors, the amalgamation of damping elements (Balakrishnan-Taylor damping, delay terms, memory term, and logarithmic nonlinearity) within a specific problem, augmented by the inclusion of the delay term $(\vartheta_2\varsigma_t(x, t-\omega))$, presents what we perceive to be a novel research challenge deserving of exploration. We posit that this problem stands apart from those previously discussed and warrants further investigation, which we aim to elucidate.
Our research is structured into distinct sections: The subsequent section delineates the lemmas, concepts, and hypotheses essential for our analysis. Following that, in Section 3, we state and prove the blow-up phenomenon observed in solutions.

2. Preliminaries

To delve into our problem, this section necessitates specific materials.
Initially, we introduce the subsequent hypothesis concerning $\mathfrak{K}$ and β_2:
(H1) $\mathfrak{K}:\mathbb{R}_+\to\mathbb{R}_+$ is non-increasing $\mathscr{D}^1$ function which holds

$$\mathfrak{K}(t)>0,\quad \chi_0-\int_0^\infty \mathfrak{K}(\varphi)d\varphi = l>0. \tag{2}$$

(H2)

$$|\vartheta_2|<\vartheta_1. \tag{3}$$

Let

$$(\mathfrak{K}\circ\varpi)(t) := \int_\Xi\int_0^t \mathfrak{K}(t-\varphi)|\varpi(t)-\varpi(\varphi)|^2 d\varphi dx.$$

Considering new variables, as in [33]

$$z(x,\varkappa,t)=\varsigma_t(x,t-\omega\varkappa),\quad (x,\varkappa,t)\in\Xi\times(0,1)\times\mathbb{R}_+,$$

which satisfy

$$\begin{cases} \omega z_t(x,\varkappa,t)+z_\varkappa(x,\varkappa,t)=0\\ z(x,0,t)=\varsigma_t(x,t). \end{cases} \tag{4}$$

Problem (1) can also be stated as

$$\begin{cases} \varsigma_{tt} - M(t)\Delta\varsigma(t) + \int_0^t \Re(t-\varphi)\Delta\varsigma(\varphi)d\varphi + \vartheta_1\varsigma_t + \vartheta_2 z(x,1,t) = \varsigma|\varsigma|^{\aleph-2}\ln|\varsigma|^{\jmath}, \\ \omega z_t(x,\varkappa,t) + z_\varkappa(x,\varkappa,t) = 0, \\ \varsigma(x,0) = \varsigma_0(x), \quad \varsigma_t(x,0) = \varsigma_1(x), \quad x \in \Xi, \\ z(x,\varkappa,0) = q_0(x,-\omega\varkappa), \quad in \ \ \Xi\times(0,1), \\ \varsigma(x,t) = 0, \quad x \in \partial\Xi. \end{cases} \tag{5}$$

Energy functional is given as follows.

Lemma 2.1. *The energy functional E, stated as*

$$\begin{aligned} E(t) &= \frac{1}{2}\|\varsigma_t\|_2^2 + \frac{1}{2}\Big(\chi_0 - \int_0^t \Re(\varphi)d\varphi\Big)\|\nabla\varsigma(t)\|_2^2 + \frac{\chi_1}{4}\|\nabla\varsigma(t)\|_2^4 \\ &+\frac{1}{2}(\Re\circ\nabla\varsigma)(t) + \frac{\jmath}{\aleph}\|\varsigma(t)\|_\aleph^\aleph - \frac{1}{\aleph}\int_\Xi |\varsigma|^\aleph \ln|\varsigma|^\jmath dx \\ &+\frac{\hbar}{2}\int_0^1 \|z\,(x,\varkappa,t)\,\|_2^2 d\varkappa. \end{aligned} \tag{6}$$

satisfies

$$\begin{aligned} E'(t) &\leq -\mathscr{D}_0\Big(\|\varsigma_t\|_2^2 + \int_\Xi z^2(x,1,t)dx\Big) + \frac{1}{2}(\Re'\circ\nabla\varsigma)(t) \\ &-\frac{1}{2}\Re(t)\|\nabla\varsigma(t)\|_2^2 - \frac{\sigma}{4}\Big(\frac{d}{dt}\Big\{\|\nabla\varsigma(t)\|_2^2\Big\}\Big)^2 \leq 0, \end{aligned} \tag{7}$$

for $\hbar$ a positive constant satisfying

$$\omega|\vartheta_2| < \hbar < \omega(2\vartheta_1 - |\vartheta_2|). \tag{8}$$

Proof. *Taking inner product of* $(5)_1$ *with* ς_t*, and then integrating over* Ξ*, we get*

$$\begin{aligned} &(\varsigma_{tt}(t),\varsigma_t(t))_{\mathscr{P}^2(\Xi)} - (M(t)\Delta\varsigma(t),\varsigma_t(t))_{\mathscr{P}^2(\Xi)} \\ &+(\int_0^t \Re(t-\varphi)\Delta\varsigma(\varphi)d\varphi,\varsigma_t(t))_{\mathscr{P}^2(\Xi)} + \vartheta_1(\varsigma_t,\varsigma_t)_{\mathscr{P}^2(\Xi)} \\ &+(\vartheta_2 z(x,1,t),\varsigma_t(t))_{\mathscr{P}^2(\Xi)} - (\jmath\varsigma|\varsigma|^{\aleph-2}\ln|\varsigma|,\varsigma_t(t))_{\mathscr{P}^2(\Xi)} = 0. \end{aligned} \tag{9}$$

Further simplification, yields

$$(\varsigma_{tt}(t),\varsigma_t(t))_{\mathscr{P}^2(\Xi)} = \frac{1}{2}\frac{d}{dt}\Big(\|\varsigma_t(t)\|_2^2\Big), \tag{10}$$

using integration by parts, we get

$$\begin{aligned} &-(M(t)\Delta\varsigma(t),\varsigma_t(t))_{\mathscr{P}^2(\Xi)} \\ =\ & -(\Big(\chi_0 + \chi_1\|\nabla\varsigma\|_2^2 + \sigma(\nabla\varsigma(t),\nabla\varsigma_t(t))_{\mathscr{P}^2(\Xi)}\Big)\Delta\varsigma(t),\varsigma_t(t))_{\mathscr{P}^2(\Xi)} \\ =\ & \Big(\chi_0 + \chi_1\|\nabla\varsigma\|_2^2 + \sigma(\nabla\varsigma(t),\nabla\varsigma_t(t))_{\mathscr{P}^2(\Xi)}\Big)\int_\Xi \nabla\varsigma(t).\nabla\varsigma_t(t)dx \\ =\ & \Big(\chi_0 + \chi_1\|\nabla\varsigma\|_2^2 + \sigma(\nabla\varsigma(t),\nabla\varsigma_t(t))_{\mathscr{P}^2(\Xi)}\Big)\frac{d}{dt}\Big\{\frac{1}{2}\int_\Xi |\nabla\varsigma(t)|^2 dx\Big\} \\ =\ & \frac{d}{dt}\Big\{\frac{1}{2}\Big(\chi_0 + \frac{\chi_1}{2}\|\nabla\varsigma\|_2^2\Big)\|\nabla\varsigma(t)\|_2^2\Big\} + \frac{\sigma}{4}\Big\{\frac{d}{dt}\|\nabla\varsigma(t)\|_2^2\Big\}^2. \end{aligned} \tag{11}$$

and we have

$$
\begin{aligned}
& \Big(\int_0^t \Re(t-\varphi)\Delta\varsigma(\varphi)d\varphi, \varsigma_t(t)\Big)_{\mathscr{P}^2(\Xi)} \\
= \; & \int_0^t \Re(t-\varphi)(\Delta\varsigma(\varphi), \varsigma_t(t))_{\mathscr{P}^2(\Xi)} d\varphi \\
= \; & -\int_0^t \Re(t-\varphi)\Big[\int_\Xi \nabla\varsigma(x,\varphi)\nabla\varsigma(x,t)dx\Big]d\varphi,
\end{aligned} \tag{12}
$$

and

$$
-\nabla\varsigma(x,\varphi).\nabla\varsigma(x,t) = \frac{1}{2}\frac{d}{dt}\Big\{|\nabla\varsigma(x,\varphi) - \nabla\varsigma(x,t)(t)|^2\Big\} - \frac{1}{2}\frac{d}{dt}\Big\{|\nabla\varsigma(x,t)|^2\Big\}, \tag{13}
$$

then

$$
\begin{aligned}
& -\int_0^t \Re(t-\varphi)(\nabla\varsigma(\varphi), \nabla\varsigma_t(t))_{\mathscr{P}^2(\Xi)} d\varphi \\
= \; & -\int_0^t \Re(t-\varphi)\int_\Xi \Big[\frac{1}{2}\frac{d}{dt}\Big\{|\nabla\varsigma(x,\varphi) - \nabla\varsigma(x,t)|^2\Big\}\Big]dxds. \\
& -\int_0^t \Re(t-\varphi)\int_\Xi \Big[\frac{1}{2}\frac{d}{dt}\Big\{|\nabla\varsigma(x,t)|^2\Big\}\Big]dxd\varphi \\
= \; & \frac{1}{2}\int_0^t \Re(t-\varphi)\Big[\frac{d}{dt}\Big\{\int_\Xi |\nabla\varsigma(x,t) - \nabla\varsigma(x,\varphi)|^2 dx\Big\}\Big]d\varphi \\
& -\frac{1}{2}\int_0^t \Re(t-\varphi)\Big[\frac{d}{dt}\Big\{\|\nabla\varsigma(x,t)\|_2^2\Big\}\Big]dxd\varphi
\end{aligned} \tag{14}
$$

We use (2), we obtain

$$
\begin{aligned}
& \frac{1}{2}\int_0^t \Re(t-\varphi)\Big[\frac{d}{dt}\Big\{\int_\Xi |\nabla\varsigma(x,t) - \nabla\varsigma(x,\varphi)|^2 dx\Big\}\Big]d\varphi \\
= \; & \frac{1}{2}\frac{d}{dt}\Big\{\int_0^t \Re(t-\varphi)\Big[\int_\Xi |\nabla\varsigma(x,t) - \nabla\varsigma(x,\varphi)|^2 dx\Big]\Big\}d\varphi \\
& -\frac{1}{2}\int_0^t \Re'(t-\varphi)\Big[\int_\Xi |\nabla\varsigma(x,t) - \nabla\varsigma(x,\varphi)|^2 dx\Big]d\varphi \\
= \; & \frac{1}{2}\frac{d}{dt}(\Re \circ \nabla\varsigma)(t) - \frac{1}{2}(\Re' \circ \nabla\varsigma)(t),
\end{aligned} \tag{15}
$$

and

$$
\begin{aligned}
& -\frac{1}{2}\int_0^t \Re(t-\varphi)\Big[\frac{d}{dt}\Big\{\|\nabla\varsigma(t)\|_2^2\Big\}\Big]dxd\varphi \\
= \; & -\frac{1}{2}\Big(\int_0^t \Re(t-\varphi)d\varphi\Big)\Big(\frac{d}{dt}\Big\{\|\nabla\varsigma(t)\|_2^2\Big\}\Big)dx \\
= \; & -\frac{1}{2}\Big(\int_0^t \Re(\varphi)d\varphi\Big)\Big(\frac{d}{dt}\Big\{\|\nabla\varsigma(t)\|_2^2\Big\}\Big)dx \\
= \; & -\frac{1}{2}\frac{d}{dt}\Big\{\Big(\int_0^t \Re(\varphi)d\varphi\Big)\|\nabla\varsigma(t)\|_2^2\Big\} + \frac{1}{2}\Re(t)\|\nabla\varsigma(t)\|_2^2,
\end{aligned} \tag{16}
$$

By substituting (15) and (16) into (14), gives

$$\begin{aligned}
&\left(\int_0^t \mathfrak{R}(t-\varphi)\Delta\varsigma(\varphi)d\varphi, \varsigma_t(t)\right)_{\mathscr{P}^2(\Xi)} \\
= \;& \frac{d}{dt}\left\{\frac{1}{2}(\mathfrak{R}\circ\nabla\varsigma)(t) - \frac{1}{2}\left(\int_0^t \mathfrak{R}(\varphi)d\varphi\right)\|\nabla\varsigma(t)\|_2^2\right\} \\
& -\frac{1}{2}(\mathfrak{R}'\circ\nabla\varsigma)(t) + \frac{1}{2}\mathfrak{R}(t)\|\nabla\varsigma(t)\|_2^2.
\end{aligned} \tag{17}$$

and we have

$$-(\jmath\varsigma|\varsigma|^{\aleph-2}\ln|\varsigma|, \varsigma_t(t))_{\mathscr{P}^2(\Xi)} = \frac{d}{dt}\left\{\frac{\jmath}{\aleph}\|\varsigma(t)\|_{\aleph}^{\aleph} - \frac{1}{\aleph}\int_{\Xi}|\varsigma|^{\aleph}\ln|\varsigma|^{\jmath}dx\right\}. \tag{18}$$

Multiplying equation $(5)_2$ *by* $\frac{\hbar}{\omega}z$*, integrating over* $\Xi\times(0,1)$ *and using* $(4)_2$*, we get*

$$\begin{aligned}
\frac{d}{dt}\frac{\hbar}{2}\int_{\Xi}\int_0^1 z^2(x,\varkappa,t)d\varkappa dx &= -\frac{\hbar}{\omega}\int_{\Xi}\int_0^1 zz_{\varkappa}(x,\varkappa,t)\,d\varkappa dx \\
&= -\frac{\hbar}{2\omega}\int_{\Xi}\int_0^1 \frac{d}{d\varkappa}z^2(x,\varkappa,t)\,d\varkappa dx \\
&= \frac{\hbar}{2\omega}\int_{\Xi}(z^2(x,0,t)-z^2(x,1,t))dx \\
&= \frac{\hbar}{2\omega}\int_{\Xi}\varsigma_t^2dx - \frac{\hbar}{2\omega}\int_{\Xi}z^2(x,1,t)\,dx,
\end{aligned} \tag{19}$$

through Young's inequality, we obtain

$$\vartheta_2(z(x,1,t),\varsigma_t(t))_{\mathscr{P}^2(\Xi)} \le \frac{|\vartheta_2|}{2}\|\varsigma_t\|_2^2 + \frac{|\vartheta_2|}{2}\|z(x,1,t)\|_2^2. \tag{20}$$

By substituting (10)-(11) and (17)-(20) into (9), we get (6) and

$$\begin{aligned}
E'(t) \le\; & -\left(\vartheta_1 - \frac{\hbar}{2\omega} - \frac{|\vartheta_2|}{2}\right)\|\varsigma_t\|_2^2 - \left(\frac{\hbar}{2\omega} - \frac{|\vartheta_2|}{2}\right)\|z(x,1,t)\|_2^2 \\
& + \frac{1}{2}(\mathfrak{R}'\circ\nabla\varsigma)(t) - \frac{1}{2}\mathfrak{R}(t)\|\nabla\varsigma(t)\|_2^2 - \frac{\sigma}{4}\left(\frac{d}{dt}\left\{\|\nabla\varsigma(t)\|_2^2\right\}\right)^2,
\end{aligned}$$

Utilizing condition (8), we achieve

$$\eta_0 = \left(\vartheta_1 - \frac{\hbar}{\omega} - \frac{|\vartheta_2|}{2}\right) > 0, \quad \textit{and} \quad \eta_1 = \left(\frac{\hbar}{\omega} - \frac{|\vartheta_2|}{2}\right) > 0. \tag{21}$$

Then, by (8),(21), (3), we get(7), where $\mathscr{D}_0 = \min\{\eta_0,\eta_1\} > 0$*.*
This completes the proof. □

Theorem 2.2. *[27] Let (2)-(3) holds. Then, there exists a weak solution* (ς, z) *of problem (5), for any* $\varsigma_0, \varsigma_1 \in H_0^1(\Xi)\cap\mathscr{P}^2(\Xi)$*, and* $q_0 \in \mathscr{P}^2(\Xi\times(0,1))$*, such that*

$$\begin{aligned}
&\varsigma \in \mathscr{D}(]0,\mathscr{T}[, H_0^1(\Xi)) \cap \mathscr{D}^1(]0,\mathscr{T}[, \mathscr{P}^2(\Xi)), \\
&\varsigma_t \in \mathscr{D}(]0,\mathscr{T}[, H_0^1(\Xi)) \cap \mathscr{P}^2(]0,\mathscr{T}[, \mathscr{P}^2(\Xi\times(0,1))).
\end{aligned}$$

Lemma 2.3. *[27] There exists a positive constant $c(\Xi) > 0$, provided that*

$$\left(\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx\right)^{\frac{s}{\aleph}} \leq c\left(\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx + \|\nabla\varsigma\|_2^2\right),$$

for any $2 \leq s \leq \aleph$, such that $\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx \geq 0$.

Corollary 2.4. *[27] A positive constant $c(\Xi) > 0$, exists in such a way that*

$$\|\varsigma\|_2^2 \leq c\left[\left(\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx\right)^{\frac{2}{\aleph}} + \|\nabla\varsigma\|_2^{\frac{4}{\aleph}}\right],$$

provided that $\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx \geq 0$.

Lemma 2.5. *[27] A positive constant $c(\Xi) > 0$, exists in such a way that*

$$\|\varsigma\|_{\aleph}^{s} \leq c\left(\|\varsigma\|_{\aleph}^{\aleph} + \|\nabla\varsigma\|_2^2\right),$$

for any $\varsigma \in \mathscr{P}^{\aleph}(\Xi)$ and $2 \leq s \leq \aleph$.

3. Blow up result

Within this section, we will prove the blow-up outcome of the solution to problem (5). Initially, we state the functional

$$\begin{aligned}\mathscr{Q}(t) = -E(t) &= -\frac{1}{2}\|\varsigma_t\|_2^2 - \frac{1}{2}\left(\chi_0 - \int_0^t \mathfrak{R}(\varphi)d\varphi\right)\|\nabla\varsigma(t)\|_2^2 - \frac{\chi_1}{4}\|\nabla\varsigma(t)\|_2^4 \\ &-\frac{1}{2}(\mathfrak{R} \circ \nabla\varsigma)(t) - \frac{\jmath}{\aleph}\|\varsigma(t)\|_{\aleph}^{\aleph} + \frac{1}{\aleph}\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx \\ &-\frac{\hbar}{2}\int_0^1 \|z(x,\varkappa,t)\|_2^2 d\varkappa. \end{aligned} \tag{22}$$

Theorem 3.1. *Let (2)-(3) hold, and $E(0) < 0$, Then, the solution of problem (5) blow up in finite time.*

Proof. From (7), we obtain

$$E(t) \leq E(0) \leq 0. \tag{23}$$

Thus

$$\mathscr{Q}'(t) = -E'(t) \geq \mathscr{D}_0\left(\|\varsigma_t\|_2^2 + \|z^2(x,1,t)\|_2^2\right), \tag{24}$$

hence

$$\begin{aligned}\mathscr{Q}'(t) &\geq \mathscr{D}_0\|\varsigma_t(t)\|_2^2 \geq 0 \\ \mathscr{Q}'(t) &\geq \mathscr{D}_0\|z^2(x,1,t)\|_2^2 \geq 0. \end{aligned} \tag{25}$$

By (22), we have

$$0 \leq \mathscr{Q}(0) \leq \mathscr{Q}(t) \leq \frac{1}{\aleph}\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx. \tag{26}$$

We set

$$\mathscr{I}(t) = \mathscr{Q}^{1-F}(t) + \wp\int_{\Xi} \varsigma\varsigma_t dx + \frac{\wp\vartheta_1}{2}\int_{\Xi} \varsigma^2 dx + \frac{\sigma}{4}\|\nabla\varsigma\|_2^4, \tag{27}$$

where $\wp > 0$ to be chosen later and

$$\frac{2(\aleph-1)}{\aleph^2} < F < \frac{\aleph-2}{2\aleph} < 1. \tag{28}$$

Multiplying $(5)_1$ by ς and derivative of (27), we achieve

$$\begin{aligned}\mathscr{I}'(t) &= (1-F)\mathscr{Q}^{-F}\mathscr{Q}'(t) + \wp\|\varsigma_t\|_2^2 + \wp\int_\Xi |\varsigma|^\aleph \ln|\varsigma|^\jmath dx \\ &-\wp\chi_0\|\nabla\varsigma\|_2^2 - \wp\chi_1\|\nabla\varsigma\|_2^4 + \underbrace{\wp\int_\Xi \nabla\varsigma\int_0^t \mathfrak{R}(t-\varphi)\nabla\varsigma(\varphi)d\varphi dx}_{J_1} \\ &\underbrace{-\wp\vartheta_2\int_\Xi \varsigma z(x,1,t)dx}_{J_2}.\end{aligned} \tag{29}$$

We have

$$\begin{aligned}J_1 &= \wp\int_0^t \mathfrak{R}(t-\varphi)d\varphi\int_\Xi \nabla\varsigma.(\nabla\varsigma(\varphi)-\nabla\varsigma(t))dxd\varphi + \wp\int_0^t \mathfrak{R}(\varphi)d\varphi\|\nabla\varsigma\|_2^2 \\ &\geq \frac{\wp}{2}\Big(\int_0^t \mathfrak{R}(\varphi)d\varphi\Big)\|\nabla\varsigma\|_2^2 - \frac{\wp}{2}(\mathfrak{R}\circ\nabla\varsigma),\end{aligned} \tag{30}$$

by (3), for $\delta_1 > 0$

$$\begin{aligned}J_2 &\geq -\wp|\vartheta_2|\delta_1\|\varsigma\|_2^2 - \frac{\wp|\vartheta_2|}{4\delta_1}\|z(x,1,t)\|_2^2 \\ &\geq -\wp\vartheta_1\delta_1\|\varsigma\|_2^2 - \frac{\wp\vartheta_1}{4\delta_1}\|z(x,1,t)\|_2^2.\end{aligned} \tag{31}$$

From (29), we find

$$\begin{aligned}\mathscr{I}'(t) &\geq (1-F)\mathscr{Q}^{-F}\mathscr{Q}'(t) + \wp\|\varsigma_t\|_2^2 + \wp\int_\Xi |\varsigma|^\aleph \ln|\varsigma|^\jmath dx \\ &-\wp\chi_1\|\nabla\varsigma\|_2^4 - \wp\Big[\Big(\chi_0 - \frac{1}{2}\int_0^t \mathfrak{R}(\varphi)d\varphi\Big)\|\nabla\varsigma\|_2^2 - \frac{\wp}{2}(\mathfrak{R}\circ\nabla\varsigma) \\ &-\wp\vartheta_1\delta_1\|\varsigma\|_2^2 - \frac{\wp\vartheta_1}{4\delta_1}\|z(x,1,t)\|_2^2.\end{aligned} \tag{32}$$

Now, setting δ_1 such that, for large κ to be assigned later

$$\frac{1}{4\delta_1\mathscr{D}_0} = \kappa\mathscr{Q}^{-F}(t),$$

by (25) and substituting in (32), we get

$$\begin{aligned}\mathscr{I}'(t) &\geq [(1-F)-\wp\kappa]\mathscr{Q}^{-F}\mathscr{Q}'(t) + \wp\|\varsigma_t\|_2^2 \\ &-\frac{\wp}{2}(\mathfrak{R}\circ\nabla\varsigma) - \wp\chi_1\|\nabla\varsigma\|_2^4 - \wp\Big(\chi_0 - \frac{1}{2}\int_0^t \mathfrak{R}(\varphi)d\varphi\Big)\|\nabla\varsigma\|_2^2 \\ &-\wp\Big(\frac{\vartheta_1\mathscr{Q}^F(t)}{4\mathscr{D}_0\kappa}\Big)\|\varsigma\|_2^2 + \wp\int_\Xi |\varsigma|^\aleph \ln|\varsigma|^\jmath dx.\end{aligned} \tag{33}$$

For $0 < a < 1$ and from (22), we get

$$
\begin{aligned}
\wp \int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx &= \wp a \int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx + \frac{\wp\aleph(1-a)}{2} \|\varsigma_t\|_2^2 + \wp\aleph(1-a)\mathscr{Q}(t) \\
&+ \wp \frac{\aleph(1-a)}{2} (\chi_0 - \int_0^t \mathfrak{R}(\varphi) d\varphi) \|\nabla\varsigma\|_2^2 + \wp\jmath(1-a)\|\varsigma\|_{\aleph}^{\aleph} \\
&+ \wp \frac{\chi_1 \aleph(1-a)}{2} \|\nabla\varsigma\|_2^4 - \wp \frac{\aleph(1-a)}{2} ((\mathfrak{R} \circ \nabla\varsigma) \\
&+ \frac{\wp\aleph(1-a)\hbar}{2} \int_0^1 \|z(x,\varkappa,t)\|_2^2 d\varkappa. \qquad (34)
\end{aligned}
$$

Putting in (33), we obtain

$$
\begin{aligned}
\mathscr{I}'(t) &\geq \Big\{(1-F) - \wp\kappa\Big\} \mathscr{Q}^{-F} \mathscr{Q}'(t) + \wp a \int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx \\
&+ \wp \Big\{ \frac{\aleph(1-a)}{2} + 1 \Big\} \|\varsigma_t\|_2^2 - \wp \Big(\frac{\vartheta_1 \mathscr{Q}^F(t)}{4\mathscr{D}_0 \kappa} \Big) \|\varsigma\|_2^2 \\
&+ \wp \Big\{ \frac{\aleph(1-a)}{2} \Big(\chi_0 - \int_0^t \mathfrak{R}(\varphi) d\varphi \Big) - \Big(\chi_0 - \frac{1}{2} \int_0^t \mathfrak{R}(\varphi) d\varphi \Big) \Big\} \|\nabla\varsigma\|_2^2 \\
&+ \wp\chi_1 \Big\{ \frac{\aleph(1-a)}{2} - 1 \Big\} \|\nabla\varsigma\|_2^4 + \wp \Big\{ \frac{\aleph(1-a)}{2} - \frac{1}{2} \Big\} (\mathfrak{R} \circ \nabla\varsigma) \\
&+ \wp\jmath(1-a)\|\varsigma\|_{\aleph}^{\aleph} + \wp\aleph(1-a)\mathscr{Q}(t) \\
&+ \frac{\wp\aleph(1-a)\hbar}{2} \int_0^1 \|z(x,\varkappa,t)\|_2^2 d\varkappa. \qquad (35)
\end{aligned}
$$

According (26), Young's inequality and Corollary 2.4, we obtain

$$
\begin{aligned}
\mathscr{Q}^F(t)\|\varsigma\|_2^2 &\leq \Big(\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx \Big)^F \|\varsigma\|_2^2 \\
&\leq c \Big[\Big(\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx \Big)^{F + \frac{2}{\aleph}} + \Big(\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx \Big)^F \|\nabla\varsigma\|_2^{\frac{4}{\aleph}} \Big] \\
&\leq c \Big[\Big(\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx \Big)^{\frac{(F\aleph+2)}{\aleph}} + \Big(\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx \Big)^{\frac{F\aleph}{(\aleph-2)}} + \|\nabla\varsigma\|_2^2 \Big]
\end{aligned}
$$

By (28), we get

$$
2 < F\aleph + 2 \leq \aleph \ \text{ and } \ 2 < \frac{F\aleph^2}{\aleph - 2} \leq \aleph.
$$

Therefore, lemma 2.3 yields

$$
\mathscr{Q}^F(t)\|\varsigma\|_2^2 \leq c \Big(\int_{\Xi} |\varsigma|^{\aleph} \ln |\varsigma|^{\jmath} dx + \|\nabla\varsigma\|_2^2 \Big). \qquad (36)
$$

Combining (35) and (36), we obtain

$$\begin{aligned}
\mathscr{I}'(t) \;\geq\; & \Big\{(1-F)-\wp\kappa\Big\}\mathscr{Q}^{-F}\mathscr{Q}'(t)+\wp\Big(a-\frac{c\vartheta_1}{4\mathscr{D}_0\kappa}\Big)\int_\Xi |\varsigma|^{\aleph}\ln|\varsigma|^{\jmath}dx \\
&+\wp\Big\{\frac{\aleph(1-a)}{2}+1\Big\}\|\varsigma_t\|_2^2+\wp\jmath(1-a)\|\varsigma\|_{\aleph}^{\aleph}+\wp\aleph(1-a)\mathscr{Q}(t) \\
&+\wp\Big\{\frac{\aleph(1-a)}{2}\Big(\chi_0-\int_0^t\Re(\varphi)d\varphi\Big)-\Big(\chi_0-\frac{1}{2}\int_0^t\Re(\varphi)d\varphi\Big)-\frac{c\vartheta_1}{2\mathscr{D}_0\kappa}\Big\}\|\nabla\varsigma\|_2^2 \\
&+\wp\chi_1\Big\{\frac{\aleph(1-a)}{2}-1\Big\}\|\nabla\varsigma\|_2^4+\wp\Big\{\frac{\aleph(1-a)}{2}-\frac{1}{2}\Big\}(\Re\circ\nabla\varsigma) \qquad (37)\\
&+\frac{\wp\aleph(1-a)\hbar}{2}\int_0^1\|z(x,\varkappa,t)\|_2^2d\varkappa,
\end{aligned}$$

Now, select $a>0$ small enough such that

$$\lambda_1=\frac{\aleph(1-a)}{2}-1>0,$$

let

$$\int_0^\infty\Re(\varphi)d\varphi<\frac{\frac{\aleph(1-a)}{2}-1}{\frac{\aleph(1-a)}{2}-\frac{1}{2}}=\frac{2\lambda_1}{2\lambda_1+1}, \qquad (38)$$

gives

$$\lambda_2 \;=\; \Big\{\Big(\frac{\aleph(1-a)}{2}-1\Big)-\Big(\int_0^t\Re(\varphi)d\varphi\Big)\Big(\frac{\aleph(1-a)}{2}-\frac{1}{2}\Big)\Big\}>0,$$

taking κ large such that

$$\begin{aligned}
\lambda_3 &= a-\frac{c\vartheta_1}{4\mathscr{D}_0\kappa}>0,\\
\lambda_4 &= \lambda_2-\frac{c\vartheta_1}{4\mathscr{D}_0\kappa}>0.
\end{aligned}$$

Selecting fixed values for κ, a, and assign $\wp$ small enough such that

$$\lambda_5=(1-F)-\wp\kappa>0$$

and

$$\mathscr{I}(0)>0.$$

Thus, for some $\eta>0$, estimate (35) yields

$$\begin{aligned}
\mathscr{I}'(t) \;\geq\; & \eta\Big\{\mathscr{Q}(t)+\|\varsigma_t\|_2^2+\|\nabla\varsigma\|_2^2+(\Re\circ\nabla\varsigma)+\|\varsigma\|_{\aleph}^{\aleph}+\int_\Xi|\varsigma|^{\aleph}\ln|\varsigma|^{\jmath}dx \\
&+\|\nabla\varsigma\|_2^4+\int_0^1\|z(x,\varkappa,t)\|_2^2d\varkappa\Big\}. \qquad (39)
\end{aligned}$$

Next, using Young's and Holder's inequalities, we get

$$\Big|\int_\Xi\varsigma\varsigma_t dx\Big|^{\frac{1}{1-F}} \;\leq\; c\Big[\|\varsigma\|_{\aleph}^{\frac{\theta}{1-F}}+\|\varsigma_t\|_2^{\frac{\vartheta}{1-F}}\Big]. \qquad (40)$$

where $\frac{1}{\vartheta} + \frac{1}{\theta} = 1$.
we take $\vartheta = 2(1-F)$, to get

$$\frac{\theta}{1-F} = \frac{2}{2(1-F)-1} \leq \aleph$$

Additionally, for $s = \frac{2}{2(1-F)-1}$, estimate (40) yields

$$\left| \int_\Xi \varsigma\varsigma_t dx \right|^{\frac{1}{1-F}} \leq c\Big[\|\varsigma\|_\aleph^s + \|\varsigma_t\|_2^2 \Big].$$

Then, Lemma 2.5 yields

$$\left| \int_\Xi \varsigma\varsigma_t dx \right|^{\frac{1}{1-F}} \leq c\Big[\|\varsigma\|_\aleph^\aleph + \|\varsigma_t\|_2^2 + \|\nabla\varsigma\|_2^2 \Big]. \tag{41}$$

Hence,

$$\begin{aligned} \mathscr{I}^{\frac{1}{1-F}}(t) &= \left(\mathscr{Q}^{1-F} + \wp \int_\Xi \varsigma\varsigma_t dx + \frac{\wp\vartheta_1}{2}\|\varsigma\|_2^2 + \wp\frac{\sigma}{4}\|\nabla\varsigma\|_2^4 \right)^{\frac{1}{1-F}} \\ &\leq c\Big(\mathscr{Q}(t) + | \int_\Xi \varsigma\varsigma_t dx|^{\frac{1}{1-F}} + \|\varsigma\|_2^{\frac{2}{1-F}} + \|\nabla\varsigma\|_2^{\frac{4}{1-F}} \Big) \\ &\leq c\Big(\mathscr{Q}(t) + \|\varsigma\|_\aleph^\aleph + \|\varsigma_t\|_2^2 + \|\nabla\varsigma\|_2^2 + \|\nabla\varsigma\|_2^4 + \|\nabla\varsigma_t\|_2^2 \Big) \\ &\leq c\Big(\mathscr{Q}(t) + \|\varsigma\|_\aleph^\aleph + \|\varsigma_t\|_2^2 + \|\nabla\varsigma\|_2^2 + \|\nabla\varsigma\|_2^4 + (\Re \circ \nabla\varsigma) \\ &\quad + \int_0^1 \|z(x,\varkappa,t)\|_2^2 d\varkappa + \int_\Xi |\varsigma|^\aleph \ln|\varsigma|^\jmath dx \Big). \end{aligned} \tag{42}$$

From (39) and (42), produces

$$\mathscr{I}'(t) \geq \Gamma \mathscr{I}^{\frac{1}{1-F}}(t), \tag{43}$$

where $\Gamma > 0$, this depends on c and η only.
From integration of (43), we achieve

$$\mathscr{I}^{\frac{F}{1-F}}(t) \geq \frac{1}{\mathscr{I}^{\frac{-F}{1-F}}(0) - \Gamma\frac{F}{(1-F)}t}.$$

Hence, $\mathscr{I}(t)$ blows up in time

$$\mathscr{T} \leq \mathscr{T}^* = \frac{1-F}{\Gamma F \mathscr{I}^{F/(1-F)}(0)}.$$

Thus proof is completed. □

References

[1] K. Agre, M. A. Rammaha, Systems of nonlinear wave equations with damping and source terms, Different. Integr. Equat., 19(2007), 1235-1270.
[2] J. Ball. Remarks on Blow-Up and Nonexistence Theorems for Nonlinear Evolutions Equation. Quarterly Journal of Mathematics, 28 (1977), 473-486.
[3] A.V. Balakrishnan, LW. Taylor. Distributed parameter nonlinear dampingmodels for flight structures. In: Proceedings (Damping 89), Flight Dynamics Lab and Air Force Wright Aeronautical Labs, Washington: WPAFB; 1989.
[4] J.D. Barrow, P. Parsons, Inflationary models with logarithmic potentials. Phys Rev D.1995;52:5576-5587.
[5] K. Bartkowski, P. Gorka, One-dimensional Klein-Gordon equation with logarithmic nonlinearities. J Phys A.2008;41(35):355201.

[6] RW. Bass D., Zes. Spillover nonlinearity, and flexible structures. The Fourth NASA Workshop on Computational Control of Flexible Aerospace Systems.Washington: NASA Conference Publication 10065; 1991:1-14.
[7] I. Bialynicki-Birula, J. Mycielski, Wave equations with logarithmic nonlinearities. Bull Acad Polon Sci Ser Sci MathAstronom Phys.1975;23(4):461-466.
[8] D.R. Bland, The theory of linear viscoelasticity. Mineola: Courier Dover Publications(2016).
[9] S. Boulaaras, A. Choucha, D. Ouchenane, B. Cherif, Blow up of solutions of two singular nonlinear viscoelastic equations with general source and localized frictional damping terms, Adv. Differ. Equ. 2020 (2020), 310.
[10] S. Boulaaras, A. Draifia, Kh. Zennir, General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping and logarithmic nonlinearity, Math Meth Appl Sci. 2019;42:4795-4814.
[11] S. Boulaaras, R. Jan, A. Choucha, A. Zeraï and M. Benzahi. Blow-up and lifespan of solutions for elastic membrane equation with distributed delay and logarithmic nonlinearity. Boundary Value Problems, 2024, 36 (2024). https://doi.org/10.1186/s13661-024-01843-5.
[12] M. M. Cavalcanti, D. Cavalcanti, J. Ferreira, Existence and uniform decay for nonlinear viscoelastic equation with strong damping, Math. Meth. Appl. Sci., 24 (2001), 1043-1053.
[13] A. Choucha, S. Boulaaras, Asymptotic behavior for a viscoelastic Kirchhoff equation with distributed delay and Balakrishnan-Taylor damping. Boundary Value Problems. (2021). DOI: 10.1186/s13661-021-01555-0.
[14] A. Choucha, D. Ouchenane, Kh. Zennir and B. Feng, Global well-posedness and exponential stability results of a class of Bresse-Timoshenko-type systems with distributed delay term. Math Meth Appl Sci. (2020);1-26. DOI: 10.1002/mma.6437.
[15] A. Choucha, S. Boulaaras, R. Jan, R. Alharbi. Blow-up and decay of solutions for a viscoelastic Kirchhoff-type equation with distributed delay and variable exponents. Math. Meth. Appl. Sci. 2024; 1-18. DOI: 10.1002/mma.9950.
[16] A. Choucha, D. Ouchenane and S. Boulaaras, Blow-up of a nonlinear viscoelastic wave equation with distributed delay combined with strong damping and source terms. J. Nonlinear Funct. Anal. (2020), Article ID 31 https://doi.org/10.23952/jnfa.2020.31.
[17] A. Choucha, D. Ouchenane, K. Zennir, Exponential growth of solution with $\mathscr{L}_p$-norm for class of non-linear viscoelastic wave equation with distributed delay term for large initial data. Open Journal of Mathematical Analysis, (2020) 4 (1), 76-83.
[18] A. Choucha, S. Boulaaras, D. Ouchenane and S. Beloul, General decay of nonlinear viscoelastic Kirchhoff equation with Balakrishnan-Taylor damping, logarithmic nonlinearity and distributed delay terms, Math Meth Appl Sci. (2020), 1-22. DOI:10.1002/mma.7121.
[19] A. Choucha, S. M. Boulaaras, D. Ouchenane, B.B. Cherif and M. Abdalla, Exponential Stability of Swelling Porous Elastic with a Viscoelastic Damping and Distributed Delay Term. JFS. V(2021), Article ID 5581634, https://doi.org/10.1155/2021/5581634.
[20] H. Chen, P. Luo, G.W. Liu, Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity. J Math Anal Appl.2015;422:84-98.
[21] B.D. Coleman and W. Noll, Foundations of linear viscoelasticity. Reviews of Modern Physics, 33(2).239 (1961).
[22] A. Djebabla, A. Choucha, D. Ouchenane, K. Zennir, Explicit Stability for a Porous Thermoelastic System with Second Sound and Distributed Delay Term. International Journal of Applied and Computational Mathematics. (2021) 7, 1-16.
[23] B. Feng, A. Soufyane. Existence and decay rates for a coupled Balakrishnan-Taylor viscoelastic system with dynamic boundary conditions. Math. Methods. Sci. 43, 3375-3391 (2020).
[24] B. Gheraibia, N. Boumaza, General decay result of solution for viscoelastic wave equation with Balakrishnan-Taylor damping and a delay term. ZAMP.(2020)71:198. Doi.org/10.1007/s00033-020-01426-1.
[25] P. Gorka, Logarithmic Klein-Gordon equation. Acta Phys Polon B.2009;40(1):59-66
[26] L. Gross, Logarithmic Sobolev inequalities. Amer J Math.1975;97(4):1061-1083.
[27] M. Kafini, S.A. Messaoudi, Local existence and blow up of solutions to a logarithmic nonlinear wave equation with delay. Applicable Analysis, August 2018. DOI: 10.1080/00036811.2018.1504029
[28] G. Kirchhoff. Vorlesungen uber Mechanik. Leipzig: Tauber; 1883.
[29] G. Liang, Y. Zhaoqin, L. Guonguang, Blow Up and Global Existence for a Nonlinear Viscoelastic Wave Equation with Strong Damping and Nonlinear Damping and Source terms, Applied Mathematics, 6 (2015), 806-816.
[30] W. Liu, B. Zhu, G. Li, D. Wang. General decay for a viscoelastic Kirchhoof equation with Balakrishnan-Taylor damping, dynamic boundary conditions and a time-varying delay term. Evol. Equ. Control Theory. 6, 239-260 (2017).
[31] F. Mesloub, S. Boulaaras, General decay for a viscoelastic problem with not necessarily decreasing kernel. J Appl Math Comput. 2018;58:647-665. doi:10.1007/S12190-017-1161-9.
[32] C. Mu, J. Ma. On a system of nonlinear wave equations with Balakrishnan-Taylor damping. Z Angew Math Phys. 2014, 65:91-113.
[33] A. S. Nicaise and C. Pignoti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, SIAM J. Control Optim., 45(5): 1561–1585, (2006).
[34] H.M. Ahmed, A.M.S. Ahmed, M.A. Ragusa, On some non-instantaneous impulsive differential equations with fractional brownian motion and Poisson jumps, TWMS Journal of Pure and Applied Mathematics, 14 (1), 125-140, (2023);
[35] N.C. Eddine, A.T. Nguyen, M.A. Ragusa, The Dirichlet problem for a class of anisotropic Schrodinger-Kirchhoff-Type equations with critical exponent, Mathematical Modelling and Analysis, 29 (2), 254–267, (2024);
[36] E. Piskin, F. Ekinci, Blow up and growth of solutions to a viscoelastic parabolic type Kirchhoff equation, Filomat, 37 (2), 519–530, (2023);