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# Simpson-like inequalities for functions whose third derivatives belong to $s$-convexity involving Atangana–Baleanu fractional integrals and their applications

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**Abstract.** The main objective of this paper is to study Simpson-like inequalities by using the Atangana–Baleanu (AB) fractional integral operators for functions whose third derivatives of absolute values are $s$-convex. To begin with, we establish the parameterized integral identity. As an effect of this outcome, we derive a series of Simpson-like integral inequalities related to functions whose third derivatives belong to the $s$-convexity in absolute values. Furthermore, an improved version of the identity is given and the estimated results are obtained by considering boundedness and Lipschitz condition. It concludes with some applications in respect of the Simpson-like quadrature formulas and special means, separately.

## 1. Introduction and preliminaries

Throughout this paper let $I \subseteq \mathbb{R}$ be a real interval, and $L_p([a,b])$ is the set of all functions $f : [a,b] \to \mathbb{R}$ such that $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \Big( \int_a^b |f(x)|^p \mathrm{d}x \Big)^{\frac{1}{p}}, & 1 \le p < \infty, \\ \sup|f(x)|, & p = \infty. \end{cases}$$

Now, let us begin by recalling the concepts of convex functions and others, which are significant in the current paper.

**Definition 1.1.** *A function $f : I \to \mathbb{R}$ is said to be convex, if the following inequality*

$$f\Big(tx + (1-t)y\Big) \le t f(x) + (1-t)y$$

*holds for all $x, y \in I$ and $t \in [0,1]$.*

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One of the most important inequalities in mathematics with regard to convex functions is the so-called Hermite–Hadamard's (abbreviated as, HH) integral inequality, which can be stated as

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}, \tag{1}$$

where $f$ is a convex function over the real-valued interval $[a, b]$ along with $a < b$.

For recent research related to integral inequalities (1), see the published papers [34, 44, 54].

The $s$-convex function described below is a generalized form of convex function, which was proposed by Hudzik and Maligranda in 1994.

**Definition 1.2.** *[28] A function $f : I \subseteq [0,\infty) \to \mathbb{R}$ is said to be $s$-convex in the second sense, if the following inequality*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y)$$

*holds for any $x, y \in I$ and for some fixed $s \in (0, 1]$.*

Clearly, taking $s = 1$ in Definition 1.2, the $s$-convex functions turn into the classical convex functions. There is a relationship between convex functions and $s$-convex functions as follows.

**Proposition 1.3.** *[6] Suppose that $f : I \subseteq [0,\infty) \to [0,\infty)$ is a non-negative convex function on $I$. Then, for $0 < s \le 1$, $f^s$ is an $s$-convex function on $[0,\infty)$.*

In 1999, Dragomir and Fitzpatrick gave the HH-type integral inequality in connection with $s$-convex functions.

**Theorem 1.4.** *[23] Suppose that $f : I \to \mathbb{R}$ is an $s$-convex function for $a, b \in I$ with $0 \le a < b$ and $f \in L_1([a, b])$. Then, the following inequality holds*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{s+1}. \tag{2}$$

As is well known, Simpson's inequality is another prominent inequality with rich geometrical significance and widespread applications, as noted below.

**Theorem 1.5.** *Suppose that $f : [a, b] \to \mathbb{R}$ is a four-order differentiable function on $(a, b)$ along with $\left\|f^{(4)}\right\|_\infty = \sup_{t\in(a,b)} \left\|f^{(4)}\right\| < \infty$. Then, the following inequality holds*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4. \tag{3}$$

To achieve the main results, we need to bear in mind the beta function, which is defined as

$$\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t, \quad p, q > 0,$$

satisfying

$$\beta(p, q) = \beta(q, p),$$

and

$$\beta(p, q+1) = \frac{q}{p+q}\beta(p, q).$$

Considering the importance of Simpson's inequality, many researchers have paid much attention to it. There are plenty of literature on the generalization of Simpson-type integral inequalities. Hereby, we enumerate several existing outcomes concerning different classes of functions, such as convexity [39], harmonically-preinvexity [33], generalized $s$-convexity [37] and $\varphi$-quasiconvexity [48].

With the development of natural science and social science, fractional calculus theory has been extensively concerned. As of late, the continuously refreshed fractional calculus theory remains widely employed in real-world problems and presents excellent solutions, for example, in chaotic dynamical systems [9], disease model [10] and chaos [42]. Meanwhile, fractional calculus has also had a remarkable impact on the field of inequalities, such as fractional integral operators becoming a prevalent approach for generalizing inequalities. Here, a series of Simpson-type inequalities, with respect to first-, second- and third-order differentiable functions, are enumerated by virtue of fractional integral operators, correspondingly. For instance, some scholars employed Riemann–Liouville (RL) fractional integral operators to develop Simpson-type inequalities for various families of differentiable functions, as for $s$-convex functions [18] and $(s, m)$-convex functions [36]. Additionally, Kermausuor [32] utilized the Katugampola fractional integral operators, as an extension of the RL- and Hadamard-fractional integrals, to investigate Simpson-type integral inequalities in relation to $s$-convex functions in the second sense. And then, Şanlı [43] offered a couple of Simpson-type integral inequalities taking advantage of the conformable fractional integrals and exhibited several applications relating to special means. In 2022, with the aid of generalized fractional integrals, also known as Sarikaya fractional integrals, Budak et al. [13] established some parameterized integral inequalities for differentiable convex functions, from which the Simpson-, midpoint- and trapezoidal-type inequalities were deduced by taking different values of the parameters. For more Simpson-type inequalities involving with first-order differentiable functions acquired by virtue of fractional integral operators, one can refer to the literature [11, 14, 15, 25, 49] and their bibliographies.

In recent years, the research of Simpson-type inequalities for twice differentiable functions has gradually enriched. For example, Hezenci et al. [26] established an identity containing twice differentiable functions with the help of RL-fractional integral operators, from which a series of Simpson-type inequalities were derived. The authors in Ref. [5] gave the generalized fractional integral identity, and they utilized the result to develop a few Simpson's-formula-type inequalities concerning twice differentiable convex functions. The outcomes can be reduced to Riemann integral, RL- and $k$-RL fractional integral forms, respectively. Afterward, by dint of the fractional integrals involving exponential kernels, Zhou et al. [55] presented several parameterized integral inequalities in connection with convex functions, which unified Simpson's inequality, the averaged midpoint-trapezoid inequality, and the trapezoid inequality. Also, Zhou et al. [53] proposed the weighted parameterized integral inequalities in relation to twice differentiable functions, unifying midpoint-, Simpson-, Bullen- and trapezoid-type inequalities. For more Simpson-type inequalities with regard to twice differentiable functions obtained by virtue of fractional integral operators, the published articles [12, 22, 24, 29] and their bibliographies are available.

Nevertheless, there are relatively few results reported in the literature on integral inequalities involving three times differentiable functions, especially Simpson-type integral inequalities. For example, in the sense of Riemann integrals, Liu and Chun et al. investigated Simpson-type inequalities with respect to functions whose third derivatives are $h$-convex [35] and extended $s$-convex [19], respectively. By means of RL-fractional integrals, Hezenci and his co-worker [27] established the Simpson-type inequalities containing three times differentiable convex functions. In addition, there are several papers [40, 41] concerning other types of fractional integral inequalities with three times differentiable functions.

Below, we recall the definition of RL-fractional integrals.

**Definition 1.6.** *Let the function $f \in L_1([a, b])$. The RL-fractional integrals $I^{\alpha}_{a^+}f(x)$ and $I^{\alpha}_{b^-}f(x)$ of order $\alpha > 0$ are defined by*

$$I^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\mathrm{d}t, \quad x > a,$$

*and*

$$I^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\mathrm{d}t, \quad x < b,$$

*respectively. Here,* $\Gamma(\alpha)$ *is the gamma function and* $I^0_{a^+}f(x) = I^0_{b^-}f(x) = f(x)$.

In 2016, a class of AB-fractional integrals was proposed by Atangana and Baleanu.

**Definition 1.7.** *[8] The left-hand side of the AB-fractional integrals with non-local kernel of a function* $f \in H^1(a, b)$, *is defined as follows*

$$ {}^{AB}_{\ a}I^{\alpha}_{t}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}\mathrm{d}y, \tag{4}$$

*where* $a < b$, $\alpha \in [0, 1]$, $\Gamma(\alpha)$ *is a gamma function,* $B(\alpha)$ *is a positive normalized function along with* $B(0) = B(1) = 1$ *and* $H^1(a, b) = \{f : f \in L_1([a, b]), f' \in L_1([a, b])\}$.

Similarly, Abdeljawad et al. [1] proposed the right-hand side of the AB-fractional integrals of a function $f \in H^1(a, b)$, that is

$$ {}^{AB}_{\ t}I^{\alpha}_{b}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_t^b f(y)(y-t)^{\alpha-1}\mathrm{d}y. \tag{5}$$

The AB-fractional integral operators, considered to be quite efficient tools, have been used by many scholars to obtain extended versions of different inequalities. For example, Set et al. [45, 46] created first- and second-order identities by way of AB-fractional integrals. Based upon the founded integral identities, the generalization of the HH-type inequalities were obtained. Nasir et al. [38] attained a series of Hermit–Hadamard–Mercer inequalities for interval-valued functions by virtue of AB-fractional integral operators. In 2023, Yuan et al. [51] established parametric integral identities and their improved versions through AB-fractional integrals, and used the resulting identities to derive a series of Simpson-like integral inequalities for functions whose second-order derivatives belong to $(s, P)$-convexity. Drawing support from the AB-fractional integral operators, Ahmad [3] et al., Ardıç [7] et al. as well as Karim [30] et al. provided certain Ostroski-type integral inequalities. In addition, it is worth noting that numerous researchers have expanded the AB-fractional integral operators to obtain extended versions of them, resulting in some interesting outcomes. For example, Kashuri [31] introduced the ABK-fractional integrals, which combine the AB- and Katugampola-fractional integrals. And then, they obtained several integral identities and related HH-type inequalities. Moreover, Butt et al. [17] not only established two integral identities involving ABK-fractional integrals, but also investigated a series of generalized HH-type inequalities, separately. In 2023, Gronwall's inequalities were established in the framework of combining the increasing function with AB-fractional integral operators by Abdeljawad et al. [2]. For other findings on the AB-fractional integral operators, one can refer to the published papers [4, 16].

Inspired from the researches referenced above, notably the methodology of the papers [27, 51], the current paper endeavors to develop the Simpson-type integral inequalities involving with the third derivatives whose absolute value belong to $s$-convexity under the settings of the AB-fractional integral operators. The current study is arranged according to the coming way. After the introduction and preliminaries in Sections 2 and 3 primary outcomes in relation to the topic are investigated. The examples concerning $s$-convex functions and applications are delivered in Sections 4 and 5, respectively. At the end, concluding remarks are provided.

## 2. Main Results

Firstly, we propose a Simpson-type integral equality for AB-fractional integrals, which is shown in the following lemma. The established lemma plays an indispensable role in the whole research.

**Lemma 2.1.** *Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentiable function on $(a,b)$ together with $a < b$. If $x \in [a,b]$, then the following equation for AB-fractional integrals holds true*

$$\begin{aligned}
&\frac{\alpha(x-a)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a)+\left[\frac{2(x-a)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}+\frac{2(b-x)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}\right]f(x)+\frac{\alpha(b-x)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)\\
&\quad-\left[{}^{AB}_{a}I^{\alpha}_{x}\{f(x)\}+{}^{AB}_{x}I^{\alpha}_{b}\{f(x)\}\right]+\frac{2(1-\alpha)}{B(\alpha)}f(x)\\
&\quad=\left[\frac{\alpha(x-a)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}-\frac{\alpha(b-x)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\right]f'(x)\\
&\quad+\frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(t)f'''\left((1-t)x+tb\right)\mathrm{d}t\\
&\quad-\frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(1-t)f'''\left((1-t)a+tx\right)\mathrm{d}t,
\end{aligned}$$

*where*

$$k(t)=\alpha t-(\alpha+1)t^2+t^{\alpha+2},$$
$$k(1-t)=\alpha(1-t)-(\alpha+1)(1-t)^2+(1-t)^{\alpha+2},$$

$\Gamma(\cdot)$ *is a gamma function and $B(\alpha)$ for $\alpha \in (0,1]$ is a positive normalized function.*

*Proof.* Using the integration by parts and appropriate substitutions, for $x \neq b$, we have that

$$\begin{aligned}
I_1 &= \frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(t)f'''\left((1-t)x+tb\right)\mathrm{d}t\\
&= \frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 \left(\alpha t-(\alpha+1)t^2+t^{\alpha+2}\right)f'''\left((1-t)x+tb\right)\mathrm{d}t\\
&= \frac{(b-x)^{\alpha+2}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\Bigg[\left(\alpha t-(\alpha+1)t^2+t^{\alpha+2}\right)f''\left((1-t)x+tb\right)\Big|_0^1\\
&\quad-\int_0^1\left(\alpha-2(\alpha+1)t+(\alpha+2)t^{\alpha+1}\right)f''\left((1-t)x+tb\right)\mathrm{d}t\Bigg]\\
&= -\frac{(b-x)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\Bigg[\left(\alpha-2(\alpha+1)t+(\alpha+2)t^{\alpha+1}\right)f'\left((1-t)x+tb\right)\Big|_0^1\\
&\quad-\int_0^1\left(-2(\alpha+1)+(\alpha+1)(\alpha+2)t^{\alpha}\right)f'\left((1-t)x+tb\right)\mathrm{d}t\Bigg]\\
&= \frac{\alpha(b-x)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}f'(x)+\frac{(b-x)^{\alpha}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\\
&\quad\times\Bigg[\left(-2(\alpha+1)+(\alpha+1)(\alpha+2)t^{\alpha}\right)f\left((1-t)x+tb\right)\Big|_0^1\\
&\quad-\int_0^1\left(\alpha(\alpha+1)(\alpha+2)t^{\alpha-1}\right)f\left((1-t)x+tb\right)\mathrm{d}t\Bigg]\\
&= \frac{\alpha(b-x)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}f'(x)+\frac{2(b-x)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(x)+\frac{\alpha(b-x)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)\\
&\quad-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_x^b(y-x)^{\alpha-1}f(y)\mathrm{d}y\\
&= \frac{\alpha(b-x)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}f'(x)+\frac{2(b-x)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(x)+\frac{\alpha(b-x)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)\\
&\quad-{}^{AB}_{x}I^{\alpha}_{b}\{f(x)\}+\frac{1-\alpha}{B(\alpha)}f(x).
\end{aligned} \tag{6}$$

Similarly, for $x \neq a$, we can show that

$$\begin{aligned}I_2 =&\frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(1-t)f'''\left((1-t)a+tx\right)\mathrm{d}t\\ =&\frac{\alpha(x-a)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}f'(x)-\frac{2(x-a)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(x)-\frac{\alpha(x-a)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a)\\ &+{}^{AB}_{\ a}I^{\alpha}_{x}\left\{f(x)\right\}-\frac{1-\alpha}{B(\alpha)}f(x).\end{aligned} \tag{7}$$

From equations (6) and (7), we find that

$$\begin{aligned}I_1-I_2 =&\frac{\alpha(x-a)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a)+\frac{\alpha(b-x)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)\\ &+\left[\frac{2(x-a)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}+\frac{2(b-x)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}\right]f(x)\\ &+\left[\frac{\alpha(b-x)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}-\frac{\alpha(x-a)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\right]f'(x)\\ &-\left[{}^{AB}_{\ a}I^{\alpha}_{x}\{f(x)\}+{}^{AB}_{\ x}I^{\alpha}_{b}\{f(x)\}\right]+\frac{2(1-\alpha)}{B(\alpha)}f(x).\end{aligned}$$

As a consequence, we obtain the required result.

For $x = a$ or $x = b$, utilizing integration by parts and appropriate variable substitutions again, we can prove the coming identities

$$\begin{aligned}&\frac{2(b-a)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a)+\frac{\alpha(b-a)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)-{}^{AB}_{\ a}I^{\alpha}_{b}\{f(a)\}+\frac{1-\alpha}{B(\alpha)}f(a)\\ &=\frac{(b-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(t)f'''\left((1-t)a+tb\right)\mathrm{d}t-\frac{\alpha(b-a)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}f'(a),\end{aligned}$$

and

$$\begin{aligned}&\frac{2(b-a)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)+\frac{\alpha(b-a)^{\alpha}}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a)-{}^{AB}_{\ a}I^{\alpha}_{b}\{f(b)\}+\frac{1-\alpha}{B(\alpha)}f(b)\\ &=-\frac{(b-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(1-t)f'''\left((1-t)a+tb\right)\mathrm{d}t+\frac{\alpha(b-a)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}f'(b).\end{aligned}$$

This completes the proof.

**Corollary 2.2.** *Assuming that all conditions raised in Lemma 2.1 are satisfied. If we consider putting* $x = \frac{a+b}{2}$, *then we attain the following equation*

$$\begin{aligned}&\frac{\alpha(b-a)^{\alpha}}{2^{\alpha}(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a)+\frac{(b-a)^{\alpha}}{2^{\alpha-2}(\alpha+2)B(\alpha)\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)+\frac{\alpha(b-a)^{\alpha}}{2^{\alpha}(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)\\ &\quad-\left[{}^{AB}_{\ a}I^{\alpha}_{\frac{a+b}{2}}\left\{f\left(\frac{a+b}{2}\right)\right\}+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}_{b}\left\{f\left(\frac{a+b}{2}\right)\right\}\right]+\frac{2(1-\alpha)}{B(\alpha)}f\left(\frac{a+b}{2}\right)\\ &=\frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(t)f'''\left((1-t)\frac{a+b}{2}+tb\right)\mathrm{d}t\\ &\quad-\frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(1-t)f'''\left((1-t)a+t\frac{a+b}{2}\right)\mathrm{d}t.\end{aligned} \tag{8}$$

**Remark 2.3.** *By Definition 1.7 of the AB-fractional integrals, we have that*

$$\frac{1-\alpha}{B(\alpha)}f\left(\frac{a+b}{2}\right)-{}^{AB}_{\ a}I^{\alpha}_{\frac{a+b}{2}}\left\{f\left(\frac{a+b}{2}\right)\right\}=-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}f(y)\left(\frac{a+b}{2}-y\right)^{\alpha-1}\mathrm{d}y, \tag{9}$$

*and*

$$\frac{1-\alpha}{B(\alpha)}f\left(\frac{a+b}{2}\right)-{}^{AB}_{\frac{a+b}{2}}I^{\alpha}_{b}\left\{f\left(\frac{a+b}{2}\right)\right\}=-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}f(y)\left(y-\frac{a+b}{2}\right)^{\alpha-1}\mathrm{d}y. \tag{10}$$

*If we substitute the above equations* (9) *and* (10) *into equation* (8), *then we have that*

$$\begin{aligned}&\frac{\alpha(b-a)^{\alpha}}{2^{\alpha}(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a)+\frac{(b-a)^{\alpha}}{2^{\alpha-2}(\alpha+2)B(\alpha)\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)+\frac{\alpha(b-a)^{\alpha}}{2^{\alpha}(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)\\&\quad-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^{\frac{a+b}{2}}f(y)\left(\frac{a+b}{2}-y\right)^{\alpha-1}\mathrm{d}y-\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_{\frac{a+b}{2}}^{b}f(y)\left(y-\frac{a+b}{2}\right)^{\alpha-1}\mathrm{d}y\\&=\frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1k(t)f'''\left((1-t)\frac{a+b}{2}+tb\right)\mathrm{d}t\\&\quad-\frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1k(1-t)f'''\left((1-t)a+t\frac{a+b}{2}\right)\mathrm{d}t.\end{aligned} \tag{11}$$

*Also, it is easy to check that*

$$\int_0^1k(1-t)f'''\left((1-t)a+t\frac{a+b}{2}\right)\mathrm{d}t=\int_0^1k(t)f'''\left(ta+(1-t)\frac{a+b}{2}\right)\mathrm{d}t. \tag{12}$$

*Substituting equation* (12) *into equation* (11), *and multiplying the resulting by* $\frac{2^{\alpha-1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha}}$, *we have the following identity for RL-fractional integrals*

$$\begin{aligned}&\frac{\alpha}{2(\alpha+2)}\left[f(a)+\frac{4}{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{a^+}f\left(\frac{a+b}{2}\right)+I^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right]\\&=\frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\int_0^1(\alpha t-(\alpha+1)t^2+t^{\alpha+2})\left[f'''\left((1-t)\frac{a+b}{2}+tb\right)-f'''\left(ta+(1-t)\frac{a+b}{2}\right)\right]\mathrm{d}t,\end{aligned} \tag{13}$$

*which is also a new equality in the sense of RL-fractional integrals. Especially, if we take* $\alpha=1$ *in equation* (13), *then we have that*

$$\begin{aligned}&\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^bf(y)\mathrm{d}y\\&=\frac{(b-a)^3}{96}\int_0^1\left(t-2t^2+t^3\right)\left[f'''\left((1-t)\frac{a+b}{2}+tb\right)-f'''\left(ta+(1-t)\frac{a+b}{2}\right)\right]\mathrm{d}t.\end{aligned} \tag{14}$$

*This is the same as Lemma 2.1 established by Chun and Qi in [19].*

**Lemma 2.4.** *The function* $k:[0,1]\to\mathbb{R}$ *is defined by* $k(t)=\alpha t-(\alpha+1)t^2+t^{\alpha+2}$ *for* $\alpha\in(0,1]$. *Then, we derive the coming result*

$$\int_0^1|k(t)|\mathrm{d}t=\frac{\alpha^2+\alpha}{6\alpha+18}.$$

*Proof.* If we consider taking $0 < \alpha \leq 1$, then $k(t) \geq 0$ for every $t \in [0,1]$. Thus, we derive that

$$\int_0^1 |k(t)|\mathrm{d}t = \int_0^1 \left(\alpha t - (\alpha+1)t^2 + t^{\alpha+2}\right)\mathrm{d}t = \frac{\alpha^2+\alpha}{6\alpha+18}.$$

This ends the proof.

By virtue of the two lemmas above, one can achieve a series of AB-fractional integral inequalities pertaining to $s$-convexity. For the convenience of expression, we denote the coming term with $\mathfrak{R}_f(a,b;\alpha,x)$:

$$\begin{aligned}
&\mathfrak{R}_f(a,b;\alpha,x)\\
&:= \frac{\alpha(x-a)^\alpha}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a) + \left[\frac{2(x-a)^\alpha}{(\alpha+2)B(\alpha)\Gamma(\alpha)} + \frac{2(b-x)^\alpha}{(\alpha+2)B(\alpha)\Gamma(\alpha)}\right]f(x)\\
&\quad + \frac{\alpha(b-x)^\alpha}{(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b) - \left[{}^{AB}_{\ a}I^\alpha_x\{f(x)\} + {}^{AB}_{\ x}I^\alpha_b\{f(x)\}\right] + \frac{2(1-\alpha)}{B(\alpha)}f(x)\\
&\quad - \left[\frac{\alpha(x-a)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)} - \frac{\alpha(b-x)^{\alpha+1}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\right]f'(x).
\end{aligned}$$

**Theorem 2.5.** *Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentiable function on $(a,b)$ together with $a < b$ and $f''' \in L_1([a,b])$. If $|f'''|$ is an s-convex function on $[a,b]$ for some fixed $s \in (0,1]$, then for each $x \in [a,b]$, we get the following inequality for AB-fractional integrals*

$$\begin{aligned}
\left|\mathfrak{R}_f(a,b;\alpha,x)\right| \leq{}& \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(v_1\left|f'''(a)\right| + v_2\left|f'''(x)\right|\right)\\
&+ \frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(v_1\left|f'''(b)\right| + v_2\left|f'''(x)\right|\right),
\end{aligned} \tag{15}$$

*where*

$$v_1 = \frac{\alpha(\alpha+1)}{(s+2)(s+3)(s+\alpha+3)}, \tag{16}$$

*and*

$$v_2 = \frac{\alpha s + \alpha - 2}{(s+1)(s+2)(s+3)} + \beta(s+1,\alpha+3). \tag{17}$$

*Proof.* Making use of the identity given in Lemma 2.1, we deduce that

$$\begin{aligned}
\left|\mathfrak{R}_f(a,b;\alpha,x)\right| &= \left|\frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(t)f'''\left((1-t)x+tb\right)\mathrm{d}t\right.\\
&\qquad \left. - \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(1-t)f'''\left((1-t)a+tx\right)\mathrm{d}t\right|\\
&= \left|\frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(t)f'''\left((1-t)x+tb\right)\mathrm{d}t\right.\\
&\qquad \left. - \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(t)f'''\left(ta+(1-t)x\right)\mathrm{d}t\right|\\
&\leq \frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 |k(t)|\cdot\left|f'''\left((1-t)x+tb\right)\right|\mathrm{d}t\\
&\qquad + \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 |k(t)|\cdot\left|f'''\left(ta+(1-t)x\right)\right|\mathrm{d}t.
\end{aligned} \tag{18}$$

To achieve the desired result, we use the $s$-convexity of $|f'''|$ on $[a,b]$. Hence, we acquire the following inequalities

$$\begin{aligned}
&\int_0^1 |k(t)| \cdot \left|f'''\left(ta+(1-t)x\right)\right| \mathrm{d}t \\
&= \int_0^1 \left|\left(\alpha t-(\alpha+1)t^2+t^{\alpha+2}\right)\right| \cdot \left|f'''\left(ta+(1-t)x\right)\right| \mathrm{d}t \\
&\le \int_0^1 \left(\alpha t-(\alpha+1)t^2+t^{\alpha+2}\right)\left(t^s\left|f'''(a)\right|+(1-t)^s\left|f'''(x)\right|\right)\mathrm{d}t \\
&= \frac{\alpha(\alpha+1)}{(s+2)(s+3)(s+\alpha+3)}|f'''(a)| + \left(\frac{\alpha s+\alpha-2}{(s+1)(s+2)(s+3)}+\beta(s+1,\alpha+3)\right)|f'''(x)|,
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
&\int_0^1 |k(t)| \cdot \left|f'''\left((1-t)x+tb\right)\right| \mathrm{d}t \\
&= \int_0^1 \left|\left(\alpha t-(\alpha+1)t^2+t^{\alpha+2}\right)\right| \cdot \left|f'''\left((1-t)x+tb\right)\right| \mathrm{d}t \\
&\le \int_0^1 \left(\alpha t-(\alpha+1)t^2+t^{\alpha+2}\right)\left((1-t)^s|f'''(x)|+t^s|f'''(b)|\right)\mathrm{d}t \\
&= \frac{\alpha(\alpha+1)}{(s+2)(s+3)(s+\alpha+3)}|f'''(b)| + \left(\frac{\alpha s+\alpha-2}{(s+1)(s+2)(s+3)}+\beta(s+1,\alpha+3)\right)|f'''(x)|.
\end{aligned} \tag{20}$$

Applying inequalities (19) and (20) to the inequality (18), we obtain the desired result. The proof is fulfilled.

**Corollary 2.6.** *Assuming that all conditions raised in Theorem 2.5 are satisfied. If we take $x=\frac{a+b}{2}$, then we have that*

$$\begin{aligned}
&\Bigg|\frac{\alpha(b-a)^\alpha}{2^\alpha(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a)+\frac{(b-a)^\alpha}{2^{\alpha-2}(\alpha+2)B(\alpha)\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)+\frac{\alpha(b-a)^\alpha}{2^\alpha(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b) \\
&\quad -\left[{}^{AB}_{\ a}I^\alpha_{\frac{a+b}{2}}\left\{f\left(\frac{a+b}{2}\right)\right\}+{}^{AB}_{\frac{a+b}{2}}I^\alpha_b\left\{f\left(\frac{a+b}{2}\right)\right\}\right]+\frac{2(1-\alpha)}{B(\alpha)}f\left(\frac{a+b}{2}\right)\Bigg| \\
&\le \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[v_1\left|f'''(a)\right|+2v_2\left|f'''\left(\frac{a+b}{2}\right)\right|+v_1\left|f'''(b)\right|\right].
\end{aligned} \tag{21}$$

**Remark 2.7.** *Plugging the equations* (9) *and* (10) *into the inequality* (21)*, and multiplying the resulting by $\frac{2^{\alpha-1}B(\alpha)\Gamma(\alpha)}{(b-a)^\alpha}$, we acquire the following RL-fractional Simpson-type integral inequality for s-convex functions*

$$\begin{aligned}
&\left|\frac{\alpha}{2(\alpha+2)}\left[f(a)+\frac{4}{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I^\alpha_{a^+}f\left(\frac{a+b}{2}\right)+I^\alpha_{b^-}f\left(\frac{a+b}{2}\right)\right]\right| \\
&\le \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\left[v_1\left|f'''(a)\right|+2v_2\left|f'''\left(\frac{a+b}{2}\right)\right|+v_1\left|f'''(b)\right|\right].
\end{aligned}$$

*Furtherly, if we consider choosing $s=1$, in this case, then the following RL-fractional Simpson-type integral inequality for convex functions exists*

$$\begin{aligned}
&\left|\frac{\alpha}{2(\alpha+2)}\left[f(a)+\frac{4}{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I^\alpha_{a^+}f\left(\frac{a+b}{2}\right)+I^\alpha_{b^-}f\left(\frac{a+b}{2}\right)\right]\right| \\
&\le \frac{\alpha(b-a)^3}{192(\alpha+2)(\alpha+4)}\left[|f'''(a)|+\frac{2\alpha+10}{\alpha+3}\left|f'''\left(\frac{a+b}{2}\right)\right|+|f'''(b)|\right].
\end{aligned}$$

*And even more, if we take $\alpha = 1$, then it yields that*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y\right| \leq \frac{(b-a)^3}{2880}\left[|f'''(a)|+3\left|f'''\left(\frac{a+b}{2}\right)\right|+|f'''(b)|\right],$$

*which is Corollary 3.1.1 established by Chun and Qi in [19].*

**Theorem 2.8.** *Let $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ be a three times differentiable function on $(a,b)$ together with $a<b$ and $f'''\in L_1([a,b])$. If for $q>1$ $|f'''|^q$ is an s-convex function on $[a,b]$ along with some fixed $s\in(0,1]$, then for each $x\in[a,b]$, we get the following inequality for AB-fractional integrals*

$$\begin{aligned}
&\left|\Re_f(a,b;\alpha,x)\right| \\
&\leq \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(\frac{\alpha^2+\alpha}{6\alpha+18}\right)^{\left(1-\frac{1}{q}\right)}\left(v_1|f'''(a)|^q+v_2|f'''(x)|^q\right)^{\frac{1}{q}} \\
&\quad+\frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(\frac{\alpha^2+\alpha}{6\alpha+18}\right)^{\left(1-\frac{1}{q}\right)}\left(v_1|f'''(b)|^q+v_2|f'''(x)|^q\right)^{\frac{1}{q}},
\end{aligned} \tag{22}$$

*where $v_1$ and $v_2$ are defined as in Theorem 2.5.*

*Proof.* Continuing from inequality (18) in the proof of Theorem 2.5, using the power mean integral inequality and the $s$-convexity of $|f'''|^q$, we can obtain that

$$\begin{aligned}
&\left|\Re_f(a,b;\alpha,x)\right| \\
&\leq \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(\int_0^1|k(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1|k(t)|\cdot\left|f'''\left(ta+(1-t)x\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad+\frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(\int_0^1|k(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1|k(t)|\cdot\left|f'''\left((1-t)x+tb\right)\right|^q\mathrm{d}t\right)^{\frac{1}{q}} \\
&\leq \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(\int_0^1|k(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1|k(t)|\cdot\left(t^s|f'''(a)|^q+(1-t)^s|f'''(x)|^q\right)\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad+\frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(\int_0^1|k(t)|\,\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1|k(t)|\cdot\left((1-t)^s|f'''(x)|^q+t^s|f'''(b)|^q\right)\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned} \tag{23}$$

It is easy to check that

$$\begin{aligned}
&\int_0^1|k(t)|\cdot\left(t^s|f'''(a)|^q+(1-t)^s|f'''(x)|^q\right)\mathrm{d}t \\
&=\int_0^1\left(\alpha t-(\alpha+1)t^2+t^{\alpha+2}\right)\left(t^s|f'''(a)|^q+(1-t)^s|f'''(x)|^q\right)\mathrm{d}t \\
&=\frac{\alpha(\alpha+1)}{(s+2)(s+3)(s+\alpha+3)}\left|f'''(a)\right|^q+\left(\frac{\alpha s+\alpha-2}{(s+1)(s+2)(s+3)}+\beta(s+1,\alpha+3)\right)\left|f'''(x)\right|^q,
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
&\int_0^1|k(t)|\cdot\left((1-t)^s|f'''(x)|^q+t^s|f'''(b)|^q\right)\mathrm{d}t \\
&=\int_0^1\left(\alpha t-(\alpha+1)t^2+t^{\alpha+2}\right)\left((1-t)^s|f'''(x)|^q+t^s|f'''(b)|^q\right)\mathrm{d}t \\
&=\frac{\alpha(\alpha+1)}{(s+2)(s+3)(s+\alpha+3)}|f'''(b)|^q+\left(\frac{\alpha s+\alpha-2}{(s+1)(s+2)(s+3)}+\beta(s+1,\alpha+3)\right)|f'''(x)|^q.
\end{aligned} \tag{25}$$

Applying equalities (24), (25) together with Lemma 2.4 to inequality (23), we obtain the desired result. The proof is fulfilled.

**Corollary 2.9.** *Assuming that all conditions raised in Theorem 2.8 are satisfied. If we take $x = \frac{a+b}{2}$, then we have that*

$$\begin{aligned}
&\Bigg|\frac{\alpha(b-a)^{\alpha}}{2^{\alpha}(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a)+\frac{(b-a)^{\alpha}}{2^{\alpha-2}(\alpha+2)B(\alpha)\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)+\frac{\alpha(b-a)^{\alpha}}{2^{\alpha}(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)\\
&\quad-\left[{}^{AB}_{\ a}I^{\alpha}_{\frac{a+b}{2}}\left\{f\left(\frac{a+b}{2}\right)\right\}+{}^{AB}_{\frac{a+b}{2}}I^{\alpha}_{b}\left\{f\left(\frac{a+b}{2}\right)\right\}\right]+\frac{2(1-\alpha)}{B(\alpha)}f\left(\frac{a+b}{2}\right)\Bigg|\\
&\quad\leq\frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(\frac{\alpha^{2}+\alpha}{6\alpha+18}\right)^{\left(1-\frac{1}{q}\right)}\\
&\quad\times\left[\left(v_1|f'''(a)|^{q}+v_2\left|f'''\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(v_1|f'''(b)|^{q}+v_2\left|f'''\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{26}$$

**Remark 2.10.** *Plugging equations* (9) *and* (10) *into the inequality* (26)*, and multiplying the resulting by $\frac{2^{\alpha-1}B(\alpha)\Gamma(\alpha)}{(b-a)^{\alpha}}$, we acquire the following RL-fractional Simpson-type integral inequality for s-convex functions*

$$\begin{aligned}
&\left|\frac{\alpha}{2(\alpha+2)}\left[f(a)+\frac{4}{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{a^{+}}f\left(\frac{a+b}{2}\right)+I^{\alpha}_{b^{-}}f\left(\frac{a+b}{2}\right)\right]\right|\\
&\quad\leq\frac{(b-a)^{3}}{16(\alpha+1)(\alpha+2)}\left(\frac{\alpha^{2}+\alpha}{6\alpha+18}\right)^{\left(1-\frac{1}{q}\right)}\\
&\quad\times\left[\left(v_1|f'''(a)|^{q}+v_2\left|f'''\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(v_1|f'''(b)|^{q}+v_2\left|f'''\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

*Furtherly, if we consider choosing $s = 1$, in this case, then the following RL-fractional Simpson-type integral inequality for convex functions exists*

$$\begin{aligned}
&\left|\frac{\alpha}{2(\alpha+2)}\left[f(a)+\frac{4}{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I^{\alpha}_{a^{+}}f\left(\frac{a+b}{2}\right)+I^{\alpha}_{b^{-}}f\left(\frac{a+b}{2}\right)\right]\right|\\
&\quad\leq\frac{\alpha(b-a)^{3}}{192(\alpha+2)}\left(\frac{2}{\alpha+3}\right)^{\left(1-\frac{1}{q}\right)}\left(\frac{1}{\alpha+4}\right)^{\frac{1}{q}}\\
&\quad\times\left[\left(|f'''(a)|^{q}+\frac{\alpha+5}{\alpha+3}\left|f'''\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(|f'''(b)|^{q}+\frac{\alpha+5}{\alpha+3}\left|f'''\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

*And even more, if we take $\alpha = 1$, then it yields that*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y\right|\\
&\quad\leq\frac{(b-a)^{3}}{1152}\left(\frac{1}{5}\right)^{\frac{1}{q}}\left[\left(2|f'''(a)|^{q}+3\left|f'''\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(2|f'''(b)|^{q}+3\left|f'''\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

*which is Corollary 3.1.2 established by Chun and Qi in [19].*

**Theorem 2.11.** *Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentiable function on $(a,b)$ together with $a < b$ and $f''' \in L_1([a,b])$. If for $q > 1$ $|f'''|^q$ is an s-convex function on $[a,b]$ along with $p^{-1} + q^{-1} = 1$, then for all $x \in [a,b]$, we get the following inequality for AB-fractional integrals*

$$\begin{aligned}
&\left|\Re_f(a,b;\alpha,x)\right| \\
&\leq \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)} w^{\frac{1}{p}} \left(\frac{1}{s+1}\right)^{\frac{1}{q}} \left(\left|f'''(a)\right|^q + \left|f'''(x)\right|^q\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)} w^{\frac{1}{p}} \left(\frac{1}{s+1}\right)^{\frac{1}{q}} \left(\left|f'''(x)\right|^q + \left|f'''(b)\right|^q\right)^{\frac{1}{q}},
\end{aligned} \tag{27}$$

*where*

$$w = \frac{2^{p-1}\alpha^p}{p+1} + \frac{2^{p-1}}{(\alpha+2)p+1} - \frac{(\alpha+1)^p}{2p+1}. \tag{28}$$

*Proof.* Continuing from inequality (18) in the proof of Theorem 2.5, using the Hölder's inequality and the $s$-convexity of $|f'''|^q$, we can obtain that

$$\begin{aligned}
&\left|\Re_f(a,b;\alpha,x)\right| \\
&\leq \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)} \left(\int_0^1 |k(t)|^p\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'''\left(ta+(1-t)x\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)} \left(\int_0^1 |k(t)|^p\right)^{\frac{1}{p}} \left(\int_0^1 \left|f'''\left((1-t)x+tb\right)\right|^q \mathrm{d}t\right)^{\frac{1}{q}} \\
&\leq \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)} \left(\int_0^1 |k(t)|^p\right)^{\frac{1}{p}} \left(\int_0^1 \left(t^s\left|f'''(a)\right|^q + (1-t)^s\left|f'''(x)\right|^q\right)\mathrm{d}t\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)} \left(\int_0^1 |k(t)|^p\right)^{\frac{1}{p}} \left(\int_0^1 \left((1-t)^s\left|f'''(x)\right|^q + t^s\left|f'''(b)\right|^q\right)\mathrm{d}t\right)^{\frac{1}{q}}.
\end{aligned} \tag{29}$$

Utilizing the inequality $(A-B)^\tau \leq A^\tau - B^\tau$ for every $A \geq B \geq 0$ as well as $\tau \geq 1$, and the inequality $(A+B)^\tau \leq 2^{\tau-1}(A^\tau + B^\tau)$ for every $A \geq 0, B \geq 0$ along with $\tau \geq 1$, we derive that

$$\begin{aligned}
\int_0^1 |k(t)|^p \mathrm{d}t &= \int_0^1 \left(\alpha t - (\alpha+1)t^2 + t^{\alpha+2}\right)^p \mathrm{d}t \\
&\leq \int_0^1 \left(\left(\alpha t + t^{\alpha+2}\right)^p - \left((\alpha+1)t^2\right)^p\right)\mathrm{d}t \\
&\leq \int_0^1 2^{p-1}\left((\alpha t)^p + \left(t^{\alpha+2}\right)^p\right)\mathrm{d}t - \int_0^1 \left((\alpha+1)t^2\right)^p \mathrm{d}t \\
&= \frac{2^{p-1}\alpha^p}{p+1} + \frac{2^{p-1}}{(\alpha+2)p+1} - \frac{(\alpha+1)^p}{2p+1}.
\end{aligned} \tag{30}$$

Direct computation yields that

$$\begin{aligned}
&\int_0^1 \left(t^s|f'''(a)|^q + (1-t)^s|f'''(x)|^q\right)\mathrm{d}t \\
&= \int_0^1 t^s|f'''(a)|^q \mathrm{d}t + \int_0^1 (1-t)^s|f'''(x)|^q \mathrm{d}t \\
&= |f'''(a)|^q \int_0^1 t^s \mathrm{d}t + |f'''(x)|^q \int_0^1 (1-t)^s \mathrm{d}t \\
&= \frac{1}{s+1}|f'''(a)|^q + \frac{1}{s+1}|f'''(x)|^q,
\end{aligned} \tag{31}$$

and

$$\int_0^1 \left((1-t)^s \left|f'''(x)\right|^q + t^s \left|f'''(b)\right|^q\right)\mathrm{d}t = \frac{1}{s+1}\left|f'''(x)\right|^q + \frac{1}{s+1}\left|f'''(b)\right|^q. \tag{32}$$

By combining (30)–(32) into (29), we obtain the desired result. The proof is fulfilled.

**Corollary 2.12.** *Assuming that all conditions raised in Theorem 2.11 are satisfied. If we take $x = \frac{a+b}{2}$, then we have that*

$$\begin{aligned}&\left|\frac{\alpha(b-a)^\alpha}{2^\alpha(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a) + \frac{(b-a)^\alpha}{2^{\alpha-2}(\alpha+2)B(\alpha)\Gamma(\alpha)}f\left(\frac{a+b}{2}\right) + \frac{\alpha(b-a)^\alpha}{2^\alpha(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)\right.\\ &\quad\left. - \left[{}^{AB}_{\ a}I^\alpha_{\frac{a+b}{2}}\left\{f\left(\frac{a+b}{2}\right)\right\} + {}^{AB}_{\frac{a+b}{2}}I^\alpha_b\left\{f\left(\frac{a+b}{2}\right)\right\}\right] + \frac{2(1-\alpha)}{B(\alpha)}f\left(\frac{a+b}{2}\right)\right|\\ &\quad\le \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}w^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\\ &\quad\times\left[\left(\left|f'''(a)\right|^q + \left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|f'''\left(\frac{a+b}{2}\right)\right|^q + \left|f'''(b)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned} \tag{33}$$

**Remark 2.13.** *Plugging equations* (9) *and* (10) *into the inequality* (33), *and multiplying the resulting by $\frac{2^{\alpha-1}B(\alpha)\Gamma(\alpha)}{(b-a)^\alpha}$, we acquire the following RL-fractional Simpson-type integral inequality for s-convex functions*

$$\begin{aligned}&\left|\frac{\alpha}{2(\alpha+2)}\left[f(a) + \frac{4}{\alpha}f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I^\alpha_{a^+}f\left(\frac{a+b}{2}\right) + I^\alpha_{b^-}f\left(\frac{a+b}{2}\right)\right]\right|\\ &\quad\le \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}w^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\\ &\quad\times\left[\left(\left|f'''(a)\right|^q + \left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|f'''\left(\frac{a+b}{2}\right)\right|^q + \left|f'''(b)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

*Furtherly, if we consider choosing $s = 1$, then the following RL-fractional Simpson-type integral inequality for convex functions exists*

$$\begin{aligned}&\left|\frac{\alpha}{2(\alpha+2)}\left[f(a) + \frac{4}{\alpha}f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} - \left[I^\alpha_{a^+}f\left(\frac{a+b}{2}\right) + I^\alpha_{b^-}f\left(\frac{a+b}{2}\right)\right]\right|\\ &\quad\le \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}w^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}\\ &\quad\times\left[\left(\left|f'''(a)\right|^q + \left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|f'''\left(\frac{a+b}{2}\right)\right|^q + \left|f'''(b)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

*And even more, if we take $\alpha = 1$, in this case, then it yields that*

$$\begin{aligned}&\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(y)\mathrm{d}y\right|\\ &\quad\le \frac{(b-a)^3}{96}\left(\frac{1}{2}\right)^{\frac{1}{q}}\left(\frac{2^p p^2}{(p+1)(2p+1)(3p+1)}\right)^{\frac{1}{p}}\\ &\quad\times\left[\left(\left|f'''(a)\right|^q + \left|f'''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\left|f'''\left(\frac{a+b}{2}\right)\right|^q + \left|f'''(b)\right|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

**Theorem 2.14.** *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentiable function on $(a, b)$ together with $a < b$ and $f''' \in L_1([a, b])$. If for $q > 1$ $|f'''|^q$ is an s-convex function on $[a, b]$ along with $p^{-1} + q^{-1} = 1$, then for all $x \in [a, b]$, we get the following inequality for AB-fractional integrals*

$$\begin{aligned}&\left|\mathfrak{R}_f(a, b; \alpha, x)\right| \\ &\leq \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p}w + \frac{1}{q(s+1)}\left(\left|f'''(a)\right|^q + \left|f'''(x)\right|^q\right)\right) \\ &\quad + \frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p}w + \frac{1}{q(s+1)}\left(\left|f'''(x)\right|^q + \left|f'''(b)\right|^q\right)\right),\end{aligned} \tag{34}$$

*where $w$ is defined as in Theorem 2.11.*

*Proof.* Continuing from inequality (18) in the proof of Theorem 2.5, using the Young inequality $xy \leq \frac{1}{p}x^p + \frac{1}{q}y^q$ for any $x, y \geq 0$ together with $p, q > 1$ satisfying $p^{-1} + q^{-1} = 1$, and the $s$-convexity of $|f'''|^q$, we can obtain that

$$\begin{aligned}&\left|\mathfrak{R}_f(a, b; \alpha, x)\right| \\ &\leq \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1\left(\frac{1}{p}\left|k(t)\right|^p + \frac{1}{q}\left|f'''\left(ta + (1-t)x\right)\right|^q\right)\mathrm{d}t \\ &\quad + \frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1\left(\frac{1}{p}\left|k(t)\right|^p + \frac{1}{q}\left|f'''\left((1-t)x + tb\right)\right|^q\right)\mathrm{d}t \\ &\leq \frac{(x-a)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p}\int_0^1\left|k(t)\right|^p\mathrm{d}t + \frac{1}{q}\int_0^1\left(t^s|f'''(a)|^q + (1-t)^s|f'''(x)|^q\right)\mathrm{d}t\right) \\ &\quad + \frac{(b-x)^{\alpha+3}}{(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(\frac{1}{p}\int_0^1\left|k(t)\right|^p\mathrm{d}t + \frac{1}{q}\int_0^1\left((1-t)^s|f'''(x)|^q + t^s|f'''(b)|^q\right)\mathrm{d}t\right).\end{aligned} \tag{35}$$

By combining (30)–(32) into (35), we obtain the desired result. The proof is fulfilled.

**Corollary 2.15.** *Assuming that all conditions raised in Theorem 2.14 are satisfied. If we take $x = \frac{a+b}{2}$, then we have that*

$$\begin{aligned}&\left|\frac{\alpha(b-a)^\alpha}{2^\alpha(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a) + \frac{(b-a)^\alpha}{2^{\alpha-2}(\alpha+2)B(\alpha)\Gamma(\alpha)}f\left(\frac{a+b}{2}\right) + \frac{\alpha(b-a)^\alpha}{2^\alpha(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)\right. \\ &\quad \left. - \left[{}^{AB}_{\ a}I^\alpha_{\frac{a+b}{2}}\left\{f\left(\frac{a+b}{2}\right)\right\} + {}^{AB}_{\frac{a+b}{2}}I^\alpha_b\left\{f\left(\frac{a+b}{2}\right)\right\}\right] + \frac{2(1-\alpha)}{B(\alpha)}f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\frac{2}{p}w + \frac{1}{q(s+1)}\left(\left|f'''(a)\right|^q + 2\left|f'''\left(\frac{a+b}{2}\right)\right|^q + \left|f'''(b)\right|^q\right)\right].\end{aligned} \tag{36}$$

**Remark 2.16.** *Plugging equations* (9) *and* (10) *into the inequality* (36)*, and multiplying the resulting by $\frac{2^{\alpha-1}B(\alpha)\Gamma(\alpha)}{(b-a)^\alpha}$, we acquire the following RL-fractional Simpson-type integral inequality for s-convex functions*

$$\begin{aligned}&\left|\frac{\alpha}{2(\alpha+2)}\left[f(a) + \frac{4}{\alpha}f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[I^\alpha_{a^+}f\left(\frac{a+b}{2}\right) + I^\alpha_{b^-}f\left(\frac{a+b}{2}\right)\right]\right| \\ &\leq \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\left[\frac{2}{p}w + \frac{1}{q(s+1)}\left(\left|f'''(a)\right|^q + 2\left|f'''\left(\frac{a+b}{2}\right)\right|^q + \left|f'''(b)\right|^q\right)\right].\end{aligned}$$

*Furtherly, if we consider choosing $s = 1$, then the following RL-fractional Simpson-type integral inequality for convex functions exists*

$$\left|\frac{\alpha}{2(\alpha+2)}\left[f(a)+\frac{4}{\alpha}f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I_{a^+}^{\alpha}f\left(\frac{a+b}{2}\right)+I_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{(b-a)^3}{16(\alpha+1)(\alpha+2)}\left[\frac{2}{p}w+\frac{1}{2q}\left(\left|f'''(a)\right|^q+2\left|f'''\left(\frac{a+b}{2}\right)\right|^q+\left|f'''(b)\right|^q\right)\right].$$

*And even more, if we take $\alpha = 1$, in this case, then it yields that*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y\right|$$
$$\leq \frac{(b-a)^3}{96}\left[\frac{2^{p+1}p^2}{p(p+1)(2p+1)(3p+1)}+\frac{1}{2q}\left(\left|f'''(a)\right|^q+2\left|f'''\left(\frac{a+b}{2}\right)\right|^q+\left|f'''(b)\right|^q\right)\right].$$

## 3. Further results

In this section, we obtain the improved version of Lemma 2.1, in which the first-order derivative of the function is eliminated. The method is to take the parameter $x = \frac{a+b}{2}$, and we deduce the following lemma.

**Lemma 3.1.** *Let $f : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentiable function on $(a, b)$ with $a < b$. Then, we achieve the following identity for AB-fractional integral operators*

$$\begin{aligned}&\frac{\alpha(b-a)^{\alpha}}{2^{\alpha}(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a)+\frac{(b-a)^{\alpha}}{2^{\alpha-2}(\alpha+2)B(\alpha)\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)+\frac{\alpha(b-a)^{\alpha}}{2^{\alpha}(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)\\&-\left[{}_{a}^{AB}I_{\frac{a+b}{2}}^{\alpha}\left\{f\left(\frac{a+b}{2}\right)\right\}+{}_{\frac{a+b}{2}}^{AB}I_{b}^{\alpha}\left\{f\left(\frac{a+b}{2}\right)\right\}\right]+\frac{2(1-\alpha)}{B(\alpha)}f\left(\frac{a+b}{2}\right)\\&=\frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(t)f'''\left((1-t)\frac{a+b}{2}+tb\right)\mathrm{d}t\\&-\frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(1-t)f'''\left((1-t)a+t\frac{a+b}{2}\right)\mathrm{d}t,\end{aligned}\tag{37}$$

*where*

$$k(t) = \alpha t - (\alpha+1)t^2 + t^{\alpha+2}, \tag{38}$$

$$k(1-t) = \alpha(1-t) - (\alpha+1)(1-t)^2 + (1-t)^{\alpha+2}, \tag{39}$$

$\Gamma(\cdot)$ *is a gamma function, and $B(\alpha)$ for $\alpha \in (0, 1]$ is a positive normalization function.*

Taking advantage of the identity established above, one obtains the estimated results by using the boundedness and the Lipschitz condition of $f'''$. For the convenience of expression, we denote the coming term with $\mathcal{T}_f(a, b; \alpha)$:

$$\begin{aligned}&\mathcal{T}_f(a,b;\alpha)\\&:=\frac{\alpha(b-a)^{\alpha}}{2^{\alpha}(\alpha+2)B(\alpha)\Gamma(\alpha)}f(a)+\frac{(b-a)^{\alpha}}{2^{\alpha-2}(\alpha+2)B(\alpha)\Gamma(\alpha)}f\left(\frac{a+b}{2}\right)+\frac{\alpha(b-a)^{\alpha}}{2^{\alpha}(\alpha+2)B(\alpha)\Gamma(\alpha)}f(b)\\&-\left[{}_{a}^{AB}I_{\frac{a+b}{2}}^{\alpha}\left\{f\left(\frac{a+b}{2}\right)\right\}+{}_{\frac{a+b}{2}}^{AB}I_{b}^{\alpha}\left\{f\left(\frac{a+b}{2}\right)\right\}\right]+\frac{2(1-\alpha)}{B(\alpha)}f\left(\frac{a+b}{2}\right).\end{aligned}$$

**Theorem 3.2.** *Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentiable function on $(a,b)$ together with $a < b$ and $f''' \in L_1([a,b])$. If there exist constants $-\infty < m < M < +\infty$ such that $m \leq f'''(x) \leq M$ for all $x \in [a,b]$, then we have that*

$$\left|\mathcal{T}_f(a,b;\alpha)\right| \leq \frac{\alpha(b-a)^{\alpha+3}(M-m)}{3\cdot 2^{\alpha+4}(\alpha+2)(\alpha+3)B(\alpha)\Gamma(\alpha)}. \tag{40}$$

*Proof.* Making use of the identity given in Lemma 3.1, we deduce that

$$\begin{aligned}
&\mathcal{T}_f(a,b;\alpha)\\
&= \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(t)\left(f'''\left((1-t)\frac{a+b}{2}+tb\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)\mathrm{d}t\\
&\quad - \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(1-t)\left(f'''\left((1-t)a+t\frac{a+b}{2}\right)-\frac{m+M}{2}+\frac{m+M}{2}\right)\mathrm{d}t\\
&= \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(t)\left(f'''\left((1-t)\frac{a+b}{2}+tb\right)-\frac{m+M}{2}\right)\mathrm{d}t\right]\\
&\quad - \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(1-t)\left(f'''\left((1-t)a+t\frac{a+b}{2}\right)-\frac{m+M}{2}\right)\mathrm{d}t\right]\\
&\quad + \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\frac{m+M}{2}\int_0^1 k(t)\mathrm{d}t-\frac{m+M}{2}\int_0^1 k(1-t)\mathrm{d}t\right]\\
&= \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(t)\left(f'''\left((1-t)\frac{a+b}{2}+tb\right)-\frac{m+M}{2}\right)\mathrm{d}t\right]\\
&\quad - \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(1-t)\left(f'''\left((1-t)a+t\frac{a+b}{2}\right)-\frac{m+M}{2}\right)\mathrm{d}t\right].
\end{aligned} \tag{41}$$

It is easy to check that

$$\int_0^1 k(t)\mathrm{d}t = \int_0^1 k(1-t)\mathrm{d}t,$$

where $k(t)$ and $k(1-t)$ are defined by (38) and (39), respectively.
Therefore, we have that

$$\begin{aligned}
&\left|\mathcal{T}_f(a,b;\alpha)\right|\\
&\leq \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(t)\cdot\left|f'''\left((1-t)\frac{a+b}{2}+tb\right)-\frac{m+M}{2}\right|\mathrm{d}t\right]\\
&\quad + \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(1-t)\cdot\left|f'''\left((1-t)a+t\frac{a+b}{2}\right)-\frac{m+M}{2}\right|\mathrm{d}t\right].
\end{aligned}$$

Since $f'''(x)$ satisfies $-\infty < m \leq f'''(x) \leq M < +\infty$ for all $x \in [a,b]$, we have that

$$\left|f'''\left((1-t)\frac{a+b}{2}+tb\right)-\frac{m+M}{2}\right| \leq \frac{M-m}{2},$$

and

$$\left|f'''\left((1-t)a+t\frac{a+b}{2}\right)-\frac{m+M}{2}\right| \leq \frac{M-m}{2}.$$

Consequently, we get that

$$\begin{aligned}
&\left|\mathcal{T}_f(a,b;\alpha)\right| \\
&\le \frac{(b-a)^{\alpha+3}(M-m)}{2^{\alpha+4}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left(\int_0^1 k(t)\mathrm{d}t+\int_0^1 k(1-t)\mathrm{d}t\right) \\
&= \frac{(b-a)^{\alpha+3}(M-m)}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(t)\mathrm{d}t \\
&= \frac{\alpha(b-a)^{\alpha+3}(M-m)}{3\cdot 2^{\alpha+4}(\alpha+2)(\alpha+3)B(\alpha)\Gamma(\alpha)}.
\end{aligned}$$

We obtain the desired result. The proof is completed.

**Corollary 3.3.** *In Theorem 3.2, if we take $\alpha = 1$, then we get that*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y\right| \le \frac{(b-a)^3(M-m)}{1152}.$$

**Theorem 3.4.** *Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be a three times differentiable function on $(a,b)$ together with $a < b$ and $f''' \in L_1([a,b])$. If $f'''$ is an L-Lipschitzian function on $[a,b]$, i.e., there exist $L > 0$ such that $|f'''(y)-f'''(x)| \le L|y-x|$, then we have that*

$$\left|\mathcal{T}_f(a,b;\alpha)\right| \le \frac{\alpha(b-a)^{\alpha+4}L}{3\cdot 2^{\alpha+4}(\alpha+2)(\alpha+3)B(\alpha)\Gamma(\alpha)}. \tag{42}$$

*Proof.* Making use of the identity given in Lemma 3.1, we deduce that

$$\begin{aligned}
&\mathcal{T}_f(a,b;\alpha) \\
&= \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(t)\left(f'''\left((1-t)\frac{a+b}{2}+tb\right)-f'''\left(\frac{a+b}{2}\right)+f'''\left(\frac{a+b}{2}\right)\right)\mathrm{d}t \\
&\quad - \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\int_0^1 k(1-t)\left(f'''\left((1-t)a+t\frac{a+b}{2}\right)-f'''(a)+f'''(a)\right)\mathrm{d}t \\
&= \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(t)\left(f'''\left((1-t)\frac{a+b}{2}+tb\right)-f'''\left(\frac{a+b}{2}\right)\right)\mathrm{d}t\right] \\
&\quad - \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(1-t)\left(f'''\left((1-t)a+t\frac{a+b}{2}\right)-f'''(a)\right)\mathrm{d}t\right] \\
&\quad + \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[f'''\left(\frac{a+b}{2}\right)\int_0^1 k(t)\mathrm{d}t - f'''(a)\int_0^1 k(1-t)\mathrm{d}t\right].
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
&\left|\mathcal{T}_f(a,b;\alpha)\right| \\
&\le \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(t)\left|\left(f'''\left((1-t)\frac{a+b}{2}+tb\right)-f'''\left(\frac{a+b}{2}\right)\right)\right|\mathrm{d}t\right] \\
&\quad + \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(1-t)\left|\left(f'''\left((1-t)a+t\frac{a+b}{2}\right)-f'''(a)\right)\right|\mathrm{d}t\right] \\
&\quad + \frac{(b-a)^{\alpha+3}}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(t)\mathrm{d}t \times \left|f'''\left(\frac{a+b}{2}\right)-f'''(a)\right|\right].
\end{aligned}$$

By using the fact that $f'''$ satisfies Lipschitz condition on $[a, b]$ for some $L > 0$, we obtain that

$$\begin{aligned}
&\left|\mathcal{T}_f(a,b;\alpha)\right| \\
&\le \frac{(b-a)^{\alpha+3}L}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(t)\left|\left((1-t)\frac{a+b}{2}+tb\right)-\frac{a+b}{2}\right|\mathrm{d}t\right] \\
&\quad + \frac{(b-a)^{\alpha+3}L}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(1-t)\left|\left((1-t)a+t\frac{a+b}{2}\right)-a\right|\mathrm{d}t\right] \\
&\quad + \frac{(b-a)^{\alpha+3}L}{2^{\alpha+3}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\int_0^1 k(t)\mathrm{d}t\times\left|\frac{a+b}{2}-a\right|\right] \\
&= \frac{(b-a)^{\alpha+4}L}{2^{\alpha+4}(\alpha+1)(\alpha+2)B(\alpha)\Gamma(\alpha)}\left[\frac{\alpha(\alpha+1)}{12(\alpha+4)}+\frac{\alpha(\alpha+1)(\alpha+5)}{12(\alpha+3)(\alpha+4)}+\frac{\alpha(\alpha+1)}{6(\alpha+3)}\right] \\
&= \frac{\alpha(b-a)^{\alpha+4}L}{3\cdot 2^{\alpha+4}(\alpha+2)(\alpha+3)B(\alpha)\Gamma(\alpha)}.
\end{aligned}$$

It is easy to check that

$$\int_0^1 (k(t)\cdot t)\,\mathrm{d}t = \frac{\alpha(\alpha+1)}{12(\alpha+4)},$$

and

$$\int_0^1 (k(1-t)\cdot t)\,\mathrm{d}t = \frac{\alpha(\alpha+1)(\alpha+5)}{12(\alpha+3)(\alpha+4)}.$$

We obtain the desired result. The proof is completed.

**Corollary 3.5.** *In Theorem 3.4, if we take $\alpha = 1$, then we get that*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(y)\mathrm{d}y\right| \le \frac{(b-a)^4 L}{1152}.$$

## 4. Examples

This section provides four examples to illustrate the correctness of the results established from different approaches in this study. For better understanding for the readers, we analyze our outcomes through numerical examples and in a pictorial way.

**Example 4.1.** *Assume that all assumptions of Theorem 2.5 are true. If one considers the function $f(x) = \frac{x^{s+3}}{(s+1)(s+2)(s+3)}$ for $0 < s \le 1$ on the interval $[0, \infty)$, then the corresponding $f'''(x) = x^s$ is an s-convex on $[0, \infty)$. Under the setting of $s = 1$, $\alpha = 1$ along with $x = \frac{a+b}{2}$, if the function $f(x) = \frac{x^{s+3}}{(s+1)(s+2)(s+3)}$ is applied to the inequality (15) for $a \in [1, 5]$ and $b \in [6, 10]$, then the left-hand side of inequality (15) can be written as*

$$f(a,b) = \frac{b-a}{144}\left(a^4+b^4\right)+\frac{b-a}{576}(a+b)^4-\frac{1}{120}\left(b^5-a^5\right).$$

*And the right-hand side of inequality (15) turns out to be*

$$F(a,b) = \frac{a+b}{1152}(b-a)^4.$$

*Figure 1 shows the graphical representation for $f(a, b)$, $-F(a, b)$ and $F(a, b)$. From it, we can easily see that $-F(a, b) \le f(a, b) \le F(a, b)$, which is consistent with the theoretical result of Theorem 2.5.*

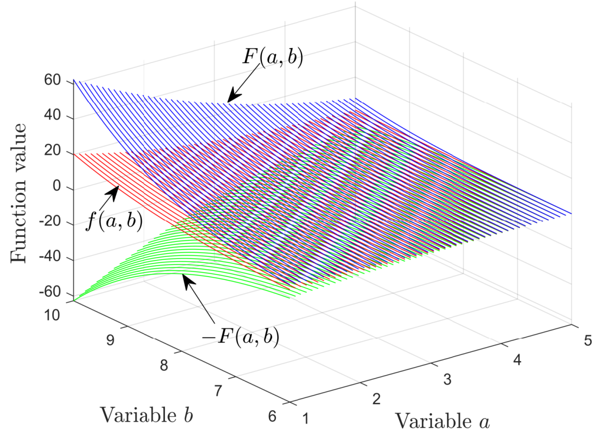



Figure 1: The graphical representation for Example 4.1 for $a \in [1, 5]$ and $b \in [6, 10]$.

**Example 4.2.** *Assume that all assumptions of Theorem 2.8 are true. If we consider the function* $f(x) = \frac{q^3}{(s+q)(s+2q)(s+3q)} x^{\frac{s}{q}+3}$ *for* $q > 1$ *and* $0 < s \leq 1$ *on the interval* $[0, \infty)$*, then the corresponding* $|f'''(x)|^q = x^s$ *is an s-convex on* $[0, \infty)$*. If the function is applied to the inequality* (22) *for* $a = 2, b = 10$, $x = \frac{a+b}{2} = 6$, $s = 1$ *and multiply* $B(\alpha)\Gamma(\alpha)$ *both sides of the inequality, then the left-hand side of inequality* (22) *can be written as*

$$h(\alpha, q) = \frac{q^3}{(1+q)(1+2q)(1+39)}\Bigg[\frac{\alpha \cdot 4^\alpha}{\alpha+2}\left(2^{\frac{1}{q}+3} + \frac{4}{\alpha} \cdot 6^{\frac{1}{q}+3} + 10^{\frac{1}{q}+3}\right) - \alpha \int_2^6 x^{\frac{1}{q}+3}(6-x)^{\alpha-1}\mathrm{d}x - \alpha \int_6^{10} x^{\frac{1}{q}+3}(x-6)^{\alpha-1}\mathrm{d}x\Bigg].$$

*And the right-hand side of inequality* (22) *turns out to be*

$$H(\alpha, q) = \frac{4^{\alpha+3}}{(\alpha+1)(\alpha+2)}\left(\frac{\alpha^2+\alpha}{6\alpha+18}\right)^{\left(1-\frac{1}{q}\right)}\left[\left(\frac{\alpha(\alpha+1)(2\alpha+9)}{3(\alpha+3)(\alpha+4)}\right)^{\frac{1}{q}} + \left(\frac{\alpha(\alpha+1)(4\alpha+15)}{3(\alpha+3)(\alpha+4)}\right)^{\frac{1}{q}}\right].$$

*We select four specific parameters* $\alpha = \frac{1}{2}, \alpha = \frac{1}{3}, \alpha = \frac{1}{4}$ *as well as* $\alpha = \frac{1}{5}$*. And the variable q is used as a variable to plot* $h(\alpha, q)$, $-H(\alpha, q)$ *and* $H(\alpha, q)$ *for different values of* $\alpha$*. The results* $-H(\alpha, q) \leq h(\alpha, q) \leq H(\alpha, q)$ *in Figure 2, which is consistent with the theoretical result of Theorem 2.8.*

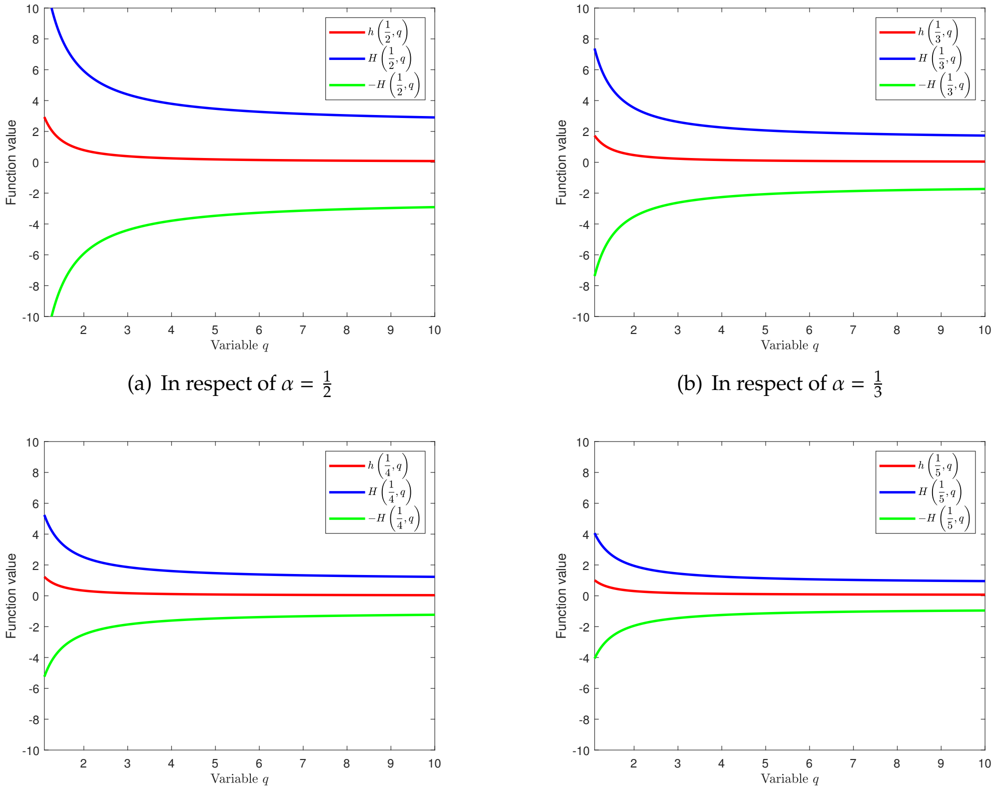
**Example 4.3.** *Assume that all assumptions of Theorem 2.11 are true. If we consider the function* $f(x) = \frac{q^3}{s^3} e^{\frac{s}{q}x}$ *for* $q > 1$ *and* $0 < s \leq 1$ *on the interval* $[0, \infty)$*, by virtue of Proposition 1.3, then we know that the corresponding* $|f'''(x)|^q = e^{xs}$ *is an s-convex on* $[0, \infty)$*. If the function* $f(x) = \frac{q^3}{s^3} e^{\frac{s}{q}x}$ *is applied to inequality* (27) *for* $a = 2, b = 6$, $x = \frac{a+b}{2} = 4$ *as well as* $\alpha = 1$*, then the left-hand side of inequality* (27) *can be written as*

$$g(s, q) = \frac{q^3}{s^3}\left[\frac{2}{3}\left(e^{\frac{2s}{q}} + 4 \cdot e^{\frac{4s}{q}} + e^{\frac{6s}{q}}\right) - \frac{q}{s}\left(e^{\frac{6s}{q}} - e^{\frac{2s}{q}}\right)\right].$$

*And the right-hand side of inequality* (27) *turns out to be*

$$G(s, q) = \frac{8}{3}\left(\frac{2^p \cdot p^2}{(p+1)(2p+1)(3p+1)}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\left(e^{2s} + e^{4s}\right)^{\frac{1}{q}} + \left(e^{4s} + e^{6s}\right)^{\frac{1}{q}}\right).$$

*Moreover, we consider* $q \in [1.1, 5]$ *and* $s \in [0.1, 1]$ *as variables to plot* $g(s, q)$, $G(s, q)$ *and* $-G(s, q)$ *in Figure 3. It is obvious that* $-G(s, q) \leq g(s, q) \leq G(s, q)$*, which is consistent with the theoretical result of Theorem 2.11.*




(a) In respect of $\alpha = \frac{1}{2}$

(b) In respect of $\alpha = \frac{1}{3}$

(c) In respect of $\alpha = \frac{1}{4}$

(d) In respect of $\alpha = \frac{1}{5}$

Figure 2: The graphical representation of Example 4.2 for $q \in [1.1, 10]$

**Example 4.4.** *Assume that all assumptions of Theorem 2.14 are true. If we consider the function* $f(x) = \frac{q^3}{(2s+q)(2s+2q)(2s+3q)} x^{\frac{2s}{q}+3}$ *for* $q > 1$ *and* $0 < s \leq 1$ *on the interval* $[0, \infty)$*, by virtue of Proposition 1.3, then we know that the corresponding* $|f'''(x)|^q = x^{2s}$ *is an s-convex on* $[0, \infty)$*. Under the setting of* $a = 2, b = 6$ *along with* $x = \frac{a+b}{2} = 4$*, if the function* $f(x)$ *is applied to inequality* (34) *for* $\alpha = 1$ *and* $s = 1$*, then the left-hand side of inequality* (34) *can be written as*

$$A(q) = \frac{q^3}{(2+q)(2+2q)(2+3q)} \left[ \frac{2}{3} \left( 2^{\frac{2}{q}+3} + 4 \cdot 4^{\frac{2}{q}+3} + 6^{\frac{2}{q}+3} \right) - \left( \frac{q}{2+4q} \right) \left( 6^{\frac{2}{q}+4} - 2^{\frac{2}{q}+4} \right) \right].$$

*And the right-hand side of inequality* (34) *turns out to be*

$$B(q) = \frac{8}{3} \left[ \frac{2}{p} \left( \frac{2^p \cdot p^2}{(p+1)(2p+1)(3p+1)} \right) + \frac{36}{q} \right].$$

*We put* $q \in [1.1, 10]$ *as the independent variable and draw the graphical representation of the functions* $A(q), B(q)$ *and* $-B(q)$ *in Figure 4. It is easy to see that* $-B(q) < A(q) < B(q)$*, which is consistent with the theoretical result of Theorem 2.14.*

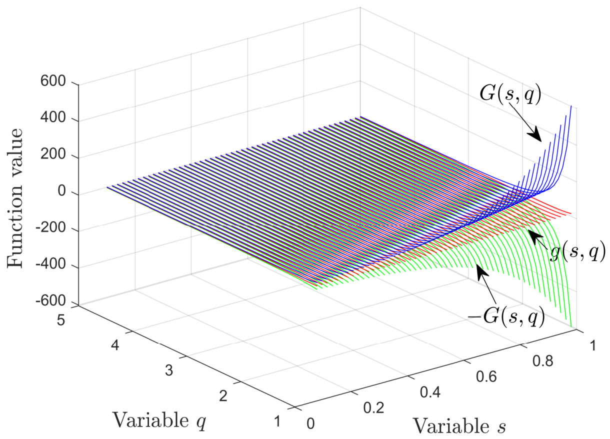



Figure 3: The graphical representation of Example 4.3 for $s \in [0.1, 1]$ and $q \in [1.1, 5]$.

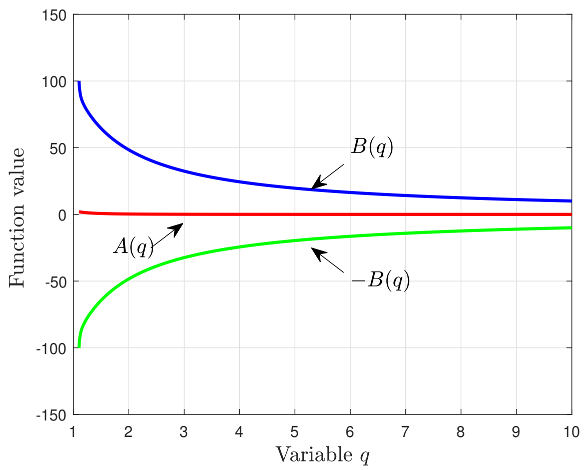



Figure 4: The graphical representation of Example 4.4 for $q \in [1.1, 10]$.

## 5. Applications

### *5.1. Simpson-type quadrature formula*

If we subdivide the interval $[a, b]$ into $n$ cells $[\mu_i, \mu_{i+1}], i = 1, 2, 3, \cdots, n-1$, then one gets a division $U : a = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = b$. Now, we consider the following quadrature formula

$$\int_a^b f(y)\mathrm{d}y = T_s(f, U) + E_s(f, U),$$

in which

$$T_s(f, U) = \frac{1}{6}\sum_{i=0}^{n-1}(\mu_{i+1} - \mu_i)\left[f(\mu_i) + 4f\left(\frac{\mu_i + \mu_{i+1}}{2}\right) + f(\mu_{i+1})\right].$$

This is a Simpson formula, and $E_s(f, U)$ means the approximate error of integral $\int_a^b f(y)\mathrm{d}y$.

Based upon the aforementioned Simpson-type integral formula, one obtains the following error estimate.

**Proposition 5.1.** *Supposing that all conditions raised in Theorem 2.5 are satisfied. Then, the error estimate of the following Simpson-type quadrature formula is given by*

$$\begin{aligned}&\left|E_s(f,U)\right|\\&\leq \frac{1}{48(s+2)(s+3)(s+4)}\sum_{i=0}^{n-1}(\mu_{i+1}-\mu_i)^4\left[\left|f'''(\mu_i)\right|+(s+2)\left|f'''\left(\frac{\mu_i+\mu_{i+1}}{2}\right)\right|+\left|f'''(\mu_{i+1})\right|\right].\end{aligned}$$

*Proof.* Employing the result of $\alpha = 1$ in Corollary 2.6 on the subinterval $[\mu_i, \mu_{i+1}] \subseteq [a, b], i = 0, 1, \cdots, n-1$, one concludes that

$$\begin{aligned}&\left|\frac{\mu_{i+1}-\mu_i}{6}\left[f(\mu_i)+4f\left(\frac{\mu_i+\mu_{i+1}}{2}\right)+f(\mu_{i+1})\right]-\int_{\mu_i}^{\mu_{i+1}}f(y)\mathrm{d}y\right|\\&\quad\leq \frac{(\mu_{i+1}-\mu_i)^4}{48(s+2)(s+3)(s+4)}\left[\left|f'''(\mu_i)\right|+(s+2)\left|f'''\left(\frac{\mu_i+\mu_{i+1}}{2}\right)\right|+\left|f'''(\mu_{i+1})\right|\right].\end{aligned}$$

If one sums $i$ from 0 to $n-1$ and takes advantage of the trigonometric inequality, then the following error estimate can be obtained:

$$\begin{aligned}&\left|E_s(f,U)\right|\\&\quad=\left|T_s(f,U)-\int_a^b f(y)\mathrm{d}y\right|\\&\quad=\left|\sum_{i=0}^{n-1}\frac{\mu_{i+1}-\mu_i}{6}\left[f(\mu_i)+4f\left(\frac{\mu_i+\mu_{i+1}}{2}\right)+f(\mu_{i+1})\right]-\sum_{i=0}^{n-1}\int_{\mu_i}^{\mu_{i+1}}f(y)\mathrm{d}y\right|\\&\quad\leq \frac{1}{48(s+2)(s+3)(s+4)}\sum_{i=0}^{n-1}(\mu_{i+1}-\mu_i)^4\left[\left|f'''(\mu_i)\right|+(s+2)\left|f'''\left(\frac{\mu_i+\mu_{i+1}}{2}\right)\right|+\left|f'''(\mu_{i+1})\right|\right].\end{aligned}$$

This fulfills the proof.

*5.2. Application to means*

For positive numbers $b > a > 0$, we consider the arithmetic mean $A(a, b) = \frac{a+b}{2}$ and the generalized logarithmic mean $L_r(a, b) = \left[\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}\right]^{\frac{1}{r}}, r \neq 0, -1$.

Now considering $f : [a, b] \to \mathbb{R}^+$ given by $f(x) = \frac{x^{\tau+3}}{(\tau+1)(\tau+2)(\tau+3)}$ with $\tau > 0$, we have the following means:

$$\frac{a^{\tau+3}+b^{\tau+3}}{2} = A(a^{\tau+3}, b^{\tau+3}),$$

$$\left(\frac{a+b}{2}\right)^{\tau+3} = A^{\tau+3}(a, b),$$

and

$$\frac{1}{b-a}\int_a^b x^{\tau+3}\mathrm{d}x = L_{\tau+3}^{\tau+3}(a, b).$$

**Proposition 5.2.** *Let $a, b \in \mathbb{R}^+$, $0 < a < b$ with $\tau > 0$. Then, we have the following inequality*

$$\left|\frac{1}{(\tau+1)(\tau+2)(\tau+3)}\left[A(a^{\tau+3}, b^{\tau+3})+2A^{\tau+3}(a, b)-3L_{\tau+3}^{\tau+3}(a, b)\right]\right| \leq \frac{(b-a)^3(b^\tau-a^\tau)}{384}.$$

*Proof.* The assertion follows Theorem 3.2. Applying to the function $f(x) = \frac{x^{\tau+3}}{(\tau+1)(\tau+2)(\tau+3)}$, $x \in [a, b]$ and taking $\alpha = 1$, we have $m = a^\tau \leq f'''(x) = x^\tau \leq b^\tau = M$. We get the desired result, and the proof is fulfilled.

**Proposition 5.3.** *Let $a, b \in \mathbb{R}^+$, $0 < a < b$ with $\tau > 0$. Then we have the following inequality*

$$\left|\frac{1}{(\tau+1)(\tau+2)(\tau+3)}\left[A(a^{\tau+3}, b^{\tau+3}) + 2A^{\tau+3}(a,b) - 3L_{\tau+3}^{\tau+3}(a,b)\right]\right| \leq \begin{cases} \frac{\tau a^{\tau-1}(b-a)^4}{384}, & 0 < \tau < 1, \\ \frac{\tau b^{\tau-1}(b-a)^4}{384}, & \tau \geq 1. \end{cases}$$

*Proof.* The assertion follows Theorem 3.4. Applying to the function $f(x) = \frac{x^{\tau+3}}{(\tau+1)(\tau+2)(\tau+3)}$, $x \in [a, b]$ and taking $\alpha = 1$, we have the Lipschitz constant $L = \sup_{x\in[a,b]} |f^{(4)}| = \sup_{x\in[a,b]} \tau x^{\tau-1}$, which is equivalent to

$$L = \begin{cases} \tau a^{\tau-1}, & 0 < \tau < 1, \\ \tau b^{\tau-1}, & \tau \geq 1. \end{cases}$$

We obtain the desired result, and the proof is fulfilled.

## 6. Conclusions

To the best of our knowledge, the current investigation is the first one with respect to the AB-fractional Simpson-type integral inequality involving with three times differentiable functions. More specifically, we develop two AB-fractional integral identities for three times differentiable functions. Taking advantage of the established identities, combining with the $s$-convexity, boundedness and the Lipschitz condition, respectively, we achieve a series of AB-fractional Simpson-type integral inequalities. It is worth noting that the main results do not contain the first derivative by taking the parameter $x = \frac{a+b}{2}$. Meanwhile, these turn into the findings for convex functions when one considers $s = 1$. Furtherly, one can get the outcomes in the sense of Riemann integrals about convexity by setting $\alpha = 1$. With the ideas developed in this paper, we hope to motivate interested researchers to further explore other types of fractional integral operators, such as the generalized fractional integrals [20], ($k$-$p$) fractional integrals [47], local fractional integrals [21, 52] and fractal-fractional integrals [50], to construct new identities similarly and to derive its related integral inequalities.

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