Filomat 38:27 (2024), 9607–9622
https://doi.org/10.2298/FIL2427607C

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Inequalities for Golden Lorentzian manifolds with gsm U-connection

Majid Ali Choudhary[a], Siraj Uddin[b]

[a] *Department of Mathematics, School of Sciences, Maulana Azad National Urdu University, Hyderabad, India*
[b] *Department of Mathematics, Faculty of Natural Science, Jamia Millia Islamia, New Delhi, India*

Abstract. This study aims to establish Wintgen and Chen-type inequalities for submanifolds of golden Lorentzian manifolds endowed with generalized symmetric metric U-connection (gsmc). Isometric immersion of warped product manifold into the same ambient space form has also been studied. Moreover, equality cases have been discussed.

1. Background

In 1970, the notion of the polynomial structure was brought to light by [18, 19] and this development inspired the geometers to study golden structure on the Riemannian manifold in [13]. Some applications of golden mean have been taken into consideration in [21] and integrability results are clubbed in [17]. For more literature, see [1, 10, 22] etc.

On the other side, Wintgen inequality credited to P. Wintgen [30] is a sharp geometrical inequality producing a relationship between intrinsic and extrinsic invariants. The famous DDVV conjecture was represented in [15]. A lot of work has been done on this so far. For more details, see [4, 8, 16, 26].

In 1993, Chen considered submanifolds of real space form [5] and introduced the basic idea for the sharp relationships between intrinsic invariants and extrinsic invariants. Later on, Chen-like inequalities were also studied in many other ambient spaces [7, 14, 23, 25] and the references therein.

While constructing an example of Riemannian manifolds with negative sectional curvature, warped product manifolds were introduced in [2]. It is known that warped products have applications in different branches of Mathematics as well as in Physics. For example, generalized Robertson-Walker space-time is a Lorentzian warped product (see [12, 28, 29] for more literature).

Let F_1 and F_2 be Riemannian manifolds of positive dimensions endowed by Riemannian metrices g_{F_1} and g_{F_2}, resp. and denote by f any positive function on F_1. Assume $F_1 \times F_2$ with its projection $\pi : F_1 \times F_2 \to F_1$ and $\eta : F_1 \times F_2 \to F_2$. The warped product $N = F_1 \times_f F_2$ is the manifold $F_1 \times F_2$ equipped with the Riemannian structure such that

$$\|X\|^2 = \|\pi_*(X)\|^2 + f^2(\pi(x))\|\eta_*(X)\|^2, X \in T_x N.$$

Hence, one obtains

$$g = g_{F_1} + f^2 g_{F_2},$$

f represents a warping function of the warped product.

2020 *Mathematics Subject Classification*. 53C40; 53B25; 53C25; 53C15
Keywords. Wintgen inequality; Chen inequality; Lorentzian structure; Scalar curvature; Warped product
Received: 05 December 2023; Accepted: 31 July 2024
Communicated by Mića S. Stanković
Email addresses: majid_alichoudhary@yahoo.co.in (Majid Ali Choudhary), siraj.ch@gmail.com (Siraj Uddin)

The present study aims to establish generalized Wintgen and Chen-type inequalities for submanifolds immersed in golden Lorentzian manifolds endowed with g.s.m. U-connection. Isometric immersion of warped product submanifolds in the same ambient space form has also been studied. Moreover, equality cases have been discussed.

2. Preliminaries

2.1.

Assume (M^m, g) stands for m-dimensional Riemannian manifold and φ be $(1,1)$-tensor field holding [10, 20, 22]

$$\varphi^2 = \varphi + I,$$

then φ introduces a golden structure on M. Suppose (M, g) represents a Riemannian manifold and φ is a golden structure on M. When

$$g(\varphi l_1, l_2) = g(l_1, \varphi l_2), \qquad \forall l_1, l_2 \in \Gamma(TM), \tag{1}$$

holds, then (M, g, φ) becomes a golden Riemannian manifold. One should also note that

$$g(\varphi l_1, \varphi l_2) = g(\varphi^2 l_1, l_2) = g(\varphi l_1, l_2) + g(l_1, l_2).$$

Assume $\mathcal{L}$ to be an almost product structure on M. In this case [20]

$$\varphi = \frac{1}{2}(\sqrt{5}\mathcal{L} + I)$$

produces a golden structure on M. On other side, when φ induces a golden structure on M, then [20]

$$\mathcal{L} = \frac{1}{\sqrt{5}}(2\varphi - I)$$

becomes an almost product structure on M.

Express the Riemannian curvature tensor of locally golden product space form M as [10]

$$\begin{aligned} R(l_1, l_2)l_3 &= \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2}{10}[g(l_2, l_3)l_1 - g(l_1, l_3)l_2] \\ &+ \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2}{10}[g(\varphi l_2, l_3)l_1 - g(\varphi l_1, l_3)l_2 \\ &+ g(l_2, l_3)\varphi l_1 - g(l_1, l_3)\varphi l_2] \\ &+ \frac{c_1 + c_2}{5}[g(\varphi l_2, l_3)\varphi l_1 - g(\varphi l_1, l_3)\varphi l_2]. \end{aligned} \tag{2}$$

Definition 2.1. *[11] Let (M^m, g) be semi-Riemannian manifold with g having signature $(-, +, +, ..., +(m-1\ times))$ and satisfying (1). In addition to this, if M is equipped with a golden structure φ, then it is known as a golden Lorentzian manifold.*

[11] The torsion tensor $\overline{T}$ for any golden Lorentzian manifold (M, g, φ) is expressed as

$$\overline{T}(l_1, l_2) = -\alpha\{u(l_1)l_2 - u(l_2)l_1\} - \beta\{u(l_1)\varphi l_2 - u(l_2)\varphi l_1\}, \tag{3}$$

in above case α, β indicate smooth functions on M and $\overline{\nabla}$ is used for generalized symmetric connection (g.s.c.). For any 1-form u and unitary vector field U, we have

$$u(l_1) = g(U, l_1).$$

$\overline{\nabla}$ represents a g.m.c. if

$$\overline{\nabla} g = 0;$$

otherwise, $\overline{\nabla}$ is known as non-metric connection.

Definition 2.2. *[11] Let (M, g, φ) be golden Lorentzian manifold and $\overline{\nabla}$ denotes the g.s.m. connection ((α, β)-type). For any parallel vector field U, $\overline{\nabla}$ is said to be a generalized symmetric metric U-connection on M.*

Let (M, g, φ) stand for locally golden product Lorentzian manifold with generalized symmetric metric U-connection. Then we write the scalar curvature concerning this connection [11]:

$$\begin{aligned}\overline{\tau} &= \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{10}(m-\varepsilon)m \\ &+\frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{10}[(2m\varepsilon - 2)trace\varphi] \\ &+\frac{c_1 + c_2 - 5\beta^2}{5}[(trace\varphi)^2 - trace\varphi - m\varepsilon]. \end{aligned} \tag{4}$$

Further, from Theorem 2.7 [11], we have

Connection type	Scalar curvature
α semi-symmetric	$(A-\alpha^2)(m-\varepsilon)m + B(2m\varepsilon-2)trace\varphi + C((trace\varphi)^2 - trace\varphi - m\varepsilon)$
β quarter symmetric	$A(m-\varepsilon)m + B(2m\varepsilon-2)trace\varphi + (C-\beta^2)((trace\varphi)^2 - trace\varphi - m\varepsilon)$
Semi-symmetric	$(A-1)(m-\varepsilon)m + B(2m\varepsilon-2)trace\varphi + C((trace\varphi)^2 - trace\varphi - m\varepsilon)$
Quarter symmetric	$A(m-\varepsilon)m + B(2m\varepsilon-2)trace\varphi + (C-1)((trace\varphi)^2 - trace\varphi - m\varepsilon)$

in above case $A = \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2}{10}$, $B = \frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2}{10}$ and $C = \frac{c_1+c_2}{5}$.

Let N^n be the submanifold of the locally golden product Lorentzian manifold M^m with g.s.m. U-connection and ∇ and $\overline{\nabla}$ be the Levi-Civita connections on N and M, respectively. Next, denote the shape operator by $\mathbb{S}_ð$ and normal connection by $\nabla^\perp$. Then, we have

$$\overline{\nabla}_{l_1} l_2 = \nabla_{l_1} l_2 + h(l_1, l_2)$$

and

$$\overline{\nabla}_{l_1} ð = -\mathbb{S}_ð l_1 + \nabla^\perp_{l_1} ð, \quad ð \in \Gamma(N),$$

in this case, h means the second fundamental form. Also

$$g(\mathbb{S}_ð l_1, l_2) = g(h(l_1, l_2), ð).$$

We have Gauss equation as [3]

$$\begin{aligned} R(l_1, l_2, l_3, l_4) &= \overline{R}(l_1, l_2, l_3, l_4) - g(h(l_1, l_4), h(l_2, l_3)) \\ &+ g(h(l_1, l_3), h(l_2, l_4)), \end{aligned} \tag{5}$$

here $l_1, l_2, l_3, l_4 \in \Gamma(TN)$, $\overline{R}$ and R mean curvature tensor of M and N, resp. Next, we recall [8]

$$g(\overline{R}(l_1, l_2)\xi_1, \xi_2) = g(R^\perp(l_1, l_2)\xi_1, \xi_2) + g([\mathbb{S}_{\xi_1}, \mathbb{S}_{\xi_2}]l_1, l_2), \tag{6}$$

where ξ_1 and ξ_2 are normal vector fields satisfying

$$[\mathbb{S}_{\xi_1}, \mathbb{S}_{\xi_2}] = \mathbb{S}_{\xi_1}\mathbb{S}_{\xi_2} - \mathbb{S}_{\xi_2}\mathbb{S}_{\xi_1}.$$

Assume that $\{u_1, \ldots, u_n\}$ and $\{u_{n+1}, \ldots, u_m\}$ be orthonormal basis of T_pN and $T_p^\perp N$, resp. Then

$$\mathcal{H} = \sum_{i=1}^{n} \frac{1}{n} h(u_i, u_i), \tag{7}$$

and

$$\|h\|^2 = \sum_{1\le i,j\le n} g\Big(h(u_i, u_j), h(u_i, u_j)\Big). \tag{8}$$

Next, for any plane section $\pi \subset T_pN$, denote the sectional curvature of N by $\mathcal{K}(\pi)$. Therefore,

$$\overline{\tau}(p) = \sum_{1\le i<j\le n} \mathcal{K}(u_i \wedge u_j) \tag{9}$$

and

$$\rho(p) = \frac{2}{n(n-1)} \sum_{1\le i<j\le n} \mathcal{K}(u_i \wedge u_j). \tag{10}$$

Represent the normalized normal scalar curvature of N by [27]

$$\begin{aligned} \rho^{\perp} &= \frac{2\tau^{\perp}}{n(n-1)} \\ &= \frac{2}{n(n-1)} \sqrt{\sum_{1\le i<j\le n} \sum_{n+1\le r<s\le m} (R^{\perp}(u_i, u_j, \xi_r, \xi_s))^2}. \end{aligned} \tag{11}$$

Similarly [8],

$$\mathcal{K}_N = \frac{1}{4} \sum_{r,s=n+1}^{m} (trace[\mathbb{S}_r, \mathbb{S}_s])^2, \tag{12}$$

in this case, $\mathbb{S}_t$ means the shape operator of N in the direction of ξ_t, $t = n+1, ..., m$. We also have [27]

$$\rho_N = \frac{2}{n(n-1)} \sqrt{\mathcal{K}_N}. \tag{13}$$

That implies

$$\begin{aligned} \mathcal{K}_N &= \frac{1}{2} \sum_{n+1\le r<s\le m} (trace[\mathbb{S}_r, \mathbb{S}_s])^2 \\ &= \sum_{n+1\le r<s\le m} \sum_{1\le i<j\le n} g([\mathbb{S}_r, \mathbb{S}_s]u_i, u_j)^2. \end{aligned}$$

One can also establishes [27]

$$\mathcal{K}_N = \sum_{n+1\le r<s\le m} \sum_{1\le i<j\le n} \Big[\sum_{k=1}^{n} (h^r_{jk} h^s_{ik} - h^r_{ik} h^s_{jk})\Big]^2. \tag{14}$$

Finally, recall the following results.

Theorem 2.3. *[16, 26] Let us suppose that $p \ge 2$ and n be two integers and $A_1, ..., A_p$ be $n \times n$ real symmetric matrices. Further, assume that $[.,.]$ means the commutator of two matrices and $\|.\|$ indicates the Hilbert-Schmidt norm of a matrix. Then we have the following inequality*

$$\sum_{\alpha,\beta=1}^{p} \|[A_\alpha, A_\beta]\|^2 \le \Big(\sum_{\alpha=1}^{p} \|A_\alpha\|^2\Big).$$

In addition to this, equality holds if and only if, under some rotations, $A_1, ..., A_p$ are null matrices, except only two of them which can be written as

$$B\begin{pmatrix} 0 & a & 0 & \dots & 0 \\ a & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}B^t, B\begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & -a & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}B^t,$$

in this case a is real number and B is an $n \times n$ orthogonal matrix.

Lemma 2.4. *[5] When one represents by $c_1, \dots, c_t, d$ the $(t+1), t \geq 2$ real numbers provided*

$$\Big(\sum_{k=1}^{t} c_k\Big)^2 = (t-1)\Big(\sum_{k=1}^{t} c_k^2 + d\Big),$$

then, $2c_1c_2 \geq d$ and equality holds if and only if

$$c_1 + c_2 = c_3 = \cdots = c_t.$$

3. Generalized Wintgen Inequality

From now on, fix M^m for the locally golden product Lorentzian manifold endowed with generalized symmetric metric U-connection.

Theorem 3.1. *For submanifold N^n isometrically immersed in M^m. We have*

$$\begin{aligned} \|\mathcal{H}\|^2 \quad \geq \quad & \rho + \rho^{\perp} - \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{10(n-1)}(n-\varepsilon) \\ & - \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{5(n^2-n)}[(n\varepsilon - 1)trace\varphi] \\ & - \frac{c_1 + c_2 - 5\beta^2}{5(n^2-n)}[(trace\varphi)^2 - trace\varphi - n\varepsilon]. \end{aligned} \tag{15}$$

Moreover, (15) satisfies equality case if and only if given some orthonormal frames $\{u_1, \dots, u_n\}$ and $\{u_{n+1}, \dots, u_m\}$, $\mathbb{S}$ reduces to

$$\mathbb{S}_{n+1} = \begin{pmatrix} ð_1 & \eth & 0 & \dots & 0 & 0 \\ \eth & ð_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & ð_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & ð_1 & 0 \\ 0 & 0 & 0 & \dots & 0 & ð_1 \end{pmatrix}, \tag{16}$$

$$\mathbb{S}_{n+2} = \begin{pmatrix} ð_2 + \eth & 0 & 0 & \dots & 0 & 0 \\ 0 & ð_2 - \eth & 0 & \dots & 0 & 0 \\ 0 & 0 & ð_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & ð_2 & 0 \\ 0 & 0 & 0 & \dots & 0 & ð_2 \end{pmatrix}, \tag{17}$$

$$\mathbb{S}_{n+3} = \begin{pmatrix} ð_3 & 0 & 0 & \dots & 0 & 0 \\ 0 & ð_3 & 0 & \dots & 0 & 0 \\ 0 & 0 & ð_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & ð_3 & 0 \\ 0 & 0 & 0 & \dots & 0 & ð_3 \end{pmatrix}, \qquad \mathbb{S}_{n+4} = \cdots = \mathbb{S}_m = 0, \tag{18}$$

where $ð_1, ð_2, ð_3$ and $\eth$ are real functions on N.

Proof. In the light of (5), we have

$$\begin{aligned}\sum_{1\le i<j\le n} R(u_i,u_j,u_j,u_i) &= \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10}(n-\varepsilon)n \\ &+n\varepsilon.\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{5}trace\varphi \\ &+\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{5}trace\varphi \\ &+\frac{c_1+c_2-5\beta^2}{5}[(trace\varphi)^2-trace\varphi-n\varepsilon] \\ &+n^2\|\mathcal{H}\|^2-\|h\|^2. \end{aligned} \tag{19}$$

We also know that

$$2\overline{\tau}=\sum_{1\le i<j\le n} R(u_i,u_j,u_j,u_i), \tag{20}$$

that produces

$$\begin{aligned}2\overline{\tau} &= \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10}(n-\varepsilon)n \\ &+\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{10}[(2n\varepsilon-2)trace\varphi] \\ &+\frac{c_1+c_2-5\beta^2}{5}[(trace\varphi)^2-trace\varphi-n\varepsilon]+n^2\|\mathcal{H}\|^2-\|h\|^2. \end{aligned} \tag{21}$$

Taking view of (10), obtained equation is

$$\begin{aligned}\rho &= \frac{1}{n(n-1)}[n^2\|\mathcal{H}\|^2-\|h\|^2] \\ &+\frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10(n-1)}(n-\varepsilon) \\ &+\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{5(n^2-n)}[(n\varepsilon-1)trace\varphi] \\ &+\frac{c_1+c_2-5\beta^2}{5(n^2-n)}[(trace\varphi)^2-trace\varphi-n\varepsilon]. \end{aligned} \tag{22}$$

Now, use μ for traceless part of h, i.e.

$$\mu=h-\mathcal{H}g,$$

then it implies

$$\|\mu\|^2=\|h\|^2-n\|\mathcal{H}\|^2.$$

This way, one obtains

$$\begin{aligned}\rho &= \|\mathcal{H}\|^2+\frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10(n-1)}(n-\varepsilon) \\ &+\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{5(n^2-n)}[(n\varepsilon-1)trace\varphi] \\ &+\frac{c_1+c_2-5\beta^2}{5(n^2-n)}[(trace\varphi)^2-trace\varphi-n\varepsilon]-\frac{1}{n(n-1)}\|\mu\|^2. \end{aligned} \tag{23}$$

On the other side, for $\alpha = 1, \ldots, m$, define $A_\alpha : TN \to TN$ as

$$g(A_\alpha X, Y) = g(\mu(X, Y), \xi_\alpha).$$

That indicates

$$A_\alpha = \mathbb{S}_\alpha - g(\mathcal{H}, \xi_\alpha) I$$

implying

$$\sum_{\alpha,\beta=1}^{m} \|[A_\alpha, A_\beta]\|^2 = \sum_{\alpha,\beta=1}^{m} \|[\mathbb{S}_\alpha, \mathbb{S}_\beta]\|^2 \tag{24}$$

and

$$\sum_{\alpha=1}^{m} \|A_\alpha\|^2 = \|\mu\|^2. \tag{25}$$

Next, taking into use (6) and (11), one obtains

$$\rho^\perp = \frac{1}{n(n-1)} \sqrt{\sum_{\alpha,\beta=1}^{m} \|[\mathbb{S}_\alpha, \mathbb{S}_\beta]\|^2}. \tag{26}$$

Thus, (24) and (26) produce

$$\sum_{\alpha,\beta=1}^{m} \|[A_\alpha, A_\beta]\|^2 = n^2(n-1)^2(\rho^\perp)^2. \tag{27}$$

Therefore, using Theorem 2.3 for $A_1, \ldots, A_m$, one obtains

$$\rho^\perp \leq \frac{1}{n(n-1)} \|\mu\|^2, \tag{28}$$

where (25) and (27) have been employed.

Also, in view of (23), one writes

$$\begin{aligned} \frac{1}{n(n-1)}\|\mu\|^2 &= \|\mathcal{H}\|^2 + \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{10(n-1)}(n-\varepsilon) \\ &+ \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{5(n^2-n)}[(n\varepsilon - 1) trace\varphi] \\ &+ \frac{c_1 + c_2 - 5\beta^2}{5(n^2-n)}[(trace\varphi)^2 - trace\varphi - n\varepsilon] - \rho, \end{aligned} \tag{29}$$

thereby establishing inequality (15) with the help of (28).

Finally, one concludes on the same lines as in [Corollary 1.2, [16]] that equality holds in (15) if and only if $\mathbb{S}$ takes the form of (16), (17) and (18) to some suitable orthonormal frames. □

Corollary 3.2. *For Riemannian manifold N^n isometrically immersed in M^m, we have these relations*

(a) M^m equips α-s.s.m. U-connection

$$\begin{aligned} \|\mathcal{H}\|^2 \geq\ & \rho + \rho^\perp - \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{10(n-1)}(n-\varepsilon) \\ & - \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2}{5(n^2-n)}[(n\varepsilon - 1) trace\varphi] \\ & - \frac{c_1 + c_2}{5(n^2-n)}[(trace\varphi)^2 - trace\varphi - n\varepsilon]. \end{aligned}$$

(b) M^m endows β-q.s.m. U-connection

$$\begin{aligned}\|\mathcal{H}\|^2 \quad \geq \quad & \rho + \rho^{\perp} - \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2}{10(n-1)}(n-\varepsilon) \\ & - \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2}{5(n^2-n)}[(n\varepsilon - 1)trace\varphi] \\ & - \frac{c_1 + c_2 - 5\beta^2}{5(n^2-n)}[(trace\varphi)^2 - trace\varphi - n\varepsilon].\end{aligned}$$

(c) M^m equips s.s.m. U-connection

$$\begin{aligned}\|\mathcal{H}\|^2 \quad \geq \quad & \rho + \rho^{\perp} - \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10}{10(n-1)}(n-\varepsilon) \\ & - \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2}{5(n^2-n)}[(n\varepsilon - 1)trace\varphi] \\ & - \frac{c_1 + c_2}{5(n^2-n)}[(trace\varphi)^2 - trace\varphi - n\varepsilon].\end{aligned}$$

(d) M^m endows q.s.m. U-connection

$$\begin{aligned}\|\mathcal{H}\|^2 \quad \geq \quad & \rho + \rho^{\perp} - \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2}{10(n-1)}(n-\varepsilon) \\ & - \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2}{5(n^2-n)}[(n\varepsilon - 1)trace\varphi] \\ & - \frac{c_1 + c_2 - 5}{5(n^2-n)}[(trace\varphi)^2 - trace\varphi - n\varepsilon].\end{aligned}$$

Additionally, equality holds in the above case if and only if with some orthonormal frame $\{u_1, \ldots, u_n, u_{n+1}, \ldots, u_m\}$, operator $\mathbb{S}$ reduces to

$$\mathbb{S}_{n+1} = \begin{pmatrix} \eth_1 & \text{Ә} & 0 & \ldots & 0 & 0 \\ \text{Ә} & \eth_1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \eth_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \eth_1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & \eth_1 \end{pmatrix}, \tag{30}$$

$$\mathbb{S}_{n+2} = \begin{pmatrix} \eth_2 + \text{Ә} & 0 & 0 & \ldots & 0 & 0 \\ 0 & \eth_2 - \text{Ә} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \eth_2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \eth_2 & 0 \\ 0 & 0 & 0 & \ldots & 0 & \eth_2 \end{pmatrix}, \tag{31}$$

$$\mathbb{S}_{n+3}=\begin{pmatrix}\eth_3 & 0 & 0 & \dots & 0 & 0\\ 0 & \eth_3 & 0 & \dots & 0 & 0\\ 0 & 0 & \eth_3 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \eth_3 & 0\\ 0 & 0 & 0 & \dots & 0 & \eth_3\end{pmatrix}, \qquad \mathbb{S}_{n+4}=\cdots=\mathbb{S}_m=0, \tag{32}$$

where $\eth_1, \eth_2, \eth_3$ *and* $\supset$ *are real functions on* N.

Remark 1. The main theorem of this section is the generalization of some Wintgen-type inequalities for submanifolds in golden product space forms equipped with different connections, including semi-symmetric, quarter-symmetric, α-semi-symmetric, etc. It also generalizes results of [8]. Some examples that can verify the equality case of Wintgen type inequality are given in [1, 8].

It is known that submanifold that attains equality in generalized Wintgen type inequality is termed as Wintgen ideal submanifold investigated in [4, 14, 24]. One knows that totally umbilical submanifolds and super-minimal surfaces provide basic examples of Wintgen ideal submanifolds in S^4 and S^6, respectively. It is a difficult task to classify these submanifolds completely. In Riemannian space forms, these have been classified to [31]: the reducible ones, the irreducible minimal ones (up to Mobius transformations), and the generic (irreducible) ones. This one is an open problem to obtain a classification for these submanifolds in locally golden product space forms equipped with several connections.

4. Chen type optimal inequality

Let N^n be Riemannian manifold isometrically immersed in M^m (locally golden product Lorentzian manifold endowed with gsm U-connection). Consider some local orthonormal frame field $\{u_1, \dots, u_n, u_{n+1}, \dots, u_m\}$ and let $\pi = \mathrm{Span}\{u_1, u_2\}$ for any $p \in M$, u_{n+1} is parallel to $\mathcal{H}(p)$. Then, we can write

$$\begin{aligned}2\overline{\tau} &= \frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10}(n-\varepsilon)n\\ &+\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{10}[(2n\varepsilon-2)trace\varphi]\\ &+\frac{2(c_1+c_2-5\beta^2)}{5}[(trace\varphi)^2-trace\varphi-n\varepsilon]+n^2\|\mathcal{H}\|^2-\|h\|^2\end{aligned} \tag{33}$$

where we have used (4) and (5).

Further, fix

$$\begin{aligned}\sqsupset &= 2\overline{\tau}(p)-\|\mathcal{H}\|^2\frac{1}{n-1}(n^3-2n^2)\\ &-\frac{(\mp\sqrt{5}+3)c_1+(\pm\sqrt{5}+3)c_2-10\alpha^2}{10}(n-\varepsilon)n\\ &-\frac{(\pm\sqrt{5}-1)c_1+(\mp\sqrt{5}-1)c_2-10\alpha\beta}{10}[(2n\varepsilon-2)trace\varphi]\\ &-\frac{2(c_1+c_2-5\beta^2)}{5}[(trace\varphi)^2-trace\varphi-n\varepsilon].\end{aligned} \tag{34}$$

Above two equations (33) and (34) produce

$$\sqsupset+\|h\|^2=\frac{n^2\|\mathcal{H}\|^2}{n-1} \tag{35}$$

simplifying to

$$\Big(\sum_{j=1}^{n} h_{jj}^{n+1}\Big)^2 = (n-1)\beth + (n-1)\Big\{\sum_{j=1}^{n}(h_{jj}^{n+1})^2 + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{s=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{s})^2\Big\}. \tag{36}$$

Taking

$$a_1 = h_{11}^{n+1}, a_2 = h_{22}^{n+1}, \dots, a_n = h_{nn}^{n+1},$$

$$b = \beth + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{s=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{s})^2,$$

implying

$$h_{11}^{n+1}h_{22}^{n+1} \geq \frac{1}{2}[\beth + \sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{s=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{s})^2], \tag{37}$$

where Lemma 2.4 has been applied.

Next, let $\sqcap(\pi_\varphi) = g(\varphi u_1, u_2)$, where $\sqcap^2(\pi_\varphi) \in [0,1]$, independent of the choice of u_1, u_2. Then, one writes the sectional curvature $\overline{K}(\pi)$ of N^n associated with π with the help of (5) as follows

$$\begin{aligned}\overline{K}(\pi) &= \|\varphi e_1\|^2[\varepsilon\{(B-\alpha\beta) - (\frac{c_1+c_2}{5} - \beta^2)\} + \beta^2] + A \\ &+\varepsilon\|\varphi e_2\|^2[(B-\alpha\beta) - (\frac{c_1+c_2}{5} - \beta^2)] - (\frac{c_1+c_2}{5})\sqcap^2(\pi_\varphi) \\ &-2\varepsilon^2(B-\alpha\beta) + (\|\varphi e_1\|^2\|\varphi e_2\|^2 + 1)(\frac{c_1+c_2}{5} - \beta^2),\end{aligned} \tag{38}$$

where $A = \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2}{5}$, $B = \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2}{5}$.

Equations (37) and (38) result

$$\begin{aligned}\overline{K}(\pi) &\geq \|\varphi e_1\|^2[\varepsilon\{(B-\alpha\beta) - (\frac{c_1+c_2}{5} - \beta^2)\} + \beta^2] + A \\ &+\varepsilon\|\varphi e_2\|^2[(B-\alpha\beta) - (\frac{c_1+c_2}{5} - \beta^2)] - (\frac{c_1+c_2}{5})\sqcap^2(\pi_\varphi) \\ &-2\varepsilon^2(B-\alpha\beta) + (\|\varphi e_1\|^2\|\varphi e_2\|^2 + 1)(\frac{c_1+c_2}{5} - \beta^2) + \frac{1}{2}\beth \\ &+\frac{1}{2}\sum_{i\neq j}(h_{ij}^{n+1})^2 + \sum_{s=n+2}^{m} h_{11}^{s}h_{22}^{s} - \sum_{s=n+1}^{m}(h_{12}^{s})^2 + \frac{1}{2}\sum_{s=n+2}^{m}\sum_{i,j=1}^{n}(h_{ij}^{s})^2 \\ &= \|\varphi e_1\|^2[\varepsilon\{(B-\alpha\beta) - (\frac{c_1+c_2}{5} - \beta^2)\} + \beta^2] + A \\ &+\varepsilon\|\varphi e_2\|^2[(B-\alpha\beta) - (\frac{c_1+c_2}{5} - \beta^2)] - (\frac{c_1+c_2}{5})\sqcap^2(\pi_\varphi) \\ &-2\varepsilon^2(B-\alpha\beta) + (\|\varphi e_1\|^2\|\varphi e_2\|^2 + 1)(\frac{c_1+c_2}{5} - \beta^2) \\ &+\frac{1}{2}\beth + \frac{1}{2}\sum_{i\neq j>2}(h_{ij}^{n+1})^2 + \frac{1}{2}\sum_{s=n+2}^{m}\sum_{i,j>2}(h_{ij}^{s})^2 \\ &+\sum_{s=n+1}^{m}\sum_{i>2}[(h_{1i}^{s})^2 + (h_{2i}^{s})^2] + \frac{1}{2}\sum_{s=n+2}^{m}(h_{11}^{s} + h_{22}^{s})^2,\end{aligned} \tag{39}$$

i.e., we have

$$\begin{aligned}\overline{K}(\pi) \geq\ & \|\varphi e_1\|^2[\varepsilon\{(B-\alpha\beta)-(\frac{c_1+c_2}{5}-\beta^2)\}+\beta^2]+A\\ &+\varepsilon\|\varphi e_2\|^2[(B-\alpha\beta)-(\frac{c_1+c_2}{5}-\beta^2)]-(\frac{c_1+c_2}{5})\sqcap^2(\pi_\varphi)\\ &-2\varepsilon^2(B-\alpha\beta)+(\|\varphi e_1\|^2\|\varphi e_2\|^2+1)(\frac{c_1+c_2}{5}-\beta^2)+\frac{1}{2}\beth. \end{aligned} \tag{40}$$

Next, we define the Riemannian invariant by

$$Ш_M(p)=\overline{\tau}(p)-\inf\{\overline{K}(\pi)|\pi\subset T_pM,\dim\pi=2\}. \tag{41}$$

Hence, (34) and (40) will conclude

$$\begin{aligned}Ш_M(p) \leq\ & n^2\|\mathcal{H}\|^2(\frac{n-2}{2(n-1)})+\frac{n(n-\varepsilon)}{2}[A(\frac{1}{2}+\frac{1}{(n^2-n\varepsilon)})-\alpha^2]\\ &+\frac{1}{2}(B-2\alpha\beta)[(n\varepsilon-1)trace\varphi-\varepsilon\{\|\varphi e_1\|^2+\|\varphi e_2\|^2-2\varepsilon\}]\\ &+(\frac{c_1+c_2}{5}-\beta^2)[(trace\varphi-1)trace\varphi+\varepsilon(\|\varphi e_1\|^2+\|\varphi e_2\|^2)]\\ &-(\frac{c_1+c_2}{5}-\beta^2)[n\varepsilon+\|\varphi e_1\|^2\|\varphi e_2\|^2+1]\\ &-\beta[\alpha(n\varepsilon-1)trace\varphi+\beta\|\varphi e_1\|^2]+(\frac{c_1+c_2}{5})\sqcap^2(\pi_\varphi). \end{aligned} \tag{42}$$

Furthermore, the equality is satisfied in (42) if and only if equality sign holds in all the previous inequalities along with Lemma 2.4:

$$\begin{aligned} h^{n+1}_{ij}&=0, i\neq j>2,\\ h^s_{1i}=h^s_{2i}=h^s_{ij}&=0, s\geq n+2, i,j>2,\\ h^{n+1}_{1i}=h^{n+1}_{2i}&=0, i>2,\\ h^s_{11}+h^s_{22}&=0, s\geq n+2,\\ h^{n+1}_{11}+h^{n+1}_{22}=h^{n+1}_{33}&=\cdots=h^{n+1}_{nn}. \end{aligned}$$

One can fix $\{e_1,e_2\}$ fulfilling $h^{n+1}_{12}=0$ and denote by $c=h^s_{11}, d=h^s_{22}, c+d=h^s_{33}=...=h^s_{nn}$. Hence one can express the shape operators $\mathbb{S}_s, s\in\{n+1,\ldots,m\}$ in the following shapes:

$$\mathbb{S}_{n+1}=\begin{pmatrix} c & 0 & 0 & \ldots & 0\\ 0 & d & 0 & \ldots & 0\\ 0 & 0 & c+d & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & 0\\ 0 & 0 & 0 & \ldots & c+d \end{pmatrix}, \tag{43}$$

and

$$\mathbb{S}_s=\begin{pmatrix} h^s_{11} & h^s_{12} & 0 & \ldots & 0\\ h^s_{12} & -h^s_{11} & 0 & \ldots & 0\\ 0 & 0 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \ldots & 0\\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad n+1\leq s\leq m. \tag{44}$$

Above discussion can be summarized as:

Theorem 4.1. *Let N^n be Riemannian manifold isometrically immersed in M^m. Then*

$$\begin{aligned}
\amalg_M(p) \leq\ & n^2\|\mathcal{H}\|^2(\frac{n-2}{2(n-1)}) + \frac{n(n-\varepsilon)}{2}[A(\frac{1}{2}+\frac{1}{(n^2-n\varepsilon)}) - \alpha^2] \\
&+\frac{1}{2}(B-2\alpha\beta)[(n\varepsilon-1)trace\varphi - \varepsilon\{\|\varphi e_1\|^2 + \|\varphi e_2\|^2 - 2\varepsilon\}] \\
&+(\frac{c_1+c_2}{5}-\beta^2)[(trace\varphi-1)trace\varphi + \varepsilon(\|\varphi e_1\|^2 + \|\varphi e_2\|^2)] \\
&-(\frac{c_1+c_2}{5}-\beta^2)[n\varepsilon + \|\varphi e_1\|^2\|\varphi e_2\|^2 + 1] \\
&-\beta[\alpha(n\varepsilon-1)trace\varphi + \beta\|\varphi e_1\|^2] + (\frac{c_1+c_2}{5})\sqcap^2(\pi_\varphi).
\end{aligned}$$

Further, equality holds in above equation if and only if for $\{u_1, \ldots, u_n, u_{n+1}, \ldots, u_m\}$, $\mathbb{S}$ reduces to (43) and (44).

From the above result, we obtain the following immediate consequences:

Corollary 4.2. *For Riemannian manifold N^n isometrically immersed in M^m, we have the following inequalities*

(a) M^m equips α-ssm U-connection

$$\begin{aligned}
\amalg_M(p) \leq\ & n^2\|\mathcal{H}\|^2(\frac{n-2}{2(n-1)}) + \frac{n(n-\varepsilon)}{2}[A(\frac{1}{2}+\frac{1}{(n^2-n\varepsilon)}) - \alpha^2] \\
&+\frac{1}{2}B[(n\varepsilon-1)trace\varphi - \varepsilon\{\|\varphi e_1\|^2 + \|\varphi e_2\|^2 - 2\varepsilon\}] \\
&+(\frac{c_1+c_2}{5})[(trace\varphi-1)trace\varphi + \varepsilon(\|\varphi e_1\|^2 + \|\varphi e_2\|^2)] \\
&-(\frac{c_1+c_2}{5})[n\varepsilon + \|\varphi e_1\|^2\|\varphi e_2\|^2 + 1] + (\frac{c_1+c_2}{5})\sqcap^2(\pi_\varphi).
\end{aligned}$$

(b) M^m endows β-qsm U-connection

$$\begin{aligned}
\amalg_M(p) \leq\ & n^2\|\mathcal{H}\|^2(\frac{n-2}{2(n-1)}) + \frac{n(n-\varepsilon)}{2}[A(\frac{1}{2}+\frac{1}{(n^2-n\varepsilon)})] \\
&+\frac{1}{2}B[(n\varepsilon-1)trace\varphi - \varepsilon\{\|\varphi e_1\|^2 + \|\varphi e_2\|^2 - 2\varepsilon\}] \\
&+(\frac{c_1+c_2}{5}-\beta^2)[(trace\varphi-1)trace\varphi + \varepsilon(\|\varphi e_1\|^2 + \|\varphi e_2\|^2)] \\
&-(\frac{c_1+c_2}{5}-\beta^2)[n\varepsilon + \|\varphi e_1\|^2\|\varphi e_2\|^2 + 1] \\
&-\beta^2\|\varphi e_1\|^2 + (\frac{c_1+c_2}{5})\sqcap^2(\pi_\varphi).
\end{aligned}$$

(c) M^m endows ssm U-connection

$$\begin{aligned}
\amalg_M(p) \leq\ & n^2\|\mathcal{H}\|^2(\frac{n-2}{2(n-1)}) + \frac{n(n-\varepsilon)}{2}[A(\frac{1}{2}+\frac{1}{(n^2-n\varepsilon)}) - 1] \\
&+\frac{1}{2}B[(n\varepsilon-1)trace\varphi - \varepsilon\{\|\varphi e_1\|^2 + \|\varphi e_2\|^2 - 2\varepsilon\}] \\
&+(\frac{c_1+c_2}{5})[(trace\varphi-1)trace\varphi + \varepsilon(\|\varphi e_1\|^2 + \|\varphi e_2\|^2)] \\
&-(\frac{c_1+c_2}{5})[n\varepsilon + \|\varphi e_1\|^2\|\varphi e_2\|^2 + 1] + (\frac{c_1+c_2}{5})\sqcap^2(\pi_\varphi).
\end{aligned}$$

(d) M^m equips qsm U-connection

$$\begin{aligned}
Ш_M(p) \quad \leq \quad & n^2\|\mathcal{H}\|^2(\frac{n-2}{2(n-1)}) + \frac{n(n-\varepsilon)}{2}[A(\frac{1}{2}+\frac{1}{(n^2-n\varepsilon)})] \\
& +\frac{1}{2}B[(n\varepsilon-1)trace\varphi - \varepsilon\{\|\varphi e_1\|^2 + \|\varphi e_2\|^2 - 2\varepsilon\}] \\
& +(\frac{c_1+c_2}{5}-1)[(trace\varphi-1)trace\varphi + \varepsilon(\|\varphi e_1\|^2 + \|\varphi e_2\|^2)] \\
& -(\frac{c_1+c_2}{5}-1)[n\varepsilon + \|\varphi e_1\|^2\|\varphi e_2\|^2 + 1] \\
& -\|\varphi e_1\|^2 + (\frac{c_1+c_2}{5})\sqcap^2(\pi_\varphi).
\end{aligned}$$

Moreover, inequalities of the above four cases become equality if and only if for $\{u_1,\ldots,u_n,u_{n+1},\ldots,u_m\}$, operator $\mathbb{S}$ appears like (43) and (44).

Remark 2. For different settings of p and q, one can define some more structures on M, see [20] and our results can be studied on manifolds equipped with these new structures.

5. Warped Product Submanifolds of M^m

In 2002, Chen [6] investigated isometric minimal immersion of n-dimensional warped product submanifolds $N = N_1 \times_f N_2$ in real space forms $M^m(c)$ and established the following inequality:

$$\frac{\triangle f}{f} \leq \frac{n^2}{4n_2}\|H\|^2 + n_1 c,$$

here $n_i = \dim N_i$, $n = n_1 + n_2$. Furthermore, equality is valid in this relation if and only if N is mixed totally geodesic.

Suppose $N = N_1 \times_f N_2$ be a warped product and D_1 and D_2 be the distributions due to vectors tangent to leaves and fibres, resp. Then

$$\nabla_{l_1} l_2 = \nabla_{l_2} l_1 = \frac{1}{f}(l_1 f) l_2, \quad l_1 \in D_1, l_2 \in D_2.$$

This way, one can write the sectional curvature for the plane spanned by l_1 and l_3 as

$$K(l_1 \wedge l_3) = g(\nabla_{l_3}\nabla_{l_1} l_1 - \nabla_{l_1}\nabla_{l_3} l_1, l_3) = \frac{1}{f}\{(\nabla_{l_1} l_1) f - {l_1}^2 f\}.$$

Hence, one writes

$$\frac{\triangle f}{f} = \sum_{j=1}^{n_1} K(u_j \wedge u_s), \quad s \in \{n_1+1, ..., n\}. \tag{45}$$

Theorem 5.1. *Let φ stands for an isometric immersion of submanifold $N_1 \times_f N_2$ of dim n into M^m (locally golden product Lorentzian manifold endowed with gsm U-connection). Then*

$$\frac{\triangle f}{f} \leq \frac{(n_1+n_2)^2}{4n_2}\|H\|^2 + n_1 c, \tag{46}$$

here $dim N_i = n_i, i = 1, 2$ and $\triangle$ stands for Laplacian operator of M_1.

In addition to the above, (46) identically holds for equality if and only if φ is a mixed totally geodesic immersion.

Proof. Put

$$\begin{aligned}\varrho &= 2\overline{\tau} - \frac{n^2}{2}\|H\|^2 - \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{10}(n-\varepsilon)n \\ &-\frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{10}[(2n\varepsilon - 2)trace\varphi] \\ &-\frac{2(c_1+c_2-5\beta^2)}{5}[(trace\varphi)^2 - trace\varphi - n\varepsilon],\end{aligned} \tag{47}$$

so that (33) may be expressed as

$$n^2\|H\|^2 = 2(\varrho + \|h\|^2). \tag{48}$$

Consider some local orthonormal frame $\{u_1, \dots, u_n\}$ of T_pN in such a way that $\{u_1, \dots, u_{n_1}\}$ are tangent to N_1 and $\{u_1, \dots, u_{n_2}\}$ to N_2. In this way, (48) produces

$$\Big(\sum_{t=1}^{3} c_t\Big)^2 = 2\Big(\sum_{t=1}^{3} c_t^2 + d\Big)^2, \tag{49}$$

here

$$c_1 = h_{11}^{n+1}, c_2 = \sum_{t=2}^{n_1} h_{tt}^{n+1}, c_3 = \sum_{t=n_1+1}^{n} h_{tt}^{n+1},$$

$$d = \varrho + \sum_{1\le t_1\ne t_2\le n}(h_{t_1t_2}^{n+1})^2 - \sum_{2\le t_2\ne t_3\le n_1} h_{t_2t_2}^{n+1}h_{t_3t_3}^{n+1} - \sum_{n_1+1\le t_4\ne t_5\le n} h_{t_4t_4}^{n+1}h_{t_5t_5}^{n+1}.$$

Using Lemma 2.4, one writes

$$\sum_{1\le t_1<t_2\le n_1} h_{t_1t_1}^{n+1}h_{t_2t_2}^{n+1} + \sum_{n_1+1\le t_3<t_4\le n} h_{t_3t_3}^{n+1}h_{t_4t_4}^{n+1} \ge \frac{1}{2}[\varrho + 2\sum_{1\le t_1<t_2\le n}(h_{t_1t_2}^{n+1})^2 + \sum_{r=n+2}^{m}\sum_{t_1,t_2=1}^{n}(h_{t_1t_2}^{r})^2], \tag{50}$$

and equality holding if and only if

$$\sum_{t_1=1}^{n_1} h_{t_1t_1}^{n+1} = \sum_{t_2=n_1+1}^{n} h_{t_2t_2}^{n+1}. \tag{51}$$

Further, the Gauss equation produces

$$\begin{aligned}n_2\frac{\triangle f}{f} &= \overline{\tau} - \sum_{1\le t_1<t_2\le n_1} K(e_{t_1}\wedge e_{t_2}) - \sum_{n_1+1\le t_1<t_2\le n} K(e_{t_1}\wedge e_{t_2}) \\ &- P_1 - \sum_{1\le t_1<t_2\le n_1}(h_{t_1t_1}^{n+1}h_{t_2t_2}^{n+1} - (h_{t_1t_2}^{n+1})^2) \\ &- P_2 - \sum_{n_1+1\le t_3<t_4\le n}(h_{t_3t_3}^{n+1}h_{t_4t_4}^{n+1} - (h_{t_3t_4}^{n+1})^2),\end{aligned}$$

where

$$\begin{aligned}P_1 &= \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{20}(n_1-\varepsilon)n_1 \\ &- \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{20}[(2n_1\varepsilon - 2)trace\varphi] \\ &- \frac{(c_1+c_2-5\beta^2)}{10}[(trace\varphi)^2 - trace\varphi - n_1\varepsilon]\end{aligned}$$

and

$$
\begin{aligned}
P_2 &= \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{20}(n_2-\varepsilon)n_2\\
&- \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{20}[(2n_2\varepsilon - 2)trace\varphi]\\
&- \frac{(c_1+c_2-5\beta^2)}{10}[(trace\varphi)^2 - trace\varphi - n_2\varepsilon].
\end{aligned}
$$

Using (50) and (52), one obtains

$$
\begin{aligned}
n_2\frac{\triangle f}{f} &\le \overline{\tau} - \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{20}(n-\varepsilon)n\\
&- \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{10}[(n\varepsilon - 1)trace\varphi]\\
&- \frac{(c_1+c_2-5\beta^2)}{10}[(trace\varphi)^2 - trace\varphi - n\varepsilon]\\
&+ n_1n_2c - \frac{1}{2}\varrho - \sum_{r=n+1}^{m}\sum_{1\le t_1\le n_1}\sum_{n_1+1\le t_2\le n}(h^r_{t_1t_2})^2\\
&- \frac{1}{2}\sum_{r=n+2}^{m}(\sum_{1\le t_1\le n_1} h^r_{t_1t_1})^2 - \frac{1}{2}\sum_{r=n+2}^{m}(\sum_{n_1+1\le t_2\le n} h^r_{t_2t_2})^2\\
&\le \overline{\tau} - \frac{(\mp\sqrt{5}+3)c_1 + (\pm\sqrt{5}+3)c_2 - 10\alpha^2}{20}(n-\varepsilon)n\\
&- \frac{(\pm\sqrt{5}-1)c_1 + (\mp\sqrt{5}-1)c_2 - 10\alpha\beta}{10}[(n\varepsilon - 1)trace\varphi]\\
&- \frac{(c_1+c_2-5\beta^2)}{10}[(trace\varphi)^2 - trace\varphi - n\varepsilon]\\
&+ n_1n_2c - \frac{1}{2}\varrho\\
&= \frac{1}{4}n^2\|H\|^2 + n_1n_2c \qquad (52)
\end{aligned}
$$

This establishes the required inequality (46).

Additionally, it is clear from the above proof that equality holds in (46) if and only if $h^{n+1}_{t_1t_2} = 0, 1 \le t_1 \le n_1, n_1 + 1 \le t_2 \le n$. This is equivalent to the fact that φ is mixed totally geodesic. The converse part is obvious.

Acknowledgements

We express our sincere thanks to the editor and the anonymous referees for their valuable suggestions and comments that helped greatly to improve the article.

References

[1] O. Bahadir and S. Uddin, Slant submanifolds of golden Riemannian manifolds, J. Math. Ext. **13**, (2019), no. 4, 1-10.
[2] R.L. Bishop and B. O'Neill, Manifolds of negative curvature, Trans. Am. Math. Soc, **145** (1969), 1-49.
[3] B. Y. Chen, Pseudo-Riemannian geometry, δ-invariants and applications, World Scientific, Hackensack, 2011.
[4] B. Y. Chen, Classification of Wintgen ideal surfaces in Euclidean 4-space with equal Gauss and normal curvatures, Ann. Global Anal. Geom., **38** (2010), 145-160.
[5] B. Y. Chen, Some pinching and classification theorems for minimal submanifolds, Arch. Math. (Basel) **60** (1993), no. 6, 568–578.
[6] B. Y. Chen, On isometric minimal immersions from warped products into real space forms, Proc. Edinb. Math. Soc., **45(3)** (2002), 579–587.
[7] B. Y. Chen, Slant immersions, Bull. Austral. Math. Soc. **41** (1990), no. 1, 135–147.

[8] M. A. Choudhary, O. Bahadir and H. Alsulami, Generalized Wintgen inequality for some submanifolds in golden Riemannian space forms, Balkan Journal of Geometry and Its Applications, **25(2)** (2020), 1-11.

[9] M. A. Choudhary and A. M. Blaga, Inequalities for generalized normalized δ-Casorati curvatures of slant submanifolds in metallic Riemannian space forms, J. Geom., **111(3)** (2020), 1-18.

[10] M. A. Choudhary and A. M. Blaga, Generalized Wintgen inequality for slant submanifolds in metallic Riemannian space forms, J. Geom. **112**(2), 26 (2021). https://doi.org/10.1007/s00022-021-00590-7

[11] M. A. Choudhary, K. M. Khedher, O. Bahadir, and M. D. Siddiqi, On golden Lorentzian manifolds equipped with generalized symmetric metric connection, Mathematics, **(9), 2430** (2021), 1-18.

[12] M. A. Choudhary, M. A. Lone, and T. Jalal, On nonexistence of warped product submanifolds of locally conformal Kaehler manifolds, Gulf J. of Mathematics, **6(2)** (2018), 31-40.

[13] M. Crasmareanu and C. Hretcanu, Golden differential geometry, Chaos Solitons and Fractals, **38(5)** (2008), 1229-1238.

[14] M. Dajczer and R. Tojeiro, Submanifolds of codimension two attaining equality in an extrinsic inequality, Math. Proc. Cambridge Philos. Soc., **146** (2009), 461–474.

[15] P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, A pointwise inequality in submanifold theory, Arch. Math. (Brno), **35** (1999), 115-128.

[16] J. Q. Ge and Z. Z. Tang, A proof of the DDVV conjecture and its equality case, Pacific J. Math., **237** (2008), 87-95.

[17] A. Gezer, N. Cengiz and A. Salimov, On integrability of golden Riemannian structures, Turkish J. Math., **37** (2013), 693-703.

[18] S. I. Goldberg and N. C. Petridids, Differentiable solutions of algebraic equations on manifolds, Kodai Math Sem Rep., **25** (1973),111–28.

[19] S. I. Goldberg and K. Yano, Polynomial structures on manifolds, Kodai Math. Sem. Rep., **22** (1970), 199-218.

[20] C. Hretcanu and M. Crasmareanu, On some invariant submanifolds in a Riemannian manifold with golden structure, An. Stiins. Univ. Al. I. Cuza Iasi. Mat. (N.S.), **53** (2007), 199-211.

[21] C. Hretcanu and M. Crasmareanu, Applications of the golden ratio on Riemannian manifolds, Turkish J. Math., **33** (2009), 179-191.

[22] C. Hretcanu, Submanifolds in Riemannian manifold with golden structure, in: Workshop on Finsler geometry and its applications, Hungary, 2007.

[23] J. S. Kim, Y. M. Song, M. M. Tripathi, and B. Y. Chen, Inequalities for submanifolds in generalized complex space forms, Bull. Korean Math. Soc., **40** (2003), no. 3, 411–423.

[24] T. Li, X. Ma, C. P. Wang and Z. X. Xie, Wintgen ideal submanifolds of codimension two, complex curves, and Mobius geometry, Tohoku Math. J., **68(4)** (2016), 621–638.

[25] A. Lotta, Slant submanifolds in contact geometry, Bull. Math. Soc. Sci. Math. Roumanie, **39** (1996), 183-198.

[26] Z. Lu, Normal scalar curvature conjecture and its applications, J. Funct. Anal., **261** (2011), 1284-1308.

[27] I. Mihai, F. R. Al-Solamy and M. H. Shahid, On Ricci curvature of a quaternion CR-submanifold in a quaternion space form, Rad. Mat., **12** (2003), no. 1, 91-98.

[28] B. O'Neill, Semi-Riemannian geometry with applications to relativity, (Academic, 1983).

[29] K. S. Park, On the estimates of warping function on isometric immersions, U.P.B. Sci. Bull. Ser. A, **82(1)** (2020), 13-26.

[30] P. Wintgen, Sur l'inegalite de Chen-Willmore, C. R. Acad. Sci. Paris Ser. A-B, **288** (1979), A993-A995.

[31] Z. X. Xie, T.Z. Li, X. Ma, and C. P. Wang, Wintgen ideal submanifolds: reduction theorems and a coarse classification, Ann. Global Anal. Geom., **53**(3) (2018), 377–403.