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On moments of order statistics from a kappa distribution

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Abstract. The kappa distribution, known as *K3D*, is a skewed generalization of the well-known logistic distribution and has many useful applications in modelling extreme value events occurred usually in hydrology and environmental sciences. The distribution also nests the Gumbel distribution in the limiting case of its shape parameter. In this paper, we consider the order statistics from the distribution and derive expressions and relations for both single and product moments of order statistics in computable forms. These relations can be used effectively in the computation of higher moments given the lower order ones. The relations also generalize those relations for the logistic distribution. The moment tables obtained can be used in the computation of the location-scale parameter estimation.

1. Introduction

Extreme value theory deals with modelling very rare or extreme events and proposes methods to estimate the probability of them. The kappa distribution was introduced by [13] to model the maximum precipitation data. It has four parameters and is known its flexibility compared to the generalized extreme value distribution. This probability model has been used, especially in the hydrology and environmental sciences literature, extensively since then. Recent works include Shin and Park [23], Anghel and Ilinca [1] and O'Shea et al. [17], among others. Further theoretical properties were also studied in the literature. These include Seenoi et al. [24], Papukdee et al. [16], Guayjarernpanishk et al. [12], Costa and Nascimento [8], among others.

If one takes one of the shape parameters 0 in the standard form of the four-parameter kappa distribution, the following distribution with the CDF (cumulative distribution function)

$$F(x;\alpha) = (1-\alpha e^{-x})^{1/\alpha},\ -\infty < x < \infty,$$

where $\alpha < 0$ is the shape parameter, is obtained. The corresponding PDF (probability density function) is

$$f(x;\alpha) = e^{-x}(1-\alpha e^{-x})^{(1-\alpha)/\alpha}.$$

The quantile function of the distribution is given by

$$F^{-1}(u;\alpha) = \ln\left(\frac{\alpha}{1-u^{\alpha}}\right),$$

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where $0 < u < 1$.

Jeong et al. [14] studied the location-scale form of this distribution. They called it the three-parameter kappa distribution (*K3D*) since it is a special case of the four-parameter kappa distribution. They studied the properties of the distribution such as the moment generating function, moments, L-moments, LH-moments and asymptotic distribution of extreme order statistics. They also estimate the parameters by both method of L-moments and maximum likelihood. From now on, we will denote both the maximum likelihood estimate and maximum likelihood estimator by MLE.

The *K3D* is also used as a model for probabilistic extreme events. The further statistical properties of the distribution defined on positive real numbers were studied by some authors (see e.g. [22]). Throughout the paper we use the symbol *K3D* for the distribution even in the standard form. The PDF of the distribution for various choices of α is sketched in Figure 1. When $\alpha = -1$, the *K3D* becomes the logistic distribution. When α tends to 0, the *K3D* becomes the Gumbel distribution. Thus, it can be viewed as a generalized version of the logistic and Gumbel distributions.

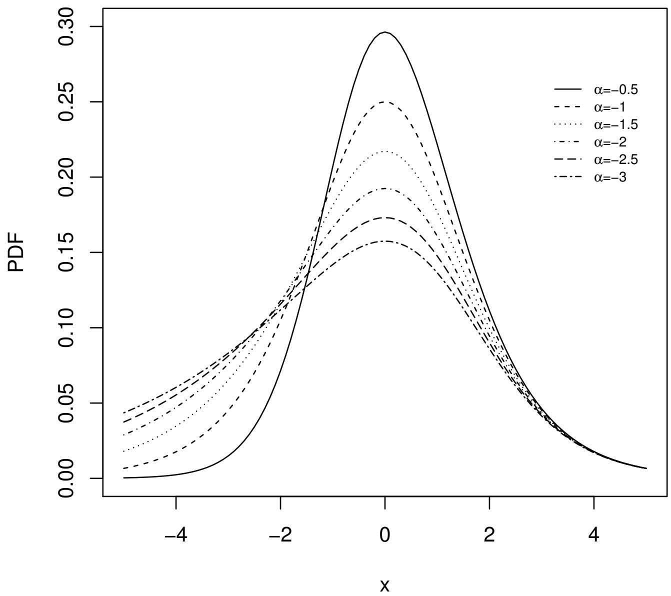

Figure 1: Plots of the density function of the *K3D* for various choices of the parameter α.

The distribution gives the following characterizing differential equations

$$\alpha f(x;\alpha) = F^{1-\alpha}(x;\alpha) - F(x;\alpha) \tag{1}$$

$$e^{x} f(x;\alpha) = F^{1-\alpha}(x;\alpha) \tag{2}$$

and

$$e^{x} f(x;\alpha)(1 - \alpha e^{-x}) = F(x;\alpha). \tag{3}$$

The identities above are especially useful when obtaining moment relations of order statistics from the distribution. Order statistics are obtained from a sample by ordering the sample items in their magnitudes. So the first order statistic is the sample minimum and the last order statistic corresponds to the sample maximum. Symbolically, if we let $X_1, X_2, \ldots, X_n$ denote the random sample from a distribution, then we let

$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the order statistics corresponding to that random sample. The order statistics have its own theory in statistics and useful applications in other disciplines from reliability to optimizing production processes (see e.g. Arnold et al. [2], David and Nagaraja [9], Esmailian and Doostparast [10], Silva et al. [21]). The statistical inference based on the best linear unbiased estimation (BLUE) and some approximate maximum likelihood methods need moments of order statistics (see e.g. Balakrishnan and Cohen [3] and Tumlinson et al. [26]).

In the literature, there are many works on moments of order statistics arising from a specific distribution. For example, Raqab [19], Balakrishnan and Aggarwala [4], Thomas and Samuel [25], Barakat and Abdelkader [6], Balakrishnan et al. [5], Roghaye et al. [20] and Castellares et al. [7]. On the other hand, the moment recurrence relations, when they are available, are especially useful and efficient in the computations since the higher moments can be obtained from the available lower order moments, without resorting to computation of a moment expression.

In this paper, we consider the *K3D* from order statistics point of view. Since the *K3D* is another useful probabilistic model especially in hydrology and environmental sciences, it deserves to be studied for further statistical properties. Our main motivation here is the *K3D* may be a useful alternative for skewed generalizations of the ordinary logistic and Gumbel distributions. This fact attracts attention of the distribution for skewed data modelling encountered in various practical situations such as environmental and actuarial sciences. Another point, characterizing differential equations between the PDF and the CDF may make it easier in deriving some distributional properties such as moment relations. Although the ML and the L-moment estimation methods for the distribution parameters are given in Jeong et al. [14], our another motivation is to search another estimation method using the moments of order statistics. With this paper, we also correct some expressions given in Jeong et al. [14]. The paper is summarized as follows. In Section 2, we present single moments and moment generating function of order statistics. In Section 3, we derive some moments relations of order statistics. In Section 4, we derive product moments of order statistics and give a relation. In Section 5, we consider the location-scale estimation problem and solve the problem by both the MLE and BLUE methods. In Section 6, we apply the theoretical results obtained to real data set. Paper is finalized with conclusions.

2. Single Moments of Order Statistics

Let $X_1, X_2, \ldots, X_n$ denote the random sample from the *K3D*. Applying the general formula for absolutely continuous models given in Arnold et al. [2], the PDF of the rth order statistic is given by

$$f_{r:n}(x;\alpha) = C_{r,n} e^{-x} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i (1-\alpha e^{-x})^{(r-\alpha+i)/\alpha}, \ -\infty < x < \infty,$$

where $C_{r,n} = n!/((r-1)!(n-r)!)$. Especially for two extremes, we have

$$f_{1:n}(x;\alpha) = n e^{-x} \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^i (1-\alpha e^{-x})^{(1-\alpha+i)/\alpha}, \ -\infty < x < \infty$$

and

$$f_{n:n}(x;\alpha) = n e^{-x} (1-\alpha e^{-x})^{(n-\alpha)/\alpha}, \ -\infty < x < \infty.$$

In this section, we will find closed form expressions for moments of order statistics. Throughout the paper let $\mathbb{N}$ and $\mathbb{Z}^-$ denote the sets of positive and negative integers, respectively. We first start with the two ordinary moments which are also meaningful in order statistics. Corrected moments in Jeong et al. [14] are given by

$$E(X) \equiv \mu_{1:1} = \gamma + \ln(-\alpha) + \psi\left(-\frac{1}{\alpha}\right)$$

and

$$E(X^2) \equiv \mu_{1:1}^{(2)} = \left[\gamma + \psi\left(-\frac{1}{\alpha}\right) + \ln(-\alpha)\right]^2 + \frac{\pi^2}{6} + \psi'\left(-\frac{1}{\alpha}\right),$$

where γ is the Euler's constant, $\psi(\cdot)$ is the digamma function defined by $\psi(x) = \Gamma'(x)/\Gamma(x)$. For the rth order statistic $X_{r:n}$, we first look at the moment generating function.

Theorem 2.1.

$$M_{r:n}(t) = C_{r,n}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\left(\frac{-1}{\alpha}\right)^{1-t}\frac{\Gamma\left(t-\frac{r+i}{\alpha}\right)\Gamma(1-t)}{\Gamma\left(1-\frac{r+i}{\alpha}\right)},$$

where $t < 1$ and $\alpha < 0$.

Proof.

$$\begin{aligned} M_{r:n}(t) &= E(e^{tX_{r:n}}) \\ &= C_{r,n}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\int_{-\infty}^{\infty} e^{-x(1-t)}(1-\alpha e^{-x})^{(r-\alpha+i)/\alpha}\,dx. \end{aligned}$$

We change the variable x in the last integral to u by $u = 1 - \alpha e^{-x}$. Then the integral becomes

$$-\frac{1}{\alpha^{1-t}}\int_1^{\infty} u^{(r+i)/\alpha-1}(1-u)^{-t}\,du.$$

It is evaluated for $t < 1$ as

$$\left(\frac{-1}{\alpha}\right)^{1-t}\frac{\Gamma\left(t-\frac{r+i}{\alpha}\right)\Gamma(1-t)}{\Gamma\left(1-\frac{r+i}{\alpha}\right)}$$

by using Formula 3.191.2 in [11]. If we put this evaluation into $M_{r:n}(t)$, the theorem follows. □

Remark 2.2. *When $\alpha = -1$, $M_{r:n}(t)$ is reduced to*

$$M_{r:n}(t) = C_{r,n}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\frac{\Gamma(t+r+i)\Gamma(1-t)}{\Gamma(1+r+i)}.$$

Using the formula 0.160.2 in [11], we can get the following nicer formula:

$$M_{r:n}(t) = \frac{\Gamma(r+t)\Gamma(n-r+1-t)}{\Gamma(r)\Gamma(n-r+1)}.$$

Differentiating once and twice of $M_{r:n}(t)$ with respect to t, and then evaluating these derivatives at $t = 0$, one can get the moments $E(X_{r:n})$ and $E(X_{r:n}^2)$, respectively. They are given in the following corollary.

Corollary 2.3. *(a) The first moment of $X_{r:n}$ is given by*

$$\begin{aligned} \mu_{r:n} &= E(X_{r:n}) \\ &= C_{r,n}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\left[\frac{\gamma + \ln(-\alpha) + \psi\left(-\frac{r+i}{\alpha}\right)}{r+i}\right], \end{aligned}$$

where $\psi(\cdot)$ is the digamma function, γ is the Euler's constant and $\alpha < 0$.

(b) *The second moment of $X_{r:n}$ is given by*

$$\begin{aligned}\mu_{r:n}^{(2)} &= E(X_{r:n}^2)\\ &= C_{r,n}\sum_{i=0}^{n-r}\binom{n-r}{i}\frac{(-1)^i}{r+i}\left\{\left[\ln(-\alpha)+\gamma+\psi\left(-\frac{r+i}{\alpha}\right)\right]^2+\frac{\pi^2}{6}+\psi'\left(-\frac{r+i}{\alpha}\right)\right\},\end{aligned}$$

where $\alpha < 0$.

We can also find an expression for the kth moment of $X_{r:n}$.

Theorem 2.4. *Let $\mu_{r:n}^{(k)}$ denote the kth moment of the rth order statistic $X_{r:n}$. Then we have*

$$\mu_{r:n}^{(k)} = C_{r,n}\left(\frac{-1}{\alpha}\right)\sum_{i=0}^{n-r}\sum_{j=0}^{k}\binom{n-r}{i}\binom{k}{j}(-1)^{i+k-j}\ln^j(-\alpha)\,\frac{\partial^{k-j}}{\partial\rho^{k-j}}\left[\frac{\Gamma\left(-\frac{r+i}{\alpha}-\rho\right)\Gamma(\rho+1)}{\Gamma\left(1-\frac{r+i}{\alpha}\right)}\right]\Bigg|_{\rho=0}. \quad (4)$$

Proof.

$$\begin{aligned}\mu_{r:n}^{(k)} &= E(X_{r:n}^k)\\ &= C_{r,n}\int_{-\infty}^{\infty}x^k f(x;\alpha)F^{r-1}(x;\alpha)(1-F(x;\alpha))^{n-r}\,dx\\ &= C_{r,n}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^i\int_{-\infty}^{\infty}x^k e^{-x}(1-\alpha e^{-x})^{(r-\alpha+i)/\alpha}\,dx.\end{aligned}$$

We change the variable x in the last integral to u by $u = 1-\alpha e^{-x}$. Then the integral becomes

$$\frac{-1}{\alpha}\sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}\ln^j(-\alpha)\int_1^{\infty}u^{(r+i)/\alpha-1}\ln^{k-j}(u-1)\,du.$$

Note that

$$\begin{aligned}\int_1^{\infty}u^{(r+i)/\alpha-1}\ln^{k-j}(u-1)\,du &= \frac{\partial^{k-j}}{\partial\rho^{k-j}}\int_1^{\infty}(u-1)^{\rho}u^{(r+i)/\alpha-1}\,du\Bigg|_{\rho=0}\\ &= \frac{\partial^{k-j}}{\partial\rho^{k-j}}\left[\frac{\Gamma\left(-\frac{r+i}{\alpha}-\rho\right)\Gamma(\rho+1)}{\Gamma\left(1-\frac{r+i}{\alpha}\right)}\right]\Bigg|_{\rho=0},\end{aligned}$$

by using Formula 3.191.2 in [11]. If we put these evaluations into $\mu_{r:n}^{(k)}$, the theorem follows. □

The derivative in (4) can be easily computed in a computer package like Mathematica [27] so that the kth moment of any order statistic can be found. These expressions are all well computable and some calculations are reported in Table 1. We observe from that table that the moments increase with increasing r for fixed n.

3. Relations for the Single Moments

In this section we derive some relations for single moments of order statistics from the $K3D$. These relations may be useful in the computation of higher moments given the lower ones. We first give a relation about moment generating function.

Theorem 3.1. *Let $\alpha < 0$ and $t \neq 1$. Then we have*

$$M_{r:n}(t) = \left(\alpha - \frac{r}{t-1}\right) M_{r:n}(t-1) + \left(\frac{r}{t-1}\right) M_{r+1:n}(t-1), \tag{5}$$

where $1 \leq r \leq n-1$.

Proof. Using (3), we have

$$M_{r:n}(t) - \alpha M_{r:n}(t-1) = C_{r,n} \int_{-\infty}^{\infty} e^{(t-1)x} F^r(x;\alpha)[1 - F(x;\alpha)]^{n-r}\, dx.$$

Then using integration by parts by treating $e^{(t-1)x}$ for integration and $F^r(x;\alpha)[1-F(x;\alpha)]^{n-r}$ for differentiation, we can get

$$\begin{aligned} M_{r:n}(t) - \alpha M_{r:n}(t-1) &= C_{r,n} \frac{n-r}{t-1} \int_{-\infty}^{\infty} e^{(t-1)x} F^r(x;\alpha)[1 - F(x;\alpha)]^{n-r-1} f(x;\alpha)\, dx \\ &\quad - C_{r,n} \frac{r}{t-1} \int_{-\infty}^{\infty} e^{(t-1)x} F^{r-1}(x;\alpha)[1 - F(x;\alpha)]^{n-r} f(x;\alpha)\, dx \\ &= \left(\frac{r}{t-1}\right) \left[M_{r+1:n}(t-1) - M_{r:n}(t-1)\right], \end{aligned}$$

as required. □

Corollary 3.2. *Let $\alpha < 0$ and $k \in \mathbb{N}$. Then we have*

$$\begin{aligned} \mu_{r+1:n}^{(k)} &= \left(\frac{C_{r,n}}{r}\right) \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^{i+1} \frac{1}{\Gamma\left(1 - \frac{r+i}{\alpha}\right)} \frac{\partial^k}{\partial t^k} (-\alpha)^t \Gamma\left(t - \frac{r+i}{\alpha} + 1\right) \Gamma(1-t) \Big|_{t=0} \\ &\quad + \mu_{r:n}^{(k)} - \left(\frac{k\alpha}{r}\right) \mu_{r:n}^{(k-1)}, \end{aligned} \tag{6}$$

where $1 \leq r \leq n-1$ *and* $\mu_{r:n}^{(0)} \equiv 1$.

Proof. From (5), we have

$$t M_{r:n}(t+1) = (\alpha t - r) M_{r:n}(t) + r M_{r+1:n}(t). \tag{7}$$

Now, using Theorem 2.1, we have

$$t M_{r:n}(t+1) = C_{r,n} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^{i+1} \frac{(-\alpha)^t \Gamma\left(t - \frac{r+i}{\alpha} + 1\right) \Gamma(1-t)}{\Gamma\left(1 - \frac{r+i}{\alpha}\right)}.$$

Putting this into (7) and differentiating k times of both sides of the resulting equation with respect to t, and then evaluating these derivatives at $t = 0$, one can get the moment recurrence relation given in the theorem. □

One can use the relation (6) appropriately to obtain the other moments by fixing $k = 1$. For example, given $\mu_{1:n}$, $\mu_{r:n}$, $2 \leq r \leq n$, can be computed recursively.

Theorem 3.3. *Let $\alpha \in \mathbb{Z}^-$. Then we have*

$$\begin{aligned} M_{r-\alpha+1:n-\alpha}(t) &= M_{r-\alpha:n-\alpha}(t) + \frac{(n-\alpha)!}{(n-r)!(r-\alpha)!} \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^{i+1} \\ &\quad \times \frac{(-\alpha)^t \Gamma\left(t - \frac{r+i}{\alpha} + 1\right) \Gamma(1-t)}{\Gamma\left(1 - \frac{r+i}{\alpha}\right)} \end{aligned}$$

where $1 \leq r \leq n-1$.

Proof. Using (2),

$$M_{r:n}(t+1) = C_{r,n}\int_{-\infty}^{\infty} e^{(t+1)x} f(x;\alpha)F^{r-1}(x;\alpha)[1-F(x;\alpha)]^{n-r}\,dx$$

becomes

$$C_{r,n}\int_{-\infty}^{\infty} e^{tx} F^{r-\alpha}(x;\alpha)[1-F(x;\alpha)]^{n-r}\,dx.$$

Then using integration by parts by treating e^{tx} for integration and $F^{r-\alpha}(x;\alpha)[1-F(x;\alpha)]^{n-r}$ for differentiation, we can get

$$M_{r:n}(t+1) = \frac{n!(r-\alpha)!}{(r-1)!(n-\alpha)!t}\left[M_{r-\alpha+1:n-\alpha}(t) - M_{r-\alpha:n-\alpha}(t)\right],$$

where $1 \le r \le n-1$.

Now, using Theorem 2.1, we have

$$tM_{r:n}(t+1) = C_{r,n}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^{i+1}\frac{(-\alpha)^t\Gamma\left(t-\frac{r+i}{\alpha}+1\right)\Gamma(1-t)}{\Gamma\left(1-\frac{r+i}{\alpha}\right)}.$$

Combining these expressions, the result in the theorem follows. □

Corollary 3.4. *Let $\alpha \in \mathbb{Z}^-$. We have*

$$\mu^{(k)}_{r-\alpha+1:n-\alpha} = \mu^{(k)}_{r-\alpha:n-\alpha} + \frac{(n-\alpha)!}{(n-r)!(r-\alpha)!}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^{i+1}\frac{1}{\Gamma\left(1-\frac{r+i}{\alpha}\right)} \times \frac{\partial^k}{\partial t^k}(-\alpha)^t\Gamma\left(t-\frac{r+i}{\alpha}+1\right)\Gamma(1-t)\Big|_{t=0}.$$

Especially,

$$\mu_{r-\alpha+1:n-\alpha} = \mu_{r-\alpha:n-\alpha} + \frac{(n-\alpha)!}{(n-r)!(r-\alpha)!}\sum_{i=0}^{n-r}\binom{n-r}{i}(-1)^{i+1}\left[\gamma+\ln(-\alpha) + \psi\left(1-\frac{r+i}{\alpha}\right)\right],$$

where γ is the Euler's constant, and $\psi(\cdot)$ is the digamma function.

Theorem 3.5. *Let $\alpha \in \mathbb{Z}^-$. We have*

$$\mu^{(k)}_{r-\alpha:n-\alpha} = \mu^{(k)}_{r-\alpha:n-\alpha-1} + \frac{r!(n-\alpha-1)!}{n!(r-\alpha-1)!}\left[\mu^{(k)}_{r:n} - \mu^{(k)}_{r+1:n} - \frac{k\alpha}{r}\mu^{(k-1)}_{r:n}\right],$$

where $1 \le r \le n-1$.

Proof. Using (1), we have

$$\begin{aligned}\mu^{(k-1)}_{r:n} &= C_{r,n}\int_{-\infty}^{\infty} x^{k-1} f(x;\alpha)F^{r-1}(x;\alpha)[1-F(x;\alpha)]^{n-r}\,dx\\ &= \frac{C_{r,n}}{\alpha}\int_{-\infty}^{\infty} x^{k-1}F^{r-\alpha}(x;\alpha)[1-F(x;\alpha)]^{n-r}\,dx\\ &\quad - \frac{C_{r,n}}{\alpha}\int_{-\infty}^{\infty} x^{k-1}F^{r}(x;\alpha)[1-F(x;\alpha)]^{n-r}\,dx.\end{aligned}$$

Then we use integration by parts by treating x^{k-1} for integration, and $F^{r-\alpha}(x;\alpha)[1-F(x;\alpha)]^{n-r}$ and $F^{r-\alpha}(x;\alpha)[1-F(x;\alpha)]^{n-r}$ for differentiation in the first and second integrals above, respectively. After some manipulations, one can get

$$\mu_{r:n}^{(k-1)} = \frac{n!(r-\alpha)!}{(r-1)!(n-\alpha)!k\alpha}\left[\mu_{r-\alpha+1:n-\alpha}^{(k)} - \mu_{r-\alpha:n-\alpha}^{(k)}\right] + \frac{r}{\alpha k}\left[\mu_{r:n}^{(k)} - \mu_{r+1:n}^{(k)}\right].$$

Now if we use the following general relation (Relation 3.3.1 in [3])

$$r\mu_{r+1:n}^{(k)} + (n-r)\mu_{r:n}^{(k)} = n\mu_{r:n-1}^{(k)} \tag{8}$$

by taking $r-\alpha$ and $n-\alpha$ for r and n, respectively, the result in the theorem follows. □

Remark 3.6. *When $\alpha = -1$, we can get the relation*

$$\mu_{r+1:n+1}^{(k)} = \mu_{r:n}^{(k)} + \frac{k}{r}\mu_{r:n}^{(k-1)}$$

by taking $r+1$ and $n+1$ for r and n, respectively, in (8).

Table 1: Moments of order statistics, $\mu_{r:n}$, for $K3D$.

r	n	$\alpha=-0.5$	$\alpha=-1$	$\alpha=-1.5$	$\alpha=-2$	$\alpha=-2.5$	$\alpha=-3$	$\alpha=-3.5$	$\alpha=-4$
1	1	0.30685	0.00000	-0.33555	-0.69315	-1.06788	-1.45621	-1.85554	-2.26394
1	2	-0.52648	-1.00000	-1.52175	-2.07944	-2.66425	-3.27001	-3.89229	-4.52789
2	2	1.14019	1.00000	0.85065	0.69315	0.52850	0.35759	0.18121	-0.00000
1	3	-0.90981	-1.50000	-2.15314	-2.85203	-3.58466	-4.34279	-5.12066	-5.91418
2	3	0.24019	0.00000	-0.25899	-0.53426	-0.82344	-1.12444	-1.43554	-1.75530
3	3	1.59019	1.50000	1.40547	1.30685	1.20447	1.09861	0.98958	0.87765
1	4	-1.15267	-1.83333	-2.58868	-3.39721	-4.24419	-5.11973	-6.01707	-6.93147
2	4	-0.18124	-0.50000	-0.84650	-1.21650	-1.60608	-2.01195	-2.43142	-2.86231
3	4	0.66161	0.50000	0.32852	0.14797	-0.04080	-0.23693	-0.43966	-0.64829
4	4	1.89971	1.83333	1.76445	1.69315	1.61955	1.54379	1.46600	1.38629
1	5	-1.32846	-2.08333	-2.92260	-3.82090	-4.76118	-5.73222	-6.72645	-7.73864
2	5	-0.44950	-0.83333	-1.25301	-1.70245	-2.17621	-2.66977	-3.17957	-3.70279
3	5	0.22114	0.00000	-0.23673	-0.48756	-0.75090	-1.02523	-1.30919	-1.60158
4	5	0.95527	0.83333	0.70536	0.57166	0.43261	0.28860	0.14002	-0.01275
5	5	2.13582	2.08333	2.02922	1.97352	1.91629	1.85759	1.79749	1.73606
1	6	-1.46541	-2.28333	-3.19391	-4.16827	-5.18738	-6.23886	-7.31451	-8.40875
2	6	-0.64376	-1.08333	-1.56606	-2.08401	-2.63021	-3.19903	-3.78612	-4.38808
3	6	-0.06097	-0.33333	-0.62691	-0.93934	-1.26820	-1.61125	-1.96649	-2.33223
4	6	0.50325	0.33333	0.15344	-0.03579	-0.23359	-0.43921	-0.65189	-0.87094
5	6	1.18128	1.08333	0.98132	0.87538	0.76571	0.65250	0.53597	0.41634
6	6	2.32673	2.28333	2.23880	2.19315	2.14641	2.09861	2.04980	2.00000
1	7	-1.57715	-2.45000	-3.42263	-4.46303	-5.55035	-6.67130	-7.81715	-8.98202
2	7	-0.79491	-1.28333	-1.82155	-2.39976	-3.00952	-3.64420	-4.29870	-4.96917
3	7	-0.26589	-0.58333	-0.92735	-1.29466	-1.68192	-2.08612	-2.50465	-2.93533
4	7	0.21226	0.00000	-0.22631	-0.46557	-0.71657	-0.97808	-1.24894	-1.52809
5	7	0.72149	0.58333	0.43825	0.28655	0.12864	-0.03505	-0.20410	-0.37808
6	7	1.36519	1.28333	1.19855	1.11091	1.02053	0.92752	0.83200	0.73411
7	7	2.48699	2.45000	2.41217	2.37352	2.33405	2.29379	2.25276	2.21098
1	8	-1.67132	-2.59286	-3.62047	-4.71920	-5.86666	-7.04872	-8.25623	-9.48307
2	8	-0.91803	-1.45000	-2.03781	-2.66982	-3.33621	-4.02938	-4.74361	-5.47464
3	8	-0.42556	-0.78333	-1.17277	-1.58957	-2.02946	-2.48866	-2.96397	-3.45277
4	8	0.00023	-0.25000	-0.51833	-0.80315	-1.10269	-1.41521	-1.73911	-2.07294
5	8	0.42428	0.25000	0.06572	-0.12799	-0.33046	-0.54096	-0.75878	-0.98324
6	8	0.89981	0.78333	0.66176	0.53528	0.40410	0.26849	0.12871	-0.01498
7	8	1.52032	1.45000	1.37748	1.30279	1.22601	1.14720	1.06643	0.98380
8	8	2.62508	2.59286	2.55999	2.52648	2.49235	2.45759	2.42224	2.38629

Table 2: Product moments of order statistics, $\mu_{r,s:n} \equiv E(X_{r:n}X_{s:n})$, for $K3D$.

r	s	n	$\alpha=-0.5$	$\alpha=-1$	$\alpha=-1.5$	$\alpha=-2$	$\alpha=-2.5$	$\alpha=-3$	$\alpha=-3.5$	$\alpha=-4$
1	1	1	2.38403	3.28987	4.82141	7.06019	10.06065	13.86107	18.48958	23.96770
1	1	2	1.53927	3.28987	6.17868	10.35006	15.89759	22.88545	31.35990	41.35567
1	2	2	0.09415	-0.00000	0.11260	0.48045	1.14036	2.12054	3.44303	5.12544
2	2	2	3.22878	3.28987	3.46413	3.77032	4.22372	4.83668	5.61927	6.57974
1	1	3	1.82068	4.28987	8.33702	14.15720	21.87386	31.56997	43.30444	57.12098
1	2	3	0.29710	0.85507	1.89907	3.51168	5.75437	8.67359	12.30513	16.67718
1	3	3	-1.07885	-1.71013	-2.26626	-2.69929	-2.97345	-3.06177	-2.94338	-2.60172
2	2	3	0.97645	1.28987	1.86199	2.73576	3.94505	5.51641	7.47082	9.82505
2	3	3	1.06420	0.85506	0.70497	0.62896	0.64017	0.74979	0.96733	1.30086
3	3	3	4.35495	4.28987	4.26520	4.28760	4.36305	4.49682	4.69349	4.95708
1	1	4	2.19625	5.28987	10.34955	17.61262	27.22619	39.28761	53.86514	71.00904
1	2	4	0.64428	1.71013	3.52010	6.19117	9.80842	14.43471	20.11753	26.89359
1	3	4	-0.45701	-0.42026	-0.09300	0.58988	1.67790	3.20938	5.21463	7.71811
1	4	4	-1.94826	-3.00000	-4.05724	-5.06472	-5.98282	-6.78284	-7.44335	-7.94779
2	2	4	0.69400	1.28987	2.29943	3.79097	5.81686	8.41705	11.62233	15.45680
2	3	4	0.35684	0.42026	0.64908	1.07449	1.72274	2.61555	3.77082	5.20344
2	4	4	0.03813	-0.42026	-0.85755	-1.25758	-1.60606	-1.89078	-2.10146	-2.22940
3	3	4	1.25889	1.28987	1.42456	1.68056	2.07324	2.61577	3.31931	4.19329
3	4	4	1.93092	1.71012	1.51417	1.34946	1.22199	1.13720	1.09998	1.11470
4	4	4	5.38697	5.28987	5.21208	5.15662	5.12632	5.12383	5.15156	5.21168
1	1	5	2.55880	6.20653	12.16940	20.72073	32.02703	46.19698	63.30613	83.40973
1	2	5	0.98634	2.49575	4.99124	8.62202	13.49419	19.68450	27.24989	36.23368
1	3	5	-0.02364	0.47287	1.45432	3.00847	5.20195	8.08612	11.70137	16.07999
1	4	5	-1.05710	-1.39314	-1.54495	-1.45123	-1.06647	-0.35604	0.70726	2.14539
1	5	5	-2.66043	-4.07124	-5.54864	-7.02746	-8.46248	-9.82218	-11.08384	-12.23044
2	2	5	0.74604	1.62320	3.07017	5.18015	8.02282	11.65011	16.10120	21.40631
2	3	5	0.28602	0.59053	1.17248	2.08134	3.35762	5.03394	7.13659	9.68689
2	4	5	-0.12356	-0.26079	-0.28373	-0.16507	0.11854	0.58673	1.25589	2.13981
2	5	5	-0.70290	-1.39314	-2.09531	-2.79002	-3.46022	-4.09163	-4.67247	-5.19303
3	3	5	0.61595	0.78987	1.14331	1.70720	2.50793	3.56748	4.90401	6.53255
3	4	5	0.66787	0.59053	0.59209	0.68743	0.88997	1.21156	1.66252	2.25180
3	5	5	0.86002	0.47287	0.09331	-0.27138	-0.61419	-0.92870	-1.20913	-1.45038
4	4	5	1.68752	1.62320	1.61206	1.66279	1.78345	1.98131	2.26284	2.63378
4	5	5	2.70891	2.49573	2.29515	2.11042	1.94473	1.80104	1.68217	1.59069
5	5	5	6.31183	6.20653	6.11209	6.03007	5.96204	5.90946	5.87374	5.85616
1	1	6	2.89237	7.03987	13.82080	23.53976	36.37944	52.45777	71.85613	94.63380
1	2	6	1.30224	3.21193	6.33539	10.85151	16.88489	24.52398	33.83334	44.86146
1	3	6	0.33631	1.21927	2.77140	5.10399	8.30065	12.42509	17.52676	23.64462
1	4	6	-0.54474	-0.42941	0.03347	0.91976	2.28673	4.17876	6.63120	9.67278
1	5	6	-1.57078	-2.21373	-2.74554	-3.10311	-3.24121	-3.12655	-2.73358	-2.04192
1	6	6	-3.27113	-5.00000	-6.84596	-8.73300	-10.60975	-12.44150	-14.20393	-15.87932
2	2	6	0.89094	2.03987	3.91241	6.62560	10.26501	14.89307	20.55612	27.28934
2	3	6	0.37277	0.90752	1.83451	3.22210	5.12493	7.58601	10.63925	14.31163
2	4	6	-0.06132	0.03822	0.34477	0.89935	1.73629	2.88392	4.36556	6.20052
2	5	6	-0.54078	-0.85881	-1.10002	-1.23759	-1.24863	-1.11409	-0.81812	-0.34748
2	6	6	-1.30731	-2.21373	-3.16245	-4.12964	-5.09463	-6.04052	-6.95375	-7.82341
3	3	6	0.45624	0.78987	1.38570	2.28924	3.53845	5.16418	7.19138	9.64026
3	4	6	0.32957	0.39119	0.59328	0.96120	1.51732	2.28084	3.26811	4.49296
3	5	6	0.23141	0.03822	-0.09593	-0.15698	-0.13204	-0.00965	0.22024	0.56640
3	6	6	0.12265	-0.42941	-0.99224	-1.55682	-2.11441	-2.65706	-3.17771	-3.67024
4	4	6	0.77566	0.78987	0.90092	1.12516	1.47741	1.97077	2.61664	3.42484
4	5	6	1.03889	0.90752	0.82065	0.78638	0.81241	0.90578	1.07289	1.31949
4	6	6	1.56113	1.21927	0.88008	0.54744	0.22524	-0.08281	-0.37320	-0.64272
5	5	6	2.14345	2.03987	1.96763	1.93161	1.93647	1.98657	2.08594	2.23825
5	6	6	3.41336	3.21190	3.01754	2.83218	2.65768	2.49582	2.34834	2.21685
6	6	6	7.14551	7.03987	6.94098	6.84976	6.76715	6.69404	6.63130	6.57974

4. Product Moments of Order Statistics

We need product moments for covariance calculations of any two order statistics from the distribution. It is well-known that the joint PDF of $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$, is given by

$$f_{r,s:n}(x,y;\alpha) = C_{r,s,n} f(x;\alpha) f(y;\alpha) F^{r-1}(x;\alpha)[1-F(y;\alpha)]^{n-s}[F(y;\alpha)-F(x;\alpha)]^{s-r-1},$$

where $x < y$, $r < s$ and $C_{r,s,n} = n!/((r-1)!(s-r-1)!(n-s)!)$. We will try to find a closed form expression for the product moment $E(X_{r:n}^k X_{s:n}^l)$, $k,l \in \mathbb{Z}^+$.

Theorem 4.1. *We have*

$$\begin{aligned} E(X_{r:n}^k X_{s:n}^l) = C_{r,s,n} \sum_{i=0}^{s-r-1} \sum_{j=0}^{n-s} \binom{s-r-1}{i}\binom{n-s}{j}(-1)^{i+j} \\ \frac{\partial^k}{\partial \rho_1^k} \frac{\partial^l}{\partial \rho_2^l} \Big\{ \frac{(-\alpha)^{\rho_1+\rho_2-2}}{\rho_1-(r+i)/\alpha} B\left(\rho_1+\rho_2-(s+j)/\alpha, 1-\rho_2\right) {}_3F_2\left(\rho_1+\rho_2-(s+j)/\alpha,\right. \\ \left. \rho_1-(r+i)/\alpha, \rho_1; 1+\rho_1-(r+i)/\alpha, \rho_1-(s+j)/\alpha+1;1\right)\Big\}\Big|_{\rho_1=\rho_2=0}, \end{aligned}$$

where $B(\cdot,\cdot)$ is the usual beta function defined by the integral $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$, $a,b>0$ and ${}_3F_2$ is the generalized hypergeometric function defined by the series

$${}_3F_2(a_1,a_2,a_3;b_1,b_2;x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k (a_3)_k}{(b_1)_k (b_2)_k} \frac{x^k}{k!}, \tag{9}$$

where $(x)_k = x(x+1)\cdots(x+k-1)$ denotes the ascending factorial and $\alpha < 0$.

Proof.

$$E(X_{r:n}^k X_{s:n}^l) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} x^k y^l f_{r,s:n}(x,y;\alpha)\,dx\,dy$$

In order to solve this double integral, we use similar change of variables that has been done for single moments. At the last step, we use the following formula 7.512.5

$$\int_0^1 u^{a-1}(1-u)^{b-1} {}_2F_1(c,d;\rho;u)\,du = B(a,b)\,{}_3F_2(a,c,d;\rho,a+b;1)$$

in [11]. □

Some calculations are reported in Table 2. Although, we cannot see any overall pattern from this table, we observe that for $r < s < n$ and fixed n and r, the product moments decrease with increasing s. Now let $\mu_{r,s:n}^{(k,l)} = E(X_{r:n}^k X_{s:n}^l)$. We can find a relation between product moments which is presented in the following theorem.

Theorem 4.2. *Let $\alpha \in \mathbb{Z}^-$ and $r < s$. Then we have*

$$\begin{aligned} \mu_{r,s:n}^{(k,l)} = \frac{n!(r-\alpha)!}{\alpha(k+1)(r-1)!(n-\alpha)!}\left[\mu_{r-\alpha+1,s-\alpha:n-\alpha}^{(k+1,l)} - \mu_{r-\alpha,s-\alpha:n-\alpha}^{(k+1,l)}\right] \\ + \frac{r}{\alpha(k+1)}\left[\mu_{r,s:n}^{(k+1,l)} - \mu_{r+1,s:n}^{(k+1,l)}\right]. \end{aligned}$$

Proof. By using (1), we have

$$\mu_{r,s:n}^{(k,l)} = \frac{C_{r,s,n}}{\alpha}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{y} x^k y^l f(y)F^{r-\alpha}(x)[1-F(y)]^{n-s}[F(y)-F(x)]^{s-r-1}\,dx\,dy\right.$$

$$\left. - \int_{-\infty}^{\infty}\int_{-\infty}^{y} x^k y^l f(y)F^{r}(x)[1-F(y)]^{n-s}[F(y)-F(x)]^{s-r-1}\,dx\,dy\right\},$$

where $F(\cdot;\alpha) \equiv F(\cdot)$ and $f(\cdot;\alpha) \equiv f(\cdot)$. After using integration by parts by treating x^k for integration and $F^{r-\alpha}(x)[F(y)-F(x)]^{s-r-1}$ for differentiation for the first integral above, and a similar substitution for the second integral, the result of the theorem is easily obtained. □

5. Location-Scale Parameter Estimation

In order to apply the theory developed in the previous sections, we refer to the statistical inference. We introduce location parameter μ and scale parameter σ into the model by transformation $Y = \mu + \sigma X$. The pdf of Y is then given by

$$f(y;\alpha,\mu,\sigma) = \frac{1}{\sigma}e^{-(y-\mu)/\sigma}\left(1-\alpha e^{-(y-\mu)/\sigma}\right)^{(1-\alpha)/\alpha},$$

where $-\infty < \mu < \infty$ and $\sigma > 0$. We use $K3D(\alpha,\mu,\sigma)$ for the distribution of Y. We assume that the shape parameter is known.

5.1. Best Linear Unbiased Estimation Method

As an application of the moments of order statistics from the $K3D$ distribution, we search for the BLUE's of the location-scale parameters. For a reference, see e.g. David and Nagaraja [9] p. 185. The BLUE's have been obtained in the literature for different models (see e.g. Balakrishnan et al. [5] and Tumlinson et al. [26].

Let $Y_1,\ldots,Y_n$ be a random sample from the $K3D(\alpha,\mu,\sigma)$ and $Y_{1:n} \le\cdots\le Y_{n:n}$ denote the corresponding ordered observations. Then the BLUE vector of μ and σ is given by

$$\begin{pmatrix}\tilde{\mu}\\ \tilde{\sigma}\end{pmatrix} = \mathbf{C}\mathbf{y},$$

where

$$\mathbf{C} = (\mathbf{A}^T\mathbf{V}^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{V}^{-1};$$

$\mathbf{A} = (\mathbf{1},\boldsymbol{\xi})$, $\mathbf{1}^T = (1,\cdots,1)$, $\mathbf{y}^T = (y_{1:n},\cdots,y_{n:n})$, $\boldsymbol{\xi}^T = (\xi_{1:n},\cdots,\xi_{n:n})$, $\xi_{r:n} = E(X_{r:n})$ and $\mathbf{V}$ is the variance-covariance matrix of the order statistics from X. All the vectors are $n\times 1$. The variance-covariance matrix of the estimates is given by

$$(\mathbf{A}^T\mathbf{V}^{-1}\mathbf{A})^{-1}\sigma^2.$$

The coefficients for the BLUE of μ and σ are computed for the $K3D$ for $\alpha = -0.5(-0.5)-4.5$ and $n = 3, 5$ and 10, and are presented in Tables 3 and 4, respectively. The coefficients of the variances and covariances of the BLUE of μ and σ are tabulated for $\alpha = -0.5(-0.5)-4.5$ and $n = 3, 5$ and 10 in Table 5 for the $K3D$. In this table, the first line gives variance coefficients for μ, the second line gives those for σ, and the last line gives the covariance coefficients multiplied by σ^2 for a given n. We observe from Table 5 that the variances increase with decreasing α. We just give such tables to show its computability, and use them in a simulation study.

Table 3: Coefficients of the BLUE of μ for the $K3D$.

n	r	$\alpha=-0.5$	$\alpha=-1$	$\alpha=-1.5$	$\alpha=-2$	$\alpha=-2.5$	$\alpha=-3$	$\alpha=-3.5$	$\alpha=-4$	$\alpha=-4.5$
3	1	0.40798	0.25000	0.16112	0.11081	0.08011	0.05941	0.04409	0.03185	0.02155
3	2	0.42240	0.50000	0.49993	0.45950	0.40476	0.34876	0.29698	0.25118	0.21147
3	3	0.16962	0.25000	0.33895	0.42968	0.51513	0.59182	0.65894	0.71697	0.76697
5	1	0.19572	0.09762	0.05908	0.04271	0.03476	0.03025	0.02725	0.02495	0.02300
5	2	0.29168	0.25017	0.18972	0.14057	0.10572	0.08168	0.06493	0.05295	0.04411
5	3	0.25717	0.30442	0.30832	0.28550	0.25188	0.21689	0.18479	0.15698	0.13350
5	4	0.17899	0.25017	0.31375	0.36141	0.39070	0.40339	0.40299	0.39324	0.37736
5	5	0.07644	0.09762	0.12914	0.16981	0.21693	0.26778	0.32003	0.37188	0.42203
10	1	0.06608	0.02615	0.01685	0.01464	0.01416	0.01406	0.01399	0.01386	0.01366
10	2	0.12598	0.07426	0.04449	0.02993	0.02276	0.01904	0.01696	0.01568	0.01480
10	3	0.14199	0.11192	0.07940	0.05597	0.04091	0.03146	0.02546	0.02154	0.01889
10	4	0.14305	0.13742	0.11477	0.09028	0.06988	0.05447	0.04325	0.03516	0.02930
10	5	0.13536	0.15026	0.14450	0.12797	0.10844	0.09004	0.07431	0.06148	0.05124
10	6	0.12154	0.15026	0.16282	0.16157	0.15137	0.13674	0.12083	0.10545	0.09149
10	7	0.10306	0.13742	0.16438	0.18103	0.18758	0.18598	0.17866	0.16784	0.15523
10	8	0.08078	0.11192	0.14438	0.17420	0.19872	0.21676	0.22820	0.23367	0.23413
10	9	0.05527	0.07426	0.09922	0.12830	0.15941	0.19070	0.22067	0.24823	0.27266
10	10	0.02689	0.02615	0.02917	0.03612	0.04676	0.06075	0.07767	0.09710	0.11859

Table 4: Coefficients of the BLUE of σ for the $K3D$.

n	r	$\alpha=-0.5$	$\alpha=-1$	$\alpha=-1.5$	$\alpha=-2$	$\alpha=-2.5$	$\alpha=-3$	$\alpha=-3.5$	$\alpha=-4$	$\alpha=-4.5$
3	1	-0.46072	-0.33333	-0.25037	-0.19751	-0.16263	-0.13844	-0.12081	-0.10740	-0.09683
3	2	0.11244	0.00000	-0.06551	-0.09700	-0.10905	-0.11097	-0.10797	-0.10277	-0.09678
3	3	0.34828	0.33333	0.31588	0.29451	0.27168	0.24941	0.22878	0.21016	0.19361
5	1	-0.30261	-0.18888	-0.13109	-0.09954	-0.08044	-0.06779	-0.05882	-0.05212	-0.04690
5	2	-0.09670	-0.12780	-0.11999	-0.10315	-0.08732	-0.07451	-0.0645	-0.05669	-0.05052
5	3	0.05944	0.00000	-0.03753	-0.05621	-0.06300	-0.06340	-0.06078	-0.05694	-0.05279
5	4	0.15630	0.12780	0.09670	0.06726	0.04212	0.02207	0.00678	-0.00447	-0.01254
5	5	0.18357	0.18888	0.19191	0.19164	0.18863	0.18364	0.17732	0.17022	0.16275
10	1	-0.15907	-0.08646	-0.05747	-0.04330	-0.03507	-0.02968	-0.02584	-0.02294	-0.02067
10	2	-0.11193	-0.08629	-0.06246	-0.04713	-0.03748	-0.03114	-0.02672	-0.02349	-0.02101
10	3	-0.06388	-0.06990	-0.06026	-0.04908	-0.04005	-0.03334	-0.02841	-0.02472	-0.02190
10	4	-0.02286	-0.04491	-0.04940	-0.04602	-0.04043	-0.03496	-0.03027	-0.02646	-0.02338
10	5	0.01167	-0.01545	-0.03000	-0.03539	-0.03564	-0.03354	-0.03063	-0.02763	-0.02486
10	6	0.04005	0.01545	-0.00341	-0.01564	-0.02243	-0.02547	-0.02619	-0.02561	-0.02437
10	7	0.06213	0.04491	0.02783	0.01316	0.00178	-0.00640	-0.01190	-0.01534	-0.01729
10	8	0.07756	0.06989	0.05952	0.04809	0.03690	0.02674	0.01796	0.01066	0.00477
10	9	0.08512	0.08629	0.08488	0.08130	0.07621	0.07018	0.06369	0.05707	0.05057
10	10	0.08121	0.08646	0.09079	0.09399	0.09621	0.09761	0.09832	0.09846	0.09813

Table 5: Variances and covariances of the BLUE's of μ and σ for the $K3D$. ($\times\sigma^2$)

n	$\alpha=-0.5$	$\alpha=-1$	$\alpha=-1.5$	$\alpha=-2$	$\alpha=-2.5$	$\alpha=-3$	$\alpha=-3.5$	$\alpha=-4$	$\alpha=-4.5$
3	0.70814	1.07247	1.46913	1.89203	2.33919	2.81113	3.31076	3.84254	4.41153
3	0.32584	0.33333	0.34775	0.36253	0.37641	0.38915	0.40065	0.41093	0.42002
3	0.02511	0.00000	-0.02571	-0.04577	-0.05873	-0.06391	-0.06116	-0.05077	-0.03332
5	0.41783	0.62824	0.85389	1.09222	1.34158	1.60082	1.86924	2.14646	2.43248
5	0.16357	0.17037	0.17800	0.18484	0.19079	0.19596	0.20050	0.20452	0.20810
5	0.02106	0.00000	-0.02021	-0.0382	-0.05382	-0.06709	-0.07812	-0.08698	-0.09379
10	0.20591	0.30745	0.41506	0.52756	0.64408	0.76400	0.88690	1.01247	1.14050
10	0.07291	0.07679	0.08035	0.08335	0.08587	0.08800	0.08983	0.09141	0.09278
10	0.01187	0.00000	-0.01125	-0.02162	-0.03107	-0.03965	-0.04743	-0.05448	-0.06087

5.2. Maximum Likelihood Method

Let the data set $x_1, x_2, \ldots, x_n$ be modeled by $K3D(\alpha, \mu, \sigma)$. Then the log-likelihood function is given by

$$l(\mu,\sigma) = -n\ln\sigma - \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right) + \left(\frac{1}{\alpha}-1\right)\sum_{i=1}^{n}\ln\left[1-\alpha e^{-(x_i-\mu)/\sigma}\right]. \tag{10}$$

Upon taking the partial derivatives of (10) with respect to the parameters, and then equating them to 0, the likelihood estimating equations are found as

$$l_\mu(\mu,\sigma) = \frac{n}{\sigma} + \left(\frac{\alpha-1}{\sigma}\right)\sum_{i=1}^{n}\frac{e^{-(x_i-\mu)/\sigma}}{1-\alpha e^{-(x_i-\mu)/\sigma}} = 0 \tag{11}$$

and

$$l_\sigma(\mu,\sigma) = -\frac{n}{\sigma} + \sum_{i=1}^{n}\frac{(x_i-\mu)}{\sigma^2} + \frac{(\alpha-1)}{\sigma^2}\sum_{i=1}^{n}\frac{(x_i-\mu)e^{-(x_i-\mu)/\sigma}}{1-\alpha e^{-(x_i-\mu)/\sigma}} = 0. \tag{12}$$

By solving (11) and (12) simultaneously, we get the MLE's of μ and σ as in the following forms:

$$\hat{\sigma} = \bar{x} + \left(\frac{\alpha-1}{n}\right)\sum_{i=1}^{n}\frac{x_i e^{-(x_i-\hat{\mu})/\hat{\sigma}}}{1-\alpha e^{-(x_i-\hat{\mu})/\hat{\sigma}}} \tag{13}$$

and

$$\hat{\mu} = (-\hat{\sigma})\ln\left[\left(\frac{1-\alpha}{n}\right)\sum_{i=1}^{n}\frac{e^{-x_i/\hat{\sigma}}}{1-\alpha e^{-(x_i-\hat{\mu})/\hat{\sigma}}}\right]. \tag{14}$$

Since the estimators were not obtained explicitly, one should write an iterative algorithm to compute the estimates. This algorithm will converge to the true values that maximize the log-likelihood globally due to the following theorem.

Theorem 5.1. *The MLE's of the parameters μ and σ which are given in (13) and (14), respectively, exist uniquely in the respective parameter spaces.*

Proof. Let $\boldsymbol{\theta}=(\mu,\sigma)$. To show the existence of a local maximum according to Theorem 2.1 in [15], one may easily show that when $\boldsymbol{\theta}$ tends to the boundary of the parameter space, the likelihood tends to zero.

To show the uniqueness of the MLE of $\boldsymbol{\theta}$ according to Corollary 2.5 in [15], we should first determine the second derivatives matrix

$$D^2 l = \begin{pmatrix} l_{\mu\mu} & l_{\mu\sigma} \\ l_{\sigma\mu} & l_{\sigma\sigma} \end{pmatrix}$$

of the log-likelihood function at the solutions of the estimating equations. We have

$$l_{\mu\mu} = \frac{(\alpha-1)}{\sigma^2}\sum_{i=1}^{n}\frac{e^{-(x_i-\mu)/\sigma}}{[1-\alpha e^{-(x_i-\mu)/\sigma}]^2},$$

$$l_{\sigma\sigma} = \frac{n}{\sigma^2} - 2\sum_{i=1}^{n}\frac{(x_i-\mu)}{\sigma^3} - \frac{2(\alpha-1)}{\sigma^3}\sum_{i=1}^{n}\frac{(x_i-\mu)e^{-(x_i-\mu)/\sigma}}{1-\alpha e^{-(x_i-\mu)/\sigma}} + \frac{(\alpha-1)}{\sigma^4}\sum_{i=1}^{n}\frac{(x_i-\mu)^2 e^{-(x_i-\mu)/\sigma}}{[1-\alpha e^{-(x_i-\mu)/\sigma}]^2},$$

$$l_{\mu\sigma} = -\frac{n}{\sigma^2} - \frac{(\alpha-1)}{\sigma^2}\sum_{i=1}^{n}\frac{e^{-(x_i-\mu)/\sigma}}{1-\alpha e^{-(x_i-\mu)/\sigma}} + \frac{(\alpha-1)}{\sigma^3}\sum_{i=1}^{n}\frac{(x_i-\mu)e^{-(x_i-\mu)/\sigma}}{[1-\alpha e^{-(x_i-\mu)/\sigma}]^2}.$$

At the solutions of the estimating equations, we have the following determinant

$$\begin{aligned}
\det(D^2 l) &= l_{\mu\mu} l_{\sigma\sigma} - l_{\mu\sigma}^2 \\
&= \left\{\frac{(\alpha-1)}{\sigma^2}\sum_{i=1}^{n}\frac{e^{-(x_i-\mu)/\sigma}}{[1-\alpha e^{-(x_i-\mu)/\sigma}]^2}\right\}\left\{\frac{-n}{\sigma^2} + \frac{(\alpha-1)}{\sigma^4}\sum_{i=1}^{n}\frac{(x_i-\mu)^2 e^{-(x_i-\mu)/\sigma}}{[1-\alpha e^{-(x_i-\mu)/\sigma}]^2}\right\} \\
&\quad - \left\{\frac{(\alpha-1)}{\sigma^3}\sum_{i=1}^{n}\frac{(x_i-\mu)e^{-(x_i-\mu)/\sigma}}{[1-\alpha e^{-(x_i-\mu)/\sigma}]^2}\right\}^2 \\
&= \frac{n(1-\alpha)}{\sigma^4}\sum_{i=1}^{n}\frac{e^{-(x_i-\mu)/\sigma}}{[1-\alpha e^{-(x_i-\mu)/\sigma}]^2} + \frac{(\alpha-1)^2}{\sigma^6}\left\{\sum_{i=1}^{n}\frac{e^{-(x_i-\mu)/\sigma}}{[1-\alpha e^{-(x_i-\mu)/\sigma}]^2}\right. \\
&\quad \left. \times \sum_{i=1}^{n}\frac{(x_i-\mu)^2 e^{-(x_i-\mu)/\sigma}}{[1-\alpha e^{-(x_i-\mu)/\sigma}]^2} - \left(\sum_{i=1}^{n}\frac{(x_i-\mu)e^{-(x_i-\mu)/\sigma}}{[1-\alpha e^{-(x_i-\mu)/\sigma}]^2}\right)^2\right\}.
\end{aligned}$$

We observe that $\det(D^2 l)$ is positive from Cauchy-Schwarz inequality. Further, it is clear that $ł_{\mu\mu} < 0$. Thus, the MLE's of μ and σ uniquely exist. □

6. Numerical Results

In this section we first report the results of a small simulation study. In order to compare the two estimation methods we generate 10,000 random samples of size 10 from the $K3D(-1.5, 2, 1)$. Then we compute the location-scale estimates from both methods. We use Tables 3 and 4 to compute the BLUE's. According to Table 6, the two estimation methods have given similar MSE's and so we may say that their performances are similar for the small data sets. However, the bias of the location estimate with the BLUE has given a smaller value.

Next we consider a small real data set from the web site https://tr.euronews.com/2019/09/23/a-dan-z-ye-turkiye-nin-yoksulluk-ve-gelir-dagilimi-esitsizligi-haritasi.

It consists of relative poverty rates of Turkey for the year 2018. The data set is 8.1, 7.7, 11.5, 10.3, 5.0, 10.1, 11.6, 9.1, 9.9, 12.7. The skewness coefficient of the data set is -0.64832. Thus, a probability distribution that can be left-skewed is needed to model this data set. We may use *K3D* for this purpose appropriately.

We assume that $\alpha = -2.5$, as a prior knowledge, in order to use the Tables 3, 4 and 5. We take this value since it is very close to the L-moment estimate -2.7 of α. At this point, we want to correct the L-skewness τ_3 measure in Jeong et al. (2014). The correct expression should be

$$\tau_3 = \frac{2\psi(-3/\alpha) - 3\psi(-2/\alpha) + \psi(-1/\alpha)}{\psi(-2/\alpha) - \psi(-1/\alpha)}.$$

We also perform some goodness-of-fit tests to support the validity of this distribution for modelling the data. Since we are mainly concerned with the location-scale estimation, we may scale the data so that we perform standard Kolmogorov-Smirnov (KS) and Cramer-Von Mises (CVM) tests. The `ks.test` and `cvm.test` functions in R program [18] are used. The results of the test statistics and their p-values within parentheses are as follows: KS=0.3637 (0.1083), CVM=0.2352 (0.5099). Both tests do not reject the hypothesis that the data come from the *K3D*.

Table 6: Means of the estimates and the corresponding mean square errors from a small simulation study.

Method	$\hat{\mu}$	$\hat{\sigma}$	$MSE(\hat{\mu})$	$MSE(\hat{\sigma})$
MLE	1.97611	0.97788	0.38206	0.06536
BLUE	1.99033	1.03861	0.38171	0.07483

After fitting the $K3D$ to the data we find the MLE's and standard errors within parentheses which are $\hat{\mu} = 10.4424(0.6122)$ and $\hat{\sigma} = 0.7629(0.2109)$. The BLUE's of the parameters are $\tilde{\mu} = 10.4761$ and $\tilde{\sigma} = 0.8129$. From Table 5, we see that the variance of $\tilde{\mu}$ is $0.64408\sigma^2$, and the variance of $\tilde{\sigma}$ is $0.08587\sigma^2$. To calculate the standard error of $\tilde{\mu}$, we use $\tilde{\sigma} = 0.8129$ and just take the square root of $0.64408\tilde{\sigma}^2$. Similarly, we do it for the standard error of $\tilde{\sigma}$. As a result, the standard errors of $\tilde{\mu}$ and $\tilde{\sigma}$ are computed as 0.6524 and 0.2382, respectively. We observe that the MLE's have slightly smaller standard errors than those for the BLUE's. We also observe from Figure 2 that the fit based on the MLE method captures the peak of the data slightly better than the BLUE method.

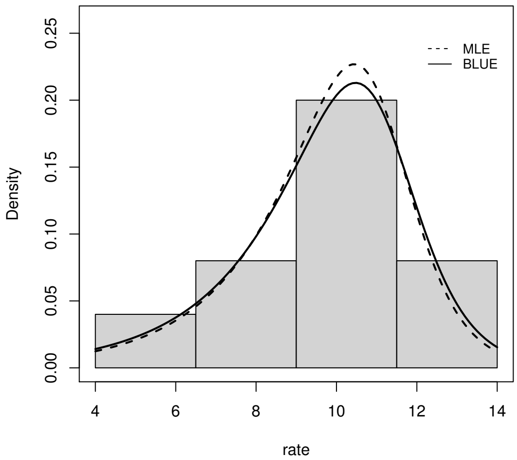

Figure 2: Fits based on the MLE and BLUE methods on the histogram of the relative poverty rates.

7. Conclusions

The kappa distribution, namely $K3D$, considered in this paper is mainly used in hydrology and environmental sciences to model hydrologic and extremal events. Although its usefulness in real applications, the theory of the distribution has been studied less in the literature. As a contribution to the theory of the distribution, this paper studied the moments of order statistics from the distribution and derived some moment relations. We observed that these expressions were in easily computable forms. The relations are also generalizations of the known results for logistic distribution which are given before. Then we showed that the BLUE's can be easily computable and a good alternative to the MLE's since their computation do not need any iterative process. Since the distribution studied here has the CDF and its inverse in nice computable forms, further applications of the distribution are challenging.

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