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On $\mathcal{I}$-statistical convergence of complex uncertain sequences

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Abstract. The concept of $\mathcal{I}$-statistical convergence almost surely of complex uncertain sequences was introduced by Halder and Debnath, which is a very recent and new approach in complex uncertainty theory. Based on that technique, in this article, we introduced the concept of $\mathcal{I}$-statistical convergence in mean, distribution, uniformly almost surely of complex uncertain sequence. Also, we explore the concept of $\mathcal{I}$-statistical convergence in p-distance, completely $\mathcal{I}$-statistical convergence, and $\mathcal{I}$-statistical convergence in metric of complex uncertain sequences. Overall, this study mainly presents a complete diagrammatic scenario of interrelationships among all $\mathcal{I}$-statistical convergence concepts of complex uncertain sequences and include some observations about the above convergence concepts.

1. Introduction

In the real world, we often face various types of indeterminacy, and sometimes it is difficult to collect observed data when some unexpected events occur, like earthquakes, wars, floods etc. In this situation, information and knowledge can't be described well by probability theory. Consequently, some domain experts are consulted to give a degree of belief that each event will take place while making a decision. To address some aspects of these uncertain events, Liu[8] initially introduced a theory named uncertainty theory in 2007. Also Liu defined different types of convergence of a sequence of real uncertain variables and identified the relationships among them. Then it has been extended to the complex uncertain variable by Peng[31]. Chen et al.[29] subsequently investigated the notion of convergence of complex uncertain sequences using complex uncertain variables. The notion of statistical convergence of complex uncertain sequences in the field of uncertainty theory was introduced by Tripathy and Nath[3]. Since then, this field has seen a lot of exciting changes; for details, see [4–6, 9, 13, 16–21, 23, 25–28].

On the other hand, the concept of $\mathcal{I}$-convergence, which is a generalization of statistical convergence, was introduced by Kostyrko et al.[22]. The idea of $\mathcal{I}$-convergence was further extended to $\mathcal{I}$-statistical convergence by Savas and Das[10]. Later on, more investigation in this direction can be found in the works of [1, 7, 11, 14, 15, 24].

Inspired by the above works and following a very recently introduced concept of $\mathcal{I}$-statistical convergence almost surely of complex uncertain sequences by Halder and Debnath[2], in this paper, we further

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investigate the concept of $\mathcal{I}$-statistical convergence of complex uncertain sequences. We also explore the concepts of $\mathcal{I}$-statistical convergence in p-distance, completely $\mathcal{I}$-statistical convergence, and $\mathcal{I}$-statistical convergence in metric of complex uncertain sequences. Finally, we try to establish the relationship among all $\mathcal{I}$-statistical convergence concepts of complex uncertain sequences with an attached diagramatic section.

2. Definitions and Preliminaries

In this section, we provide some basic ideas and results on generalized convergence concepts and the theory of uncertainty that will be used throughout the article.

Definition 2.1. *[12] A sequence (x_n) is said to be statistically convergent to ℓ provided that each $\epsilon > 0$ such that $\mathcal{D}(A(\epsilon)) = 0$, where $A(\epsilon) = \{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\}$. We denote $\mathcal{D}(A(\epsilon))$ as the density of the set $A(\epsilon)$.*

Definition 2.2. *[22] Let X be a non-empty set. A family of subsets $\mathcal{I} \subset P(X)$ is called an ideal on X if and only if*
(i) for each $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$;
(ii) for each $A \in \mathcal{I}$ and $B \subset A \Rightarrow B \in \mathcal{I}$.
An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\phi\}$ and $X \notin \mathcal{I}$.
A non-trivial ideal $\mathcal{I}$ is called an admissible ideal in X if and only if $\{\{x\} : x \in X\} \subset \mathcal{I}$.

Example 2.3. *(i) $\mathcal{I}_f$:= The set of all finite subsets of $\mathbb{N}$ forms a non-trivial admissible ideal.*
(ii) $\mathcal{I}_d$:= The set of all subsets of $\mathbb{N}$ whose natural density is zero forms a non-trivial admissible ideal.

Definition 2.4. *[22] A sequence (x_n) is said to be $\mathcal{I}$-convergent to ℓ, if for every $\varepsilon > 0$, the set*

$$\{n \in \mathbb{N} : |x_n - \ell| \geq \varepsilon\} \in \mathcal{I}.$$

The usual convergence of sequences is a special case of $\mathcal{I}$-convergence ($\mathcal{I}$=$\mathcal{I}_f$-the ideal of all finite subsets of $\mathbb{N}$). The statistical convergence of sequences is also a special case of $\mathcal{I}$-convergence. In this case, $\mathcal{I}$=$\mathcal{I}_d = \{A \subseteq \mathbb{N} : \lim\limits_{n\to\infty} \frac{|A\cap\{1,2,\ldots,n\}|}{n} = 0\}$, where $|A|$ is the cardinality of the set A.

Definition 2.5. *[10] A sequence $x = (x_n)$ is said to be $\mathcal{I}$-statistically convergent to $\ell \in \mathbb{R}$, if for every $\varepsilon, \delta > 0$ such that $\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |x_k - \ell| \geq \varepsilon\}| \geq \delta\right\} \in \mathcal{I}$.*
For $\mathcal{I} = \mathcal{I}_{fin}$, $\mathcal{I}$-statistical convergence coincides with statistical convergence.

Definition 2.6. *[8] Let $\mathcal{L}$ be a σ-algebra on a nonempty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:*
Axiom 1 (Normality): $\mathcal{M}\{\Gamma\} = 1$;
Axiom 2 (Duality): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ for any $\Lambda \in \mathcal{L}$;
Axiom 3 (Subadditivity): For every countable sequence of $\{\Lambda_j\} \in \mathcal{L}$,

$$\mathcal{M}\{\bigcup_{j=1}^{\infty} \Lambda_j\} \leq \sum_{j=1}^{\infty} \mathcal{M}\{\Lambda_j\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in $\mathcal{L}$ is called an event.

Definition 2.7. *[8] A variable $\zeta = \xi + i\eta$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers is a complex uncertain variable if and only if ξ and η are uncertain variables, where ξ and η are the real and imaginary parts of ζ, respectively.*

Definition 2.8. *[30] An uncertain measure $\mathcal{M}$ is called continuous if for any sequence of events Λ_k with $k \to \infty$, we have $\mathcal{M}\left\{\lim\limits_{k\to\infty} \Lambda_k\right\} = \lim\limits_{k\to\infty} \mathcal{M}\{\Lambda_k\}$.*

Definition 2.9. *[31] Let $\zeta = \xi + i\eta$ be a complex uncertain variable, where ξ and η are real and imaginary part of ζ, respectively. Then the complex uncertainty distribution of ζ is a function from $\mathbb{C}$ to $[0, 1]$ defined by $\Phi(z) = \mathcal{M}\{\xi \leq x, \eta \leq y\}$ for any complex number $z = x + iy$.*

Definition 2.10. *[31] Let $\zeta = \xi + i\eta$ be a complex uncertain variable. If the expected value of ξ and η i.e., $E[\xi]$ and $E[\eta]$ exists, then the expected value of ζ is defined by*

$$E[\zeta] = E[\xi] + iE[\eta].$$

Definition 2.11. *[25] Let ζ and ζ^* be two complex uncertain variables. Then the p-distance between them is defined as*

$$d_p(\zeta, \zeta^*) = \left(E\left[\|\zeta - \zeta^*\|^p\right]\right)^{\frac{1}{p+1}}, p > 0.$$

Definition 2.12. *[5] Let ζ and ζ^* be two complex uncertain variables, then the metric between them is defined as follows*

$$D(\zeta, \zeta^*) = \inf\{r : \mathcal{M}\{\|\zeta - \zeta^*\| \le r\} = 1\}.$$

Definition 2.13. *[5] A complex uncertain sequence (ζ_n) is said to be convergent in metric to ζ if*

$$\lim_{n\to\infty} D(\zeta_n, \zeta) = 0.$$

Definition 2.14. *[2] A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-statistically convergent almost surely to ζ if for every $\delta, \upsilon > 0$, there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \le n : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \ge \delta\right\}\right| \ge \upsilon\right\} \in \mathcal{I} \text{ for every } \gamma \in \Lambda.$$

For $\mathcal{I} = \mathcal{I}_f$, $\mathcal{I}$-statistical convergence almost surely of complex uncertain sequences coincides with statistical convergence almost surely of complex uncertain sequences, which was studied by Tripathy and Nath[3].

Definition 2.15. *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-statistically convergent in measure to ζ if for every $\varepsilon, \delta, \upsilon > 0$ such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \le n : \mathcal{M}\left(\left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \ge \varepsilon\right) \ge \delta\right\}\right| \ge \upsilon\right\} \in \mathcal{I}.$$

For $\mathcal{I} = \mathcal{I}_f$, $\mathcal{I}$-statistical convergence in measure of complex uncertain sequences coincides with statistical convergence in measure of complex uncertain sequences, which was studied by Tripathy and Nath[3].

Throughout the paper, we consider $\mathcal{I}$ to be a non-trivial admissible ideal of $\mathbb{N}$.

3. Main Results

Definition 3.1. *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-statistically convergent in mean to ζ if for every $\delta, \upsilon > 0$ such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \le n : E\left[\left\|\zeta_k(\gamma) - \zeta(\gamma)\right\|\right] \ge \delta\right\}\right| \ge \upsilon\right\} \in \mathcal{I}.$$

For $\mathcal{I} = \mathcal{I}_f$, $\mathcal{I}$-statistical convergence in mean of complex uncertain sequences coincides with statistical convergence in mean of complex uncertain sequences, which was studied by Tripathy and Nath[3].

Definition 3.2. *Let $\Phi, \Phi_1, \Phi_2, ...$ be the complex uncertainty distributions of complex uncertain variables $\zeta, \zeta_1, \zeta_2, ...$ respectively. Then the complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-statistically convergent in distribution to ζ if for every $\delta, \upsilon > 0$ such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \le n : \|\Phi_k(z) - \Phi(z)\| \ge \delta\right\}\right| \ge \upsilon\right\} \in \mathcal{I}.$$

for all z at which $\Phi(z)$ is continuous.
For $\mathcal{I} = \mathcal{I}_f$, $\mathcal{I}$-statistical convergence in distribution of complex uncertain sequences coincides with statistical convergence in distribution of complex uncertain sequences, which was studied by Tripathy and Nath[3].

Definition 3.3. *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-statistically convergent uniformly almost surely to ζ if for every $\varepsilon, \delta, \vartheta > 0$ and a sequence (X_n) of events such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \le n : |\mathcal{M}(X_k)| \ge \vartheta\right\}\right| \ge \delta\right\} \in \mathcal{I}$$
$$\Rightarrow \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \le n : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \ge \varepsilon\right\}\right| \ge \delta\right\} \in \mathcal{I} \text{ for all } \gamma \in \Gamma \setminus X_n.$$

For $\mathcal{I} = \mathcal{I}_f$, $\mathcal{I}$-statistical convergence uniformly almost surely of complex uncertain sequences coincides with statistical convergence uniformly almost surely of complex uncertain sequences, which was studied by Tripathy and Nath[3].

Proposition 3.4. *Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. Then (ζ_n) is $\mathcal{I}$-statistically convergent almost surely to ζ if and only if for any $\varepsilon, \delta, \upsilon > 0$, we have*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcap_{k=1}^{\infty}\bigcup_{m=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_m(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Proof. *From the definition of $\mathcal{I}$-statistical convergence almost surely of complex uncertain sequence, there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I} \text{ for every } \varepsilon, \delta > 0.$$

Then for any $\varepsilon, \upsilon > 0$, there exists k such that $\left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| < \varepsilon$ where $n > k$ and for any $\gamma \in \Lambda$, that is equivalent to

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{k=1}^{\infty}\bigcap_{m=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_m(\gamma) - \zeta(\gamma)\right\| < \varepsilon\right\}\right) \geq 1\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

It follows from the duality axiom of an uncertain measure that

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcap_{k=1}^{\infty}\bigcup_{m=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_m(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}. \quad \square$$

Proposition 3.5. *Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. Then (ζ_n) is $\mathcal{I}$-statistically convergent uniformly almost surely to ζ if and only if for every $\varepsilon, \delta, \upsilon > 0$ such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_m(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Proof. *Let (ζ_n) be $\mathcal{I}$-statistically convergent uniformly almost surely to ζ, then for every $\varepsilon, \delta, \upsilon > 0$ and a sequence (X_n) of events such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : |\mathcal{M}(X_k)| \geq \delta\}\right| \geq \upsilon\right\} \in \mathcal{I}$$
$$\Rightarrow \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right| \geq \upsilon\right\} \in \mathcal{I} \text{ for all } \gamma \in \Gamma \setminus X_n.$$

Thus for any $\varepsilon > 0$, there exists $k > 0$ such that $\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| < \varepsilon$ where $n \geq k$ and for all $\gamma \in \Gamma \setminus X_n$.

That is $\bigcup_{n=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\} \subset X_n$.

Then from the subadditivity axiom of uncertain measure, we have

$$\mathcal{M}\left(\bigcup_{n=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \leq \mathcal{M}(X_n).$$

Then for every $\delta > 0$ and $m \in \mathbb{N}$, we have

$$\left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_m(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\} \subseteq \{k \leq n : \mathcal{M}(X_k) \geq \delta\}$$
$$\Rightarrow \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_m(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \leq \frac{1}{n}\left|\{k \leq n : \mathcal{M}(X_k) \geq \delta\}\right|.$$

For every $\upsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_m(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \geq \upsilon\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \mathcal{M}(X_k) \geq \delta\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

To prove converse part, let us take for any $\varepsilon, \delta, \upsilon > 0$ such that

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_m(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Then for given $\delta > 0$ and in particula if $\varepsilon = \frac{1}{p}(p \geq 1)$, there exists k_p such that

$\mathcal{D}(A(p)) < \frac{\delta}{2^p}$, where $A(p) = \left\{n \in \mathbb{N} : \mathcal{M}\left(\bigcup_{n=k_p}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \frac{1}{p}\right\}\right)\right\}$.

If we take $X_n = \bigcup_{p=1}^{\infty}\bigcup_{n=k_p}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \frac{1}{p}\right\}$. Then

$$\mathcal{M}(X_n) \leq \sum_{n=1}^{\infty} \mathcal{M}\left(\bigcup_{n=k_p}^{\infty} \left\{ \gamma \in \Gamma : \left\| \zeta_n(\gamma) - \zeta(\gamma) \right\| \geq \frac{1}{p} \right\} \right).$$

For every $\delta > 0$, *we have*

$$\mathcal{D}\{n \in \mathbb{N} : |\mathcal{M}(X_n)| \geq \delta\} \leq \sum_{n=1}^{\infty} \mathcal{D}\left\{ n \in \mathbb{N} : \mathcal{M}\left(\bigcup_{n=k_p}^{\infty} \left\{ \gamma \in \Gamma : \left\| \zeta_n(\gamma) - \zeta(\gamma) \right\| \geq \frac{1}{p} \right\} \right) \geq \delta \right\} < \delta.$$

Hence for every $\upsilon > 0$ *and* $m \in \mathbb{N}$, *we have*
$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |\mathcal{M}(X_k)| \geq \delta\}| \geq \upsilon \right\} \in \mathcal{I}$.
Furthermore, for every $\upsilon > 0$ *and* $n \geq k_p (p \geq 1)$, *we have*

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \left\| \zeta_k(\gamma) - \zeta(\gamma) \right\| \geq \frac{1}{p} \right\} \right| \geq \upsilon \right\} \in \mathcal{I} \text{ for all } \gamma \in \Gamma \setminus X_n. \quad \square$$

Theorem 3.6. *The complex uncertain sequence* (ζ_n) *where* $\zeta_n = \xi_n + i\eta_n$ *is* $\mathcal{I}$*-statistically convergent in measure to* $\zeta = \xi + i\eta$ *if and only if the uncertain sequence* (ξ_n) *and* (η_n) *are* $\mathcal{I}$*-statistically convergent in measure to* ξ *and* η*, respectively.*

Proof. *Let the uncertain sequence* (ξ_n) *and* (η_n) *are* $\mathcal{I}$*-statistically convergent in measure to* ξ *and* η*, respectively. Then from the definition of* $\mathcal{I}$*-statistical convergence in measure of uncertain sequences, it follows that for any* $\varepsilon, \delta, \upsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \mathcal{M}\left(|\xi_k - \xi| \geq \frac{\varepsilon}{\sqrt{2}} \right) \geq \frac{\delta}{2} \right\} \right| \geq \upsilon \right\} \in \mathcal{I}$$

and

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \mathcal{M}\left(\left|\eta_k - \eta\right| \geq \frac{\varepsilon}{\sqrt{2}} \right) \geq \frac{\delta}{2} \right\} \right| \geq \upsilon \right\} \in \mathcal{I}.$$

Note that $||\zeta_n - \zeta|| = \sqrt{|\xi_n - \xi|^2 + |\eta_n - \eta|^2}$.
Thus we have $\{||\zeta_n - \zeta|| \geq \varepsilon\} \subset \{|\xi_n - \xi| \geq \frac{\varepsilon}{\sqrt{2}}\} \cup \{|\eta_n - \eta| \geq \frac{\varepsilon}{\sqrt{2}}\}$
$\Rightarrow \mathcal{M}\{||\zeta_n - \zeta|| \geq \varepsilon\} \leq \mathcal{M}\{|\xi_n - \xi| \geq \frac{\varepsilon}{\sqrt{2}}\} + \mathcal{M}\{|\eta_n - \eta| \geq \frac{\varepsilon}{\sqrt{2}}\}$.
Then for every $\delta > 0$,

$$\{k \leq n : \mathcal{M}(||\zeta_k - \zeta|| \geq \varepsilon) \geq \delta\} \subseteq \left\{ k \leq n : \mathcal{M}\left(|\xi_k - \xi| \geq \frac{\varepsilon}{\sqrt{2}} \right) \geq \frac{\delta}{2} \right\} \cup \left\{ k \leq n : \mathcal{M}\left(\left|\eta_k - \eta\right| \geq \frac{\varepsilon}{\sqrt{2}} \right) \geq \frac{\delta}{2} \right\}$$

$$\Rightarrow \frac{1}{n} |\{k \leq n : \mathcal{M}(||\zeta_k - \zeta|| \geq \varepsilon) \geq \delta\}| \leq \frac{1}{n} \left| \left\{ k \leq n : \mathcal{M}\left(|\xi_k - \xi| \geq \frac{\varepsilon}{\sqrt{2}} \right) \geq \frac{\delta}{2} \right\} \right| + \frac{1}{n} \left| \left\{ k \leq n : \mathcal{M}\left(\left|\eta_k - \eta\right| \geq \frac{\varepsilon}{\sqrt{2}} \right) \geq \frac{\delta}{2} \right\} \right|.$$

For every $\upsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \mathcal{M}(||\zeta_k - \zeta|| \geq \varepsilon) \geq \delta\}| \geq \upsilon \right\} \subseteq \left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \mathcal{M}\left(|\xi_k - \xi| \geq \frac{\varepsilon}{\sqrt{2}} \right) \geq \frac{\delta}{2} \right\} \right| \geq \upsilon \right\} \cup \left\{ n \in \mathbb{N} : \frac{1}{n} \left| \left\{ k \leq n : \mathcal{M}\left(||\zeta_k - \zeta|| \geq \frac{\varepsilon}{2q} \right) \geq \frac{\delta}{2} \right\} \right| \geq \upsilon \right\} \in \mathcal{I}.$$

Thus (ζ_n) *is* $\mathcal{I}$*-statistically convergent in measure to* ζ.
Conversely, let the complex uncertain sequence (ζ_n) *is* $\mathcal{I}$*-statistically convergent in measure to* ζ*. Then from the definition of* $\mathcal{I}$*-statistically convergent in measure to* ζ *of complex uncertain sequences, it follows that for any* $\varepsilon, \delta, \upsilon > 0$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : \mathcal{M}(||\zeta_k - \zeta|| \geq \varepsilon) \geq \delta\}| \geq \upsilon \right\} \in \mathcal{I}.$$

Note that $|\xi_n - \xi| \leq \left|(\xi_n - \xi) + i(\eta_n - \eta)\right| = \left|(\xi_n + i\eta_n) - (\xi + i\eta)\right| = ||\zeta_n - \zeta||$.
Thus we have $\{|\xi_n - \xi| \geq \varepsilon\} \subseteq \{||\zeta_n - \zeta|| \geq \varepsilon\}$
$\Rightarrow \mathcal{M}\{|\xi_n - \xi| \geq \varepsilon\} \leq \mathcal{M}\{||\zeta_n - \zeta|| \geq \varepsilon\}$.
Then for every $\delta > 0$,
$\{k \leq n : \mathcal{M}(|\xi_k - \xi| \geq \varepsilon) \geq \delta\} \subseteq \{k \leq n : \mathcal{M}(||\zeta_n - \zeta|| \geq \varepsilon) \geq \delta\}$
$\Rightarrow \frac{1}{n} |\{k \leq n : \mathcal{M}(|\xi_k - \xi| \geq \varepsilon) \geq \delta\}| \leq \frac{1}{n} |\{k \leq n : \mathcal{M}(||\zeta_k - \zeta|| \geq \varepsilon) \geq \delta\}|$.

For every $\upsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : \mathcal{M}\left(|\xi_k - \xi| \geq \varepsilon\right) \geq \delta\}\right| \geq \upsilon\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : \mathcal{M}\left(\|\zeta_k - \zeta\| \geq \varepsilon\right) \geq \delta\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Similarly $\left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : \mathcal{M}\left(\left|\eta_k - \eta\right| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \upsilon\right\}$

$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : \mathcal{M}\left(\|\zeta_k - \zeta\| \geq \varepsilon\right) \geq \delta\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Hence (ξ_n) and (η_n) are $\mathcal{I}$-statistically convergent in measure to ξ and η, respectively. □

Theorem 3.7. *If a complex uncertain sequence (ζ_n) is $\mathcal{I}$-statistically convergent in mean to ζ, then it is $\mathcal{I}$-statistically convergent in measure to ζ.*

Proof. *Let the complex uncertain sequence (ζ_n) is $\mathcal{I}$-statistically convergent in mean to ζ. Then from the definition of $\mathcal{I}$-statistical convergence in mean of complex uncertain sequences, it follows that for every $\delta, \upsilon > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : E\left[\left\|\zeta_k(\gamma) - \zeta(\gamma)\right\|\right] \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Using Markov inequality we can see that for given $\varepsilon \geq 1$, we have

$$\mathcal{M}\{\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\} \leq \frac{E\left[\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\|\right]}{\varepsilon} \leq E\left[\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\|\right].$$

Then for every $\delta > 0$,

$$\left\{k \leq n : \mathcal{M}\left(\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right) \geq \delta\right\} \subseteq \left\{k \leq n : E\left[\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\|\right] \geq \delta\right\}$$
$$\Rightarrow \frac{1}{n} \left|\left\{k \leq n : \mathcal{M}\left(\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right) \geq \delta\right\}\right| \leq \frac{1}{n} \left|\left\{k \leq n : E\left[\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\|\right] \geq \delta\right\}\right|.$$

For every $\upsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : \mathcal{M}\left(\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \upsilon\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : E\left[\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\|\right] \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Hence (ζ_n) is $\mathcal{I}$-statistically convergent in measure to ζ and the theorem is proved. □

Remark 3.8. *But the converse of the above theorem is not true in general.*

Example 3.9. *Consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ and*

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{3}{(2n+1)}, & if \sup\limits_{\gamma_n \in \Lambda} \frac{3}{(2n+1)} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{3}{(2n+1)}, & if \sup\limits_{\gamma_n \in \Lambda^c} \frac{3}{(2n+1)} < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases} \quad for\ n = 1, 2, 3, \cdots$$

Also, the complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} ni, & if\ \gamma = \gamma_n \\ 0, & otherwise \end{cases} \quad for\ n = 1, 2, 3, \cdots$$

and $\zeta \equiv 0$. Take $\mathcal{I} = \mathcal{I}_d$.

For every $\varepsilon, \delta, \upsilon > 0$ and $n \geq 3$ we have,

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : \mathcal{M}\left(\left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\} \geq \delta\right\}\right| \geq \upsilon\right\}$$
$$= \left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : \mathcal{M}\left(\gamma : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \upsilon\right\}$$
$$= \left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : \mathcal{M}\{\gamma_k\} \geq \delta\right\}\right| \geq \upsilon\right\} = \left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \leq n : \frac{3}{2k+1} \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Thus the sequence (ζ_n) is $\mathcal{I}$-statistically convergent in measure to ζ.

However, for each $n \geq 3$, we have the complex uncertainty distributions of uncertain variable $\|\zeta_n - \zeta\|$ is

$$\Phi_n(r) = \begin{cases} 0, & if\ r < 0 \\ 1 - \frac{3}{2n+1}, & if\ 0 \leq r < n \\ 1, & if\ r \geq n \end{cases} \quad for\ n = 1, 2, 3, \cdots.$$

Now $E\left[\|\zeta_n - \zeta\|\right] = \int_0^{+\infty} (1 - \Phi_n(r))\, dr = \int_0^n \frac{3}{2n+1} dr = \frac{3n}{2n+1}$.
Then for every $\varepsilon, \delta > 0$ *and for each* $n \geq 3$,
$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : E\left[\left\|\zeta_k(\gamma) - \zeta(\gamma)\right\|\right] \geq \varepsilon\right\}\right| \geq \delta\right\} = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \frac{3k}{2k+1} \geq \varepsilon\right\}\right| \geq \delta\right\} \notin \mathcal{I}$.
Hence the sequence (ζ_n) *is nt* $\mathcal{I}$*-statistically convergent in mean to* ζ.

Theorem 3.10. *Assume that a complex uncertain sequence* (ζ_n) *with real part* (ξ_n) *and imaginary part* (η_n) *are* $\mathcal{I}$*-statistically convergent in measure to* ξ *and* η*, respectively. Then the complex uncertain sequence* (ζ_n) *is* $\mathcal{I}$*-statistically convergent in distribution to* $\zeta = \xi + i\eta$.

Proof. Let $z = x + iy$ be a given continuity point of the complex uncertainty distribution Φ. On the other hand, for any $\alpha > x, \beta > y$, we have

$$\begin{aligned}\{\xi_n \leq x, \eta_n \leq y\} &= \{\xi_n \leq x, \eta_n \leq y, \xi \leq \alpha, \eta \leq \beta\} \cup \{\xi_n \leq x, \eta_n \leq y, \xi > \alpha, \eta > \beta\} \\ &\cup \{\xi_n \leq x, \eta_n \leq y, \xi \leq \alpha, \eta > \beta\} \cup \{\xi_n \leq x, \eta_n \leq y, \xi > \alpha, \eta \leq \beta\} \\ &\subset \{\xi \leq \alpha, \eta \leq \beta\} \cup \{|\xi_n - \xi| \geq \alpha - x\} \cup \{\left|\eta_n - \eta\right| \geq \beta - y\}.\end{aligned}$$

It follows from the subadditivity axiom that

$$\Phi_n(z) = \Phi_n(x + iy) \leq \Phi(\alpha + i\beta) + \mathcal{M}\{|\xi_n - \xi| \geq \alpha - x\} + \mathcal{M}\{\left|\eta_n - \eta\right| \geq \beta - y\}.$$

Since (ξ_n) and (η_n) are $\mathcal{I}$-statistically convergent in measure to ξ and η, respectively. So for any $\varepsilon, \delta, \upsilon > 0$ we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \mathcal{M}\left(|\xi_k - \xi| \geq \varepsilon\right) \geq \delta\}\right| \geq \upsilon\right\} \in \mathcal{I}$$
$$\text{and } \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\left|\eta_k - \eta\right| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Thus we obtain $\mathcal{I}$-S-$\limsup\limits_{n\to\infty} \Phi_n(z) \leq \Phi(\alpha + i\beta)$ for any $\alpha > x, \beta > y$. Letting $\alpha + i\beta \to x + iy$, we get

$$\mathcal{I} - S - \limsup_{n\to\infty} \Phi_n(z) \leq \Phi(z). \tag{1}$$

On the other hand, for any $\gamma < x, \kappa < y$ we have,

$$\begin{aligned}\{\xi \leq \gamma, \eta \leq \kappa\} &= \{\xi_n \leq x, \eta_n \leq y, \xi \leq \gamma, \eta \leq \kappa\} \cup \{\xi_n > x, \eta_n > y, \xi \leq \gamma, \eta \leq \kappa\} \\ &\cup \{\xi_n > x, \eta_n \leq y, \xi \leq \gamma, \eta \leq \kappa\} \cup \{\xi_n \leq x, \eta_n > y, \xi \leq \gamma, \eta \leq \kappa\} \\ &\subset \{\xi_n \leq x, \eta_n \leq y\} \cup \{|\xi_n - \xi| \geq x - \gamma\} \cup \{\left|\eta_n - \eta\right| \geq y - \kappa\}.\end{aligned}$$

It follows from the subadditivity axiom that
$\Phi(\gamma + i\kappa) \leq \Phi_n(x + iy) + \mathcal{M}\{|\xi_n - \xi| \geq x - \gamma\} + \mathcal{M}\{\left|\eta_n - \eta\right| \geq y - \kappa\}$.
Since (ξ_n) and (η_n) are $\mathcal{I}$-statistically convergent in measure to ξ and η, respectively. So for any $\varepsilon, \delta, \upsilon > 0$ we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \mathcal{M}\left(|\xi_k - \xi| \geq \varepsilon\right) \geq \delta\}\right| \geq \upsilon\right\} \in \mathcal{I}$$
$$\text{and } \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\left|\eta_k - \eta\right| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Thus we obtain $\Phi(\gamma + i\kappa) \leq \mathcal{I}$-S-$\liminf\limits_{n\to\infty} \Phi_n(x + iy)$ for any $\gamma < x, \kappa < y$. Letting $\gamma + i\kappa \to x + iy$, we get

$$\Phi(z) \leq \mathcal{I} - S - \liminf_{n\to\infty} \Phi_n(z). \tag{2}$$

It follows from (1) and (2), that the complex uncertain sequence (ζ_n) is $\mathcal{I}$-statistically convergent in distribution to ζ. □

Remark 3.11. *But the converse of the above theorem is not true in general.*

Example 3.12. *Consider the uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to be* $\{\gamma_1, \gamma_2\}$ *with* $\mathcal{M}(\gamma_1) = \mathcal{M}(\gamma_1) = \frac{1}{2}$. *We define a complex uncertain variable as*

$$\zeta(\gamma) = \begin{cases} i, & if\ \gamma = \gamma_1, \\ -i, & if\ \gamma = \gamma_2. \end{cases}$$

We also define $\zeta_n = -\zeta$ *for* $n = 1, 2, \cdots$ *and take* $\mathcal{I}=\mathcal{I}_d$.
Then the sequence (ζ_n) *and* ζ *have the same distribution as:*

$$\Phi_n(z) = \Phi_n(x+iy) = \begin{cases} 0, & if\ x < 0, -\infty < y < +\infty, \\ 0, & if\ x \geq 0, y < -1, \\ \frac{1}{2}, & if\ x \geq 0, -1 \leq y < 1, \\ 1, & if\ x \geq 0, y \geq 1. \end{cases}$$

So the sequence (ζ_n) *is* $\mathcal{I}$*-statistically convergent in distribution to* ζ.
However, for a given $\varepsilon, \delta, \upsilon > 0$, *we have* $\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \upsilon\right\} \notin \mathcal{I}$.
Thus the sequence (ζ_n) *is not* $\mathcal{I}$*-statistically convergent in measure to* ζ.

Theorem 3.13. *Let* $\zeta, \zeta_1, \zeta_2, \ldots$ *be complex uncertain variables defined on uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$. *If* (ζ_n) *is* $\mathcal{I}$*-statistically convergent uniformly almost surely to* ζ, *then it is* $\mathcal{I}$*-statistically convergent almost surely to* ζ.

Proof. *Let the complex uncertain sequence* (ζ_n) *be* $\mathcal{I}$*-statistically convergent uniformly almost surely to* ζ, *then for every* $\varepsilon, \delta, \upsilon > 0$, *we have*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_m(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Since $\mathcal{M}\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\right) \leq \mathcal{M}\left(\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\right)$, *so for every* $\delta > 0$,

$$\left\{k \leq n : \mathcal{M}\left(\bigcap_{k=1}^{\infty}\bigcup_{m=k}^{\infty}\{\|\zeta_m - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\} \subseteq \left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\{\|\zeta_m - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}$$

$$\Rightarrow \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcap_{k=1}^{\infty}\bigcup_{m=k}^{\infty}\{\|\zeta_m - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}\right| \leq \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\{\|\zeta_m - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}\right|.$$

For every $\upsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcap_{k=1}^{\infty}\bigcup_{m=k}^{\infty}\{\|\zeta_m - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}\right| \geq \upsilon\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\{\|\zeta_m - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Hence (ζ_n) *is* $\mathcal{I}$*-statistically convergent almost surely to* ζ. □

Remark 3.14. *But the converse of the above theorem is not true in general.*

Example 3.15. *Consider the uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to be* $\{\gamma_1, \gamma_2, \ldots\}$ *with power set and* $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ *and*

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup_{\gamma_n \in \Lambda} \frac{n\beta_n}{2n+1}, & if\ \sup_{\gamma_n \in \Lambda} \frac{n\beta_n}{2n+1} < \frac{1}{2} \\ 1 - \sup_{\gamma_n \in \Lambda^c} \frac{n\beta_n}{2n+1}, & if\ \sup_{\gamma_n \in \Lambda^c} \frac{n\beta_n}{2n+1} < \frac{1}{2} \\ \frac{1}{2}, & otherwise, \end{cases}$$

where $\beta_n = \begin{cases} 1, & if\ n\ is\ odd \\ 0, & if\ n\ is\ even \end{cases}$ $for\ n = 1, 2, 3, \cdots$.

Also, the complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, \ if\ \gamma = \gamma_n \\ 0, \quad otherwise \end{cases} \qquad for\ n = 1, 2, 3, \cdots$$

and $\zeta \equiv 0$. *Take* $\mathcal{I}=\mathcal{I}_d$.
Now for every $\varepsilon > 0$, *we have*

$$\mathcal{M}\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) = \mathcal{M}\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\{\gamma_n\}\right).$$

Then for any $\delta, \upsilon > 0$, *we have*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcap_{k=1}^{\infty}\bigcup_{m=k}^{\infty}\{\|\zeta_m - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}\right| \geq \upsilon\right\}$$
$$= \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcap_{k=1}^{\infty}\bigcup_{m=k}^{\infty}\{\gamma_m\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Again for every $\varepsilon > 0$, *we have*

$$\mathcal{M}\left(\bigcup_{n=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) = \mathcal{M}\left(\bigcup_{n=k}^{\infty}\{\gamma_n\}\right).$$

Then for every $\delta, \upsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_m(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \geq \upsilon\right\}$$
$$= \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\{\gamma_m\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \notin \mathcal{I}.$$

Hence the sequence (ζ_n) *is* $\mathcal{I}$*-statistically convergent almost surely to* $\zeta \equiv 0$ *but it is not* $\mathcal{I}$*-statistically convergent uniformly almost surely to* $\zeta \equiv 0$.

Theorem 3.16. *Let* $\zeta, \zeta_1, \zeta_2, ...$ *be complex uncertain variables defined on the same continuous uncertainty space. If* (ζ_n) *is* $\mathcal{I}$*-statistically convergent almost surely to* ζ*, then it is* $\mathcal{I}$*-statistically convergent uniformly almost surely to* ζ.

Proof. Let the complex uncertain sequence (ζ_n) be $\mathcal{I}$-statistically convergent almost surely to ζ, then for every $\varepsilon, \delta, \upsilon > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcap_{k=1}^{\infty}\bigcup_{m=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_m(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Note that $\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\} = \lim_{k\to\infty}\bigcup_{n=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}$.
Therefore

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\lim_{k\to\infty}\bigcup_{n=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Again $\mathcal{M}\left(\lim_{k\to\infty}\bigcup_{n=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta$

$\Rightarrow \lim_{k\to\infty}\mathcal{M}\left(\bigcup_{n=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta$, since uncertainty space is continuous

$\Rightarrow \mathcal{M}\left(\bigcup_{n=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta + \psi(n)$, where $\psi(k) \to 0$ as $k \to \infty$

$\Rightarrow \mathcal{M}\left(\bigcup_{n=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta$.

Thus $\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_m(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}$.
Hence the sequence (ζ_n) is $\mathcal{I}$-statistically convergent uniformly almost surely to ζ. □

Theorem 3.17. *Let* $\zeta, \zeta_1, \zeta_2, ...$ *be complex uncertain variables. If* (ζ_n) *is* $\mathcal{I}$*-statistically convergent uniformly almost surely to* ζ*, then it is* $\mathcal{I}$*-statistically convergent in measure to* ζ.

Proof. *Let the complex uncertain sequence* (ζ_n) *be* $\mathcal{I}$*-statistically convergent uniformly almost surely to* ζ*, then for every* $\varepsilon, \delta, \upsilon > 0$*, we have*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{k=n}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Since, $\mathcal{M}\left\{\gamma \in \Gamma : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\} \leq \mathcal{M}\left(\bigcup_{k=n}^{\infty}\{\gamma \in \Gamma : \|\zeta_k - \zeta\| \geq \varepsilon\}\right)$ *so for every* $\delta > 0$*,*

$$\left\{k \leq n : \mathcal{M}\left\{\gamma \in \Gamma : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\} \geq \delta\right\} \subseteq \left\{k \leq n : \mathcal{M}\left(\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}$$

$$\Rightarrow \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left\{\gamma \in \Gamma : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\} \geq \delta\right\}\right| \leq \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}\right|.$$

For every $\upsilon > 0$*,*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left\{\gamma \in \Gamma : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\} \geq \delta\right\}\right| \geq \upsilon\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{n=k}^{\infty}\{\|\zeta_n - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Hence the sequence (ζ_n) *is* $\mathcal{I}$*-statistically convergent in measure to* ζ*.* □

Remark 3.18. *But the converse of the above theorem is not true in general.*

Example 3.19. *Consider the uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to be* $\{\gamma_1, \gamma_2, ...\}$ *with power set and* $\mathcal{M}\{\Gamma\} = 1$*,* $\mathcal{M}\{\phi\} = 0$ *and*

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \frac{n\beta_n}{2n+1}, & if \sup\limits_{\gamma_n \in \Lambda} \frac{n\beta_n}{2n+1} < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \frac{n\beta_n}{2n+1}, & if \sup\limits_{\gamma_n \in \Lambda^c} \frac{n\beta_n}{2n+1} < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases}$$

where $\beta_n = \begin{cases} 1, & if\ n = k^2, k \in \mathbb{N} \\ 0, & otherwise \end{cases} \quad for\ n = 1, 2, 3, \cdots.$

Also, the complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, & if\ \gamma = \gamma_n \\ 0, & otherwise \end{cases} \quad for\ n = 1, 2, 3, \cdots$$

and $\zeta \equiv 0$*. Take* $\mathcal{I} = \mathcal{I}_d$*.*

It can be shown that the sequence (ζ_n) *is* $\mathcal{I}$*-statistically convergent in measure to* $\zeta \equiv 0$ *but it is not* $\mathcal{I}$*-statistically convergent uniformly almost surely to* $\zeta \equiv 0$*.*

Theorem 3.20. *Suppose* (ζ_n) *where* $\zeta_n = \xi_n + i\eta_n$ *be a complex uncertain variable sequence and* ζ *where* $\zeta = \xi + i\eta$ *be a complex uncertain variable such that* $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_n \geq \cdots \geq \zeta$ *in the sense that* $\xi_k \geq \xi_n \geq \xi$ *and* $\eta_k \geq \eta_n \geq \eta$ *for* $n \geq k$*. Then* (ζ_n) *is* $\mathcal{I}$*-statistically convergent uniformly almost surely to* ζ *if it is* $\mathcal{I}$*-statistically convergent in measure to* ζ*.*

Proof. *Let* $\xi_k \geq \xi_n \geq \xi$ *and* $\eta_k \geq \eta_n \geq \eta$ *for* $n \geq k$*, then*
$\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \leq \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\|$ *for* $n \geq k$*.*
Now for every $\varepsilon > 0$*, we have*
$\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\} \subseteq \left\{\gamma \in \Gamma : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}.$
Therefore $\bigcup_{n=k}^{\infty}\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\} = \left\{\gamma \in \Gamma : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}.$
Since the complex uncertain sequence (ζ_n) *is* $\mathcal{I}$*-statistically convergent in measure to* ζ*, then for every* $\varepsilon, \delta, \upsilon > 0$ *we have*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}$$

$$\Rightarrow \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\bigcup_{m=k}^{\infty}\{\gamma \in \Gamma : \|\zeta_m - \zeta\| \geq \varepsilon\}\right) \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Hence the sequence (ζ_n) is $\mathcal{I}$-statistically convergent uniformly almost surely to ζ. □

Theorem 3.21. *Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables. If (ζ_n) is $\mathcal{I}$-statistically convergent uniformly almost surely to ζ, then it is $\mathcal{I}$-statistically convergent in distribution to ζ.*

Proof. Let (ζ_n) be $\mathcal{I}$-statistically convergent uniformly almost surely to ζ, then it is $\mathcal{I}$-statistically convergent in distribution to ζ by theorem 3.17, 3.6 and 3.10. □

Remark 3.22. *But the converse of the above theorem is not true in general.*

Example 3.23. *In example 3.19, the complex uncertainty distributions of (ζ_n) are*

$$\Phi_n(z) = \Phi_n(x+iy) = \begin{cases} 0, & \text{if } x < 0, y < \infty \\ 0, & \text{if } x \geq 0, y < 0 \\ 1 - \frac{n\beta_n}{2n+1}, & \text{if } x \geq 0, 0 \leq y < (n+1) \\ 1, & \text{if } x \geq 0, y \geq (n+1) \end{cases} \quad \text{for } n = 1, 2, 3, \cdots$$

and the complex uncertainty distributions of ζ is

$$\Phi(z) = \Phi(x+iy) = \begin{cases} 0, & \text{if } x < 0, y < \infty \\ 0, & \text{if } x \geq 0, y < 0 \\ 1, & \text{if } x \geq 0, y \geq 0. \end{cases}$$

Then for every $\varepsilon, \delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{k \leq n : \|\Phi_k(z) - \Phi(z)\| \geq \varepsilon\}\right| \geq \delta\right\} = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \frac{n\beta_n}{2n+1} \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I} \text{ for } x \geq 0, y \geq 0.$$

Thus the sequence (ζ_n) is $\mathcal{I}$-statistically convergent in distribution to $\zeta \equiv 0$ but it is not $\mathcal{I}$-statistically convergent uniformly almost surely to $\zeta \equiv 0$.

Theorem 3.24. *Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables defined on the same continuous uncertainty space. If (ζ_n) is $\mathcal{I}$-statistically convergent almost surely to ζ, then it is $\mathcal{I}$-statistically convergent in measure to ζ.*

Proof. Let (ζ_n) be $\mathcal{I}$-statistically convergent almost surely to ζ, then it is $\mathcal{I}$-statistically convergent in measure to ζ by theorem 3.16 and 3.17. □

Theorem 3.25. *Suppose (ζ_n) where $\zeta_n = \xi_n + i\eta_n$ be a complex uncertain variable sequence and ζ where $\zeta = \xi + i\eta$ be a complex uncertain variable such that $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_n \geq \cdots \geq \zeta$ in the sense that $\xi_k \geq \xi_n \geq \xi$ and $\eta_k \geq \eta_n \geq \eta$ for $n \geq k$. Then (ζ_n) is $\mathcal{I}$-statistically convergent almost surely to ζ if it is $\mathcal{I}$-statistically convergent in measure to ζ.*

Proof. Let (ζ_n) be $\mathcal{I}$-statistically convergent in measure to ζ, then it is $\mathcal{I}$-statistically convergent almost surely to ζ by theorem 3.20 and 3.13. □

Theorem 3.26. *Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables defined on the same continuous uncertainty space. If (ζ_n) is $\mathcal{I}$-statistically convergent almost surely to ζ, then it is $\mathcal{I}$-statistically convergent in distribution to ζ.*

Proof. Let (ζ_n) be $\mathcal{I}$-statistically convergent almost surely to ζ, then it is $\mathcal{I}$-statistically convergent in distribution to ζ by theorem 3.24 and 3.10. □

Definition 3.27. *A complex uncertain sequence (ζ_n) is said to be $\mathcal{I}$-statistically convergent in p-distance to ζ if for every $\delta, \upsilon > 0$ such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : (E\left[\|\zeta_k - \zeta\|^p\right])^{\frac{1}{p+1}} \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

For $\mathcal{I} = \mathcal{I}_f$, $\mathcal{I}$-statistical convergence in p-distance of complex uncertain sequences coincide with statistical convergence in p-distance of complex uncertain sequences, which was studied by Saha et al. [26].

Theorem 3.28. *Let $\zeta, \zeta_1, \zeta_2, \ldots$ be complex uncertain variables defined on uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. If (ζ_n) is $\mathcal{I}$-statistically convergent in p-distance to ζ, then it is $\mathcal{I}$-statistically convergent in measure to ζ.*

Proof. Let the complex uncertain sequence (ζ_n) be $\mathcal{I}$-statistically convergent in p-distance to ζ, then for every $\delta, \upsilon > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : (E\left[\|\zeta_k - \zeta\|^p\right])^{\frac{1}{p+1}} \geq \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Then for any given $\varepsilon, p > 0$, we have

$\mathcal{M}\left(\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right) \leq \dfrac{E\left[\|\zeta_n - \zeta\|^p\right]}{\varepsilon^p}$ (Using Markov Inequality).

So for every $\delta > 0$,

$$\left\{k \leq n : \mathcal{M}\left(\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\} \geq \delta\right\} \subseteq \left\{k \leq n : \frac{E\left[\|\zeta_n - \zeta\|^p\right]}{\varepsilon^p} \geq \delta\right\}$$

$$= \left\{k \leq n : (E\left[\|\zeta_n - \zeta\|^p\right])^{\frac{1}{p+1}} \geq (\delta \cdot \varepsilon^p)^{\frac{1}{p+1}}\right\}$$

$$= \left\{k \leq n : (E\left[\|\zeta_n - \zeta\|^p\right])^{\frac{1}{p+1}} \geq \delta'\right\}, \text{ where } \delta' = (\varepsilon \cdot \delta^p)^{\frac{1}{p+1}}$$

$$\Rightarrow \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\} \geq \delta\right\}\right| \leq \frac{1}{n}\left|\left\{k \leq n : (E\left[\|\zeta_n - \zeta\|^p\right])^{\frac{1}{p+1}} \geq \delta'\right\}\right|.$$

For every $\upsilon > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\} \geq \delta\right\}\right| \geq \upsilon\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : (E\left[\|\zeta_n - \zeta\|^p\right])^{\frac{1}{p+1}} \geq \delta'\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Hence the sequence (ζ_n) is $\mathcal{I}$-statistically convergent in measure to ζ. □

Remark 3.29. *But the converse of the above theorem is not true in general.*

Example 3.30. *Let $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$, where $D_j = \{2^{j-1}j^* : 2$ does not divide $j^*, j^* \in \mathbb{N}\}$ be the decomposition of $\mathbb{N}$ such that each D_j is infinite and $D_j \cap D_{j^*} = \phi$, for $j \neq j^*$. Let $\mathcal{I}$ be the class of all subsets of $\mathbb{N}$ that can intersect only finite number of D_j's. Then $\mathcal{I}$ is a nontrivial admissible ideal of $\mathbb{N}$ [22].*

Now we consider the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \ldots\}$ with power set and $\mathcal{M}\{\Gamma\} = 1$, $\mathcal{M}\{\phi\} = 0$ and

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_n \in \Lambda} \beta_n, & if \sup\limits_{\gamma_n \in \Lambda} \beta_n < \frac{1}{2} \\ 1 - \sup\limits_{\gamma_n \in \Lambda^c} \beta_n, & if \sup\limits_{\gamma_n \in \Lambda^c} \beta_n < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases}$$

where $\beta_n = \frac{1}{j+1}$, $if\ n \in D_j\ for\ n = 1, 2, 3, \cdots$.

Also, the complex uncertain variables are defined by

$$\zeta_n(\gamma) = \begin{cases} (n+1)i, & if\ \gamma = \gamma_n \\ 0, & otherwise \end{cases} \quad for\ n = 1, 2, 3, \cdots$$

and $\zeta \equiv 0$.

For any $\varepsilon > 0$ and $n \in \mathbb{N} \setminus D_1$, we have

$\mathcal{M}\left(\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) = \mathcal{M}(\gamma_n) = \beta_n.$

Then for every $\delta > 0$, we have

$\left\{n \in \mathbb{N} : \mathcal{M}\left(\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\} = \{n \in \mathbb{N} : \beta_n \geq \delta\} \in \mathcal{I}.$

Now $\left\{k \leq n : \mathcal{M}\left(\left\{\gamma \in \Gamma : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\} \subseteq \left\{n \in \mathbb{N} : \mathcal{M}\left(\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}$

$$\Rightarrow \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\left\{\gamma \in \Gamma : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \leq \left|\left\{n \in \mathbb{N} : \mathcal{M}\left(\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right|.$$

Then for every $\upsilon > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \mathcal{M}\left(\left\{\gamma \in \Gamma : \left\|\zeta_k(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\}\right| \geq \upsilon\right\}$$
$$\subseteq \left\{n \in \mathbb{N} : \mathcal{M}\left(\left\{\gamma \in \Gamma : \left\|\zeta_n(\gamma) - \zeta(\gamma)\right\| \geq \varepsilon\right\}\right) \geq \delta\right\} \in \mathcal{I}.$$

Therefore the sequence (ζ_n) is $\mathcal{I}$-statistically convergent in measure to $\zeta \equiv 0$.
Now for $p > 0$, we have

$$\left\|\zeta_n(\gamma) - \zeta(\gamma)\right\|^p = \begin{cases} (n+1)^p, \ if\ \gamma = \gamma_n \\ 0, \ \ otherwise \end{cases} \quad for\ n = 1, 2, 3, \cdots.$$

Then for each $n \in \mathbb{N} \setminus D_1$, we have the uncertainty distribution of uncertain variable $\|\zeta_n - \zeta\|^p$ is

$$\Phi_n(r) = \begin{cases} 0, & if\ r < 0 \\ 1 - \beta_n, & if\ 0 \le r < (n+1)^p \\ 1, & if\ r \ge (n+1)^p \end{cases} \quad for\ n = 1, 2, 3, \cdots.\ and\ p > 0.$$

So for $n \in \mathbb{N} \setminus D_1$, we have

$$E\left[\|\zeta_n - \zeta\|^p\right] = \int_0^{(n+1)^p} (1 - (1 - \beta_n))\, dr = (n+1)^p \beta_n \Rightarrow (E\left[\|\zeta_n - \zeta\|^p\right])^{\frac{1}{p+1}} = ((n+1)^p \beta_n)^{\frac{1}{p+1}}.$$

Then for any $\delta, \upsilon > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \le n : (E\left[\|\zeta_k - \zeta\|^p\right])^{\frac{1}{p+1}} \ge \delta\right\}\right| \ge \upsilon\right\} = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \le n : ((k+1)^p \beta_k)^{\frac{1}{p+1}} \ge \delta\right\}\right| \ge \upsilon\right\} \notin \mathcal{I}.$$

Hence the sequence (ζ_n) is not $\mathcal{I}$-statistically convergent in p-distance to $\zeta \equiv 0$.

Theorem 3.31. *Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables defined on uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. If (ζ_n) is $\mathcal{I}$-statistically convergent in p-distance to ζ, then it is $\mathcal{I}$-statistically convergent in distribution to ζ.*

Proof. *Let (ζ_n) be $\mathcal{I}$-statistically convergent in p-distance to ζ, then it is $\mathcal{I}$-statistically convergent in distribution to ζ by theorem 3.28 and 3.10.* □

Remark 3.32. *But the converse of the above theorem is not true in general.*

Example 3.33. *In example 3.30, the complex uncertainty distributions of (ζ_n) are*

$$\Phi_n(z) = \Phi_n(x+iy) = \begin{cases} 0, & if\ x < 0, y < \infty \\ 0, & if\ x \ge 0, y < 0 \\ 1 - \beta_n, & if\ x \ge 0, 0 \le y < (n+1) \\ 1, & if\ x \ge 0, y \ge (n+1) \end{cases} \quad for\ n = 1, 2, 3, \cdots$$

and the complex uncertainty distributions of ζ is

$$\Phi(z) = \Phi(x+iy) = \begin{cases} 0, & if\ x < 0, y < \infty \\ 0, & if\ x \ge 0, y < 0 \\ 1, & if\ x \ge 0, y \ge 0. \end{cases}$$

It can be shown that the complex uncertain sequence (ζ_n) is $\mathcal{I}$-convergent in distribution to $\zeta \equiv 0$ but it is not $\mathcal{I}$-convergent in p-distance to $\zeta \equiv 0$.

Definition 3.34. *A complex uncertain sequence (ζ_n) is said to be completely $\mathcal{I}$-statistically convergent to ζ if for every $\varepsilon, \delta, \upsilon > 0$ such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \le n : \sum_{m=k}^{\infty} \mathcal{M}\left(\{\gamma \in \Gamma : \|\zeta_m - \zeta\| \ge \varepsilon\}\right) \ge \delta\right\}\right| \ge \upsilon\right\} \in \mathcal{I}.$$

For $\mathcal{I} = \mathcal{I}_f$, completely $\mathcal{I}$-statistical convergence of complex uncertain sequences coincide with completely statistical convergence of complex uncertain sequences, which was studied by Saha et al.[26].

Theorem 3.35. *Let $\zeta, \zeta_1, \zeta_2, ...$ be complex uncertain variables defined on uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. If (ζ_n) is completely $\mathcal{I}$-statistically convergent to ζ, then it is $\mathcal{I}$-statistically convergent uniformly almost surely to ζ.*

Proof. *Let the complex uncertain sequence (ζ_n) be completely $\mathcal{I}$-statistically convergent to ζ, then for every $\varepsilon, \delta, \upsilon > 0$ we have*

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \le n : \sum_{m=k}^{\infty} \mathcal{M}\left(\{\gamma \in \Gamma : \|\zeta_m - \zeta\| \ge \varepsilon\}\right) \ge \delta\right\}\right| \ge \upsilon\right\} \in \mathcal{I}.$$

It follows from Axiom 3 that,

$$\mathcal{M}\left(\bigcup_{n=k}^{\infty}\{\gamma\in\Gamma:\|\zeta_n-\zeta\|\geq\varepsilon\}\right)\leq\sum_{n=k}^{\infty}\mathcal{M}\left(\{\gamma\in\Gamma:\|\zeta_n-\zeta\|\geq\varepsilon\}\right).$$

So for every $\delta > 0$,

$$\left\{k\leq n:\mathcal{M}\left(\bigcup_{m=k}^{\infty}\{\gamma\in\Gamma:\|\zeta_m-\zeta\|\geq\varepsilon\}\right)\geq\delta\right\}\subseteq\left\{k\leq n:\sum_{m=k}^{\infty}\mathcal{M}\left(\{\gamma\in\Gamma:\|\zeta_m-\zeta\|\geq\varepsilon\}\right)\geq\delta\right\}$$

$$\Rightarrow\frac{1}{n}\left|\left\{k\leq n:\mathcal{M}\left(\bigcup_{m=k}^{\infty}\{\gamma\in\Gamma:\|\zeta_m-\zeta\|\geq\varepsilon\}\right)\geq\delta\right\}\right|\leq\frac{1}{n}\left|\left\{k\leq n:\sum_{m=k}^{\infty}\mathcal{M}\left(\{\gamma\in\Gamma:\|\zeta_m-\zeta\|\geq\varepsilon\}\right)\geq\delta\right\}\right|.$$

Then for every $\upsilon > 0$,

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\mathcal{M}\left(\bigcup_{m=k}^{\infty}\{\gamma\in\Gamma:\|\zeta_m-\zeta\|\geq\varepsilon\}\right)\geq\delta\right\}\right|\geq\upsilon\right\}$$
$$\subseteq\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\sum_{m=k}^{\infty}\mathcal{M}\left(\{\gamma\in\Gamma:\|\zeta_m-\zeta\|\geq\varepsilon\}\right)\geq\delta\right\}\right|\geq\upsilon\right\}\in\mathcal{I}.$$

Hence the sequence (ζ_n) *is* $\mathcal{I}$*-statistically convergent uniformly almost surely to* ζ. □

Theorem 3.36. *Let* $\zeta, \zeta_1, \zeta_2, \ldots$ *be complex uncertain variables defined on uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$. *If* (ζ_n) *is completely* $\mathcal{I}$*-statistically convergent to* ζ, *then it is* $\mathcal{I}$*-statistically convergent almost surely to* ζ.

Proof. *Let* (ζ_n) *be completely* $\mathcal{I}$*-statistically convergent to* ζ, *then it is* $\mathcal{I}$*-statistically convergent almost surely to* ζ *by theorem 3.35 and 3.13.* □

Remark 3.37. *But the converse of the above theorem is not true in general.*

Example 3.38. *From example 3.15, we see that the complex uncertain sequence* (ζ_n) *is* $\mathcal{I}$*-statistically convergent almost surely to* $\zeta \equiv 0$.

Now $\sum_{n=k}^{\infty}\mathcal{M}\left(\{\gamma\in\Gamma:\|\zeta_n-\zeta\|\geq\varepsilon\}\right)=\sum_{n=k}^{\infty}\mathcal{M}\{\gamma_n\}=\sum_{n=k}^{\infty}\frac{n\beta_n}{2n+1}$.

Then for every $\delta, \upsilon > 0$,

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\sum_{m=k}^{\infty}\mathcal{M}\left(\{\gamma\in\Gamma:\|\zeta_m-\zeta\|\geq\varepsilon\}\right)\geq\delta\right\}\right|\geq\upsilon\right\}=\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\sum_{m=k}^{\infty}\frac{m\beta_m}{2m+1}\geq\delta\right\}\right|\geq\upsilon\right\}\notin\mathcal{I}.$$

Hence the sequence (ζ_n) *is not completely* $\mathcal{I}$*-statistically convergent to* $\zeta \equiv 0$.

Theorem 3.39. *Let* $\zeta, \zeta_1, \zeta_2, \ldots$ *be complex uncertain variables defined on uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$. *If* (ζ_n) *is completely* $\mathcal{I}$*-statistically convergent to* ζ, *then it is* $\mathcal{I}$*-statistically convergent in measure to* ζ.

Proof. *Let* (ζ_n) *be completely* $\mathcal{I}$*-statistically convergent to* ζ, *then it is* $\mathcal{I}$*-statistically convergent in measure to* ζ *by theorem 3.35 and 3.17.* □

Remark 3.40. *But the converse of the above theorem is not true in general.*

Example 3.41. *From example 3.19, we see that the complex uncertain sequence* (ζ_n) *is* $\mathcal{I}$*-convergent in measure to* $\zeta \equiv 0$.

Now $\sum_{n=k}^{\infty}\mathcal{M}\left(\{\gamma\in\Gamma:\|\zeta_n-\zeta\|\geq\delta\}\right)=\sum_{n=k}^{\infty}\mathcal{M}\{\gamma_n\}=\sum_{n=k}^{\infty}\frac{n\beta_n}{2n+1}$.

Then for every $\delta, \upsilon > 0$,

$$\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\sum_{m=k}^{\infty}\mathcal{M}\left(\{\gamma\in\Gamma:\|\zeta_m-\zeta\|\geq\varepsilon\}\right)\geq\delta\right\}\right|\geq\upsilon\right\}=\left\{n\in\mathbb{N}:\frac{1}{n}\left|\left\{k\leq n:\sum_{m=k}^{\infty}\frac{m\beta_m}{2m+1}\geq\delta\right\}\right|\geq\upsilon\right\}\notin\mathcal{I}.$$

Hence the sequence (ζ_n) *is not completely* $\mathcal{I}$*-statistically convergent to* $\zeta \equiv 0$.

Theorem 3.42. *Let* $\zeta, \zeta_1, \zeta_2, \ldots$ *be complex uncertain variables defined on uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$. *If* (ζ_n) *is completely* $\mathcal{I}$*-statistically convergent to* ζ, *then it is* $\mathcal{I}$*-statistically convergent in distribution to* ζ.

Proof. *Let* (ζ_n) *be completely* $\mathcal{I}$*-statistically convergent to* ζ, *then it is* $\mathcal{I}$*-statistically convergent in distribution to* ζ *by theorem 3.39 and 3.10.* □

Remark 3.43. *But the converse of the above theorem is not true in general.*

Example 3.44. *From example 3.23 and 3.41, we see that the complex uncertain sequence* (ζ_n) *is* $\mathcal{I}$*-convergent in distribution to* $\zeta \equiv 0$ *but it is not completely* $\mathcal{I}$*-statistically convergent to* $\zeta \equiv 0$.

Definition 3.45. *A complex uncertain sequence* (ζ_n) *is said to be* $\mathcal{I}$*-statistically convergent in metric to* ζ *if for every* $\delta, \upsilon > 0$ *such that*

$$\left\{n \in \mathbb{N} : \tfrac{1}{n} \left|\{k \le n : D(\zeta_k, \zeta) \ge \delta\}\right| \ge \upsilon\right\} \in \mathcal{I}.$$

Theorem 3.46. *Let* $\zeta, \zeta_1, \zeta_2, ...$ *be complex uncertain variables defined on uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$*. If* (ζ_n) *is* $\mathcal{I}$*-statistically convergent in metric to* ζ*, then it is* $\mathcal{I}$*-statistically convergent in mean to* ζ*.*

Proof. *Let the complex uncertain sequence* (ζ_n) *be* $\mathcal{I}$*-statistically convergent in metric to* ζ*, then for every* $\varepsilon > 0$ *we have,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \le n : D(\zeta_k, \zeta) \ge \varepsilon\}\right| \ge \delta\right\} \in \mathcal{I},$$

where $D(\zeta_n, \zeta) = \inf\{r : \mathcal{M}\{\|\zeta_n - \zeta\| \le r\} = 1\}$.
Let $\Phi_n(r)$ *be the complex uncertainty distributions of uncertain variable* $\|\zeta_n - \zeta\|$ *and* $D(\zeta_n, \zeta) = D$*, then* $D(\zeta_n, \zeta) = \inf\{r : \Phi_n(r) = 1\}$.
Now for any positive number ε',
$E[\|\zeta_n - \zeta\|] = \int_0^{+\infty} (1 - \Phi_n(r))\, dr = \int_0^{D+\varepsilon'} (1 - \Phi_n(r))\, dr + \int_{D+\varepsilon'}^{+\infty} (1 - \Phi_n(r))\, dr$
$\quad = \int_0^{D+\varepsilon'} (1 - \Phi_n(r))\, dr < 1 \cdot (D + \varepsilon') = D + \varepsilon'$
$\Rightarrow E[\|\zeta_n - \zeta\|] \le D \Rightarrow E[\|\zeta_n - \zeta\|] \le D(\zeta_n, \zeta)$.
So for every $\delta > 0$,
$\{k \le n : E[\|\zeta_k - \zeta\|] \ge \delta\} \subseteq \{k \le n : D(\zeta_k, \zeta) \ge \delta\}$
$\Rightarrow \frac{1}{n} |\{k \le n : E[\|\zeta_k - \zeta\|] \ge \delta\}| \le \frac{1}{n} |\{k \le n : D(\zeta_k, \zeta) \ge \delta\}|$.
Then for every $\upsilon > 0$,
$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \le n : E[\|\zeta_k - \zeta\|] \ge \delta\}| \ge \upsilon\right\} \subseteq \left\{n \in \mathbb{N} : \frac{1}{n} |\{k \le n : D(\zeta_k, \zeta) \ge \delta\}| \ge \upsilon\right\} \in \mathcal{I}$.
Hence the sequence (ζ_n) *is* $\mathcal{I}$*-statistically convergent in mean to* ζ. □

Remark 3.47. *But the converse of the above theorem is not true in general.*

Example 3.48. *From 3.19, we have the complex uncertainty distributions of uncertain variable* $\|\zeta_n - \zeta\|$ *is*

$$\Phi_n(r) = \begin{cases} 0, & \text{if } r < 0 \\ 1 - \frac{n\beta_n}{2n+1}, & \text{if } 0 \le r < (n+1) \\ 1, & \text{if } r \ge (n+1) \end{cases} \quad \text{for } n = 1, 2, 3, \cdots.$$

Now $E[\|\zeta_n - \zeta\|] = \int_0^{+\infty} (1 - \Phi_n(r))\, dr = \int_0^{(n+1)} \frac{n\beta_n}{2n+1} dr = \frac{n(n+1)\beta_n}{2n+1}$.
Then for every $\varepsilon, \delta > 0$*, we have*
$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \le n : E\left[\left\|\zeta_k(\gamma) - \zeta(\gamma)\right\|\right] \ge \varepsilon\right\}\right| \ge \delta\right\} = \left\{n \in \mathbb{N} : \frac{1}{n} \left|\left\{k \le n : \frac{k(k+1)\beta_k}{2k+1} \ge \varepsilon\right\}\right| \ge \delta\right\} \in \mathcal{I}$.
Again the metric between complex uncertain veriables ζ_n *and* ζ *is given by*
$D(\zeta_n, \zeta) = \inf\{r : \mathcal{M}\{\|\zeta_n - \zeta\| \le r\} = 1\} = \inf\{r : \Phi_n(r) = 1\} = n + 1$.
Thus for every $\varepsilon, \delta > 0$,
$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \le n : D(\zeta_k, \zeta) \ge \varepsilon\}| \ge \delta\right\} = \left\{n \in \mathbb{N} : \frac{1}{n} |\{k \le n : (k + 1) \ge \varepsilon\}| \ge \delta\right\} \notin \mathcal{I}$.
Hence the complex uncertain sequence (ζ_n) *is* $\mathcal{I}$*-statistically convergent in mean to* $\zeta \equiv 0$ *but it is not* $\mathcal{I}$*-statistically convergent in metric to* $\zeta \equiv 0$.

Theorem 3.49. *Let* $\zeta, \zeta_1, \zeta_2, ...$ *be complex uncertain variables defined on uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$*. If* (ζ_n) *is* $\mathcal{I}$*-statistically convergent in metric to* ζ*, then it is* $\mathcal{I}$*-statistically convergent in measure to* ζ*.*

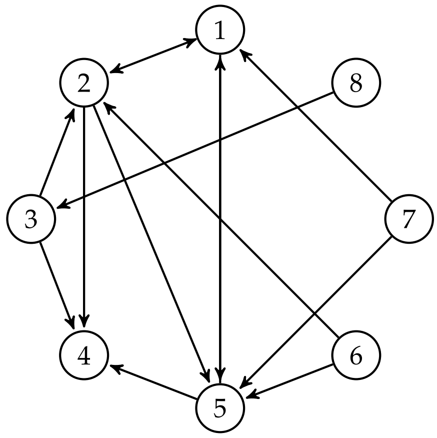
Proof. *Let* (ζ_n) *be* $\mathcal{I}$*-statistically convergent in metric to* ζ*, then it is* $\mathcal{I}$*-statistically convergent in measure to* ζ *by theorem 3.46 and 3.7.* □

Remark 3.50. *But the converse of the above theorem is not true in general.*

Example 3.51. *From example 3.19 and 3.48, we see that the complex uncertain sequence* (ζ_n) *is* $\mathcal{I}$*-statistically convergent in measure to* $\zeta \equiv 0$ *but it is not* $\mathcal{I}$*-statistically convergent in metric to* $\zeta \equiv 0$.

4. Interrelationships among all convergence concepts

1. $\mathcal{I}$-statistically convergence almost surely.
2. $\mathcal{I}$-statistically convergence in measure
3. $\mathcal{I}$-statistically convergence in mean
4. $\mathcal{I}$-statistically convergence in distribution
5. $\mathcal{I}$-statistically convergence uniformly almost surely
6. $\mathcal{I}$-statistically convergence in p-distance
7. Completely $\mathcal{I}$-statistically convergence
8. $\mathcal{I}$-statistically convergence in metric.



5. Conclusion

This paper has mainly discussed some $\mathcal{I}$-statistical convergence concepts of complex uncertain sequences, such as $\mathcal{I}$-statistical convergence in mean, distribution, uniformly almost surely, and established the relationships among them. Also, we initiate the notion of $\mathcal{I}$-statistical convergence in p-distance, completely $\mathcal{I}$-statistical convergence, and $\mathcal{I}$-statistical convergence in metric of complex uncertain sequences and include some interesting examples related to the notion. Furthermore, this paper is a more generalized form of $\mathcal{I}$-statistical convergence almost surely of complex uncertain sequences was introduced by Halder and Debnath, which is a very recent and new approach in complex uncertainty theory. In this paper, we try to establish relationships among all $\mathcal{I}$-statistical convergence concepts of complex uncertain sequences, but we see that some of them are not related to each other. It may attract future researchers in this direction.

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