Filomat 38:29 (2024), 10463–10483
https://doi.org/10.2298/FIL2429463B

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Jensen-Mercer variant of Hermite-Hadamard type inequalities via generalized fractional operator

Saad Ihsan Butt[a], Mehroz Nadeem[a], Jamshed Nasir[a], Ahmet Ocak Akdemir[b,*], Malahat Sh. Orujova[c]

[a]*COMSATS University Islamabad, Lahore Campus, Pakistan*
[b]*Department of Mathematics, Faculty of Sciences and Letters, Agri Ibrahim Çeçen University, Agri, Türkiye*
[c]*Department of Digital Technologies and Applied Informatics, Azerbaijan State University of Economics, Baku, Azerbaijan*

Abstract. The main motivation of this study is to present new Hermite-Hadamard-Mercer type inequalities via a certain fractional operators. We establish several new identities and give Jensen-Mercer variants of Hermite-Hadamard- type inequalities for differentiable and convex mappings via Katugampola-fractional operators. We establish connections of our results with several renowned results in literature.
Here, we gave new Lemmas having identities for differentiable functions and construct related inequalities. Main findings of this study would provide elegant connections and general variants of well known results established recently. In future, we are going to extend this work for coordinate convex functions. This research is open for further work by investigating such results for other class of convex functions.

1. Introduction

Convexity is a very functional concept in programming, statistics and numerical analysis as in many different branches of mathematics. In theory of inequality, the concept of convexity exists in the proof of many classical inequalities, but has been a source of inspiration for many new and useful inequalities.
Let $0 < \mu_1 \leq \mu_2 \leq ... \leq \mu_n$ and let $\xi = \left(\xi_1, \xi_2, ..., \xi_n\right)$ be non-negative weights such that $\sum_{i=1}^{n} \xi_i = 1$. The famous Jensen inequality (see [8]) in the literature states that if f is convex function on the interval $\left[a, b\right]$, then

$$\Upsilon\left(\sum_{i=1}^{n} \xi_i \, \mu_i\right) \leq \left(\sum_{i=1}^{n} \xi_i \, \Upsilon\left(\mu_i\right)\right) \tag{1}$$

for all $\mu_i \in [a, b]$ and all $\xi_i \in [0, 1]$, $(i = 1, 2, ..., n)$.
In (2003) Mercer gave a variant of Jensen's inequality (see [3]) as:

2020 *Mathematics Subject Classification*. Primary 26D15; Secondary 26A51

Keywords. Convex functions; Hermite-Hadamard inequalities; Jensen inequality; Jensen-Mercer inequality, Katugampola-fractional integrals.
Received: 23 March 2024; Revised: 03 May 2024; Accepted: 31 May 2024
Communicated by Miodrag Spalević

* Corresponding author: Ahmet Ocak Akdemir
Email address: ahmetakdemir@agri.edu.tr (Ahmet Ocak Akdemir)

Theorem 1.1. *If Υ is a convex function on $[a,b]$, then*

$$\Upsilon\left(a+b-\sum_{i=1}^{n}\xi_i\,\mu_i\right)\leq\Upsilon(a)+\Upsilon(b)-\sum_{i=1}^{n}\xi_i\,\Upsilon(\mu_i) \tag{2}$$

$\forall\ \mu_i\in[a,b]$ *and all* $\xi_i\in[0,1]$, $(i=1,2,...,n)$.

Pecaric et al. in (2006) worked on Jensen's inequality of Mercer's type for operators with applications [2]. In an another study performed by Niezgoda in (2009) and it includes some generalizations of Mercer inequality in terms of higher dimensions [15]. In recent years, notable contributions have been made on Jensen-Mercer's type inequality. In (2014) M. Kian gave concept of Jensen inequality for superquadratic functions [14]. Further, E. Anjidani worked on Reverse Jensen-Mercer type operator inequalities and Jensen-Mercer operator inequalities for superquadratic functions (see [6], [7]). In [13], the authors gave Mercer's inequality as a general form and integral means .

Another important inequality that characterize convex function is Hermite-Hadamard inequality, that is if a mapping $\Upsilon:J\subseteq\Re\to\Re$ is a convex function and $r,s\in J$, $r<s$, then

$$\Upsilon\left(\frac{r+s}{2}\right)\leq\frac{1}{s-r}\int_r^s\Upsilon(\lambda)d\lambda\leq\frac{\Upsilon(r)+\Upsilon(s)}{2}.$$

Although fractional analysis is basically a generalization of classical analysis, it has developed rapidly with the definition of fractional integral and derivative operators. Fractional analysis has recently become a popular topic with its applications in many fields such as statictics, economics, engineering and mathematical biology, based on applied mathematics problems (see [1], [9], [19], [27], [28], [29] and [30]).

Recently in [24], the author introduced a new concept to unify Riemann-Liouville and Hadamard fractional integral operators which a certain general form fro fractional integral operators. Also the conditions are given so that the operator is bounded in an extended Lebesgue measurable space. The corresponding fractional derivative approach to this new generalized operator can be seen in [25]. Moreover, Katugampola worked for the Mellin transforms of the fractional integrals and derivatives (see [26]). For further applications and related results (see [4], [5], [23]) and references therein.

Definition 1.2. *[24] Let $[a,b]\subset\Re$ be a finite interval. Then the left-sided and right-sided Katugampola fractional integrals of order $\alpha>0$ of $\Upsilon\in X_c^{\rho}(a^{\rho},b^{\rho})$ are defined as follows:*

$$({}^{\rho}I_{a+}^{\alpha}\Upsilon)(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{\Upsilon(\lambda)}{(x-\lambda)^{1-\alpha}}\lambda^{\rho-1}d\lambda,\ x>a$$

with $a<x<b$ and $\rho>0$, if the integrals exist.

$$({}^{\rho}I_{b-}^{\alpha}\Upsilon)(x)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b\frac{\Upsilon(\lambda)}{(\lambda-x)^{1-\alpha}}\lambda^{\rho-1}d\lambda,\ x<b.$$

Theorem 1.3. *[24]*
If $\alpha>0$ and $\rho>0$, then for $x>a$

1) $\lim\rho\to1({}^{\rho}I_{a^+}^{\alpha}\Upsilon)(x)=(J_{a^+}^{\alpha}\Upsilon)(x)$

2) $\lim\rho\to(0^+)({}^{\rho}I_{a^+}^{\alpha}\Upsilon)(x)=(H_{a^+}^{\alpha}\Upsilon)(x)$.

The main motivation of the paper is to establish new and useful HH- type inequalities with the help of Jensen-Mercer's inequality via a certain fractional integral operator. The main findings includes new approaches and estimations for differentiable and convex mappings.

2. Hermite-Jensen-Mercer Type Inequalities for Katugampola-Fractional Integrals

We will start with the following result that includes HH-type inequalities by using Jensen-Mercer's results via Katugampola integrals:

Theorem 2.1. *Suppose that if $\Upsilon : [a^\rho, b^\rho] \to \Re$ is positive function with $0 \le a < b$ and $\Upsilon \in X_c^\rho(a^\rho, b^\rho)$. If Υ is convex on $[a^\rho, b^\rho]$, then the following result is valid:*

$$\begin{aligned} &\frac{1}{\rho}\Upsilon\left(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2}\right) \le \frac{1}{\rho}[\Upsilon(a^\rho) + \Upsilon(b^\rho)] \\ -\;& \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{2\left(z_2^\rho - z_1^\rho\right)^\alpha}\left\{\left({}^\rho I^\alpha_{z_1^+}\right)\Upsilon\left(z_2^\rho\right) + \left({}^\rho I^\alpha_{z_2^-}\right)\Upsilon\left(z_1^\rho\right)\right\} \\ \le\;& \frac{1}{\rho}[\Upsilon(a^\rho) + \Upsilon(b^\rho)] - \frac{1}{\rho}\Upsilon\left(\frac{z_1^\rho + z_2^\rho}{2}\right) \end{aligned} \tag{3}$$

for all $\alpha > 0$, $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$ and $\Gamma(.)$ is the Gamma function.

Proof. Since Υ is convex on $[a^\rho, b^\rho]$, so by Jensen-Mercer's inequality (2)

$$\Upsilon\left(a^\rho + b^\rho - \frac{u^\rho + v^\rho}{2}\right) \le \Upsilon(a^\rho) + \Upsilon(b^\rho) - \frac{\Upsilon(u^\rho) + \Upsilon(v^\rho)}{2} \tag{4}$$

$\forall\, u^\rho, v^\rho \in [a^\rho, b^\rho]$.

Now by change of variables $u^\rho = \lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho$ and $v^\rho = (1-\lambda^\rho) z_1^\rho + \lambda^\rho z_2^\rho$, $\forall z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$ and $\lambda \in [0,1]$ in (4), we have

$$\Upsilon\left(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2}\right) \le \Upsilon(a^\rho) + \Upsilon(b^\rho) - \frac{\Upsilon\left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right) + \Upsilon\left((1-\lambda^\rho) z_1^\rho + \lambda^\rho z_2^\rho\right)}{2}$$

If we product the above result by $\lambda^{\rho\alpha-1}$ and then apply the integration with respect to λ on $[0,1]$, we have

$$\begin{aligned} &\frac{1}{\rho\alpha}\Upsilon\left(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2}\right) \le \frac{1}{\rho\alpha}\{\Upsilon(a^\rho) + \Upsilon(b^\rho)\} \\ -\;& \frac{1}{2}\left\{\int_0^1 \lambda^{\rho\alpha-1}\left(\Upsilon\left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right) + \Upsilon\left((1-\lambda^\rho) z_1^\rho + \lambda^\rho z_2^\rho\right)\right) d\lambda\right\}. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{1}{\rho}\Upsilon\left(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2}\right) \le \frac{1}{\rho}[\Upsilon(a^\rho) + \Upsilon(b^\rho)] \\ -\;& \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{2\left(z_2^\rho - z_1^\rho\right)^\alpha}\left\{\left({}^\rho I^\alpha_{z_1^+}\right)\Upsilon\left(z_2^\rho\right) + \left({}^\rho I^\alpha_{z_2^-}\right)\Upsilon\left(z_1^\rho\right)\right\}. \end{aligned}$$

This shows the proof of the first inequality of (3).

We will proceed with the second part of (3), we remind that Υ is convex on $[a^\rho, b^\rho]$, then for $\lambda \in [0,1]$, it yields

$$\begin{aligned} &\Upsilon\left(\frac{z_1^\rho + z_2^\rho}{2}\right) = \Upsilon\left(\frac{\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho + (1-\lambda^\rho) z_1^\rho + \lambda^\rho z_2^\rho}{2}\right) \\ \le\;& \frac{\Upsilon\left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right) + \Upsilon\left((1-\lambda^\rho) z_1^\rho + \lambda^\rho z_2^\rho\right)}{2} \end{aligned} \tag{5}$$

If we product the above expression by $\lambda^{\rho\alpha-1}$ and then apply integration with respect to λ on $\lambda \in [0,1]$, we get

$$\frac{1}{\rho\alpha}\Upsilon\left(\frac{z_1^\rho+z_2^\rho}{2}\right) \le \frac{1}{2}\left\{\int_0^1 \lambda^{\rho\alpha-1}\left(\Upsilon\left(\lambda^\rho z_1^\rho+(1-\lambda^\rho)z_2^\rho\right)+\Upsilon\left((1-\lambda^\rho)z_1^\rho+\lambda^\rho z_2^\rho\right)\right)\right\}$$
$$\Upsilon\left(\frac{z_1^\rho+z_2^\rho}{2}\right) \le \frac{\rho^\alpha\Gamma(\alpha+1)}{2(z_2-z_1)^\alpha}\left\{\left({}^\rho I^\alpha_{z_1^+}\right)\Upsilon\left(z_2^\rho\right)+\left({}^\rho I^\alpha_{z_2^-}\right)\Upsilon\left(z_1^\rho\right)\right\}$$

Multiplying by (-1), we obtain

$$-\frac{\rho^\alpha\Gamma(\alpha+1)}{2\left(z_2^\rho-z_1^\rho\right)^\alpha}\left\{\left({}^\rho I^\alpha_{z_1^+}\right)\Upsilon\left(z_2^\rho\right)+\left({}^\rho I^\alpha_{z_2^-}\right)\Upsilon\left(z_1^\rho\right)\right\} \le -\Upsilon\left(\frac{z_1^\rho+z_2^\rho}{2}\right) \tag{6}$$

By adding $\Upsilon(a^\rho)+\Upsilon(b^\rho)$ both sides in (6), we provide the desired inequality of (3). □

Remark 2.2. *For $\rho=1$ in (3), we will get*

$$\begin{aligned}&\Upsilon\left(a+b-\frac{z_1+z_2}{2}\right)\\ \le\ & [\Upsilon(a)+\Upsilon(b)]-\frac{\Gamma(\alpha+1)}{2(z_2-z_1)^\alpha}\left\{\left(J^\alpha_{z_1^+}\right)\Upsilon(z_2)+\left(J^\alpha_{z_2^-}\right)\Upsilon(z_1)\right\}\\ \le\ & [\Upsilon(a)+\Upsilon(b)]-\Upsilon\left(\frac{z_1+z_2}{2}\right).\end{aligned}$$

proved in [[10], Theorem 2].

Remark 2.3. *For $\rho=1$ and $\alpha=1$ in (3) we will get*

$$\Upsilon\left(a+b-\frac{z_1+z_2}{2}\right)\le\frac{1}{(z_2-z_1)}\int_{z_1}^{z_2}\Upsilon(a+b-t)\,d\lambda\le\Upsilon(a)+\Upsilon(b)-\left(\frac{\Upsilon(z_1)+\Upsilon(z_2)}{2}\right) \tag{7}$$

for all $z_1,z_2\in[a,b]$ proved by Kian and Moslehian in [16].

Theorem 2.4. *Suppose that if $\Upsilon:[a^\rho,b^\rho]\to\Re$ is positive function with $0\le a<b$. If Υ is convex on $[a^\rho,b^\rho]$, then the following result is valid:*

$$\begin{aligned}&\Upsilon\left(a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}\right)\le\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{\left(z_2^\rho-z_1^\rho\right)^\alpha}\\ \times\ &\left\{\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}\right)^+}\right)\Upsilon\left(a^\rho+b^\rho-z_1^\rho\right)+\left({}^\rho I^\alpha_{\left(a+b-\frac{z_1+z_2}{2}\right)^-}\right)\Upsilon\left(a^\rho+b^\rho-z_2^\rho\right)\right\}\\ \le\ &\Upsilon(a^\rho)+\Upsilon(b^\rho)-\left(\frac{\Upsilon\left(z_1^\rho\right)+\Upsilon\left(z_2^\rho\right)}{2}\right)\end{aligned}\tag{8}$$

for all $z_1^\rho,z_2^\rho\in[a^\rho,b^\rho]$, $\alpha>0$.

Proof. To see the proof of first part of (8), we consider convexity of Υ and write

$$\Upsilon\left(a^\rho+b^\rho-\frac{u^\rho+v^\rho}{2}\right)=\Upsilon\left(\frac{a^\rho+b^\rho-u^\rho+a^\rho+b^\rho-v^\rho}{2}\right)$$
$$\le\Upsilon\left(a^\rho+b^\rho-u^\rho\right)+\Upsilon\left(a^\rho+b^\rho-v^\rho\right)$$

$\forall u^\rho, v^\rho \in [a^\rho, b^\rho]$. Now by change of variables $u^\rho = \frac{\lambda^\rho}{2} z_1^\rho + \frac{2-\lambda^\rho}{2} z_2^\rho$ and $v^\rho = \frac{2-\lambda^\rho}{2} z_1^\rho + \frac{\lambda^\rho}{2} z_2^\rho$, $\lambda \in [0,1]$ we get

$$\begin{aligned} & 2\Upsilon\left(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2}\right) \\ \leq\ & \Upsilon\left(a^\rho + b^\rho - \left(\frac{\lambda^\rho}{2} z_1^\rho + \frac{2-\lambda^\rho}{2} z_2^\rho\right)\right) + \Upsilon\left(a^\rho + b^\rho - \left(\frac{2-\lambda^\rho}{2} z_1^\rho + \frac{\lambda^\rho}{2} z_2^\rho\right)\right) \end{aligned}$$

If we product the above expression by $\lambda^{\rho\alpha-1}$ and then apply integration with respect to λ on $[0,1]$, we get

$$\begin{aligned} & \frac{2}{\rho\alpha}\Upsilon\left(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2}\right)\int_0^1 \lambda^{\rho\alpha-1} d\lambda \\ & \leq \int_0^1 \lambda^{\rho\alpha-1}\left[\Upsilon\left(a^\rho + b^\rho - \left(\frac{\lambda^\rho}{2} z_1^\rho + \frac{2-\lambda^\rho}{2} z_2^\rho\right)\right) + \Upsilon\left(a^\rho + b^\rho - \left(\frac{2-\lambda^\rho}{2} z_1^\rho + \frac{\lambda^\rho}{2} z_2^\rho\right)\right)\right] d\lambda \end{aligned}$$

$$\begin{aligned} \Upsilon\left(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2}\right) \leq & \frac{2^{\alpha-1}\rho^\alpha \Gamma(\alpha+1)}{\left(z_2^\rho - z_1^\rho\right)^\alpha} \\ & \times \left\{\left({}^\rho I^\alpha_{\left(a+b-\frac{z_1+z_2}{2}\right)^+}\right)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \left({}^\rho I^\alpha_{\left(a+b-\frac{z_1+z_2}{2}\right)^-}\right)\Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)\right\}. \end{aligned} \tag{9}$$

Thus, this implies the first part of (8).

We will try to prove the second part of (3), we remind that if Υ is convex, then for $\lambda \in [0,1]$, it gives

$$\Upsilon\left(a^\rho + b^\rho - \left(\frac{\lambda^\rho}{2} z_1^\rho + \frac{2-\lambda^\rho}{2} z_2^\rho\right)\right) \leq \Upsilon(a^\rho) + \Upsilon(b^\rho) - \left[\frac{\lambda^\rho}{2}\Upsilon\left(z_1^\rho\right) + \frac{2-\lambda^\rho}{2}\Upsilon\left(z_2^\rho\right)\right] \tag{10}$$

and

$$\Upsilon\left(a^\rho + b^\rho - \left(\frac{2-\lambda^\rho}{2} z_1^\rho + \frac{\lambda^\rho}{2} z_2^\rho\right)\right) \leq \Upsilon(a^\rho) + \Upsilon(b^\rho) - \left[\frac{2-\lambda^\rho}{2}\Upsilon\left(z_1^\rho\right) + \frac{\lambda^\rho}{2}\Upsilon\left(z_2^\rho\right)\right] \tag{11}$$

By adding the inequalities of (10) and (11), we have

$$\begin{aligned} & \Upsilon\left(a^\rho + b^\rho - \left(\frac{\lambda^\rho}{2} z_1^\rho + \frac{2-\lambda^\rho}{2} z_2^\rho\right)\right) + \Upsilon\left(a^\rho + b^\rho - \left(\frac{2-\lambda^\rho}{2} z_1^\rho + \frac{\lambda^\rho}{2} z_2^\rho\right)\right) \\ \leq\ & 2\left(\Upsilon(a^\rho) + \Upsilon(b^\rho)\right) - \left(\Upsilon\left(z_1^\rho\right) + \Upsilon\left(z_2^\rho\right)\right) \end{aligned}$$

If we product the above expression by $\lambda^{\rho\alpha-1}$ and then apply integration with respect λ on $[0,1]$, we have

$$\begin{aligned} & \int_0^1 \lambda^{\rho\alpha-1}\left[\Upsilon\left(a^\rho + b^\rho - \left(\frac{\lambda^\rho}{2} z_1^\rho + \frac{2-\lambda^\rho}{2} z_2^\rho\right)\right) + \Upsilon\left(a^\rho + b^\rho - \left(\frac{2-\lambda^\rho}{2} z_1^\rho + \frac{\lambda^\rho}{2} z_2^\rho\right)\right)\right] d\lambda \\ & \qquad \leq \left\{2\left(\Upsilon(a^\rho) + \Upsilon(b^\rho)\right) - \left(\Upsilon\left(z_1^\rho\right) + \Upsilon^\rho\left(z_2^\rho\right)\right)\right\} \int_0^1 \lambda^{\rho\alpha-1} d\lambda \end{aligned}$$

Therefore,

$$\frac{2^\alpha \rho^{\alpha-1}\Gamma(\alpha)}{\left(z_2^\rho - z_1^\rho\right)^\alpha}\left[\left({}^\rho I^\alpha_{\left(a+b-\frac{z_1+z_2}{2}\right)^+}\right)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \left({}^\rho I^\alpha_{\left(a+b-\frac{z_1+z_2}{2}\right)^-}\right)\Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)\right]$$

$$\leq \left\{2\left(\Upsilon\left(a^{\rho}\right)+\Upsilon\left(b^{\rho}\right)\right)-\left(\Upsilon\left(z_{1}^{\rho}\right)+\Upsilon\left(z_{2}^{\rho}\right)\right)\right\}\frac{1}{\rho\alpha}. \tag{12}$$

Multiplying (12) by $\frac{\rho\alpha}{2}$, we get

$$\frac{2^{\alpha-1}\rho^{\alpha}\,\Gamma(\alpha+1)}{\left(z_{2}^{\rho}-z_{1}^{\rho}\right)^{\alpha}}\left\{\left({}^{\rho}I^{\alpha}_{\left(a+b-\frac{z_1+z_2}{2}\right)^{+}}\right)\Upsilon\left(a^{\rho}+b^{\rho}-z_{1}^{\rho}\right)+\left({}^{\rho}I^{\alpha}_{\left(a+b-\frac{z_1+z_2}{2}\right)^{-}}\right)\Upsilon\left(a^{\rho}+b^{\rho}-z_{2}^{\rho}\right)\right\}$$

$$\leq \left(\Upsilon\left(a^{\rho}\right)+\Upsilon\left(b^{\rho}\right)\right)-\frac{\Upsilon\left(z_{1}^{\rho}\right)+\Upsilon\left(z_{2}^{\rho}\right)}{2}. \tag{13}$$

Collecting (9) and (13), we get (8). □

The following interesting cases can be deduced

Remark 2.5. *For $\rho = 1$ in Theorem 2.4, we will get Theorem 3 proved in [10].*

Remark 2.6. *For $\rho = 1$ and $\alpha = 1$, Theorem 2.4 recapture* (8) *given in Remark 2.3.*

3. New Identities and Related Results for differentiable function λ'

In this section, we first introduce new identities and then give related results:

Lemma 3.1. *Suppose that $\Upsilon : [a^{\rho}, b^{\rho}] \to \mathfrak{R}$ is a differentiable mapping on (a^{ρ}, b^{ρ}) with $0 \leq a < b$, then the following equality for Katugampola-fractional integrals holds:*

$$\frac{\Upsilon\left(a^{\rho}+b^{\rho}-z_{1}^{\rho}\right)+\Upsilon\left(a^{\rho}+b^{\rho}-z_{2}^{\rho}\right)}{\alpha\rho}-\frac{\rho^{\alpha-1}\Gamma\left(\alpha\right)}{\left(z_{2}^{\rho}-z_{1}^{\rho}\right)^{\alpha}}\left(\left({}^{\rho}I^{\alpha}_{\left(a^{\rho}+b^{\rho}-z_{2}^{\rho}\right)^{+}}\right)\Upsilon\left(a^{\rho}+b^{\rho}-z_{1}^{\rho}\right)\right.$$

$$\left.+\left({}^{\rho}I^{\alpha}_{\left(a^{\rho}+b^{\rho}-z_{1}^{\rho}\right)^{-}}\right)\Upsilon\left(a^{\rho}b^{\rho}-z_{2}^{\rho}\right)\right)=\frac{(z_{2}^{\rho}-z_{1}^{\rho})}{\alpha}\int_{0}^{1}\lambda^{\rho(\alpha+1)-1}$$

$$\times\left[-\Upsilon'\left(a^{\rho}+b^{\rho}-\left((1-\lambda^{\rho})z_{1}^{\rho}+\lambda^{\rho}z_{2}^{\rho}\right)\right)+\Upsilon'\left(a^{\rho}+b^{\rho}-\left(\lambda^{\rho}z_{1}^{\rho}+(1-\lambda^{\rho})\,z_{2}^{\rho}\right)\right)\right]d\lambda \tag{14}$$

for all $z_{1}^{\rho}, z_{2}^{\rho} \in [a^{\rho}, b^{\rho}]$, $\alpha > 0$, $\lambda \in [0, 1]$.

Proof. We have used the fact that

$$K = \frac{(z_{2}^{\rho}-z_{1}^{\rho})}{\alpha}\left(K_{2}-K_{1}\right) \tag{15}$$

where

$$K_{1}=\int_{0}^{1}(\lambda^{\rho})^{\alpha}\,\lambda^{\rho-1}\,\Upsilon'\left(a^{\rho}+b^{\rho}-\left((1-\lambda^{\rho})z_{1}^{\rho}+(\lambda^{\rho})\,z_{2}^{\rho}\right)\right)d\lambda$$

$$=-\frac{\Upsilon\left(a^{\rho}+b^{\rho}-z_{2}^{\rho}\right)}{\rho(z_{2}^{\rho}-z_{1}^{\rho})}+\frac{\alpha}{z_{2}^{\rho}-z_{1}^{\rho}}\int_{0}^{1}(1-\lambda^{\rho})^{\alpha-1}\;\Upsilon\left(a^{\rho}+b^{\rho}-\left(\lambda^{\rho}z_{1}^{\rho}+(1-\lambda^{\rho})\,z_{2}^{\rho}\right)\right)d\lambda$$

$$=-\frac{\Upsilon\left(a^{\rho}+b^{\rho}-z_{2}^{\rho}\right)}{\rho(z_{2}^{\rho}-z_{1}^{\rho})}+\frac{\rho^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(z_{2}^{\rho}-z_{1}^{\rho}\right)^{\alpha+1}}\left\{\left({}^{\rho}I^{\alpha}_{\left(a^{\rho}+b^{\rho}-z_{2}^{\rho}\right)^{+}}\right)\Upsilon\left(a^{\rho}+b^{\rho}-z_{1}^{\rho}\right)\right\}$$

and

$$\begin{aligned}K_2 &= \int_0^1 \left(\lambda^{\rho\alpha}\lambda^{\rho-1}\right) \Upsilon'\left(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right)\right) d\lambda \\ &= \frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)}{\rho(z_2^\rho - z_1^\rho)} - \frac{\alpha}{z_2^\rho - z_1^\rho}\int_0^1 \lambda^{\rho\alpha-1}\lambda^{\rho-1}\,\Upsilon\left(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right)\right) d\lambda \\ &= \frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)}{\rho(z_2^\rho - z_1^\rho)} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{\left(z_2^\rho - z_1^\rho\right)^{\alpha+1}}\left\{\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\right)\Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)\right\}\end{aligned}$$

Substituting the values of the K_1 and K_2 in (15), we will get (14). □

Theorem 3.2. *Suppose that* $\Upsilon : [a^\rho, b^\rho] \to \Re$ *is a differentiable mapping on* (a^ρ, b^ρ) *with* $0 \le a < b$. *If* Υ' *is differentiable function on* $[a^\rho, b^\rho]$, *then one has the inequality for Katugampola-fractional integrals:*

$$\begin{aligned}&\Bigg|\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{2} - \frac{\alpha\rho^\alpha\Gamma(\alpha)}{2\left(z_2^\rho - z_1^\rho\right)^\alpha}\Bigg(\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\right)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) \\ &+ \left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\right)\Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)\Bigg)\Bigg| \le \frac{(z_2^\rho - z_1^\rho)^2}{2(\alpha+1)(\alpha+2)}\left(\alpha + \frac{1}{2^\alpha}\right)\sup_{\xi\in[a,b]}\left|\Upsilon''(\xi)\right| \qquad (16)\end{aligned}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$.

Proof. By following Lemma 3.1, we have the following equality

$$\begin{aligned}&\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\left(z_2^\rho - z_1^\rho\right)^\alpha}\Bigg(\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\right)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) \\ &+ \left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\right)\Upsilon\left(a^\rho b^\rho - z_2^\rho\right)\Bigg) = \frac{(z_2^\rho - z_1^\rho)}{\alpha}\Bigg[\int_0^1 \lambda^{\rho(\alpha+1)-1}\Big(\Upsilon'\left(a^\rho + b^\rho - \left((1-\lambda^\rho)z_1^\rho + \lambda^\rho z_2^\rho\right)\right) \\ &- \Upsilon'\left(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right)\right)\Big)d\lambda\Bigg]\end{aligned}$$

Now applying Mean value theorem for the function Υ' on the right side of above equality, we have

$$\begin{aligned}&\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \left(a^\rho + b^\rho - z_2^\rho\right)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\left(z_2^\rho - z_1^\rho\right)^\alpha}\Bigg(\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\right)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) \\ &+ \left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\right)\Upsilon\left(a^\rho b^\rho - z_2^\rho\right)\Bigg) = \frac{(z_2^\rho - z_1^\rho)^2}{\alpha}\left\{\int_0^1 \lambda^{\rho(\alpha+1)-1}(2\lambda^\rho - 1)\Upsilon''\left(\xi(\lambda)\right) d\lambda\right\}\end{aligned}$$

where $\xi(\lambda) \in (a^\rho, b^\rho)$. This leads us to

$$\begin{aligned}&\Bigg|\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \left(a^\rho + b^\rho - z_2^\rho\right)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\left(z_2^\rho - z_1^\rho\right)^\alpha}\Bigg(\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\right)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) \\ &+ \left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\right)\Upsilon\left(a^\rho b^\rho - z_2^\rho\right)\Bigg)\Bigg| = \frac{(z_2^\rho - z_1^\rho)^2}{\alpha}\left\{\int_0^1 \lambda^{\rho(\alpha+1)-1}\left|2\lambda^\rho - 1\right|\left|\Upsilon''\left(\xi(\lambda)\right)\right| d\lambda\right\}\end{aligned}$$

$$\leq \frac{(z_2^\rho - z_1^\rho)^2}{\alpha} \sup_{\xi\in[a,b]} \left|\Upsilon''(\xi)\right| \left(\int_0^{\frac{1}{\sqrt[\rho]{2}}} \lambda^{\rho(\alpha+1)-1}(1-2\lambda^\rho)d\lambda + \int_{\frac{1}{\sqrt[\rho]{2}}}^1 \lambda^{\rho(\alpha+1)-1}(2\lambda^\rho - 1)d\lambda \right)$$
$$= \frac{(z_2^\rho - z_1^\rho)^2}{\alpha\rho(\alpha+1)(\alpha+2)} \left(\alpha + \frac{1}{2^\alpha}\right) \sup_{\xi\in[a,b]} \left|\Upsilon''\left(\xi\right)\right|.$$

After simple re-arrangement, we will get (16). □

Remark 3.3. *For $z_1 = a$ and $z_2 = b$ in Theorem 3.2, we will get Theorem 2.2 proved in [11].*

Theorem 3.4. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to \Re$ is a differentiable mapping on (a^ρ, b^ρ) with $0 \leq a < b$. If $|\Upsilon'|$ is convex function on $[a^\rho, b^\rho]$, then one has the inequality for Katugampola-fractional integrals:*

$$\left| \frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{2} - \frac{\rho^\alpha \Gamma\left(\alpha\right)}{2\left(z_2^\rho - z_1^\rho\right)^\alpha} \left(\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+} \right) \Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) \right.\right.$$
$$\left.\left. + \left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-} \right) \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right) \right) \right|$$
$$\leq \frac{(z_2^\rho - z_1^\rho)}{\alpha+1} \left\{ \left|\Upsilon'(a^\rho)\right| + \left|\Upsilon'(b^\rho)\right| - \left(\frac{\left|\Upsilon'\left(z_1^\rho\right)\right| + \left|\Upsilon'\left(z_2^\rho\right)\right|}{2} \right) \right\} \tag{17}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$.

Proof. By taking into account Lemma 3.1 and employing Jensen-Mercer's inequality for the convex function $|\Upsilon'|$, we have

$$\left| \frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{\alpha\rho} - \frac{\rho^{\alpha-1} \Gamma\left(\alpha\right)}{\rho\left(z_2^\rho - z_1^\rho\right)^\alpha} \left(\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+} \right) \Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) \right.\right.$$
$$\left.\left. + \left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-} \right) \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right) \right) \right|$$
$$\leq \frac{(z_2^\rho - z_1^\rho)}{\alpha} \left(\int_0^1 \lambda^{\rho(\alpha+1)-1} \left| \Upsilon'\left(a^\rho + b^\rho - \left((1-\lambda^\rho)z_1^\rho + \lambda^\rho z_2^\rho\right)\right) \right.\right.$$
$$\left.\left. - \Upsilon'\left(a^\rho + b^\rho - \left((\lambda^\rho)z_1^\rho + (1-\lambda^\rho) z_2^\rho\right)\right) \right| d\lambda \right)$$
$$\leq \frac{(z_2^\rho - z_1^\rho)}{\alpha} \int_0^1 \lambda^{\rho(\alpha+1)-1} \left(\left| \Upsilon'\left(a^\rho + b^\rho - \left((1-\lambda^\rho)z_1^\rho + \lambda^\rho z_2^\rho\right)\right) \right| \right.$$
$$\left. + \left| \Upsilon'\left(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right)\right) \right| \right) d\lambda$$
$$\leq \frac{(z_2^\rho - z_1^\rho)}{\alpha} \left\{ 2\left(\left|\Upsilon'(a^\rho)\right| + \left|\Upsilon'(b^\rho)\right| \right) - \left(\left|\Upsilon'\left(z_1^\rho\right)\right| + \left|\Upsilon'\left(z_2^\rho\right)\right| \right) \right\} \int_0^1 \lambda^{\rho(\alpha+1)-1} d\lambda.$$

Simplifying the integral gives the required inequality (17). □

Remark 3.5. *For $z_1 = a$ and $z_2 = b$ in Theorem 3.4 , we recapture Theorem 2.3 proved in [11].*

Now, we give the follownig new identity.

Lemma 3.6. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to \Re$ is a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$, then the following equality for Katugampola-fractional integrals holds:*

$$\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2\left(z_2^\rho - z_1^\rho\right)^\alpha}$$
$$\times \left\{\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\right)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\right)\Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)\right\}$$
$$= \frac{(z_2^\rho - z_1^\rho)}{2}\int_0^1 \left(\lambda^{\rho\alpha} - (1-\lambda^\rho)^\alpha\right)\lambda^{\rho-1}\,\Upsilon'\left(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right)\right)d\lambda. \tag{18}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$.

Proof. It is obvious to see that

$$K = \frac{(z_2^\rho - z_1^\rho)}{2}\left(K_1 - K_2\right) \tag{19}$$

where

$$K_1 = \int_0^1 \left(\lambda^{\rho\alpha}\lambda^{\rho-1}\right)\Upsilon'\left(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right)\right)d\lambda$$
$$= \frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)}{\rho(z_2^\rho - z_1^\rho)} - \frac{\alpha}{z_2^\rho - z_1^\rho}\int_0^1 \lambda^{\rho\alpha-1}\lambda^{\rho-1}\,\Upsilon\left(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right)\right)d\lambda$$
$$= \frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)}{\rho(z_2^\rho - z_1^\rho)} - \frac{\rho^{\alpha-1}\alpha\Gamma(\alpha+1)}{\left(z_2^\rho - z_1^\rho\right)^{\alpha+1}}\left\{\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\right)\Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)\right\}$$

and

$$K_2 = \int_0^1 (1-\lambda^\rho)^\alpha\lambda^{\rho-1}\,\Upsilon'\left(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right)\right)d\lambda$$
$$= -\frac{\Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{\rho(z_2^\rho - z_1^\rho)} + \frac{\alpha}{z_2^\rho - z_1^\rho}\int_0^1 (1-\lambda^\rho)^{\alpha-1}\,\Upsilon\left(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right)\right)d\lambda$$
$$= -\frac{\Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{\rho(z_2^\rho - z_1^\rho)} + \frac{\rho^{\alpha-1}\alpha\Gamma(\alpha+1)}{\left(z_2^\rho - z_1^\rho\right)^{\alpha+1}}\left\{\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\right)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)\right\}.$$

Putting the respective values of K_1 and K_2 in (19), we will get (18). □

We deduce the following important connections of our result:

Remark 3.7. *For $z_1 = a$ and $z_2 = b$ in Lemma 3.6, we will get Lemma 2.4 proved in [11].*

Remark 3.8. *For $\rho = 1$ in Lemma 3.6, we will get Lemma 1 proved in [10].*

Remark 3.9. *For $\rho = 1$, $\alpha = 1$ and $z_1 = a$ and $z_2 = b$ in Lemma 3.6, the equality reduces to the equality*

$$\frac{\Upsilon(a) + \Upsilon(b)}{2} - \frac{1}{(b-a)}\int_a^b \Upsilon(u)\,du = \frac{b-a}{2}\int_0^1 (2\lambda - 1)\Upsilon((1-\lambda)a + \lambda b)\,d\lambda. \tag{20}$$

proved by Dragomir and Agarwal in [21].

Theorem 3.10. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to \Re$ is a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|\Upsilon'|$ is convex on $[a^\rho, b^\rho]$, then one has the new result as:*

$$\begin{aligned}&\Bigg|\frac{\Upsilon\left(a^\rho+b^\rho-z_1^\rho\right)+\Upsilon\left(a^\rho+b^\rho-z_2^\rho\right)}{2}-\frac{\rho^\alpha\alpha\Gamma\left(\alpha+1\right)}{2\left(z_2^\rho-z_1^\rho\right)^\alpha}\Big(\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\right)\Upsilon\left(a^\rho+b^\rho-z_1^\rho\right)\\&+\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\right)\Upsilon\left(a^\rho+b^\rho-z_2^\rho\right)\Big)\Bigg|\\&\le\frac{z_2^\rho-z_1^\rho}{\rho(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left\{\left|\Upsilon'\left(a^\rho\right)\right|+\left|\Upsilon'\left(b^\rho\right)\right|-\left(\frac{\left|\Upsilon'\left(z_1^\rho\right)\right|+\left|\Upsilon'\left(z_2^\rho\right)\right|}{2}\right)\right\}\end{aligned}\tag{21}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$.

Proof. From Lemma 3.6 and Jensen-Mercer's inequality, we have

$$\begin{aligned}&\Bigg|\frac{\Upsilon\left(a^\rho+b^\rho-z_1^\rho\right)+\Upsilon\left(a^\rho+b^\rho-z_2^\rho\right)}{2}-\frac{\rho^\alpha\alpha\Gamma\left(\alpha\right)}{2\left(z_2^\rho-z_1^\rho\right)^\alpha}\Big(\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\right)\Upsilon\left(a^\rho+b^\rho-z_1^\rho\right)\\&+\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\right)\Upsilon\left(a^\rho b^\rho-z_2^\rho\right)\Big)\Bigg|\\&\le\frac{(z_2^\rho-z_1^\rho)}{2}\int_0^1\left|(\lambda^{\rho\alpha}-(1-\lambda^\rho)^\alpha)\lambda^{\rho-1}\right|\left|\Upsilon'\left(a^\rho+b^\rho-\left(\lambda^\rho z_1^\rho+(1-\lambda^\rho)\,z_2^\rho\right)\right)\right|d\lambda\\&\le\frac{(z_2^\rho-z_1^\rho)}{2}\int_0^1\left|\lambda^{\rho\alpha}-(1-\lambda^\rho)^\alpha\right|\lambda^{\rho-1}\Big\{\left|\Upsilon'\left(a^\rho\right)\right|+\left|\Upsilon'\left(b^\rho\right)\right|-\left(\lambda^\rho\left|\Upsilon'\left(z_1^\rho\right)\right|+(1-\lambda^\rho)\left|\Upsilon'\left(z_2^\rho\right)\right|\right)\Big\}d\lambda\\&=\frac{\rho(z_2^\rho-z_1^\rho)}{2}\left[I_1+I_2\right]\end{aligned}$$

where

$$\begin{aligned}I_1&=\int_0^{\frac{1}{\sqrt[\rho]{2}}}((1-\lambda^\rho)^\alpha-\lambda^{\rho\alpha})\,\lambda^{\rho-1}\Big\{\left|\Upsilon'\left(a^\rho\right)\right|+\left|\Upsilon'\left(b^\rho\right)\right|-\left(\lambda^\rho\left|\Upsilon'\left(z_1^\rho\right)\right|+(1-\lambda^\rho)\left|\Upsilon'\left(z_2^\rho\right)\right|\right)\Big\}d\lambda\\&=\left(\left|\Upsilon'\left(a^\rho\right)\right|+\left|\Upsilon'\left(b^\rho\right)\right|\right)\left(\frac{1}{\rho(\alpha+1)}-\frac{2^{-\alpha}}{\rho(\alpha+1)}\right)\\&\quad-\left\{\left|\Upsilon'\left(z_1^\rho\right)\right|\left(\frac{1}{(\rho(\alpha+1))\,(\alpha+2)}-\frac{2^{-\alpha-1}}{\rho(\alpha+1)}\right)+\left|\Upsilon'\left(z_2^\rho\right)\right|\left(\frac{1}{(\rho(\alpha+2))}-\frac{2^{-\alpha-1}}{\rho(\alpha+1)}\right)\right\}\end{aligned}$$

and

$$\begin{aligned}I_2&=\int_{\frac{1}{\sqrt[\rho]{2}}}^1(\lambda^{\rho\alpha}-(1-\lambda^\rho)^\alpha)\,\lambda^{\rho-1}\Big\{\left|\Upsilon'\left(a^\rho\right)\right|+\left|\Upsilon'\left(b^\rho\right)\right|-\left(\lambda^\rho\left|\Upsilon'\left(z_1^\rho\right)\right|+(1-\lambda^\rho)\left|\Upsilon'\left(z_2^\rho\right)\right|\right)\Big\}d\lambda\\&=\left(\left|\Upsilon'\left(a^\rho\right)\right|+\left|\Upsilon'\left(b^\rho\right)\right|\right)\left(\frac{1}{(\rho(\alpha+1))}-\frac{2^{-\alpha}}{\rho(\alpha+1)}\right)\\&\quad-\left\{\left|\Upsilon'\left(z_1^\rho\right)\right|\left(\frac{1}{(\rho(\alpha+2))}-\frac{2^{-\alpha-1}}{\rho(\alpha+1)}\right)+\left|\Upsilon'\left(z_2^\rho\right)\right|\left(\frac{1}{(\rho(\alpha+1))\,(\alpha+2)}-\frac{2^{-\alpha-1}}{\rho(\alpha+1)}\right)\right\}\end{aligned}$$

Placing the values of the evaluated integrals I_1 and I_2 gives inequality (21). □

It is pertinent to give the following elegant connections of our obtained result:

Remark 3.11. *For $z_1 = a$ and $z_2 = b$ in Theorem 3.10, we will get Theorem 2.5 proved in [11].*

Remark 3.12. *For $\rho = 1$ in Theorem 3.10, we will get Theorem 4 proved in [10].*

Remark 3.13. *For $\rho = 1$, $z_1 = a$ and $z_2 = b$ in Theorem 3.10, we will get Theorem 3 proved by Sarikaya et al. in [17].*

Next we formulate another important lemma that is useful to obtain further results:

Lemma 3.14. *Let $\Upsilon : [a^\rho, b^\rho] \to \Re$ be a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If Υ' is convex, then the following equation hold:*

$$\begin{aligned}
&\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{2} - \frac{2^\alpha \rho^\alpha \Gamma(\alpha+1)}{2\left(z_2^\rho - z_1^\rho\right)^\alpha}\Bigg\{\left({}^\rho I^\alpha_{(a^\rho+b^\rho-z_2^\rho)^+}\right)\\
&\times\left(\Upsilon(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2})\right) + \left({}^\rho I^\alpha_{(a^\rho+b^\rho-z_1^\rho)^-}\right)\left(\Upsilon(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2})\right)\Bigg\}\\
&= \frac{\rho(z_2^\rho - z_1^\rho)}{4}\int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\,\Upsilon'\left(a^\rho + b^\rho - \left(\frac{1+\lambda^\rho}{2}z_1^\rho + \frac{1-\lambda^\rho}{2}z_2^\rho\right)\right)d\lambda\\
&- \frac{\rho(z_2^\rho - z_1^\rho)}{4}\int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\,\Upsilon'\left(a^\rho + b^\rho - \left(\frac{1-\lambda^\rho}{2}z_1^\rho + \frac{1+\lambda^\rho}{2}z_2^\rho\right)\right)d\lambda
\end{aligned}\tag{22}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$ and $\Gamma(.)$ is the Gamma function.

Proof. To make things simple, let us denote

$$K = \frac{\rho(z_2^\rho - z_1^\rho)}{4}\left(K_1 - K_2\right)\tag{23}$$

where

$$\begin{aligned}
K_1 &= \int_0^1 (\lambda^{\rho\alpha})\,\lambda^{\rho-1}\,\Upsilon'\left(a^\rho + b^\rho - \left(\frac{1+\lambda^\rho}{2}z_1^\rho + \frac{1-\lambda^\rho}{2}z_2^\rho\right)\right)d\lambda\\
&= \frac{2\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)}{\rho(z_2^\rho - z_1^\rho)} - \frac{2^{\alpha+1}\alpha}{(z_2^\rho - z_1^\rho)^{\alpha+1}}\int_{a^\rho+b^\rho-\left(\frac{z_1^\rho+z_2^\rho}{2}\right)}^{a^\rho+b^\rho-z_1^\rho}\left(w^\rho - \left(a^\rho + b^\rho - \frac{z_1^\rho+z_2^\rho}{2}\right)\right)^{\alpha-1}\Upsilon(w^\rho)dw\\
&= \frac{2\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)}{\rho(z_2^\rho - z_1^\rho)} - \frac{2^{\alpha+1}\Gamma(\alpha+1)}{\rho^{1-\alpha}\left(z_2^\rho - z_1^\rho\right)^{\alpha+1}}\left\{{}^\rho I^\alpha_{(a^\rho+b^\rho-z_1^\rho)^-}\Upsilon(a^\rho + b^\rho - \frac{z_1^\rho+z_2^\rho}{2})\right\}
\end{aligned}$$

and

$$\begin{aligned}
K_2 &= \int_0^1 (\lambda)^{\rho\alpha}\,\lambda^{\rho-1}\,\Upsilon'\left(a^\rho + b^\rho - \left(\frac{1-\lambda^\rho}{2}z_1^\rho + \frac{1+\lambda^\rho}{2}z_2^\rho\right)\right)d\lambda\\
&= -\frac{2\Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{\rho(z_2^\rho - z_1^\rho)} + \frac{2^{\alpha+1}\alpha}{(z_2^\rho - z_1^\rho)^{\alpha+1}}\int_{a^\rho+b^\rho-z_2^\rho}^{a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}}\left((a^\rho + b^\rho - \frac{z_1^\rho+z_2^\rho}{2}) - w^\rho\right)^{\alpha-1}\Upsilon(w^\rho)dw\\
&= -\frac{2\Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{\rho(z_2^\rho - z_1^\rho)} + \frac{2^{\alpha+1}\Gamma(\alpha+1)}{\rho^{1-\alpha}\left(z_2^\rho - z_1^\rho\right)^{\alpha+1}}\left\{{}^\rho I^\alpha_{(a^\rho+b^\rho-z_2^\rho)^+}\Upsilon(a^\rho + b^\rho - \frac{z_1^\rho+z_2^\rho}{2})\right\}
\end{aligned}$$

Putting the respective values of K_1 and K_2 in (23), we get (22). □

Remark 3.15. *For $\rho = 1$ in Lemma 3.3, we will get Lemma 2.1 proved in [20].*

Theorem 3.16. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to R$ is a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|\Upsilon'|$ is convex on $[a^\rho, b^\rho]$, then one has the following statement:*

$$\Bigg|\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{2} - \frac{2^\alpha \rho^\alpha \Gamma(\alpha+1)}{2\left(z_2^\rho - z_1^\rho\right)^\alpha}\Bigg\{\left({}^\rho I^\alpha_{(a^\rho+b^\rho-z_2^\rho)^+}\right)$$
$$\times\left(\Upsilon(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2})\right) + \left({}^\rho I^\alpha_{(a^\rho+b^\rho-z_1^\rho)^-}\right)\left(\Upsilon(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2})\right)\Bigg\}\Bigg|$$
$$\le \frac{z_2^\rho - z_1^\rho}{2\left(\alpha+1\right)}\Bigg\{\left|\Upsilon'\left(a^\rho\right)\right| + \left|\Upsilon'\left(b^\rho\right)\right| - \left(\frac{\left|\Upsilon'\left(z_1^\rho\right)\right| + \left|\Upsilon'\left(z_2^\rho\right)\right|}{2}\right)\Bigg\} \tag{24}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$.

Proof. By considering Lemma 3.14 and Jensen-Mercer's inequality along with the convexity of $|\Upsilon'|$, we have

$$\Bigg|\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{2} - \frac{2^\alpha \rho^\alpha \Gamma(\alpha+1)}{2\left(z_2^\rho - z_1^\rho\right)^\alpha}\Bigg\{\left({}^\rho I^\alpha_{(a^\rho+b^\rho-z_2^\rho)^+}\right)$$
$$\times\left(\Upsilon(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2})\right) + \left({}^\rho I^\alpha_{(a^\rho+b^\rho-z_1^\rho)^-}\right)\left(\Upsilon(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2})\right)\Bigg\}\Bigg|$$
$$\le \frac{\rho(z_2^\rho - z_1^\rho)}{4}\int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\left|\Upsilon'\left(a^\rho + b^\rho - \left(\frac{1+\lambda^\rho}{2}z_1^\rho + \frac{1-\lambda^\rho}{2}z_2^\rho\right)\right)\right|d\lambda$$
$$+ \frac{\rho(z_2^\rho - z_1^\rho)}{4}\int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\left|\Upsilon'\left(a^\rho + b^\rho - \left(\frac{1-\lambda^\rho}{2}z_1^\rho + \frac{1+\lambda^\rho}{2}z_2^\rho\right)\right)\right|d\lambda$$
$$\le \frac{\rho(z_2^\rho - z_1^\rho)}{4}\Bigg[\int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\Bigg\{\left|\Upsilon'\left(a^\rho\right)\right| + \left|\Upsilon'\left(b^\rho\right)\right| - \left(\frac{1+\lambda^\rho}{2}\left|\Upsilon'\left(z_1^\rho\right)\right| + \frac{\left(1-\lambda^\rho\right)}{2}\left|\Upsilon'\left(z_2^\rho\right)\right|\right)\Bigg\}d\lambda$$
$$+ \int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\Bigg\{\left|\Upsilon'\left(a^\rho\right)\right| + \left|\Upsilon'\left(b^\rho\right)\right| - \left(\frac{\left(1-\lambda^\rho\right)}{2}\left|\Upsilon'\left(z_1^\rho\right)\right| + \frac{1+\lambda^\rho}{2}\left|\Upsilon'\left(z_2^\rho\right)\right|\right)\Bigg\}d\lambda\Bigg]$$

Evaluating the integrals leads to inequality (3.16). □

Remark 3.17. *For $\rho = 1$, $\alpha = 1$, $z_1 = a$ and $z_2 = b$ in Theorem 3.16, we will get Theorem 2.2 proved in [21].*

Remark 3.18. *For $\rho = 1$ in Theorem 3.16, we will get Theorem 2.2 proved in [20].*

Lemma 3.19. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to R$ is a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$ and $\Upsilon \in L[a^\rho, b^\rho]$, then the following equality for Katugampola-fractional integrals holds:*

$$\Upsilon\left(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2}\right) - \frac{2^{\alpha-1}\rho^\alpha\Gamma\left(\alpha+1\right)}{\left(z_2^\rho - z_1^\rho\right)^\alpha}\Bigg[\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}\right)^+}\right)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)$$
$$+ \left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}\right)^-}\right)\Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)\Bigg]$$
$$= \frac{\rho(z_2^\rho - z_1^\rho)}{4}\Bigg[\int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\,\Upsilon'\left(a^\rho + b^\rho - \left(\frac{\lambda^\rho}{2}z_1^\rho + \frac{2-\lambda^\rho}{2}z_2^\rho\right)\right)d\lambda$$

$$- \int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\, \Upsilon'\Big(a^\rho + b^\rho - \Big(\frac{2-\lambda^\rho}{2}z_1^\rho + \frac{\lambda^\rho}{2}z_2^\rho\Big)\Big)d\lambda\Big]. \tag{25}$$

Proof. The technique to prove this Lemma is similar to that of Lemma 3.14. □

Remark 3.20. *For $z_1 = a$ and $z_2 = b$ in Lemma 4.1, we will get Lemma 1 proved in [12].*

Remark 3.21. *For $\rho = 1$ in Lemma 4.1, we will get Lemma 2 proved in [10].*

Remark 3.22. *For $\rho = 1$, $z_1 = a$ and $z_2 = b$ in this Lemma 4.1, we will get Lemma 3 proved in [17].*

Theorem 3.23. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to R$ is a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|\Upsilon'|$ is convex on $[a^\rho, b^\rho]$, then one has the following statement:*

$$\begin{aligned}
&\Bigg|\Upsilon\Big(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2}\Big) - \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{\big(z_2^\rho - z_1^\rho\big)^\alpha}\Big[\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}\right)^+}\Big)\Upsilon\big(a^\rho + b^\rho - z_1^\rho\big) \\
&\quad + \Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}\right)^-}\Big)\Upsilon\big(a^\rho + b^\rho - z_2^\rho\big)\Big]\Bigg| \\
&\le \frac{z_2^\rho - z_1^\rho}{2(\alpha+1)}\Big\{\big|\Upsilon'(a^\rho)\big| + \big|\Upsilon'(b^\rho)\big| - \Big(\frac{\big|\Upsilon'(z_1^\rho)\big| + \big|\Upsilon'(z_2^\rho)\big|}{2}\Big)\Big\}
\end{aligned} \tag{26}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$.

Proof. From Lemma 4.1 and Jensen-Mercer's inequality, we have

$$\begin{aligned}
&\Bigg|\Upsilon\Big(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2}\Big) - \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{\big(z_2^\rho - z_1^\rho\big)^\alpha}\Big[\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}\right)^+}\Big)\Upsilon\big(a^\rho + b^\rho - z_1^\rho\big) \\
&\quad + \Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}\right)^-}\Big)\Upsilon\big(a^\rho + b^\rho - z_2^\rho\big)\Big]\Bigg| \\
&\le \frac{\rho(z_2^\rho - z_1^\rho)}{4}\Big[\int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\Big|\Upsilon'\Big(a^\rho + b^\rho - \Big(\frac{\lambda^\rho}{2}z_1^\rho + \frac{2-\lambda^\rho}{2}z_2^\rho\Big)\Big)\Big|d\lambda \\
&\quad + \int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\Big|\Upsilon'\Big(a^\rho + b^\rho - \Big(\frac{2-\lambda^\rho}{2}z_1^\rho + \frac{\lambda^\rho}{2}z_2^\rho\Big)\Big)\Big|d\lambda\Big] \\
&\le \frac{\rho(z_2^\rho - z_1^\rho)}{4}\Big[\int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\Big\{\big|\Upsilon'(a^\rho)\big| + \big|\Upsilon'(b^\rho)\big| - \Big(\frac{\lambda^\rho}{2}\big|\Upsilon'(z_1^\rho)\big| + \frac{(2-\lambda^\rho)}{2}\big|\Upsilon'(z_2^\rho)\big|\Big)\Big\}d\lambda \\
&\quad + \int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\Big\{\big|\Upsilon'(a^\rho)\big| + \big|\Upsilon'(b^\rho)\big| - \Big(\frac{(2-\lambda^\rho)}{2}\big|\Upsilon'(z_1^\rho)\big| + \frac{\lambda^\rho}{2}\big|\Upsilon'(z_2^\rho)\big|\Big)\Big\}d\lambda\Big].
\end{aligned}$$

Now simplifying the integrals gives (27). □

Remark 3.24. *For $\rho = 1$ in Theorem 3.26, we will get Theorem 5 proved in [10].*

Remark 3.25. *For $z_1 = a$ and $z_2 = b$ and $\rho = 1$ in Theorem 3.26, we will get following inequality proved by Sarikaya and Yildrim in [18].*

$$\Big|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Big[I^\alpha_{\left(\frac{a+b}{2}\right)^+}\Upsilon(b) + I^\alpha_{\left(\frac{a+b}{2}\right)^-}\Upsilon(a)\Big| \le \frac{b-a}{2(\alpha+1)}\Big\{\frac{\big|\Upsilon'(a)\big| + \big|\Upsilon'(b)\big|}{2}\Big\}.$$

Now, we will state following result of our paper for function $|\Upsilon'|^q$ to be convex as:

Theorem 3.26. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to R$ is a differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|\Upsilon'|^q$ is convex on $[a^\rho, b^\rho]$, $q > 1$, then one has the following statement:*

$$\Bigg|\Upsilon\Big(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2}\Big) - \frac{2^{\alpha-1}\rho^\alpha\Gamma\big(\alpha+1\big)}{\big(z_2^\rho - z_1^\rho\big)^\alpha}\Big[\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}\right)^+}\Big)\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big)$$
$$+\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}\right)^-}\Big)\Upsilon\Big(a^\rho + b^\rho - z_2^\rho\Big)\Big]\Bigg| \le \frac{z_2^\rho - z_1^\rho}{16}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}$$
$$\times\left[4^{\frac{1}{q}}.2\left(\left|\Upsilon'\left(a^\rho\right)\right| + \left|\Upsilon'\left(b^\rho\right)\right|\right) - \left(3^{\frac{1}{q}}+1\right)\left(\left|\Upsilon'\left(z_1^\rho\right)\right| + \left|\Upsilon'\left(z_2^\rho\right)\right|\right)\right] \tag{27}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$.

Proof. By Lemma 3.6 and if we use Hölder's integral inequality along with Jensen-Mercer's inequality for the convex function $|\Upsilon'|^q$, we have

$$\Bigg|\Upsilon\Big(a^\rho + b^\rho - \frac{z_1^\rho + z_2^\rho}{2}\Big) - \frac{2^{\alpha-1}\rho^\alpha\Gamma\big(\alpha+1\big)}{\big(z_2^\rho - z_1^\rho\big)^\alpha}\Big[\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}\right)^+}\Big)\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big)$$
$$+\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-\frac{z_1^\rho+z_2^\rho}{2}\right)^-}\Big)\Upsilon\Big(a^\rho + b^\rho - z_2^\rho\Big)\Big]\Bigg|$$

$$\le \frac{\rho(z_2^\rho - z_1^\rho)}{4}\Big[\Big(\int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\,d\lambda\Big)\Big(\int_0^1\Big|\Upsilon'\Big(a^\rho + b^\rho - \Big(\frac{\lambda^p}{2}z_1^\rho + \frac{2-\lambda^\rho}{2}z_2^\rho\Big)\Big)\Big|\,d\lambda\Big)$$
$$+\Big(\int_0^1 \lambda^{\rho\alpha}\lambda^{\rho-1}\,d\lambda\Big)\Big(\int_0^1\Big|\Upsilon'\Big(a^\rho + b^\rho - \Big(\frac{2-\lambda^\rho}{2}z_1^\rho + \frac{\lambda^\rho}{2}z_2^\rho\Big)\Big)\Big|d\lambda\Big)\Big]$$
$$\le \frac{(z_2^\rho - z_1^\rho)}{4}\Big[\Big(\int_0^1 \lambda^{\rho\left(\alpha p\right)}\rho\lambda^{\rho-1}\,d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|\Upsilon'\Big(a^\rho + b^\rho - \Big(\frac{\lambda^p}{2}z_1^\rho + \frac{2-\lambda^\rho}{2}z_2^\rho\Big)\Big)\Big|^q\rho\lambda^{\rho-1}\,d\lambda\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1 \lambda^{\rho\left(\alpha p\right)}\rho\lambda^{\rho-1}\,d\lambda\Big)^{\frac{1}{p}}\Big(\int_0^1\Big|\Upsilon'\Big(a^\rho + b^\rho - \Big(\frac{2-\lambda^\rho}{2}z_1^\rho + \frac{\lambda^\rho}{2}z_2^\rho\Big)\Big)\Big|^q\rho\lambda^{\rho-1}\,d\lambda\Big)^{\frac{1}{q}}\Big]$$
$$\le \frac{(z_2^\rho - z_1^\rho)}{4}\Big(\frac{1}{\alpha p+1}\Big)^{\frac{1}{p}}$$
$$\times\Big[\Big\{\int_0^1\Big(\Big|\Upsilon'\Big(a^\rho\Big)\Big|^q + \Big|\Upsilon'\Big(b^\rho\Big)\Big|^q - \Big(\frac{\lambda^\rho}{2}\Big|\Upsilon'\Big(z_1^\rho\Big)\Big|^q + \frac{2-\lambda^\rho}{2}\Big|\Upsilon'\Big(z_2^\rho\Big)\Big|^q\Big)\Big)\rho\lambda^{\rho-1}d\lambda\Big\}^{\frac{1}{q}}$$
$$+\Big\{\int_0^1\Big(\Big|\Upsilon'\Big(a^\rho\Big)\Big|^q + \Big|\Upsilon'\Big(b^\rho\Big)\Big|^q - \Big(\frac{2-\lambda^\rho}{2}\Big|\Upsilon'\Big(z_1^\rho\Big)\Big|^q + \frac{\lambda^\rho}{2}\Big|\Upsilon'\Big(z_2^\rho\Big)\Big|^q\Big)\Big)\rho\lambda^{\rho-1}d\lambda\Big\}^{\frac{1}{q}}\Big]$$
$$\le \frac{z_2^\rho - z_1^\rho}{4}\Big(\frac{1}{\alpha p+1}\Big)^{\frac{1}{p}}\Big[\Big(\Big|\Upsilon'\Big(a^\rho\Big)\Big|^q + \Big|\Upsilon'\Big(b^\rho\Big)\Big|^q - \Big(\frac{\Big|\Upsilon'\Big(z_1^\rho\Big)\Big|^q + 3\Big|\Upsilon'\Big(z_2^\rho\Big)\Big|^q}{4}\Big)\Big)^{\frac{1}{q}}$$
$$+\Big(\Big|\Upsilon'\Big(a^\rho\Big)\Big|^q + \Big|\Upsilon'\Big(b^\rho\Big)\Big|^q - \Big(\frac{3\Big|\Upsilon'\Big(z_1^\rho\Big)\Big|^q + \Big|\Upsilon'\Big(z_2^\rho\Big)\Big|^q}{4}\Big)\Big)^{\frac{1}{q}}\Big].$$

After simplification we will get inequality 27. □

Remark 3.27. *For $\rho = 1$ in Theorem 3.26, we will get Theorem 6 proved in [10].*

Remark 3.28. *For $\rho = 1$, $z_1 = a$ and $z_2 = b$ in Theorem 3.26, we will get Theorem 6 proved in [17].*

4. New Identities and Related Results for Differentiable Function λ''

Now, we will state the final result of our paper for function $|\Upsilon''|^q$ to be convex by using previous results:

Lemma 4.1. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to \Re$ is a differentiable mapping on (a^ρ, b^ρ) with $0 \leq a < b$, then the following equality for Katugampola-fractional integrals holds:*

$$\frac{\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big) + \Upsilon\Big(a^\rho + b^\rho - z_2^\rho\Big)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma\Big(\alpha\Big)}{\Big(z_2^\rho - z_1^\rho\Big)^\alpha}\Bigg(\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\Big)\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big)$$
$$+ \Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\Big)\Upsilon\Big(a^\rho b^\rho - z_2^\rho\Big)\Bigg)$$
$$= \frac{(z_2^\rho - z_1^\rho)^2}{\alpha\Big(\alpha+1\Big)}\Bigg[\int_0^1 \Big[1 - \lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}\Upsilon''\Big(a^\rho + b^\rho - \Big((1-\lambda^\rho)z_1^\rho + \lambda^\rho z_2^\rho\Big)\Big)d\lambda$$
$$- \int_0^1 \lambda^{\rho(\alpha+2)-1}\Upsilon''\Big(a^\rho + b^\rho - \Big(\lambda^\rho z_1^\rho + (1-\lambda^\rho)\, z_2^\rho\Big)\Big)d\lambda\Bigg] \tag{28}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$ and $\Gamma(.)$ is the Gamma function.

Proof. Let

$$K_1 = \int_0^1 \Big[1 - \lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}\Upsilon''\Big(a^\rho + b^\rho - \Big((1-\lambda^\rho)z_1^\rho + \lambda^\rho z_2^\rho\Big)\Big)d\lambda$$

and

$$K_2 = \int_0^1 \lambda^{\rho(\alpha+2)-1}\Upsilon''\Big(a^\rho + b^\rho - \Big(\lambda^\rho z_1^\rho + (1-\lambda^\rho)\, z_2^\rho\Big)\Big)d\lambda$$

By using integration by parts, we have

$$K_1 = \int_0^1 \Big[1 - \lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}\Upsilon''\Big(a^\rho + b^\rho - \Big((1-\lambda^\rho)z_1^\rho + \lambda^\rho z_2^\rho\Big)\Big)d\lambda$$
$$= \frac{\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big)}{\rho(z_2^\rho - z_1^\rho)} - \frac{\alpha+1}{z_2^\rho - z_1^\rho}\int_0^1 \lambda^{\rho(\alpha+1)-1}\, \Upsilon'\Big(a^\rho + b^\rho - \Big((1-\lambda^\rho)z_1^\rho + \lambda^\rho z_2^\rho\Big)\Big)d\lambda \tag{29}$$

and

$$K_2 = \int_0^1 \lambda^{\rho(\alpha+2)-1}\Upsilon''\Big(a^\rho + b^\rho - \Big(\lambda^\rho z_1^\rho + (1-\lambda^\rho)\, z_2^\rho\Big)\Big)d\lambda$$
$$= \frac{\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big)}{\rho(z_2^\rho - z_1^\rho)} - \frac{\alpha+1}{z_2^\rho - z_1^\rho}\int_0^1 \lambda^{\rho(\alpha+1)-1}\, \Upsilon'\Big(a^\rho + b^\rho - \Big(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\Big)\Big)d\lambda \tag{30}$$

Using (29) and (30), we have

$$K_1 - K_2 = \frac{\alpha+1}{(z_2^\rho - z_1^\rho)}\int_0^1 \lambda^{\rho(\alpha+1)-1}$$
$$\times\Bigg[\Upsilon'\Big(a^\rho + b^\rho - \Big(\lambda^\rho z_1^\rho + (1-\lambda^\rho)\, z_2^\rho\Big)\Big) - \Upsilon'\Big(a^\rho + b^\rho - \Big((1-\lambda^\rho)z_1^\rho + \lambda^\rho z_2^\rho\Big)\Big)\Bigg]d\lambda \tag{31}$$

The desired identity in (28) follows from (31) by using (14) and rearranging the terms. □

Remark 4.2. *For $z_1 = a$ and $z_2 = b$ in Lemma 4.1, we will get Lemma 3 proved in [22].*

Theorem 4.3. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to R$ is twice differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|\Upsilon''|^q$ is convex on $[a^\rho, b^\rho]$, $q > 1$, then one has the following statement:*

$$\Bigg|\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2\left(z_2^\rho - z_1^\rho\right)^\alpha}\Big[\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\Big)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)$$
$$+ \Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\Big)\Upsilon\left(a^\rho b^\rho - z_2^\rho\right)\Big]\Bigg| \le \frac{\rho(z_2^\rho - z_1^\rho)^2}{2(\alpha+1)}\Big[\Big(\frac{s(\alpha+1)}{\rho\left(s(\alpha+1)+1\right)}\Big)^{\frac{1}{s}}\Big(\frac{1}{\rho}\left|\Upsilon''\left(a^\rho\right)\right|^q$$
$$+ \frac{1}{\rho}\left|\Upsilon''\left(b^\rho\right)\right|^q - \frac{1}{2\rho}\left|\Upsilon''\left(x^\rho\right)\right|^q - \frac{1}{2\rho}\left|\Upsilon''\left(y^\rho\right)\right|^q\Big)^{\frac{1}{q}} + \Big(\frac{1}{s\rho(\alpha+2) - s + 1}\Big)^{\frac{1}{s}}\Big(\left|\Upsilon''\left(a^\rho\right)\right|^q$$
$$+ \left|\Upsilon''\left(b^\rho\right)\right|^q - \frac{1}{(\rho+1)}\left|\Upsilon''\left(x^\rho\right)\right|^q - \frac{\rho}{(\rho+1)}\left|\Upsilon''\left(y^\rho\right)\right|^q\Big)^{\frac{1}{q}}\Big] \tag{32}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$, where $\frac{1}{s} + \frac{1}{q} = 1$.

Proof. From Lemma 4.1 and if we use Hölder's integral inequality along with Jensen-Mercer's inequality for the convex function $|\Upsilon''|^q$, we have

$$\Bigg|\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{2} - \frac{\rho^\alpha \Gamma(\alpha+1)}{2\left(z_2^\rho - z_1^\rho\right)^\alpha}\Big[\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\Big)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)$$
$$+ \Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\Big)\Upsilon\left(a^\rho b^\rho - z_2^\rho\right)\Big]\Bigg|$$

$$\le \frac{\rho(z_2^\rho - z_1^\rho)^2}{2(\alpha+1)}\Big[\Big(\int_0^1 \left|1 - \lambda^{\rho(\alpha+1)}\right|^s \lambda^{\rho-1} d\lambda\Big)^{\frac{1}{s}}\Big(\int_0^1 \lambda^{\rho-1}\Big|\Upsilon''\Big(a^\rho + b^\rho$$
$$- \left((1-\lambda^\rho)z_1^\rho + \lambda^\rho z_2^\rho\right)\Big|^q d\lambda\Big)^{\frac{1}{q}} + \Big(\int_0^1 \lambda^{s\rho(\alpha+2)-s} d\lambda\Big)^{\frac{1}{s}}$$
$$\times\Big(\int_0^1 \Big|\Upsilon''\Big(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho)z_2^\rho\right)\Big|^q d\lambda\Big)^{\frac{1}{q}}\Big]$$
$$\le \frac{\rho(z_2^\rho - z_1^\rho)^2}{2(\alpha+1)}\Big[\Big(\int_0^1 \left|1 - \lambda^{\rho(\alpha+1)}\right|^s \lambda^{\rho-1} d\lambda\Big)^{\frac{1}{s}}\Big\{\left|\Upsilon''\left(a^\rho\right)\right|^q \int_0^1 \lambda^{\rho-1} d\lambda$$
$$+ \left|\Upsilon''\left(b^\rho\right)\right|^q \int_0^1 \lambda^{\rho-1} d\lambda - \left|\Upsilon''\left(z_1^\rho\right)\right|^q \int_0^1 \lambda^{\rho-1}(1-\lambda^\rho) d\lambda - \left|\Upsilon''\left(z_2^\rho\right)\right|^q \int_0^1 \lambda^{\rho-1}\lambda^\rho d\lambda\Big\}^{\frac{1}{q}}$$
$$+ \Big(\int_0^1 \lambda^{s\rho(\alpha+2)-s} d\lambda\Big)^{\frac{1}{s}}\Big\{\left|\Upsilon''\left(a^\rho\right)\right|^q + \left|\Upsilon''\left(b^\rho\right)\right|^q$$
$$- \left|\Upsilon''\left(z_1^\rho\right)\right|^q \int_0^1 \lambda^\rho d\lambda - \left|\Upsilon''\left(z_2^\rho\right)\right|^q \int_0^1 (1-\lambda^\rho) d\lambda\Big\}^{\frac{1}{q}}\Big]$$

After simplification we will get inequality 32. □

Remark 4.4. *For $z_1 = a$ and $z_2 = b$ in Theorem 4.3, we will get Corollary 2 proved in [22].*

Theorem 4.5. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to R$ is twice differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|\Upsilon''|^q$ is convex on $[a^\rho, b^\rho]$, $q > 1$, then one has the following statement:*

$$\Bigg|\frac{\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big) + \Upsilon\Big(a^\rho + b^\rho - z_2^\rho\Big)}{2} - \frac{\rho^\alpha \Gamma\big(\alpha+1\big)}{2\Big(z_2^\rho - z_1^\rho\Big)^\alpha}\Big[\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\Big)\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big)$$
$$+\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\Big)\Upsilon\Big(a^\rho b^\rho - z_2^\rho\Big)\Big]\Bigg| \le \frac{(z_2^\rho - z_1^\rho)^2}{2(\alpha+1)}\Big[\Big(\frac{(\alpha+1)}{(\alpha+1)}\Big)^{1-\frac{1}{q}}\Big(\frac{\alpha+1}{\alpha+2}\Big|\Upsilon''\Big(a^\rho\Big)\Big|^q$$
$$+\frac{\alpha+1}{\alpha+2}\Big|\Upsilon''\Big(b^\rho\Big)\Big|^q - \Big(\frac{(\alpha+1)(\alpha+4)}{2(\alpha+3)(\alpha+2)}\Big)\Big|\Upsilon''\Big(x^\rho\Big)\Big|^q - \Big(\frac{(\alpha+1)}{2(\alpha+3)}\Big)\Big|\Upsilon''\Big(y^\rho\Big)\Big|^q\Big)^{\frac{1}{q}}$$
$$+\Big(\frac{1}{(\alpha+2)}\Big)^{1-\frac{1}{q}}\Big(\frac{1}{\alpha+2}\Big|\Upsilon''\Big(a^\rho\Big)\Big|^q$$
$$+\frac{1}{\alpha+2}\Big|\Upsilon''\Big(b^\rho\Big)\Big|^q - \frac{1}{(\alpha+3)}\Big|\Upsilon''\Big(x^\rho\Big)\Big|^q - \frac{1}{(\alpha+2)(\alpha+3)}\Big\|\Upsilon''\Big(y^\rho\Big)\Big|^q\Big)^{\frac{1}{q}}\Big] \tag{33}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$, where $\frac{1}{s} + \frac{1}{q} = 1$

Proof. From Lemma 4.1 and if we use Power mean inequality along with Jensen-Mercer's inequality for the convex function $|\Upsilon''|^q$, we have

$$\Bigg|\frac{\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big) + \Upsilon\Big(a^\rho + b^\rho - z_2^\rho\Big)}{\alpha\rho} - \frac{\rho^{\alpha-1}\Gamma\big(\alpha\big)}{2\Big(z_2^\rho - z_1^\rho\Big)^\alpha}\Big[\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\Big)\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big)$$
$$+\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\Big)\Upsilon\Big(a^\rho b^\rho - z_2^\rho\Big)\Big]\Bigg|$$

$$\le \frac{(z_2^\rho - z_1^\rho)^2}{\alpha(\alpha+1)}\Big[\Big(\int_0^1 \Big[1-\lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}d\lambda\Big)^{1-\frac{1}{q}}\Big(\int_0^1 \Big[1-\lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}\Big|\Upsilon''\Big(a^\rho + b^\rho$$
$$-\Big((1-\lambda^\rho)z_1^\rho + \lambda^\rho z_2^\rho\Big)\Big|^q d\lambda\Big)^{\frac{1}{q}} + \Big(\int_0^1 \lambda^{\rho(\alpha+2)-1}d\lambda\Big)^{1-\frac{1}{q}}$$
$$\times\Big(\int_0^1 \lambda^{\rho(\alpha+2)-1}\Big|\Upsilon''\Big(a^\rho + b^\rho - \Big(\lambda^\rho z_1^\rho + (1-\lambda^\rho)z_2^\rho\Big)\Big|^q d\lambda\Big)^{\frac{1}{q}}\Big]$$
$$\le \frac{(z_2^\rho - z_1^\rho)^2}{\alpha(\alpha+1)}\Big[\Big(\int_0^1 \Big[1-\lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}d\lambda\Big)^{1-\frac{1}{q}}\Big\{\Big|\Upsilon''\Big(a^\rho\Big)\Big|^q \int_0^1 \Big[1-\lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}d\lambda$$
$$+\Big|\Upsilon''\Big(b^\rho\Big)\Big|^q \int_0^1 \Big[1-\lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}d\lambda - \Big|\Upsilon''\Big(z_1^\rho\Big)\Big|^q \int_0^1 \Big[1-\lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}(1-\lambda^\rho)d\lambda$$
$$-\Big|\Upsilon''\Big(z_2^\rho\Big)\Big|^q \int_0^1 \Big[1-\lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}\lambda^\rho d\lambda\Big\}^{\frac{1}{q}}$$
$$+\Big(\int_0^1 \lambda^{\rho(\alpha+2)-1}d\lambda\Big)^{1-\frac{1}{q}}\Big\{\Big|\Upsilon''\Big(a^\rho\Big)\Big|^q \int_0^1 \lambda^{\rho(\alpha+2)-1}d\lambda + \Big|\Upsilon''\Big(b^\rho\Big)\Big|^q \int_0^1 \lambda^{\rho(\alpha+2)-1}d\lambda$$
$$-\Big|\Upsilon''\Big(z_1^\rho\Big)\Big|^q \int_0^1 \lambda^{\rho(\alpha+2)-1}\lambda^\rho d\lambda - \Big|\Upsilon''\Big(z_2^\rho\Big)\Big|^q \int_0^1 \lambda^{\rho(\alpha+2)-1}(1-\lambda^\rho)d\lambda\Big\}^{\frac{1}{q}}\Big]$$

After simplification we will get inequality 33. □

Remark 4.6. *For $z_1 = a$ and $z_2 = b$ in Theorem 4.5, we will get Corollary 1 proved in [22].*

Lemma 4.7. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to \mathfrak{R}$ is twice differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$, then the following equality for Katugampola-fractional integrals holds:*

$$\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{2} - \frac{\rho^\alpha \Gamma\left(\alpha+1\right)}{\left(z_2^\rho - z_1^\rho\right)^\alpha}\Big(\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\right)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)$$
$$+\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\right)\Upsilon\left(a^\rho b^\rho - z_2^\rho\right)\Big)$$
$$= \frac{(z_2^\rho - z_1^\rho)^2}{\alpha\left(\alpha+1\right)}\int_0^1 \left[1-(1-\lambda^\rho)^{\alpha+1} - \lambda^{\rho(\alpha+1)}\right]\lambda^{\rho-1}\Upsilon''\left(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right)\right)d\lambda \tag{34}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$ and $\Gamma(.)$ is the Gamma function.

Proof. We start by considering the following computation which is a direct application of integration by parts.

$$= \int_0^1 \left[1-(1-\lambda^\rho)^{\alpha+1} - \lambda^{\rho(\alpha+1)}\right]\lambda^{\rho-1}\Upsilon''\left(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\right)\right)d\lambda$$
$$= \frac{\alpha+1}{(z_2^\rho - z_1^\rho)}\int_0^1 \left[\lambda^{\rho\alpha} - (1-\lambda^\rho)^\alpha\right]\lambda^{\rho-1}\,\Upsilon'\left(a^\rho + b^\rho - \left(\lambda^\rho z_1^\rho + (1-\lambda^\rho)\, z_2^\rho\right)\right)d\lambda \tag{35}$$

The intended identity in (34) follows from (35) by using (18) and rearranging the terms. □

Remark 4.8. *For $z_1 = a$ and $z_2 = b$ in Lemma 4.7, we will get Lemma 4 proved in [22].*

Theorem 4.9. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to R$ is twice differentiable mapping on (a^ρ, b^ρ) with $0 \le a < b$. If $|\Upsilon''|^q$ is convex on $[a^\rho, b^\rho]$, $q > 1$, then one has the following statement:*

$$\Bigg|\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{2} - \frac{\rho^\alpha \Gamma\left(\alpha+1\right)}{2\left(z_2^\rho - z_1^\rho\right)^\alpha}\Big[\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\right)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)$$
$$+\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\right)\Upsilon\left(a^\rho b^\rho - z_2^\rho\right)\Big]\Bigg| \le \frac{(z_2^\rho - z_1^\rho)^2}{2\rho(\alpha+1)}\Bigg[\left(\frac{s(\alpha+1)-1}{s(\alpha+1)+1}\right)^{\frac{1}{s}}\Big(\left|\Upsilon''\left(a^\rho\right)\right|^q$$
$$+\left|\Upsilon''\left(b^\rho\right)\right|^q - \frac{1}{2}\left|\Upsilon''\left(x^\rho\right)\right|^q - \frac{1}{2}\left|\Upsilon''\left(y^\rho\right)\right|^q\Big)^{\frac{1}{q}}\Bigg] \tag{36}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$, where $\frac{1}{s} + \frac{1}{q} = 1$.

Proof. From Lemma 4.7 and if we use the Hölder's integral inequality along with Jensen-Mercer's inequality for the convex function $|\Upsilon''|^q$, we have

$$\Bigg|\frac{\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right) + \Upsilon\left(a^\rho + b^\rho - z_2^\rho\right)}{2} - \frac{\rho^\alpha \Gamma\left(\alpha+1\right)}{2\left(z_2^\rho - z_1^\rho\right)^\alpha}\Big[\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\right)\Upsilon\left(a^\rho + b^\rho - z_1^\rho\right)$$
$$+\left({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\right)\Upsilon\left(a^\rho b^\rho - z_2^\rho\right)\Big]\Bigg|$$

$$\leq \frac{(z_2^\rho - z_1^\rho)^2}{2(\alpha+1)}\Big(\int_0^1 \Big|1-(1-\lambda^\rho)^{\alpha+1} - \lambda^{\rho(\alpha+1)}\Big|^s \lambda^{\rho-1} d\lambda\Big)^{\frac{1}{s}}$$

$$\times\Big(\int_0^1 \lambda^{\rho-1}\Big|\Upsilon''\Big(a^\rho + b^\rho - \Big(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\Big)\Big|^q d\lambda\Big)^{\frac{1}{q}}\Big)$$

$$\leq \frac{(z_2^\rho - z_1^\rho)^2}{2(\alpha+1)}\Big(\int_0^1 \Big|1-(1-\lambda^\rho)^{\alpha+1} - \lambda^{\rho(\alpha+1)}\Big|^s \lambda^{\rho-1} d\lambda\Big)^{\frac{1}{s}}\Big\{\int_0^1 \lambda^{\rho-1}\Big(\Big|\Upsilon''\Big(a^\rho\Big)\Big|^q$$

$$-\Big(\lambda^\rho\Big|\Upsilon''\Big(z_1^\rho\Big)\Big|^q + (1-\lambda^\rho)\Big|\Upsilon''\Big(z_2^\rho\Big)\Big|^q\Big)\Big)d\lambda\Big\}^{\frac{1}{q}}\Big]$$

$$\leq \frac{(z_2^\rho - z_1^\rho)^2}{2(\alpha+1)}\Big(\int_0^1 \Big|1-(1-\lambda^\rho)^{\alpha+1} - \lambda^{\rho(\alpha+1)}\Big|^s \lambda^{\rho-1} d\lambda\Big)^{\frac{1}{s}}\Big\{\Big|\Upsilon''\Big(a^\rho\Big)\Big|^q$$

$$+\Big|\Upsilon''\Big(b^\rho\Big)\Big|^q - \Big|\Upsilon''\Big(z_1^\rho\Big)\Big|^q - \int_0^1 \lambda^{\rho-1}\lambda^\rho d\lambda - \Big|\Upsilon''\Big(z_2^\rho\Big)\Big|^q \int_0^1 (1-\lambda^\rho)\lambda^{\rho-1} d\lambda\Big\}^{\frac{1}{q}}$$

After simplification we will get inequality 36. □

Remark 4.10. *For $z_1 = a$ and $z_2 = b$ in Theorem 4.9, we will get Corollary 4 proved in [22].*

Theorem 4.11. *Suppose that $\Upsilon : [a^\rho, b^\rho] \to R$ is twice differentiable mapping on (a^ρ, b^ρ) with $0 \leq a < b$. If $|\Upsilon''|^q$ is convex on $[a^\rho, b^\rho]$, $q > 1$, then one has the following statement:*

$$\Big|\frac{\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big) + \Upsilon\Big(a^\rho + b^\rho - z_2^\rho\Big)}{2} - \frac{\rho^\alpha \Gamma\Big(\alpha+1\Big)}{2\Big(z_2^\rho - z_1^\rho\Big)^\alpha}\Big[\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\Big)\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big)$$

$$+\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\Big)\Upsilon\Big(a^\rho b^\rho - z_2^\rho\Big)\Big]\Big| \leq \frac{(z_2^\rho - z_1^\rho)^2}{2\rho(\alpha+1)}\Big[\Big(\frac{(\alpha)}{(\alpha+2)}\Big)^{1-\frac{1}{q}}\Big(\frac{\alpha}{\alpha+2}\Big|\Upsilon''\Big(a^\rho\Big)\Big|^q$$

$$+\frac{\alpha}{\alpha+2}\Big|\Upsilon''\Big(b^\rho\Big)\Big|^q - \Big(\frac{(\alpha+1)}{2(\alpha+3)} - B\Big(2,\alpha+2\Big)\Big)\Big|\Upsilon''\Big(x^\rho\Big)\Big|^q$$

$$-\Big(\frac{(\alpha^2 - 2 + 3\alpha)}{2(\alpha+3)(\alpha+2)} - B\Big(2,\alpha+2\Big)\Big)\Big|\Upsilon''\Big(y^\rho\Big)\Big|^q\Big)^{\frac{1}{q}}\Big] \tag{37}$$

for all $z_1^\rho, z_2^\rho \in [a^\rho, b^\rho]$, $\alpha > 0$, $\lambda \in [0,1]$ and $B(.,.)$ is the Beta function.

Proof. From Lemma 4.7 and if we use the Power mean inequality along with Jensen-Mercer's inequality for the convex function $|\Upsilon''|^q$, we have

$$\Big|\frac{\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big) + \Upsilon\Big(a^\rho + b^\rho - z_2^\rho\Big)}{2} - \frac{\rho^\alpha \Gamma\Big(\alpha\Big)}{2\Big(z_2^\rho - z_1^\rho\Big)^\alpha}\Big[\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_2^\rho\right)^+}\Big)\Upsilon\Big(a^\rho + b^\rho - z_1^\rho\Big)$$

$$+\Big({}^\rho I^\alpha_{\left(a^\rho+b^\rho-z_1^\rho\right)^-}\Big)\Upsilon\Big(a^\rho b^\rho - z_2^\rho\Big)\Big]\Big|$$

$$\leq \frac{(z_2^\rho - z_1^\rho)^2}{2(\alpha+1)}\Big[\Big(\int_0^1 \Big[1-(1-\lambda^\rho)^{\alpha+1} - \lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1} d\lambda\Big)^{1-\frac{1}{q}}$$

$$\times\Big(\int_0^1 \Big[1-(1-\lambda^\rho)^{\alpha+1} - \lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}\Big|\Upsilon''\Big(a^\rho + b^\rho - \Big(\lambda^\rho z_1^\rho + (1-\lambda^\rho) z_2^\rho\Big)\Big|^q d\lambda\Big)^{\frac{1}{q}}$$

$$\leq \frac{(z_2^\rho - z_1^\rho)^2}{2(\alpha+1)}\Bigg[\bigg(\int_0^1\Big[1-(1-\lambda^\rho)^{\alpha+1}-\lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}d\lambda\bigg)^{1-\frac{1}{q}}$$

$$\times\Bigg\{\Big|\Upsilon''\big(a^\rho\big)\Big|^q\int_0^1\Big[1-(1-\lambda^\rho)^{\alpha+1}-\lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}d\lambda\ +\Big|\Upsilon''\big(b^\rho\big)\Big|^q$$

$$\times\int_0^1\Big[1-(1-\lambda^\rho)^{\alpha+1}-\lambda^{\rho(\alpha+1)}\Big]\lambda^{\rho-1}d\lambda-\Big|\Upsilon''\big(z_1^\rho\big)\Big|^q\int_0^1\Big[1-(1-\lambda^\rho)^{\alpha+1}-\lambda^{\rho(\alpha+1)}\Big]\lambda^\rho d\lambda$$

$$-\Big|\Upsilon''\big(z_2^\rho\big)\Big|^q\int_0^1\Big[1-(1-\lambda^\rho)^{\alpha+1}-\lambda^{\rho(\alpha+1)}\Big]\lambda^\rho(1-\lambda^\rho)d\lambda$$

After simplification we will get inequality 37. □

Remark 4.12. *For $z_1 = a$ and $z_2 = b$ in Theorem 4.11, we will get Corollary 4 proved in [22].*

5. Conclusion

In this paper, we have obtained some new Hermite-Jensen-Mercer type integral inequalities for Katugampola fractional integral operators. The results can be performed for different kinds of convexity and operators. However it is not easy to give Jensen-Mercer's inequality for other classes of convex functions. These results can be applied in convex analysis, optimization and different areas of pure and applied sciences. The authors hope that these results will serve as a motivation for future work in this fascinating area.

References

[1] A. Kilbas, H.M. Srivastava and J.J. Trujillo,*Theory and applications of fractional differential equations*, Elsevier B.V., Amsterdam, Netherlands, (2006).

[2] A. Matkovic, J. Pečarić and I. Perić, *A variant of Jensens inequality of Mercers type for operators with applications*Linear Algebra Appl., 418(2006), 551-564.

[3] A.McD. Mercer , *A Varient of Jensen's Inequality*. J.Ineq. Pure and Appl. Math 4(4) (2003), Article 73.

[4] A.B. Malinowska, T. Odzijewicz and D.F.M. Torres *Advanced Methods in the Fractional Calculus of Variations*, Springer, Berlin (2015).

[5] A.G. Butkovskii, S.S. Postnov and E.A. Postnova, *Fractional Integro-Differential Calculus and its Control-Theoretical Applications in Mathematical Fundamentals and the Problem of interpretation*, Autom Control, 74, (2013), 543-574.

[6] E. Anjidani and M.R. Changalvaiy, *Reverse Jensen-Mercer type operator inequalities*, Electron. J. Linear Algebra, 31 (2016), 87-99.

[7] E. Anjidani, *Jensen-Mercer Operator Inequalities Involving Superquadratic Functions*, Mediterr. J. Math., 18 (2018), 1660-5446.

[8] J. E. Pečarić, D. S. Mitrinovic and A. M. Fink, *Classical and New Inequalities in Analysis*, (1993): 1-2.

[9] K. S. Miller and B. Ross, *An introduction to fractional calculus and fractional differential equations*, A Wiley-Interscience Publication, John Wiley and Sons, Inc., New York, (1993).

[10] H. Ogulmus, M.Z. Sarikaya, *Hermite-Hadamard-Mercer Type Inequalities for fractional integrals* 2019.

[11] H. Chen, U.N. Katugampola, *Hermite-Hadamard and Hermite-Hadamar-Fejer type inequalityies for generalized fractional integrals*, J.Math. Anal. Appl. , 26(2013), 742-753.

[12] H. Yaldiz, A.O. Akdemir, *Katugampola Fractional Integrals within the Class of s-convex function*, Turkish Journal of Science, 3 (1), 2018, 40-50.

[13] M. M. Ali and A. R. Khan, *Generalized integral Mercer's inequality and integral means*, Journal of Inequalities and Special Functions, 10 (1) (2019), pp. 60-76.

[14] M. Kian, *Operator Jensen inequality for superquadratic functions*, Linear Algebra Appl. 456 (2014), 82-87.

[15] M. Niezgoda, *A generalization of Mercer's result on convex functions*, Nonlinear Analysis. 71 (2009), 277.

[16] M. Kian and M.S. Moslehian, *Refinements of the operator Jensen-Mercer inequality*Electron. J. Linear Algebra, 26(2013), 742-753.

[17] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, *Hermite-Hadamards inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, 57 (2013) 2403-2407, doi:10.1016/j.mcm.2011.12.048.

[18] M.Z. Sarikaya and H. Yildirim, *On Hermite-Hadamard type inequalities for Riemann Liouville fractional integrals*, Miskolc Mathematical Notes, 17 (2016), No. 2, 1049-1059.

[19] R. Gorenflo, *Fractional calculus: Some numerical methods. In: A. Carpinteri, F. Mainardi (Eds), Fractals and Fractional Calculus in Continuum Mechanics*, Springer-Verlag, Wien- New York, (1997), 277-290.

[20] S.I. Butt, M. Nadeem and J. Nasir *New Variant of Hermite-Jensen-Mercer Inequalities Via Riemann-Liouville Fractional Integral Operators* (submitted).

[21] S.S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula* Appl. Math. lett. 11(5),(1998), 91-95.

[22] S. Kermausuor, E.R. Nwaeze and A.M. Tameru, *New Integral Inequalities via the Katugampola Fractional Integrals for Functions Whose Second Derivatives Are Strongly η-Convex* Mathematics 2019, 7, 183; doi:10.3390/math7020183.

[23] S. Gaboury, R. Tremblay and B.J. Fugere, *Some Relations Involving a Generalized Fractional Derivative Operator*, J. Inequal. Appl. Article ID 167, (2013).

[24] U.N. Katugampola , *New approach to a Generalized Fractional Integral*, Appl. Math. Comput. (2011), 218(3), 860-865.

[25] U.N. Katugampola , *A New approach to a Generalized Fractional Derivatives*, Bull. Math. Anal. Appl. (2014), 6(4), 1-15.

[26] U.N. Katugampola , *Mellin transforms of the Generalized Fractional Integrals and Derivatives* , Appl. Math. Comput. (2015), 257, 566-580.

[27] M.Z. Sarikaya, N. Alp, On Hermite-Hadamard-Fejer type integral inequalities for generalized convex functions via local fractional integrals, Open Journal of Mathematical Sciences, Vol. 3 (2019), Issue 1, pp. 273-284.

[28] L.L. Dinh, D.P. Nguyen, M.A. Ragusa, On a non-local Sobolev-Galpern-type equa- tion associated with random noise, Bulletin of the Malaysian Mathematical Sciences So- ciety, 46 (6), (2023).

[29] E. Ilhan, Analysis of the spread of Hookworm infection with Caputo-Fabrizio fractional derivative, Turkish Journal of Science, 7 (1), 43–52, (2022).

[30] M.N. Cakaloglu, S. Aslan, A.O. Akdemir Hadamard type integral inequalities for differentiable $(h, m)-$convex functions, Eastern Anatolian Journal of Science, 7 (1), 12–18, (2021).