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Further results on $\mathcal{I}-$deferred statistical convergence

Chiranjib Choudhury[a], Shyamal Debnath[a], Ayhan Esi[b,*]

[a]*Department of Mathematics, Tripura University (A Central University), Suryamaninagar-799022, Agartala, India*
[b]*Department of Basic Eng. Sci.(Math.Sect), Malatya Turgut Ozal University, 44100, Malatya, Turkey*

Abstract. For a non-empty set X, an ideal $\mathcal{I}$ represents a family of subsets of X that is closed under taking finite unions and subsets of its elements. Considering $X = \mathbb{N}$, in the present study, we set forth with the new notion of $\mathcal{I}-$deferred statistical limit point, $\mathcal{I}-$deferred statistical cluster point and study various properties of the newly introduced notion. For a real valued sequence $x = (x_n)$, we prove that every $\mathcal{I}-$deferred statistical limit point is an $\mathcal{I}-$deferred statistical cluster point. Moreover, the collection of all $\mathcal{I}-$deferred statistical cluster points of x is a closed subset of $\mathbb{R}$. We also introduce the notion of $\mathcal{I}-$deferred statistical limit superior and inferior for real valued sequences and prove several interesting properties. In the end, we establish a necessary and sufficient condition under which a $\mathcal{I}-$deferred statistically bounded real valued sequence is $\mathcal{I}-$deferred statistically convergent.

1. Introduction

The notion of statistical convergence was first introduced by Fast [11] and Steinhaus [28] independently in the year 1951. Later on, it was further investigated and studied from the sequence space point of view by Fridy [12, 13], Šalát [22], and many others. For more details on statistical convergence, one may refer [14, 20] where one can find many more references.

In 2016, Küçükaslan and Yilmaztürk [17] introduced the notion of deferred statistical convergence as a generalization of statistical convergence. They used the notion of deferred Cesàro mean [1] to define such concept. Several investigations in this direction have been occurred due to Şengül et al. [27], and many others [7–10].

On the other hand, in 2001, the idea of $\mathcal{I}-$convergence was developed by Kostyrko et al. [16] mainly as an extension of statistical convergence. They showed that many other known notions of convergence were a particular type of $\mathcal{I}-$convergence by considering particular ideals. Consequently, this direction gradually gets more attention of the researchers and became one of the most active areas of research. Several investigations and extensions of $\mathcal{I}-$convergence can be found from the works of Demirci [6], Kostyrko et al. [15], Lahiri and Das [18], Mohiuddine and Hazarika [19], Šalát et al. [23], Tripathy and Hazarika [29–31], and many others.

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* Corresponding author: Ayhan Esi
Email addresses: chiranjibchoudhury123@gmail.com, chiranjib.mathematics@tripurauniv.in (Chiranjib Choudhury), shyamalnitamath@gmail.com, debnathshyamal@tripurauniv.ac.in (Shyamal Debnath), ayhan.esi@ozal.edu.tr, aesi23@hotmail.com (Ayhan Esi)

Combining the notion of statistical convergence and $\mathcal{I}-$convergence, in 2011, Das et al. [2] introduced the notion of $\mathcal{I}-$statistical convergence. Later on, several investigations in this direction has been occurred due to Debnath and Rakshit [5], Mursaleen et al. [21], and many others. For an extensive view of $\mathcal{I}-$statistical convergence, one may refer [3, 4, 24, 25].

Recently, Şengül et al. [26] extended the notion of $\mathcal{I}-$statistical convergence to $\mathcal{I}-$ deferred statistical convergence using deferred density. Motivated by their work, in this paper, we introduce the notion of $\mathcal{I}-$deferred statistical limit point, cluster point, limit superior, limit inferior and analyzed various properties of these concepts.

2. Definitions and Preliminaries

Definition 2.1. *[12] If K is a subset of the positive integers $\mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$. The natural density of K is given by*

$$d(K) = \lim_{n\to\infty} \frac{|K_n|}{n},$$

provided that the limit exists.

Definition 2.2. *[12] A sequence $x = (x_n)$ is said to be statistically convergent to l if for every $\varepsilon > 0$, the set*

$$A(\varepsilon) = \{n \in \mathbb{N} : |x_n - l| \geq \varepsilon\}$$

has natural density zero. l is called the statistical limit of the sequence (x_n) and symbolically, $st - lim\, x = l$.

Definition 2.3. *[17] Let $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ denote the sequences of whole numbers satisfying*

$$q(n) - p(n) \geq 1 \text{ and } \lim_{n\to\infty} q(n) = \infty.$$

A sequence $x = (x_n)$ is said to be deferred statistically convergent to l if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))} \left|\{p(n) < k < q(n) : |x_k - l| \geq \varepsilon\}\right| = 0.$$

Symbolically, $DS_{p,q} \lim x = l$ or $\lim_{n\to\infty} x_n = l(DS_{p,q})$.

Definition 2.4. *[16] A family $\mathcal{I} \subset 2^X$ of subsets of a nonempty set X is said to be an ideal in X if and only if (i) $\emptyset \in \mathcal{I}$ (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (Additive) and (iii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$ (Hereditary).*

If $\forall x \in X$, $\{x\} \in \mathcal{I}$ then $\mathcal{I}$ is said to be admissible. Also $\mathcal{I}$ is said to be non-trivial if $X \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$.

Definition 2.5. *[16] A family $\mathcal{F} \subset 2^X$ of subsets of a nonempty set X is said to be a filter in X if and only if (i) $\emptyset \notin \mathcal{F}$ (ii) $M, N \in \mathcal{F}$ implies $M \cap N \in \mathcal{F}$ and (iii) $M \in \mathcal{F}, N \supset M$ implies $N \in \mathcal{F}$.*

If $\mathcal{I}$ is a proper non-trivial ideal in X, then

$$\mathcal{F}(\mathcal{I}) = \{M \subset X : \exists A \in \mathcal{I} \text{ such that } M = X \setminus A\}$$

is a filter in X. It is called the filter associated with the ideal $\mathcal{I}$.

Definition 2.6. *[16] A sequence $x = (x_n)$ is said to be $\mathcal{I}-$convergent to l if and only if for every $\varepsilon > 0$, the set*

$$\{k \in \mathbb{N} : |x_n - l| \geq \varepsilon\}$$

belongs to $\mathcal{I}$. The real number l is called the $\mathcal{I}-$limit of the sequence $x = (x_n)$. Symbolically, $\mathcal{I} - \lim x = l$.

Definition 2.7. *[2] A sequence $x = (x_n)$ is said to be $\mathcal{I}-$statistically convergent to l if and only if for every $\varepsilon > 0, \delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : |x_k - l| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

If a sequence $x = (x_n)$ is $\mathcal{I}-$statistically convergent to l, then it is denoted by $\mathcal{I} - st - \lim x = l$.

Definition 2.8. *[5] An element x_0 is said to be an $\mathcal{I}-$statistical limit point of a sequence $x = (x_n)$ if there exists $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $st - \lim x_{m_k} = x_0$.*

For a sequence $x = (x_n)$, the set of all $\mathcal{I}-$statistical limit points is denoted by $\mathcal{I} - S(\Lambda_x)$.

Definition 2.9. *[21] An element x_0 is said to be an $\mathcal{I}-$statistical cluster point of a sequence $x = (x_n)$ if for every $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : |x_k - x_0| \geq \varepsilon\}\right| < \delta\right\} \notin \mathcal{I}.$$

For a sequence $x = (x_n)$, the set of all $\mathcal{I}-$statistical cluster points is denoted by $\mathcal{I} - S(\Gamma_x)$.

Definition 2.10. *[21] A sequence $x = (x_n)$ is said to be $\mathcal{I}-$statistically bounded ($\mathcal{I}-st$ bounded), if there exists a number B such that*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : |x_k| > B\}\right| > \delta\right\} \in \mathcal{I}.$$

Definition 2.11. *[21] Let $x = (x_n)$ be a real valued sequence. Then $\mathcal{I}-$statistical limit superior of x is defined as*

$$\mathcal{I} - st \limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \emptyset \\ -\infty, & \text{if } B_x = \emptyset \end{cases};$$

where B_x stands for the set

$$\left\{b \in \mathbb{R} : \left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : x_k > b\}\right| > \delta\right\} \notin \mathcal{I}\right\}.$$

Definition 2.12. *[21] Let $x = (x_n)$ be a real valued sequence. Then $\mathcal{I}-$statistical limit inferior of x is defined as*

$$\mathcal{I} - st \liminf x = \begin{cases} \inf A_x, & \text{if } A_x \neq \emptyset \\ +\infty, & \text{if } A_x = \emptyset \end{cases};$$

where A_x stands for the set

$$\left\{a \in \mathbb{R} : \left\{n \in \mathbb{N} : \frac{1}{n} \left|\{k \leq n : x_k < a\}\right| > \delta\right\} \notin \mathcal{I}\right\}.$$

Definition 2.13. *[26] A sequence $x = (x_n)$ is said to be $\mathcal{I}-$deferred statistically convergent (or $\mathcal{I} - DS_{p,q}$ convergent) to l if for any $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{p(n) - q(n)} \left|\{p(n) < k \leq q(n) : |x_k - l| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Symbolically, $\mathcal{I} - DS_{p,q} \lim x = l$.

3. Main Results

The entire study is divided into two subsections. Throughout the subsections $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ will be used to denote the sequences of whole numbers satisfying $q(n) - p(n) \geq 1$ and $\lim_{n\to\infty} q(n) = \infty$. Also $\mathcal{I}$ stands for non-trivial admissible ideal of $\mathbb{N}$.

3.1. $\mathcal{I}$−deferred statistical limit points, cluster points

Definition 3.1. *A real number x_0 is said to be an $\mathcal{I}$−deferred statistical limit point of a sequence $x = (x_n)$ if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_{k\to\infty} x_{m_k} = x_0(DS_{p,q})$.*

If we take $p(n) = 0$ and $q(n) = n$, then the above definition is turned to the definition of $\mathcal{I}-$ statistical limit point [20].

Definition 3.2. *A real number x_0 is said to be an $\mathcal{I}$−deferred statistical cluster point of a sequence $x = (x_n)$ if for every $\varepsilon > 0$ and $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\{p(n) < k \le q(n) : |x_k - x_0| \ge \varepsilon\}\right| < \delta\right\} \notin \mathcal{I}.$$

If we take $p(n) = 0$ and $q(n) = n$, then the above definition is turned to the definition of $\mathcal{I}-$ statistical cluster point [21].

Throughout the paper we will use $\mathcal{I} - DS_{p,q}(\Lambda_x)$ and $\mathcal{I} - DS_{p,q}(\Gamma_x)$ to denote the set of all $\mathcal{I}$−deferred statistical limit points and $\mathcal{I}$−deferred statistical cluster points of a sequence $x = (x_n)$.

Theorem 3.3. *If $x = (x_n)$ is any sequence such that $\mathcal{I} - DS_{p,q} \lim x = x_0$, then $\mathcal{I} - DS_{p,q}(\Lambda_x) = \{x_0\}$.*

Proof. Since $\mathcal{I} - DS_{p,q} \lim x = x_0$, so for any $\varepsilon, \delta > 0$, the set

$$A = \left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\{p(n) < k \le q(n) : |x_k - x_0| \ge \varepsilon\}\right| \ge \delta\right\} \in \mathcal{I}. \tag{1}$$

If possible suppose there exists $y_0 \in \mathcal{I} - DS_{p,q}(\Lambda_x)$ with $x_0 \ne y_0$. Then there exists $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and for every $\varepsilon > 0, \delta > 0$, the set

$$B = \left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\{p(n) < k \le q(n) : |x_{m_k} - y_0| \ge \varepsilon\}\right| \ge \delta\right\}$$

is finite. Since $\mathcal{I}$ is admissible, so we have $\mathbb{N} \setminus C \in \mathcal{F}(\mathcal{I})$ where

$$C = \left\{n \in M : \frac{1}{(q(n)-p(n))}\left|\{p(n) < k \le q(n) : |x_k - y_0| \ge \varepsilon\}\right| \ge \delta\right\} \subseteq B.$$

Again from (1) we have, $\mathbb{N} \setminus D \in \mathcal{F}(\mathcal{I})$ where

$$D = \left\{n \in M : \frac{1}{(q(n)-p(n))}\left|\{p(n) < k \le q(n) : |x_k - x_0| \ge \varepsilon\}\right| \ge \delta\right\} \subseteq A.$$

Clearly, $(\mathbb{N} \setminus C) \cap (\mathbb{N} \setminus D) \ne \emptyset$ since $(\mathbb{N} \setminus C) \cap (\mathbb{N} \setminus D) \in \mathcal{F}(\mathcal{I})$. Choose $s \in (\mathbb{N} \setminus C) \cap (\mathbb{N} \setminus D)$ and a particular $\varepsilon > 0$ satisfying $\varepsilon < |x_0 - y_0|$. Then the following inequations are true

$$\frac{1}{(q(s)-p(s))}\left|\left\{p(s) < k \le q(s) : |x_k - y_0| \ge \frac{\varepsilon}{2}\right\}\right| < \delta$$

and

$$\frac{1}{(q(s)-p(s))}\left|\left\{p(s) < k \le q(s) : |x_k - x_0| \ge \frac{\varepsilon}{2}\right\}\right| < \delta.$$

Now choosing δ sufficiently small, we can ensure the existence of an element $\xi \in \mathbb{N}$ for which the following properties holds good

$$p(s) < \xi < q(s),\ |x_\xi - y_0| < \frac{\varepsilon}{2} \text{ and } |x_\xi - x_0| < \frac{\varepsilon}{2}.$$

But then

$$\varepsilon < |x_0 - y_0| \le |x_\xi - x_0| + |x_\xi - y_0| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ a contradiction.}$$

Hence $\mathcal{I} - DS_{p,q}(\Lambda_x) = \{x_0\}$. □

Theorem 3.4. *For any sequence $x = (x_n)$, the set $\mathcal{I} - DS_{p,q}(\Gamma_x)$ is a closed subset of $\mathbb{R}$.*

Proof. Suppose $y_0 \in \overline{\mathcal{I} - DS_{p,q}(\Gamma_x)}$. Then for any $\varepsilon > 0$,

$$\mathcal{I} - DS_{p,q}(\Gamma_x) \cap (y_0 - \varepsilon, y_0 + \varepsilon) \neq \emptyset.$$

Let $z_0 \in \mathcal{I} - DS_{p,q}(\Gamma_x) \cap (y_0 - \varepsilon, y_0 + \varepsilon)$ and put $\varepsilon_1 > 0$ in such a manner that

$$(z_0 - \varepsilon_1, z_0 + \varepsilon_1) \subseteq (y_0 - \varepsilon, y_0 + \varepsilon).$$

Then, the following inequation holds:

$$\left|\{p(n) < k \leq q(n) : |x_k - z_0| \geq \varepsilon_1\}\right| \geq \left|\{p(n) < k \leq q(n) : |x_k - y_0| \geq \varepsilon_1\}\right|.$$

As a consequence, for any $\delta > 0$, the set

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : |x_k - y_0| \geq \varepsilon\}\right| < \delta\right\}$$

is a superset of the set

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : |x_k - z_0| \geq \varepsilon_1\}\right| < \delta\right\}.$$

Now since $z_0 \in \mathcal{I} - DS_{p,q}(\Gamma_x)$, we must have

$$A = \left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : |x_k - y_0| \geq \varepsilon\}\right| < \delta\right\} \notin \mathcal{I}.$$

This completes the proof. □

Theorem 3.5. *For any sequence $x = (x_n)$, $\mathcal{I} - DS_{p,q}(\Lambda_x) \subseteq \mathcal{I} - DS_{p,q}(\Gamma_x)$.*

Proof. Let $x_0 \in \mathcal{I} - DS_{p,q}(\Lambda_x)$. Then there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and

$$\lim_{n\to\infty} \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : |x_{m_k} - x_0| \geq \varepsilon\}\right| = 0.$$

Therefore for any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\forall n > n_0,\ \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : |x_{m_k} - x_0| \geq \varepsilon\}\right| < \delta.$$

Let $A = \left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))} \left|\{p(n) < k \leq q(n) : |x_k - x_0| \geq \varepsilon\}\right| < \delta\right\}$. Then $A \supset M \setminus \{m_1, m_2, \cdots, m_{k_0}\}$ and eventually $A \notin \mathcal{I}$ since $\mathcal{I}$ is admissible. Hence $x_0 \in \mathcal{I} - DS_{p,q}(\Gamma_x)$. □

Theorem 3.6. *Let $x = (x_n)$ and $y = (y_n)$ be two sequences such that $\{n \in \mathbb{N} : x_n \neq y_n\} \in \mathcal{I}$. Then,*
(i) $\mathcal{I} - DS_{p,q}(\Lambda_x) = \mathcal{I} - DS_{p,q}(\Lambda_y)$ and (ii) $\mathcal{I} - DS_{p,q}(\Gamma_x) = \mathcal{I} - DS_{p,q}(\Gamma_y)$.

Proof. (i) Let $x_0 \in \mathcal{I} - DS_{p,q}(\Lambda_x)$. Then there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_{k\to\infty} x_{m_k} = x_0(DS_{p,q})$. Put $N = M \cap \{n \in \mathbb{N} : x_n = y_n\}$. Then since $M \notin \mathcal{I}$, so we must have $N \notin \mathcal{I}$. Suppose $N = \{n_1 < n_2 < \cdots < n_k < \cdots\}$. Then we must have $\lim_{k\to\infty} y_{n_k} = x_0(DS_{p,q})$ and therefore $x_0 \in \mathcal{I} - DS_{p,q}(\Lambda_y)$. Thus the inclusion $\mathcal{I} - DS_{p,q}(\Lambda_x) \subseteq \mathcal{I} - DS_{p,q}(\Lambda_y)$ holds. By symmetry, we have $\mathcal{I} - DS_{p,q}(\Lambda_y) \subseteq \mathcal{I} - DS_{p,q}(\Lambda_x)$.

Hence $\mathcal{I}-DS_{p,q}(\Lambda_x)=\mathcal{I}-DS_{p,q}(\Lambda_y)$.
(ii) Let $x_0\in\mathcal{I}-DS_{p,q}(\Gamma_x)$. So by definition for any $\varepsilon>0$ and $\delta>0$,

$$A=\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))}\left|\{p(n)<k\leq q(n):|x_k-x_0|\geq\varepsilon\}\right|<\delta\right\}\notin\mathcal{I}.$$

To complete the proof, it is enough to show that the set $B\notin\mathcal{I}$ where

$$B=\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))}\left|\{p(n)<k\leq q(n):|y_k-x_0|\geq\varepsilon\}\right|<\delta\right\}.$$

If possible let $B\in\mathcal{I}$. Put $C=\{n\in\mathbb{N}:x_n=y_n\}$. Then by additivity of $\mathcal{I}$, we have $(\mathbb{N}\setminus C)\cup B\in\mathcal{I}$. But this leads us to the contradiction $A\in\mathcal{I}$ because of the inclusion $A\subset B\cup(\mathbb{N}\setminus C)$. Hence we must have $B\notin\mathcal{I}$ and the proof is complete. □

3.2. *$\mathcal{I}-$deferred statistical limit superior, limit inferior*

In this subsection we introduce the notion of $\mathcal{I}-$deferred statistical limit superior, limit inferior which are natural generalizations of $\mathcal{I}-$statistical limit superior, limit inferior introduced by Mursaleen et al. [21].

Throughout this section, for a real sequence $x=(x_n)$, A_x and B_x will denote the sets

$$\left\{a\in\mathbb{R}:\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))}\left|\{p(n)<k\leq q(n):x_k<a\}\right|>\delta\right\}\notin\mathcal{I}\right\}$$

and

$$\left\{b\in\mathbb{R}:\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))}\left|\{p(n)<k\leq q(n):x_k>b\}\right|>\delta\right\}\notin\mathcal{I}\right\}$$

respectively.

Definition 3.7. *Let $x=(x_n)$ be any real number sequence. Then $\mathcal{I}-$deferred statistical limit superior of x is defined as*

$$\mathcal{I}-DS_{p,q}\limsup x=\begin{cases}\sup B_x, & \text{if } B_x\neq\emptyset\\ -\infty, & \text{if } B_x=\emptyset\end{cases}.$$

Also $\mathcal{I}-$deferred statistical limit inferior of x is defined as

$$\mathcal{I}-DS_{p,q}\liminf x=\begin{cases}\inf A_x, & \text{if } A_x\neq\emptyset\\ +\infty, & \text{if } A_x=\emptyset\end{cases}.$$

Remark 3.8. *If we consider $p(n)=0$ and $q(n)=n$, then the above definition coincides with the definition of $\mathcal{I}-$statistical limit superior and limit inferior respectively introduced in [21].*

Theorem 3.9. *For any real number sequence $x=(x_n)$, if $\alpha=\mathcal{I}-DS_{p,q}\liminf x$ is finite then for any $\varepsilon>0$,*

$$\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))}\left|\{p(n)<k\leq q(n):x_k<\alpha+\varepsilon\}\right|>\delta\right\}\notin\mathcal{I}$$

and

$$\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))}\left|\{p(n)<k\leq q(n):x_k<\alpha-\varepsilon\}\right|>\delta\right\}\in\mathcal{I}.$$

Similarly, if $\beta=\mathcal{I}-DS_{p,q}\limsup x$ is finite then for any $\varepsilon>0$,

$$\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))}\left|\{p(n)<k\leq q(n):x_k>\beta-\varepsilon\}\right|>\delta\right\}\notin\mathcal{I}$$

and

$$\left\{n\in\mathbb{N}:\frac{1}{(q(n)-p(n))}\left|\{p(n)<k\leq q(n):x_k>\beta+\varepsilon\}\right|>\delta\right\}\in\mathcal{I}.$$

Proof. Proof is trivial and therefore is omitted. □

Theorem 3.10. *For any real number sequence $x = (x_n)$, $\mathcal{I} - DS_{p,q} \liminf x \leq \mathcal{I} - DS_{p,q} \limsup x$.*

Proof. **Case-I:** If $\mathcal{I} - DS_{p,q} \limsup x = \infty$, then there is nothing to prove.

Case-II: If $\mathcal{I} - DS_{p,q} \limsup x = -\infty$, then we have $B_x = \emptyset$. So for every $b \in \mathbb{R}$,

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : x_k > b\}\right| > \delta\right\} \in \mathcal{I}$$

which immediately implies,

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : x_k > b\}\right| < \delta\right\} \in \mathcal{F}(\mathcal{I}).$$

$$\text{i.e., } \left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : x_k < b\}\right| > \delta\right\} \in \mathcal{F}(\mathcal{I}).$$

In other words,

$$\forall a \in \mathbb{R}, \left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : x_k < a\}\right| > \delta\right\} \notin \mathcal{I}.$$

Therefore we have $A_x = \mathbb{R}$ and hence $\mathcal{I} - DS_{p,q} \liminf x = -\infty$.

Case-III: If $-\infty < \mathcal{I} - DS_{p,q} \limsup x < \infty$, then suppose $\beta = \mathcal{I} - DS_{p,q} \limsup x$ and $\alpha = \mathcal{I} - DS_{p,q} \liminf x$. Then by Theorem 3.9, for any $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : x_k > \beta + \varepsilon\}\right| > \delta\right\} \in \mathcal{I}.$$

Which implies,

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : x_k < \beta + \varepsilon\}\right| > \delta\right\} \in \mathcal{F}(\mathcal{I}).$$

$$\textit{i.e.,} \left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : x_k < \beta + \varepsilon\}\right| > \delta\right\} \notin \mathcal{I}.$$

So we have $\beta + \varepsilon \in A_x$. Now since ε was arbitrary and $\alpha = \inf A_x$, so we must have $\alpha < \beta + \varepsilon$. Hence $\alpha \leq \beta$ and the proof is complete. □

Definition 3.11. *A sequence $x = (x_n)$ is said to be $\mathcal{I} - DS_{p,q}$ bounded if there exists a number B such that for every $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : |x_k| > B\}\right| > \delta\right\} \in \mathcal{I}.$$

Note that, for $p(n) = 0$ and $q(n) = n$, the above definition turns to the definition of $\mathcal{I}-$st boundedness [21].

Remark 3.12. *If a sequence is $\mathcal{I} - DS_{p,q}$ bounded then $\mathcal{I} - DS_{p,q} \liminf x$ and $\mathcal{I} - DS_{p,q} \limsup x$ are finite.*

Theorem 3.13. *An $\mathcal{I} - DS_{p,q}$ bounded sequence is $\mathcal{I} - DS_{p,q}$ convergent iff $\mathcal{I} - DS_{p,q} \liminf x = \mathcal{I} - DS_{p,q} \limsup x$.*

Proof. Suppose $\alpha = \mathcal{I} - DS_{p,q} \liminf x$ and $\beta = \mathcal{I} - DS_{p,q} \limsup x$. Let $\mathcal{I} - DS_{p,q} \lim x = l$. Then for all $\varepsilon > 0$ and $\delta > 0$,

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : |x_k - l| \geq \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

$$\textit{i.e.,} \left\{n \in \mathbb{N} : \frac{1}{(q(n) - p(n))} \left|\{p(n) < k \leq q(n) : x_k > l + \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I}.$$

Which implies $\beta \leq l$. Also we have,

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\{p(n) < k \leq q(n) : x_k < l - \varepsilon\}\right| \geq \delta\right\} \in \mathcal{I},$$

which yields $l \leq \alpha$ and hence we have $\beta \leq \alpha$. But by Theorem 3.10, we have $\beta \geq \alpha$, so we must have $\alpha = \beta$ i.e, $\mathcal{I} - DS_{p,q} \liminf x = \mathcal{I} - DS_{p,q} \limsup x$.

For the converse part, suppose $\alpha = \beta$ and define $l = \alpha$. Now for any $\varepsilon > 0, \delta > 0$, from Theorem 3.9, we obtain

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\left\{p(n) < k \leq q(n) : x_k > l + \frac{\varepsilon}{2}\right\}\right| > \delta\right\} \in \mathcal{I}$$

and

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\left\{p(n) < k \leq q(n) : x_k < l - \frac{\varepsilon}{2}\right\}\right| > \delta\right\} \in \mathcal{I}.$$

This immediately implies that, $\mathcal{I} - DS_{p,q} \lim x = l$. □

Theorem 3.14. *Suppose $x = (x_n)$ and $y = (y_n)$ be two $\mathcal{I} - DS_{p,q}$ bounded sequences. Then,*
(i) $\mathcal{I} - DS_{p,q} \limsup(x + y) \leq \mathcal{I} - DS_{p,q} \limsup x + \mathcal{I} - DS_{p,q} \limsup y$.
(ii) $\mathcal{I} - DS_{p,q} \liminf(x + y) \geq \mathcal{I} - DS_{p,q} \liminf x + \mathcal{I} - DS_{p,q} \liminf y$.

Proof. Let $\beta_1 = \mathcal{I} - DS_{p,q} \limsup x$ and $\beta_2 = \mathcal{I} - DS_{p,q} \limsup y$. Then for every $\varepsilon > 0$, $\delta > 0$, we have

$$P = \left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\left\{p(n) < k \leq q(n) : x_k > \beta_1 + \frac{\varepsilon}{2}\right\}\right| > \delta\right\} \in \mathcal{I}$$

and

$$Q = \left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\left\{p(n) < k \leq q(n) : y_k > \beta_2 + \frac{\varepsilon}{2}\right\}\right| > \delta\right\} \in \mathcal{I}.$$

Now as the inclusion

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\{p(n) < k \leq q(n) : x_k + y_k > \beta_1 + \beta_2 + \varepsilon\}\right| > \delta\right\} \subset P \cup Q$$

is true, we must have,

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\{p(n) < k \leq q(n) : x_k + y_k > \beta_1 + \beta_2 + \varepsilon\}\right| > \delta\right\} \in \mathcal{I}.$$

If $c \in B_{x+y}$, then

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\{p(n) < k \leq q(n) : x_k + y_k > c\}\right| > \delta\right\} \notin \mathcal{I}.$$

We claim that $c < \beta_1 + \beta_2 + \varepsilon$. For if $c \geq \beta_1 + \beta_2 + \varepsilon$, then the inclusion

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\{p(n) < k \leq q(n) : x_k + y_k > \beta_1 + \beta_2 + \varepsilon\}\right| > \delta\right\}$$
$$\supseteq \left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\{p(n) < k \leq q(n) : x_k + y_k > c\}\right| > \delta\right\}$$

leads us to the contradiction that

$$\left\{n \in \mathbb{N} : \frac{1}{(q(n)-p(n))}\left|\{p(n) < k \leq q(n) : x_k + y_k > c\}\right| > \delta\right\} \in \mathcal{I}.$$

Hence, we must have $c < \beta_1 + \beta_2 + \varepsilon$. As this is true for every $c \in B_{x+y}$, so $\mathcal{I} - DS_{p,q} \limsup(x+y) = \sup B_{x+y} < \beta_1 + \beta_2 + \varepsilon$. Now as $\varepsilon > 0$ was arbitrary, so $\mathcal{I} - DS_{p,q} \limsup(x + y) \leq \mathcal{I} - DS_{p,q} \limsup x + \mathcal{I} - DS_{p,q} \limsup y$.

(ii) The proof is analogous to that of (i) and so is omitted. □

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