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Almost automorphic colombeau generalized ultradistributions and linear ordinary differential systems

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Abstract. The aim of this work is to introduce and to study the concept of almost automorphy in the setting of generalized ultradistributions. Such generalized ultradistributions contain their analogue of almost automorphic ultraditributions and generalized functions. A result on the existence of almost automorphic ultradistributional solutions of a linear ordinary differential systems is given.

1. Introduction

Almost automorphic functions was defined explicitly in the papers [5], [6] and [22], where also some of their basic properties are given. It is well known that the concept of almost automorphy is strictly more general than the almost periodicity studied in a full generality by H. Bohr, see [7], [3], [28] and [23]. Almost periodicity in the setting of distributions established in [33], extends Bohr almost periodicity and Stepanov almost periodicity [36]. Ultradistributions in the sense of Komatsu [25] are strictly larger than distributions. Almost periodic ultradistributions were considered in [19], see also [24].

The result of [34] on the impossibility of multiplication of distributions motivated the introduction of algebras of generalized functions, see [20], [29]. Almost periodic generalized functions are the main work of [9], such generalized functions contains their classical analogue of almost periodic functions and distributions. An application to a linear ordinary diffierential equations in the framework of almost periodic generalized functions was given in [10].

As the problem of multiplication of distributions, algebras of ultradistributions were constructed and studied in [1] and [2], see also the paper [21]. Almost periodicity in [12] was tackled in the context of generalized ultradistributions, they contain almost periodic ultradistributions and almost periodic generalized functions. A study of the existence of almost periodic generalized ultradistributional solutions of a linear ordinary differential systems in [13] is considered.

It is well known that the concept of almost automorphy is more general than the concept of almost periodicity, see [37]. S. Bochner studied in [6] linear difference differential equations in the framework of almost periodic and almost automorphic functions. S. Zaidman gave a condition on the existence of almost automorphic solutions of a linear ordinary differential systems in [38]. Almost automorphy in the setting of distributions due to [15] extends almost automorphic functions and Stepanov almost automorphic

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functions [18]. The paper [26] deals with almost automorphic ultradistributions. C. Bouzar and col. in [11] have introduced and studied almost automorphic generalized functions containing their classical analogue of almost automorphic functions and distributions.

This paper introduces and studies an algebra of almost automorphic Colombeau generalized ultradistributions containing almost automorphic ultradistributions and almost automorphic generalized functions. A generalization of the result of [39] on the primitive of an almost automorphic function to Colombeau generalized ultradistributions is given. Also we establish an existence result on almost automorphic Colombeau generalized ultradistributional solutions of linear ordinary differential systems.

The paper is organized as follows : section two deals with preliminary results needed in the sequel. In section three we introduce an algebra of almost automorphic Colombeau generalized ultradistributions and also we study their main properties. Non-linear operations on the introduced algebra are performed in section four. The last section is aimed to study systems of linear ordinary differential equations in the framework of almost automorphic Colombeau generalized ultradistributions Conditions on the existence of generalized ultradistributional solutions are given.

2. Preliminaries

This section recalls some preliminary results needed in the sequel. The space of continuous and bounded complex valued functions defined on $\mathbb{R}$, is denoted by C_b. It is well known that $(C_b, \|\cdot\|_{L^\infty})$ is a Banach algebra. For the definition of almost automorphic functions and their properties, see [4], [5], [6] and [22].

Definition 2.1. *A complex-valued function f defined and continuous on $\mathbb{R}$ is called almost automorphic, if for any sequence $(s_m)_{m\in\mathbb{N}} \subset \mathbb{R}$, one can extract a subsequence $(s_{m_k})_k$ such that*

$$g(x) := \lim_{k\to+\infty} f(x + s_{m_k}) \text{ is well-defined for every } x \in \mathbb{R},$$

and

$$\lim_{k\to+\infty} g(x - s_{m_k}) = f(x) \text{ for every } x \in \mathbb{R}.$$

Denote by C_{aa} the set of almost automorphic functions on $\mathbb{R}$.

Remark 2.2. *The function g is not necessary continuous but $g \in L^\infty$. Furthermore, we have*

$$\|f\|_{L^\infty} = \|g\|_{L^\infty}.$$

Let $\mathcal{E}$ be the algebra of space of smooth functions on $\mathbb{R}$, and define the space

$$\mathcal{D}_{L^p} := \left\{\varphi \in \mathcal{E} : \forall j \in \mathbb{Z}_+,\ \varphi^{(j)} \in L^p\right\}, \quad p \in [1, +\infty],$$

that we endow with the topology defined by the family of semi-norms

$$|\varphi|_{k,p} := \sum_{j\le k} \left\|\varphi^{(j)}\right\|_{L^p}, \quad k \in \mathbb{Z}_+.$$

So, $\mathcal{D}_{L^p}$ is a Fréchet subalgebra of $\mathcal{E}$. Denote $\mathcal{B} := \mathcal{D}_{L^\infty}$.

The set of smooth almost automorphic functions on $\mathbb{R}$, denoted by $\mathcal{B}_{aa}$, is defined by

$$\mathcal{B}_{aa} := \left\{\varphi \in \mathcal{E} :\ \forall i \in \mathbb{Z}_+,\ \varphi^{(i)} \in C_{aa}\right\}.$$

It is clear that $\mathcal{B}_{aa} \subset \mathcal{B}$ and we endow $\mathcal{B}_{aa}$ with the topology induced by $\mathcal{B}$.

The following properties of the space $\mathcal{B}_{aa}$ were proved in [15].

Proposition 2.3. *1. The space $\mathcal{B}_{aa}$ is a Fréchet subalgebra of $\mathcal{B}$ stable under translation..*
2. $\mathcal{B}_{aa} * L^1 \subset \mathcal{B}_{aa}$.
3. $\mathcal{B}_{aa} = C_{aa} \cap \mathcal{B}$.

In order to introduce some classes of functions, we need definitions and results from [25].
Let $M = (M_k)_{k\in\mathbb{Z}_+}$ be a sequence of positive numbers, define the following properties
Logarithmic convexity

$$\forall k \in \mathbb{N},\ M_k^2 \leq M_{k-1}M_{k+1}. \tag{$M.1$}$$

Stability under ultradifferential operators

$$\exists A > 0,\ \exists H > 0,\ \forall k,\ l \in \mathbb{Z}_+,\ M_{k+l} \leq AH^{k+l}M_kM_l. \tag{$M.2$}$$

Strong non quasi-analyticity

$$\exists A > 0,\ \forall k \in \mathbb{Z}_+,\ \sum_{l=k+1}^{+\infty} \frac{M_{l-1}}{M_l} \leq Ak\frac{M_k}{M_{k+1}}. \tag{$M.3$}$$

Non quasi-analyticity

$$\sum_{k=0}^{+\infty} \frac{M_{k-1}}{M_k} < \infty. \tag{$M.3$)$'$}$$

Definition 2.4. *The associated function of the sequence M is defined by*

$$M(t) = \sup_{k\in\mathbb{Z}_+} \ln \frac{t^k M_0}{M_k}, \quad t > 0.$$

Example 2.5. *If the sequence M_k is the Gevrey sequence $(k!^{\sigma})$, $\sigma > 1$, then it satisfies $(M.1)$, $(M.2)$, $(M.3)$ and its associated function $M(t)$ is equivalent to $t^{\frac{1}{\sigma}}$.*

The next result shows that the conditions $(M.1)$ and $(M.2)$ can expressed in the term of the associated function, see Propositions 3.1 and 3.6 of [25].

Proposition 2.6. 1. *The sequence M satisfies $(M.1)$ if and only if $\forall k \in \mathbb{Z}_+$,*

$$M_k = \sup_{t>0} \frac{t^k M_0}{e^{M(t)}}.$$

2. *Let M satisfies $(M.1)$, then M satisfies $(M.2)$ if and only if $\exists A > 0,\ \exists H > 0,\ \forall t > 0$,*

$$2M(t) \leq M(Ht) + \ln(AM_0).$$

As a consequence of Proposition 2.6− (2), the following result was obtained in Lemma 4.2 of [12].

Lemma 2.7. *If M satisfies $(M.2)$ then $\exists A > 0,\ \exists H > 0,\ \forall t_1, \ldots, t_n > 0,\ \forall n \in \mathbb{N}$,*

$$M(t_1) + \cdots + M(t_n) \leq M\left(H^{\frac{(n-1)(n+2)}{2n}} \max(t_1, \ldots, t_n)\right) + (n-1)\ln(AM_0).$$

Remark 2.8. *Throughout the paper we assume that the sequence M satisfies $(M.1)$, $(M.2)$ and $(M.3)'$.*

We recall from [17] some needed spaces. Let $p \in [1,+\infty]$, $h > 0$, the space

$$\mathcal{D}_{L^p}^{M,h} := \left\{ \varphi \in \mathcal{E} : \left\|\varphi\right\|_{p,h,M} := \sup_{j\in\mathbb{Z}_+} \frac{\left\|\varphi^{(j)}\right\|_{L^p}}{h^j M_j} < \infty \right\},$$

endowed with the norm $\|\cdot\|_{p,h,M}$ is a Banach space.

The space of L^p-Beurling ultradifferentiable functions is

$$\mathcal{D}_{L^p}^{(M)} := proj \lim_{h\to 0} \mathcal{D}_{L^p}^{M,h}.$$

The space of Beurling ultradifferentiable functions

$$\mathcal{D}^{(M)} := \left\{ \varphi \in \mathcal{E} : \forall K \text{ compact of } \mathbb{R},\ \forall h > 0,\ \exists c > 0,\ \forall j \in \mathbb{Z}_+,\ \sup_{x\in K} \left|\varphi^{(j)}(x)\right| \le c h^j M_j \right\}$$

is dense in $\mathcal{D}_{L^p}^{(M)}, p \in [1,+\infty[$. The space $\dot{\mathcal{B}}^{(M)}$ is the closure of the space $\mathcal{D}^{(M)}$ in $\mathcal{B}^{(M)} := \mathcal{D}_{L^\infty}^{(M)}$.

Definition 2.9. *Let $p \in]1,+\infty]$, the space of L^p-Beurling ultradistributions denoted by $\mathcal{D}'_{L^p,(M)}$ is the topological dual of $\mathcal{D}_{L^q}^{(M)}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Denotes by $\mathcal{D}'_{L^1,(M)}$ the topological dual of $\dot{\mathcal{B}}^{(M)}$. The elements of $\mathcal{D}'_{L^\infty,(M)}$ are said to be bounded ultradistributions.*

3. Almost automorphic Colombeau generalized ultradistributions

In this section we introduce an algebra of almost automophic Colombeau generalized ultradistributions and also we study some of their main properties.

Let $I :=]0,1]$ and $(u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^p})^I$, $p \in [1,+\infty]$, $j \in \mathbb{Z}_+$ and $k > 0$, we mean by the notation

$$\left\|u_\varepsilon^{(j)}\right\|_{L^p} = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right),\ \varepsilon \to 0,$$

that

$$\exists c > 0,\ \exists \varepsilon_0 > 0,\ \forall \varepsilon < \varepsilon_0,\ \left\|u_\varepsilon^{(j)}\right\|_{L^p} \le c e^{M\left(\frac{k}{\varepsilon}\right)}.$$

Definition 3.1. 1. *The space of almost automorphic moderate elements is denoted and defined by*

$$\mathcal{M}_{aa}^{M} := \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{aa})^I : \forall j \in \mathbb{Z}_+,\ \exists k > 0,\ \left\|u_\varepsilon^{(j)}\right\|_{L^\infty} = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right),\ \ \varepsilon \to 0 \right\}.$$

2. *The space of almost automorphic null elements is denoted and defined by*

$$\mathcal{N}_{aa}^{M} := \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{aa})^I : \forall j \in \mathbb{Z}_+,\ \forall k > 0,\ \left\|u_\varepsilon^{(j)}\right\|_{L^\infty} = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right),\ \ \varepsilon \to 0 \right\}.$$

We give some properties of the spaces $\mathcal{M}_{aa}^{M}$ and $\mathcal{N}_{aa}^{M}$.

Proposition 3.2. 1. *We have the null characterization of $\mathcal{N}_{aa}^{M}$, i.e.*

$$\mathcal{N}_{aa}^{M} := \left\{ (u_\varepsilon)_\varepsilon \in \mathcal{M}_{aa}^{M} : \forall k > 0,\ \|u_\varepsilon\|_{L^\infty} = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right),\ \ \varepsilon \to 0 \right\}.$$

2. *The space $\mathcal{M}_{aa}^{M}$ is an algebra stable under translation and derivation.*

3. *The space $\mathcal{N}_{aa}^{M}$ is an ideal of $\mathcal{M}_{aa}^{M}$.*

Proof. 1. Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{aa}^{M}$, i.e.

$$\forall j \in \mathbb{Z}_+,\ \exists k_j > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j,\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty} \leq c_j e^{M\left(\frac{k_j}{\varepsilon}\right)}, \tag{1}$$

and $(u_\varepsilon)_\varepsilon$ satisfies the null estimate of zero order, i.e.

$$\forall k > 0,\ \exists c' > 0,\ \exists \varepsilon_0' \in I,\ \forall \varepsilon < \varepsilon_0',\ \|u_\varepsilon\|_{L^\infty} \leq c' e^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{2}$$

In order to show that $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{aa}^{M}$, we use the Landau-Kolmogorov inequality which states that for any $f \in C^n,\ n \in \mathbb{Z}_+$, we have for every $1 \leq p \leq n$,

$$\left\| f^{(p)} \right\|_{L^\infty} \leq 2\pi \left\| f \right\|_{L^\infty}^{1-\frac{p}{n}} \left\| f^{(n)} \right\|_{L^\infty}^{\frac{p}{n}}.$$

For every $j \in \mathbb{Z}_+$, by using the Landau-Kolmogorov inequality for $p = j$ and $n = 2j$, and due to the estimates $(1), (2)$, we obtain

$$\begin{aligned} \left\| u_\varepsilon^{(j)} \right\|_{L^\infty} &\leq 2\pi \|u_\varepsilon\|_{L^\infty}^{1-\frac{1}{2}} \left\| u_\varepsilon^{(2j)} \right\|_{L^\infty}^{\frac{1}{2}} \\ &\leq 2\pi \left(c' e^{-M\left(\frac{k}{\varepsilon}\right)} \right)^{\frac{1}{2}} \left(c_{2j} e^{M\left(\frac{k_{2j}}{\varepsilon}\right)} \right)^{\frac{1}{2}} \\ &\leq 2\pi \left(c' c_{2j} \right)^{\frac{1}{2}} e^{-\frac{1}{2}M\left(\frac{k}{\varepsilon}\right) + \frac{1}{2}M\left(\frac{k_{2j}}{\varepsilon}\right)}. \end{aligned}$$

By Lemma 2.7, let $k' > 0$ and taking $k > 0$ such that $\frac{k}{\varepsilon} = H \max\left(\frac{k_{2j}}{\varepsilon}, \frac{k'}{\varepsilon}\right)$, then

$$e^{-M\left(\frac{k}{\varepsilon}\right) + M\left(\frac{k_{2j}}{\varepsilon}\right)} \leq A M_0 e^{-M\left(\frac{k'}{\varepsilon}\right)}.$$

Consequently, $\forall j \in \mathbb{Z}_+,\ \forall k' > 0,\ \exists C_j = \left(2\pi \left(c' c_{2j} \right)^{\frac{1}{2}} A M_0 \right) > 0,\ \forall \varepsilon < \min\left(\varepsilon_j, \varepsilon_0' \right)$,

$$\left\| u_\varepsilon^{(j)} \right\|_{L^\infty} \leq C_j e^{-M\left(\frac{k'}{\varepsilon}\right)},$$

which means that $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{aa}^{M}$.

2. The stability under translation and derivation of the space $\mathcal{M}_{aa}^{M}$ is obvious. Let $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon \in \mathcal{M}_{aa}^{M}$, i.e. they satisfy the estimate (1), for any $j \in \mathbb{Z}_+$, we have

$$\begin{aligned} \left\| (u_\varepsilon v_\varepsilon)^{(j)} \right\|_{L^\infty} &\leq \sum_{i+l=j} \frac{j!}{i!l!} \left\| u_\varepsilon^{(i)} \right\|_{L^\infty} \left\| v_\varepsilon^{(l)} \right\|_{L^\infty} \\ &\leq \sum_{i+l=j} \frac{j!}{i!l!} c_i c_l e^{M\left(\frac{k_i}{\varepsilon}\right) + M\left(\frac{k_l}{\varepsilon}\right)}, \end{aligned}$$

due to Lemma 2.7, taking $k > 0$ such that $\frac{k}{\varepsilon} = H \max\limits_{i+l=j} \left(\frac{k_i}{\varepsilon}, \frac{k_l}{\varepsilon}\right)$, then

$$e^{M\left(\frac{k_i}{\varepsilon}\right) + M\left(\frac{k_l}{\varepsilon}\right)} \leq A M_0 e^{M\left(\frac{k}{\varepsilon}\right)}.$$

Hence, $\forall j \in \mathbb{Z}_+,\ \exists k > 0,\ \exists C_j = \left(A M_0 \sum\limits_{i+l=j} \frac{j!}{i!l!} c_i c_l \right) > 0,\ \forall \varepsilon < \min\limits_{i+l=j} (\varepsilon_i, \varepsilon_l)$,

$$\left\| (u_\varepsilon v_\varepsilon)^{(j)} \right\|_{L^\infty} \leq C_j e^{M\left(\frac{k}{\varepsilon}\right)},$$

which gives $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{M}_{aa}^M$.

3. Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{aa}^M$ and $(v_\varepsilon)_\varepsilon \in \mathcal{N}_{aa}^M$, i.e. $(u_\varepsilon)_\varepsilon$ satisfies the estimate (1) and $(v_\varepsilon)_\varepsilon$ satisfies

$$\forall j \in \mathbb{Z}_+,\ \forall k > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j,\ \left\| v_\varepsilon^{(j)} \right\|_{L^\infty} \leq c_j e^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{3}$$

For every $j \in \mathbb{Z}_+$ and by (1) and (3) , we have

$$\begin{aligned} \left\| (u_\varepsilon v_\varepsilon)^{(j)} \right\|_{L^\infty} &\leq \sum_{i+l=j} \frac{j!}{i!l!} \left\| u_\varepsilon^{(i)} \right\|_{L^\infty} \left\| v_\varepsilon^{(l)} \right\|_{L^\infty} \\ &\leq \sum_{i+l=j} \frac{j!}{i!l!} c_i c_l e^{M\left(\frac{k_i}{\varepsilon}\right)} e^{-M\left(\frac{k}{\varepsilon}\right)}, \end{aligned}$$

by Lemma 2.7, let $k' > 0$ and taking $k > 0$ such that $\frac{k}{\varepsilon} = H \max\left(\frac{k_i}{\varepsilon}, \frac{k'}{\varepsilon}\right)$, then

$$e^{M\left(\frac{ki}{\varepsilon}\right)} e^{-M\left(\frac{k}{\varepsilon}\right)} \leq A M_0 e^{-M\left(\frac{k'}{\varepsilon}\right)}.$$

Hence, $\forall j \in \mathbb{Z}_+,\ \forall k' > 0,\ \exists C_j = \left(A M_0 \sum_{i+l=j} \frac{j!}{i!l!} c_i c_l \right) > 0,\ \forall \varepsilon < \min_{i+l=j} (\varepsilon_i, \varepsilon_l)$,

$$\left\| (u_\varepsilon v_\varepsilon)^{(j)} \right\|_{L^\infty} \leq C_j e^{-M\left(\frac{k'}{\varepsilon}\right)},$$

so, $(u_\varepsilon v_\varepsilon)_\varepsilon \in \mathcal{N}_{aa}^M$. □

Now we give the definition of almost automorphic Colombeau generalized ultradistributions.

Definition 3.3. *The set of almost automorphic Colombeau generalized ultradistributions, is denoted and defined by the quotient*

$$\mathcal{G}_{aa}^M := \frac{\mathcal{M}_{aa}^M}{\mathcal{N}_{aa}^M}.$$

The next result follows from Proposition 3.2.

Proposition 3.4. *The set of almost automorphic Colombeau generalized ultradistributions is an algebra.*

Example 3.5. *We have $\mathcal{G}_{ap}^M \subsetneq \mathcal{G}_{aa}^M$, where $\mathcal{G}_{ap}^M$ is the algebra of almost periodic generalized ultradistributions of [12]. It is easy to see that $\mathcal{G}_{ap}^M$ is embedded canonically into $\mathcal{G}_{aa}^M$.*

Example 3.6. *We have $\mathcal{G}_{aa} \subsetneq \mathcal{G}_{aa}^M$, where $\mathcal{G}_{aa}$ is the algebra of almost automorphic generalized functions defined as the quotient algebra*

$$\mathcal{G}_{aa} := \frac{\mathcal{M}_{aa}}{\mathcal{N}_{aa}},$$

where

$$\mathcal{M}_{aa} := \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{aa})^I : \forall j \in \mathbb{Z}_+,\ \exists k > 0,\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty} = O\left(\varepsilon^{-k}\right),\ \ \varepsilon \to 0 \right\}$$

and

$$\mathcal{N}_{aa} := \left\{ (u_\varepsilon)_\varepsilon \in (\mathcal{B}_{aa})^I : \forall j \in \mathbb{Z}_+,\ \forall k > 0,\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty} = O\left(\varepsilon^{k}\right),\ \ \varepsilon \to 0 \right\}.$$

For more details on $\mathcal{G}_{aa}$ see [11].

Moreover, we have the following canonical embedding of $\mathcal{G}_{aa}$ into $\mathcal{G}_{aa}^{M}$.

Proposition 3.7. *The map*

$$\begin{array}{llll} I_{aa}: & \mathcal{G}_{aa} & \longrightarrow & \mathcal{G}_{aa}^{M} \\ & (u_\varepsilon)_\varepsilon + \mathcal{N}_{aa} & \longmapsto & (u_\varepsilon)_\varepsilon + \mathcal{N}_{aa}^{M} \end{array}$$

is a linear embedding.

Proof. It remains to prove $\mathcal{M}_{aa} \subset \mathcal{M}_{aa}^{M}$ and $\mathcal{M}_{aa} \cap \mathcal{N}_{aa}^{M} \subset \mathcal{N}_{aa}$. Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{aa}$, i.e.

$$\forall j \in \mathbb{Z}_+,\ \exists k_j > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j,\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty} \leq c_j \varepsilon^{-k_j}. \tag{4}$$

Due to Proposition 2.6– (1),

$$M_n = \sup_{t>0} \frac{t^n M_0}{e^{M(t)}},\ \forall n \in \mathbb{Z}_+,$$

hence, $\forall n \in \mathbb{Z}_+,\ \forall k > 0,\ \forall \varepsilon > 0,$

$$e^{M\left(\frac{k}{\varepsilon}\right)} \geq \frac{k^n M_0}{M_n} \varepsilon^{-n}. \tag{5}$$

By (4) and (5), taking $k_j' = -\left[k_j\right] - 1$, then $\forall j \in \mathbb{Z}_+,\ \exists k_j > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j,$

$$\left\| u_\varepsilon^{(j)} \right\|_{L^\infty} \leq c_j \varepsilon^{-k_j} \leq c_j \varepsilon^{-[k_j]-1} \leq c_j \frac{M_{k_j'}}{k_j'^{k_j'} M_0} e^{M\left(\frac{k_j'}{\varepsilon}\right)},$$

so, $(u_\varepsilon)_\varepsilon \in \mathcal{M}_{aa}^{M}$. Thus, $\mathcal{M}_{aa} \subset \mathcal{M}_{aa}^{M}$. Let $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{aa}^{M}$, i.e.

$$\forall j \in \mathbb{Z}_+,\ \forall k > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j,\ \left\| u_\varepsilon^{(j)} \right\|_{L^\infty} \leq c_j e^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{6}$$

We have, $\forall n \in \mathbb{Z}_+,\ \forall k > 0,\ \forall \varepsilon > 0,$

$$\varepsilon^n \geq \frac{k^n M_0}{M_n} e^{-M\left(\frac{k}{\varepsilon}\right)},$$

from (6), taking $k' = [k]$, then $\forall j \in \mathbb{Z}_+,\ \forall k > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j,$

$$\left\| u_\varepsilon^{(j)} \right\|_{L^\infty} \leq c_j e^{-M\left(\frac{k}{\varepsilon}\right)} \leq c_j e^{-M\left(\frac{k'}{\varepsilon}\right)} \leq c_j \frac{M_{k'}}{k'^{k'} M_0} \varepsilon^{k'},$$

so $(u_\varepsilon)_\varepsilon \in \mathcal{N}_{aa}$. Thus, $\mathcal{N}_{aa}^{M} \subset \mathcal{N}_{aa}$. Moreover, $\mathcal{M}_{aa} \cap \mathcal{N}_{aa}^{M} \subset \mathcal{N}_{aa}^{M} \subset \mathcal{N}_{aa}$. □

Now, we give another example of almost automorphic Colombeau generalized ultradistributions.

Definition 3.8. *Let $T \in \mathcal{D}'_{L^\infty,(M)}$ such that there exist an ultradifferential operator P of class (M), $f \in C_{aa}$ and $g \in C_{aa}$ such that $T = P(D) f + g$. We denote by $E'_{aa,(M)}$ the space of such ultradistributions.*

Let $\rho \in \mathcal{D}_{L^1}^{\{N\}} := ind \lim\limits_{h \to +\infty} \mathcal{D}_{L^p}^{N,h}$ and set $\rho_\varepsilon(\cdot) = \frac{1}{\varepsilon} \rho\left(\frac{\cdot}{\varepsilon}\right),\ \varepsilon > 0.$

Proposition 3.9. *Let M and N be two sequences satisfying* $(M.1)$, $(M.2)$ *and* $(M.3)'$, *then the map*

$$\begin{array}{llll} J_{aa}: & E'_{aa,(MN)} & \longrightarrow & \mathcal{G}^M_{aa} \\ & T & \longmapsto & (T*\rho_\varepsilon)_\varepsilon + \mathcal{N}^M_{aa} \end{array}$$

is a linear embedding.

Proof. Let $T \in E'_{aa,(MN)}$ then $T = P(D) f + g$, where P is an ultradifferential operator of class (MN), $f \in C_{aa}$ and $g \in C_{aa}$. Due to Young inequality, we have $\forall j \in \mathbb{Z}_+$, $\forall x \in \mathbb{R}$,

$$\begin{aligned} \left|(T*\rho_\varepsilon)^{(j)}(x)\right| &\le \left|f * P(D)\rho_\varepsilon^{(j)}(x)\right| + \left|g*\rho_\varepsilon^{(j)}(x)\right| \\ &\le \|f\|_{L^\infty} \sum_{i\in\mathbb{Z}_+} |a_i| \frac{1}{\varepsilon^{i+j}} \int_{\mathbb{R}} \left|\rho^{(i+j)}(y)\right| dy + \|g\|_{L^\infty} \frac{1}{\varepsilon^j} \int_{\mathbb{R}} \left|\rho^{(j)}(y)\right| dy. \end{aligned}$$

On the other hand, as $\rho \in \mathcal{D}^{\{N\}}_{L^1}$ then $\exists h > 0$ such that $\|\rho\|_{1,h,N} < \infty$. As $P(D) = \sum_{i\in\mathbb{Z}_+} a_i D^i$, is an ultradifferential operator of class (MN), so $\exists L > 0$ and $\exists c > 0$ such that $\forall i \in \mathbb{Z}_+$, $|a_i| \le c L^i (M_i N_i)^{-1}$. It follows that

$$\left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty} \le c\|f\|_{L^\infty} \sum_{i\in\mathbb{Z}_+} \frac{L^i}{M_i N_i} \frac{h^{i+j}}{\varepsilon^{i+j}} \frac{\left\|\rho^{(i+j)}\right\|_{L^1}}{h^{i+j}} + \|g\|_{L^\infty} \frac{h^j}{\varepsilon^j} \frac{\left\|\rho^{(j)}\right\|_{L^1}}{h^j}.$$

Since M and N satisfy $(M.2)$, there exist $A,\ A' > 0$ and $H,\ H' > 0$ such that

$$M_{i+j} \le A H^{i+j} M_i M_j \quad \text{and} \quad N_{i+j} \le A' H'^{i+j} N_i N_j,$$

which gives

$$\frac{1}{M_i N_i} \le \frac{AA' (HH')^{i+j}}{M_{i+j} N_{i+j}} M_j N_j.$$

Therefore,

$$\begin{aligned} \left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty} \le\ & cAA' \|f\|_{L^\infty} \sum_{i\in\mathbb{Z}_+} \frac{L^i}{M_{i+j} N_{i+j}} (HH')^{i+j} \frac{h^{i+j}}{\varepsilon^{i+j}} M_j N_j \frac{\left\|\rho^{(i+j)}\right\|_{L^1}}{h^{i+j}} \\ & + \|g\|_{L^\infty} \frac{h^j}{\varepsilon^j} \frac{\left\|\rho^{(j)}\right\|_{L^1}}{h^j}, \end{aligned}$$

hence,

$$\begin{aligned}\frac{1}{M_jN_j}\left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty} &\leq cAA'\left\|f\right\|_{L^\infty}\sum_{i\in\mathbb{Z}_+}\frac{L^i}{M_{i+j}N_{i+j}}(HH')^{i+j}\frac{h^{i+j}}{\varepsilon^{i+j}}\frac{\left\|\rho^{(i+j)}\right\|_{L^1}}{h^{i+j}}\\ &\quad+\left\|g\right\|_{L^\infty}\frac{h^j}{\varepsilon^j}\frac{1}{M_jN_j}\frac{\left\|\rho^{(j)}\right\|_{L^1}}{h^j},\\ &\leq cAA'\left\|f\right\|_{L^\infty}\sum_{i\in\mathbb{Z}_+}\frac{L^i}{M_{i+j}}(HH')^{i+j}\frac{h^{i+j}}{\varepsilon^{i+j}}\frac{\left\|\rho^{(i+j)}\right\|_{L^1}}{h^{i+j}N_{i+j}}\\ &\quad+\left\|g\right\|_{L^\infty}\frac{h^j}{\varepsilon^j}\frac{1}{M_j}\frac{\left\|\rho^{(j)}\right\|_{L^1}}{h^jN_j},\\ &\leq \left\|\rho\right\|_{1,h,N}\left(cAA'\left\|f\right\|_{L^\infty}\sum_{i\in\mathbb{Z}_+}\frac{L^i}{M_{i+j}}(HH')^{i+j}\frac{h^{i+j}}{\varepsilon^{i+j}}+\left\|g\right\|_{L^\infty}\frac{h^j}{\varepsilon^j}\frac{1}{M_j}\right).\end{aligned}$$

Thus,

$$\frac{(2L)^j}{M_jN_j}\left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty}\leq\left\|\rho\right\|_{1,h,N}\begin{pmatrix} cAA'\left\|f\right\|_{L^\infty}\sum_{i\in\mathbb{Z}_+}2^{-i}\frac{(2L)^{i+j}}{M_{i+j}}(HH')^{i+j}\frac{h^{i+j}}{\varepsilon^{i+j}}+\\ +\left\|g\right\|_{L^\infty}(2L)^j\frac{h^j}{\varepsilon^j}\frac{1}{M_j}\end{pmatrix},$$

i.e.

$$\frac{(2L)^j}{M_jN_j}\left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty}\leq\left\|\rho\right\|_{1,h,N}\left(cAA'\left\|f\right\|_{L^\infty}\sum_{i\in\mathbb{Z}_+}2^{-i}\frac{\left(\frac{2LHH'h}{\varepsilon}\right)^{i+j}}{M_{i+j}}+\left\|g\right\|_{L^\infty}\frac{\left(\frac{2Lh}{\varepsilon}\right)^j}{M_j}\right).$$

The fact that M satisfies $(M.1)$, so Proposition 2.6– (1), gives

$$\frac{\left(\frac{2LHH'h}{\varepsilon}\right)^{i+j}}{M_{i+j}}\leq\frac{1}{M_0}e^{M\left(\frac{2LHH'h}{\varepsilon}\right)}\quad\text{and}\quad\frac{\left(\frac{2Lh}{\varepsilon}\right)^j}{M_j}\leq\frac{1}{M_0}e^{M\left(\frac{2Lh}{\varepsilon}\right)}.$$

Consequently,

$$\begin{aligned}\frac{(2L)^j}{M_jN_j}\left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty} &\leq \frac{1}{M_0}\left\|\rho\right\|_{1,h,N}\left(2cAA'\left\|f\right\|_{L^\infty}e^{M\left(\frac{2LHH'h}{\varepsilon}\right)}+\left\|g\right\|_{L^\infty}e^{M\left(\frac{2Lh}{\varepsilon}\right)}\right)\\ &\leq \frac{C}{M_0}\left(e^{M\left(\frac{2LHH'h}{\varepsilon}\right)}+e^{M\left(\frac{2Lh}{\varepsilon}\right)}\right),\end{aligned}$$

where $C=\left\|\rho\right\|_{1,h,N}\max\left(2cAA'\left\|f\right\|_{L^\infty},\left\|g\right\|_{L^\infty}\right)$. Hence,

$$\frac{(2L)^j}{M_jN_j}\left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty}\leq\frac{2C}{M_0}e^{M\left(\frac{2LHH'h}{\varepsilon}\right)+M\left(\frac{2Lh}{\varepsilon}\right)}.$$

Due to Lemma 2.7, let $k>0$ such that $\frac{k}{\varepsilon}=H\max\left(\frac{2LHH'h}{\varepsilon},\frac{2Lh}{\varepsilon}\right)$ and $C'_j=\left(2AC\frac{M_jN_j}{(2L)^j}\right)>0$, we get

$$\left\|(T*\rho_\varepsilon)^{(j)}\right\|_{L^\infty}\leq C'_je^{M\left(\frac{k}{\varepsilon}\right)},$$

which means that $(T * \rho_\varepsilon) \in \mathcal{M}_{aa}^M$. The linearity follows from the fact that the convolution is linear. Let $\rho \in \mathcal{D}_{L^1}^{\{N\}}$ such that $\int_{\mathbb{R}} \rho(x)\,dx = 1$. If $(T * \rho_\varepsilon) \in \mathcal{N}_{aa}^M$, then

$$\forall k > 0,\ \exists c > 0,\ \exists \varepsilon_0 \in I,\ \forall \varepsilon < \varepsilon_0,\ \left\|T * \rho_\varepsilon\right\|_{L^\infty} \leq ce^{-M\left(\frac{k}{\varepsilon}\right)}. \tag{7}$$

Let $\psi \in \mathcal{D}_{L^1}^{(MN)}$, we have

$$\langle T, \psi \rangle = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} (T * \rho_\varepsilon)(x)\, \psi(x)\, dx.$$

From (7), we obtain

$$\left| \int_{\mathbb{R}} (T * \rho_\varepsilon)(x)\, \psi(x)\, dx \right| \leq \left\|\psi\right\|_{L^1} \left\|T * \rho_\varepsilon\right\|_{L^\infty} \leq c \left\|\psi\right\|_{L^1} e^{-M\left(\frac{k}{\varepsilon}\right)},$$

let $\varepsilon \to 0$, thus $\langle T, \psi \rangle = 0, \forall \psi \in \mathcal{D}_{L^1}^{(MN)}$. Hence, J_{aa} is injective. □

Remark 3.10. *In addition, if the sequence M satisfies the condition (M.3) then due to Theorem 3.1 of [26], the space $E'_{aa,(M)}$ coincides with the space of almost automorphic Beurling ultradistributions studied in [26]. Therefore, in view of Proposition 3.9 the space of almost automorphic Beurling ultradistributions is embedded into the algebra of almost automorphic Colombeau generalized ultradistributions.*

In order to establish some properties of the algebra $\mathcal{G}_{aa}^M$, we recall some needed algebras. The algebra of $C_{aa}-$generalized functions defined by

$$\mathcal{G}_{C_{aa}}^M := \frac{\mathcal{M}_{C_{aa}}^M}{\mathcal{N}_{C_{aa}}^M},$$

where

$$\mathcal{M}_{C_{aa}}^M := \left\{(f_\varepsilon)_{\varepsilon\in I} \in (C_{aa})^I : \exists k > 0, \left\|f_\varepsilon\right\|_{L^\infty} = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right),\ \varepsilon \to 0\right\},$$

and

$$\mathcal{N}_{C_{aa}}^M := \left\{(f_\varepsilon)_{\varepsilon\in I} \in (C_{aa})^I : \forall k > 0, \left\|f_\varepsilon\right\|_{L^\infty} = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right),\ \varepsilon \to 0\right\}.$$

The algebra of L^p-Colombeau generalized ultradistributions, $p \in [1, +\infty]$, is denoted and defined by the quotient algebra

$$\mathcal{G}_{L^p}^M := \frac{\mathcal{M}_{L^p}^M}{\mathcal{N}_{L^p}^M},$$

where

$$\mathcal{M}_{L^p}^M := \left\{(u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^p})^I : \forall j \in \mathbb{Z}_+,\ \exists k > 0,\ \left\|u_\varepsilon^{(j)}\right\|_{L^p} = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right),\ \ \varepsilon \to 0\right\}$$

and

$$\mathcal{N}_{L^p}^M := \left\{(u_\varepsilon)_\varepsilon \in (\mathcal{D}_{L^p})^I : \forall j \in \mathbb{Z}_+,\ \forall k > 0,\ \left\|u_\varepsilon^{(j)}\right\|_{L^p} = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right),\ \ \varepsilon \to 0\right\}.$$

Remark 3.11. *The elements of* $\mathcal{G}^M_{L^\infty}$ *are said to be bounded generalized ultradistributions. We denote* $\mathcal{G}^M_{\mathcal{B}} := \mathcal{G}^M_{L^\infty}$.

The following result summarise some properties of $\mathcal{G}^M_{aa}$.

Proposition 3.12. *1.* $\mathcal{G}^M_{aa}$ *is a subalgebra of* $\mathcal{G}^M_{\mathcal{B}}$ *stable under translation and derivation.*
2. $\mathcal{G}^M_{aa} * \mathcal{G}^M_{L^1} \subset \mathcal{G}^M_{aa}$.
3 $\mathcal{G}^M_{aa}$ *is embedded canonically into* $\mathcal{G}^M_{\mathcal{B}}$ *and* $\mathcal{G}^M_{C_{aa}}$.

Proof. 1. It follows from Proposition 3.2− (2) that $\mathcal{G}^M_{aa}$ is an algebra stable under translation and derivation.

2. Let $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{aa}$ and $(v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{L^1}$ be a respective representatives of $\widetilde{u} \in \mathcal{G}^M_{aa}$ and $\widetilde{v} \in \mathcal{G}^M_{L^1}$. If $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{aa}$ so it satisfies the estimate (1) and $(v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{L^1}$, i.e. satisfies

$$\forall j \in \mathbb{Z}_+,\ \exists k'_j > 0,\ \exists c'_j > 0,\ \exists \varepsilon'_j \in I,\ \forall \varepsilon < \varepsilon'_j,\ \left\| v_\varepsilon^{(j)} \right\|_{L^1} \leq c'_j e^{M\left(\frac{k'_j}{\varepsilon}\right)}.$$

In view of Proposition 2.3− (2) , $\forall \varepsilon \in I,\ (u_\varepsilon * v_\varepsilon) \in \mathcal{B}_{aa}$ and due to Young inequality, we obtain for every $j \in \mathbb{Z}_+$,

$$\left\| (u_\varepsilon * v_\varepsilon)^{(j)} \right\|_{L^\infty} \leq \left\| u_\varepsilon^{(j)} \right\|_{L^\infty} \|v_\varepsilon\|_{L^1} \leq c_j c'_0 e^{M\left(\frac{k_j}{\varepsilon}\right)} e^{M\left(\frac{k'_0}{\varepsilon}\right)},$$

by Lemma 2.7, let $k > 0$ such that $\frac{k}{\varepsilon} = H \max\left(\frac{k_j}{\varepsilon}, \frac{k'_0}{\varepsilon}\right)$, then

$$e^{M\left(\frac{k_j}{\varepsilon}\right)} e^{M\left(\frac{k'_0}{\varepsilon}\right)} \leq A M_0 e^{M\left(\frac{k}{\varepsilon}\right)}.$$

Consequently, $\forall j \in \mathbb{Z}_+,\ \exists k > 0,\ \exists C_j = \left(c_j c'_0 A M_0\right) > 0,\ \forall \varepsilon < \min\left(\varepsilon_j, \varepsilon'_0\right)$,

$$\left\| (u_\varepsilon * v_\varepsilon)^{(j)} \right\|_{L^\infty} \leq C_j e^{M\left(\frac{k}{\varepsilon}\right)},$$

this gives that $(u_\varepsilon * v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{aa}$. It is easy to prove that the convolution does not depend on the representatives $(u_\varepsilon)_\varepsilon$ and $(v_\varepsilon)_\varepsilon$.

3. We clearly have $\mathcal{G}^M_{aa} \subset \mathcal{G}^M_{\mathcal{B}}$. Indeed let $\widetilde{u} = [(u_\varepsilon)_\varepsilon] \in \mathcal{G}^M_{aa}$, i.e. $(u_\varepsilon)_\varepsilon$ satisfies (1) , as $u_\varepsilon \in \mathcal{B}_{aa} = \mathcal{B} \cap C_{aa} \subset \mathcal{B}, \forall \varepsilon > 0$, then $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{\mathcal{B}}$. In the same way, if $(u_\varepsilon)_\varepsilon \in \mathcal{N}^M_{aa}$, then $(u_\varepsilon)_\varepsilon \in \mathcal{N}^M_{\mathcal{B}}$. As obviously $\mathcal{N}^M_{\mathcal{B}} \cap \mathcal{M}^M_{aa} \subset \mathcal{N}^M_{aa}$, then the embedding is clear. In the same way, considering (1) with $i = 0$ we obtain that $\mathcal{G}^M_{aa} \subset \mathcal{G}^M_{C_{aa}}$. The inclusion $\mathcal{N}^M_{C_{aa}} \cap \mathcal{M}^M_{aa} \subset \mathcal{N}^M_{aa}$, giving the embedding, is obtained from the null characterisation of $\mathcal{N}^M_{aa}$, i.e. Proposition 3.2. □

As a consequence we have a characterization of elements of $\mathcal{G}^M_{aa}$ similar to the result characterizing almost automorphic ultradistributions.

Corollary 3.13. *Let* $\widetilde{u} \in \mathcal{G}^M_{\mathcal{B}}$*, the following assertions are equivalent :*
i) $\widetilde{u} \in \mathcal{G}^M_{aa}$.
ii) $\widetilde{u} * \varphi \in \mathcal{G}^M_{C_{aa}}, \forall \varphi \in \mathcal{D}^M$.

Proof. If $\widetilde{u} \in \mathcal{G}^M_{aa}$, then by the results 2 and 3 of Proposition 3.12 we obtain $\widetilde{u} * \varphi \in \mathcal{G}^M_{C_{aa}}, \forall \varphi \in \mathcal{D}^M$. Conversely, let $(u_\varepsilon)_\varepsilon \in \mathcal{M}^M_{\mathcal{B}}$ be a representative of $\widetilde{u}$ and $\forall \varphi \in \mathcal{D}^M,\ \forall \varepsilon \in I, u_\varepsilon * \varphi \in C_{aa}$. In view of Proposition 5 −(4) of [11] it follows that $\forall \varepsilon \in I, u_\varepsilon \in \mathcal{B}_{aa}$. Since $u_\varepsilon \in \mathcal{M}^M_{\mathcal{B}}$ then it holds that $u_\varepsilon \in \mathcal{M}^M_{aa}$. If $u_\varepsilon \in \mathcal{N}^M_{\mathcal{B}}$ and $\widetilde{u} * \varphi \in \mathcal{G}^M_{C_{aa}}, \forall \varphi \in \mathcal{D}^M$, we obtain the same result as we have $\mathcal{N}^M_{\mathcal{B}} \subset \mathcal{M}^M_{\mathcal{B}}$. □

4. Non-linear operation

We show in this section that the composition of a tempered generalized ultradistribution with an almost automorphic Colombeau generalized ultradistribution is an almost automorphic Colombeau generalized ultradistribution. First, recall from [12], the algebra of tempered generalized ultradistributions on $\mathbb{C}$, denoted and defined as the quotient algebra

$$\mathcal{G}_{\tau}^{M}(\mathbb{C}) := \frac{\mathcal{M}_{\tau}^{M}(\mathbb{C})}{\mathcal{N}_{\tau}^{M}(\mathbb{C})},$$

where

$$\mathcal{M}_{\tau}^{M}(\mathbb{C}) := \left\{ \begin{array}{c} (f_{\varepsilon})_{\varepsilon} \in \left(\mathcal{E}\left(\mathbb{R}^{2}\right)\right)^{I} : \forall j \in \mathbb{Z}_{+}^{2},\ \exists k > 0, \\ \sup\limits_{x\in\mathbb{R}^{2}} (1+|x|)^{-k} \left| f_{\varepsilon}^{(j)}(x) \right| = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right),\quad \varepsilon \to 0 \end{array} \right\}$$

and

$$\mathcal{N}_{\tau}^{M}(\mathbb{C}) := \left\{ \begin{array}{c} (f_{\varepsilon})_{\varepsilon} \in \left(\mathcal{E}\left(\mathbb{R}^{2}\right)\right)^{I} : \forall j \in \mathbb{Z}_{+}^{2},\ \exists m > 0,\ \forall k > 0, \\ \sup\limits_{x\in\mathbb{R}^{2}} (1+|x|)^{-m} \left| f_{\varepsilon}^{(j)}(x) \right| = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right),\quad \varepsilon \to 0 \end{array} \right\}.$$

Proposition 4.1. *Let* $\widetilde{u} = [(u_{\varepsilon})_{\varepsilon}] \in \mathcal{G}_{aa}^{M}$ *and* $\widetilde{F} = \left[(f_{\varepsilon})_{\varepsilon}\right] \in \mathcal{G}_{\tau}^{M}(\mathbb{C})$ *then*

$$\widetilde{F} \circ \widetilde{u} := \left[(f_{\varepsilon} \circ u_{\varepsilon})_{\varepsilon}\right]$$

is well-defined element of $\mathcal{G}_{aa}^{M}$.

Proof. Let $(u_{\varepsilon})_{\varepsilon} \in \mathcal{M}_{aa}^{M}$ and $(f_{\varepsilon})_{\varepsilon} \in \mathcal{M}_{\tau}^{M}(\mathbb{C})$ be a respective representatives of $\widetilde{u}$ and $\widetilde{F}$, then due to Proposition 1 − (4) of [8] we obtain $\forall \varepsilon \in I,\ f_{\varepsilon} \circ u_{\varepsilon} \in \mathcal{B}_{aa}$. Moreover, we have

$$\forall j \in \mathbb{Z}_{+},\ \exists k_j > 0,\ \exists c_j > 0,\ \exists \varepsilon_j \in I,\ \forall \varepsilon < \varepsilon_j,\ \left\| u_{\varepsilon}^{(j)} \right\|_{L^{\infty}} \leq c_j e^{M\left(\frac{k_j}{\varepsilon}\right)} \tag{8}$$

and

$$\forall j \in \mathbb{Z}_{+},\ \exists k'_j > 0,\ \exists c'_j > 0,\ \exists \varepsilon'_j \in I,\ \forall \varepsilon < \varepsilon'_j,\ \left\| f_{\varepsilon}^{(j)}(u_{\varepsilon}) \right\|_{L^{\infty}} \leq c'_j e^{M\left(\frac{k'_j}{\varepsilon}\right)} \|1 + u_{\varepsilon}\|_{L^{\infty}}^{k'_j}. \tag{9}$$

By using the classical Faà di Bruno formula, we have $\forall j \in \mathbb{Z}_{+}$,

$$\frac{(f_{\varepsilon} \circ u_{\varepsilon})^{(j)}(x)}{j!} = \sum_{\substack{l_1+2l_2+\cdots+jl_j=j \\ r=l_1+\cdots+l_j}} \frac{f_{\varepsilon}^{(r)}(u_{\varepsilon}(x))}{l_1! \cdots l_j!} \prod_{i=1}^{j} \left(\frac{u_{\varepsilon}^{(i)}(x)}{i!} \right)^{l_i}, \tag{10}$$

from (8) and (9), we get

$$\begin{aligned} \frac{\left\| (f_{\varepsilon} \circ u_{\varepsilon})^{(j)} \right\|_{L^{\infty}}}{j!} &\leq \sum_{\substack{l_1+2l_2+\cdots+jl_j=j \\ r=l_1+\cdots+l_j}} \frac{\left\| f_{\varepsilon}^{(r)}(u_{\varepsilon}) \right\|_{L^{\infty}}}{l_1! \cdots l_j!} \prod_{i=1}^{j} \left(\frac{\left\| u_{\varepsilon}^{(i)} \right\|_{L^{\infty}}}{i!} \right)^{l_i} \\ &\leq \sum_{\substack{l_1+2l_2+\cdots+jl_j=j \\ r=l_1+\cdots+l_j}} \frac{c'_r e^{M\left(\frac{k'_r}{\varepsilon}\right)} \|1 + u_{\varepsilon}\|_{L^{\infty}}^{k'_r}}{l_1! \cdots l_j!} \prod_{i=1}^{j} \left(\frac{\left\| u_{\varepsilon}^{(i)} \right\|_{L^{\infty}}}{i!} \right)^{l_i}, \end{aligned}$$

so

$$\frac{\left\|(f_\varepsilon\circ u_\varepsilon)^{(j)}\right\|_{L^\infty}}{j!}\leq\sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}}\frac{c_r'e^{M\left(\frac{k_r'}{\varepsilon}\right)}\left(1+c_0e^{M\left(\frac{k_0}{\varepsilon}\right)}\right)^{k_r'}}{l_1!\cdots l_j!}\prod_{i=1}^{j}\left(\frac{c_ie^{M\left(\frac{k_i}{\varepsilon}\right)}}{i!}\right)^{l_i},$$

hence there exists $C_r>0$,

$$\begin{aligned}\frac{\left\|(f_\varepsilon\circ u_\varepsilon)^{(j)}\right\|_{L^\infty}}{j!} &\leq \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}}\frac{C_r}{l_1!\cdots l_j!}e^{M\left(\frac{k_r'}{\varepsilon}\right)}e^{k_r'M\left(\frac{k_0}{\varepsilon}\right)}\prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}e^{l_iM\left(\frac{k_i}{\varepsilon}\right)}\\ &\leq \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}}\frac{C_r}{l_1!\cdots l_j!}e^{M\left(\frac{k_r'}{\varepsilon}\right)}e^{([k_r']+1)M\left(\frac{k_0}{\varepsilon}\right)}\prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}e^{l_iM\left(\frac{k_i}{\varepsilon}\right)}.\end{aligned}$$

Set $m=[k_r']+1$, due to Lemma 2.7, we have

$$e^{([k_r']+1)M\left(\frac{k_0}{\varepsilon}\right)}\leq(AM_0)^{m-1}\,e^{M\left(\frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right)}$$

and

$$e^{l_iM\left(\frac{k_i}{\varepsilon}\right)}\leq(AM_0)^{l_i-1}\,e^{M\left(\frac{k_i}{\varepsilon}H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right)}.$$

Therefore,

$$\begin{aligned}\frac{\left\|(f_\varepsilon\circ u_\varepsilon)^{(j)}\right\|_{L^\infty}}{j!} &\leq \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}}\frac{C_r}{l_1!\cdots l_j!}(AM_0)^{m-1}\,e^{M\left(\frac{k_r'}{\varepsilon}\right)}e^{M\left(\frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right)}\\ &\quad\times\prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}(AM_0)^{l_i-1}\,e^{M\left(\frac{k_i}{\varepsilon}H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right)}\\ &\leq \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}}\frac{C_r}{l_1!\cdots l_j!}(AM_0)^{m-1}\,e^{M\left(\frac{k_r'}{\varepsilon}\right)}e^{M\left(\frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right)}\\ &\quad\times\left((AM_0)^{r-j}\,e^{\sum_{i=1}^{j}M\left(\frac{k_i}{\varepsilon}H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right)}\right)\prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}\\ &\leq \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}}\frac{C_r}{l_1!\cdots l_j!}(AM_0)^{m+r-j-1}\,e^{M\left(\frac{k_r'}{\varepsilon}\right)}e^{M\left(\frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right)}\\ &\quad\times e^{\sum_{i=1}^{j}M\left(\frac{k_i}{\varepsilon}H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right)}\prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}.\end{aligned}$$

By Lemma 2.7, we get

$$\begin{aligned}\sum_{i=1}^{j} M\left(\frac{k_i}{\varepsilon} H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right) &\le M\left(H^{\frac{(j-1)(j+2)}{2j}} \max_{1\le i\le j}\left(\frac{k_i}{\varepsilon} H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right)\right) + (j-1)\ln(AM_0)\\ &\le M\left(\frac{m_j}{\varepsilon}\right) + (j-1)\ln(AM_0),\end{aligned}$$

where $m_j := H^{\frac{(j-1)(j+2)}{2j}} \max_{1\le i\le j}\left(k_i H^{\frac{(l_i-1)(l_i+2)}{2l_i}}\right)$, so

$$\begin{aligned}\frac{\left\|(f_\varepsilon \circ u_\varepsilon)^{(j)}\right\|_{L^\infty}}{j!} &\le \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!}(AM_0)^{m+r-j-1} e^{M\left(\frac{k'_r}{\varepsilon}\right)} e^{M\left(\frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right)}\\ &\quad\times (AM_0)^{j-1} e^{M\left(\frac{m_j}{\varepsilon}\right)} \prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i},\end{aligned}$$

by using again Lemma 2.7, let $N_j > 0$, such that $\frac{N_j}{\varepsilon} = H^{\frac{5}{3}} \max\left(\frac{k'_r}{\varepsilon}, \frac{m_j}{\varepsilon}, \frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right)$, thus

$$e^{M\left(\frac{k'_r}{\varepsilon}\right)} e^{M\left(\frac{k_0}{\varepsilon}H^{\frac{(m-1)(m+2)}{2m}}\right)} e^{M\left(\frac{m_j}{\varepsilon}\right)} \le (AM_0)^2 e^{M\left(\frac{N_j}{\varepsilon}\right)}.$$

It follows

$$\begin{aligned}\frac{\left\|(f_\varepsilon \circ u_\varepsilon)^{(j)}\right\|_{L^\infty}}{j!} &\le \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!}(AM_0)^{m+r-j-1}(AM_0)^{j-1}(AM_0)^2 e^{M\left(\frac{N_j}{\varepsilon}\right)} \prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}\\ &\le \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!}(AM_0)^{m+r} e^{M\left(\frac{N_j}{\varepsilon}\right)} \prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\left\|(f_\varepsilon \circ u_\varepsilon)^{(j)}\right\|_{L^\infty} &\le j! \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!}(AM_0)^{m+r} e^{M\left(\frac{N_j}{\varepsilon}\right)} \prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}\\ &\le C'_j e^{M\left(\frac{N_j}{\varepsilon}\right)},\end{aligned}$$

where $C'_j := j! \sum_{\substack{l_1+2l_2+\cdots+jl_j=j\\ r=l_1+\cdots+l_j}} \frac{C_r}{l_1!\cdots l_j!}(AM_0)^{m+r} \prod_{i=1}^{j}\left(\frac{c_i}{i!}\right)^{l_i}$. Then, we deduce that $(f_\varepsilon \circ u_\varepsilon)_\varepsilon \in \mathcal{M}_{aa}^{M}$. It is easy to prove that the composition $\widetilde{F} \circ \widetilde{u}$ is independent on the representatives $(u_\varepsilon)_\varepsilon$ and $(f_\varepsilon)_\varepsilon$. □

5. Linear systems of ordinary differential equations

Consider the system of linear ordinary differential equations

$$\dot{\widetilde{u}} = A\widetilde{u} + \widetilde{f}, \tag{11}$$

where $\widetilde{f} = \left(\left[(f_{1,\varepsilon})_\varepsilon\right], ..., \left[(f_{n,\varepsilon})_\varepsilon\right]\right) \in \left(\mathcal{G}_{aa}^M\right)^n$ and $A = \left(a_{ij}\right)_{0\leq i,j\leq n}$ is a square matrix of order n of elements of $\mathbb{C}$. The unknown generalized ultradistribution is $\widetilde{u} = \left(\left[(u_{1,\varepsilon})_\varepsilon\right], ..., \left[(u_{n,\varepsilon})_\varepsilon\right]\right)$.

Remark 5.1. *We say that* $\widetilde{u} = \left(\left[(u_{1,\varepsilon})_\varepsilon\right], ..., \left[(u_{n,\varepsilon})_\varepsilon\right]\right)$ *is bounded (resp. almost automorphic) if each component* $(u_{i,\varepsilon})_\varepsilon$ *,* $0 \leq i \leq n$*, is bounded (resp. almost automorphic).*

Algebra of generalized numbers of type M was introduced in [12], which is defined by the quotient

$$\widetilde{\mathbb{C}}^M := \frac{\mathcal{M}^M[\mathbb{K}]}{\mathcal{N}^M[\mathbb{K}]},$$

where

$$\mathcal{M}^M[\mathbb{C}] := \left\{(z_\varepsilon)_\varepsilon \in \mathbb{C}^I, \exists k \in \mathbb{Z}_+, |z_\varepsilon| = O\left(e^{M\left(\frac{k}{\varepsilon}\right)}\right),\ \varepsilon \to 0\right\}$$

and

$$\mathcal{N}^M[\mathbb{C}] := \left\{(z_\varepsilon)_\varepsilon \in \mathbb{C}^I, \forall k \in \mathbb{Z}_+, |z_\varepsilon| = O\left(e^{-M\left(\frac{k}{\varepsilon}\right)}\right),\ \varepsilon \to 0\right\}.$$

The following result is a generalized version of the Bohr-Neugebauer Theorem.

Proposition 5.2. *Let a bounded gerneralized ultradistribution* $\widetilde{u} \in \left(\mathcal{G}_{\mathcal{B}}^M\right)^n$ *satisfy*

$$(\dot{u}_\varepsilon)_\varepsilon - A(u_\varepsilon)_\varepsilon - (f_\varepsilon)_\varepsilon \in \left(\mathcal{N}_{aa}^M\right)^n, \tag{12}$$

where $(u_\varepsilon)_\varepsilon$ *and* $(f_\varepsilon)_\varepsilon$ *are respectively representatives of* $\widetilde{u}$ *and* $\widetilde{f}$*. Then* $\widetilde{u}$ *is an almost automorphic Colombeau generalized ultradistribution.*

Proof. from [38] , there exist an invertible matrix $P = \left(P_{ij}\right)_{0\leq i,j\leq n}$ such that $A = PTP^{-1}$, where

$$T = \begin{pmatrix} \lambda_1 & b_{12} & \cdots & b_{1n} \\ 0 & \lambda_2 & \cdots & b_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix}$$

and $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of the matrix A. Let $\widetilde{v} = \left(\left[(v_{1,\varepsilon})_\varepsilon\right], ..., \left[(v_{n,\varepsilon})_\varepsilon\right]\right) = P^{-1}\widetilde{u}$ and $\widetilde{g} = \left(\left[(g_{1,\varepsilon})_\varepsilon\right], ..., \left[(g_{n,\varepsilon})_\varepsilon\right]\right) = P^{-1}\widetilde{f}$, then 12 is equivalent to the system

$$\begin{cases} \dot{v}_{1,\varepsilon}(t) - \lambda_1 v_{1,\varepsilon}(t) + b_{12}v_{2,\varepsilon}(t) + \cdots + b_{1n}v_{n,\varepsilon}(t) + g_{1,\varepsilon}(t) = h_{1,\varepsilon}(t). \\ \dot{v}_{2,\varepsilon}(t) - \lambda_2 v_{2,\varepsilon}(t) + b_{23}v_{2,\varepsilon}(t) + \cdots + b_{2n}v_{n,\varepsilon}(t) + g_{2,\varepsilon}(t) = h_{2,\varepsilon}(t). \\ \vdots \\ \dot{v}_{n,\varepsilon}(t) - \lambda_n v_{n,\varepsilon}(t) + g_{n,\varepsilon}(t) = h_{n,\varepsilon}(t) \end{cases}, \tag{13}$$

where $\left(\left[(h_{1,\varepsilon})_\varepsilon\right], ..., \left[(h_{n,\varepsilon})_\varepsilon\right]\right) \in \left(\mathcal{N}_{aa}^M\right)^n$. The result is then reduced to prove that if $\widetilde{v} = [(v_\varepsilon)_\varepsilon] \in \mathcal{G}_{\mathcal{B}}^M$ satisfies

$$(\dot{v}_\varepsilon)_\varepsilon - \lambda(v_\varepsilon)_\varepsilon - (g_\varepsilon)_\varepsilon \in \mathcal{N}_{aa}^M, \tag{14}$$

where $\widetilde{g} = \left[(g_\varepsilon)_\varepsilon\right] \in \mathcal{G}_{aa}^M$ and $\lambda \in \mathbb{C}$, then $\widetilde{v} \in \mathcal{G}_{aa}^M$. The general solution of 14 is given by the representative

$$(v_\varepsilon(t))_\varepsilon = \left(e^{\lambda t}\left(C_\varepsilon + \int_0^t e^{-\lambda s}(g_\varepsilon(s) + h_\varepsilon(s))\,ds\right)\right)_\varepsilon,$$

where $(C_\varepsilon)_\varepsilon \in \mathcal{M}^M[\mathbb{C}]$ and $(h_\varepsilon)_\varepsilon \in \mathcal{N}^M_{aa}$. Since $\widetilde{v} \in \mathcal{G}^M_{\mathcal{B}}$, then we have three cases :

1) $v_\varepsilon(t) = -\int\limits_t^{+\infty} e^{\lambda(t-s)}(g_\varepsilon(s) + h_\varepsilon(s))\,ds$, if $\operatorname{Re}\lambda > 0$.

2) $v_\varepsilon(t) = \int\limits_{-\infty}^{t} e^{\lambda(t-s)}(g_\varepsilon(s) + h_\varepsilon(s))\,ds$, if $\operatorname{Re}\lambda < 0$.

3) $v_\varepsilon(t) = e^{i\operatorname{Im} t}\left(C_\varepsilon + \int\limits_0^t e^{-i\operatorname{Im} s}(g_\varepsilon(s) + h_\varepsilon(s))\,ds\right)$, if $\operatorname{Re}\lambda = 0$.

In the case $\operatorname{Re}\lambda > 0$, since $\forall\varepsilon > 0, g_\varepsilon, h_\varepsilon \in \mathcal{B}_{aa}$ and due to ([16], Proposition 9) so, for every $(s_m)_{m\in\mathbb{N}} \subset \mathbb{R}$, there exist a subsequence $\left(s_{m_{k(\varepsilon)}}\right)_k$ of $(s_m)_{m\in\mathbb{N}}$ and $\overline{g}, \overline{h}$ such that for any $j \in \mathbb{Z}_+, t \in \mathbb{R}, \varepsilon \in I$,

$$\begin{aligned}
\overline{g}_\varepsilon^{(j)} \quad &:= \lim_{k\to+\infty} g_\varepsilon^{(j)}\left(t + s_{m_{k(\varepsilon)}}\right) \text{ and } \lim_{k\to+\infty} \overline{g}_\varepsilon^{(j)}\left(t - s_{m_{k(\varepsilon)}}\right) = g_\varepsilon^{(j)}(t) \\
\overline{h}_\varepsilon^{(j)} \quad &:= \lim_{k\to+\infty} h_\varepsilon^{(j)}\left(t + s_{m_{k(\varepsilon)}}\right) \text{ and } \lim_{k\to+\infty} \overline{h}_\varepsilon^{(j)}\left(t - s_{m_{k(\varepsilon)}}\right) = h_\varepsilon^{(j)}(t)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
v_\varepsilon^{(j)}\left(t + s_{m_{k(\varepsilon)}}\right) &= -\int\limits_{t+s_{m_{k(\varepsilon)}}}^{+\infty} e^{\lambda\left(t+s_{m_{k(\varepsilon)}}-s\right)}\left(g_\varepsilon^{(j)}(s) + h_\varepsilon^{(j)}(s)\right)ds \\
&= -\int\limits_t^{+\infty} e^{\lambda(t-s)}\left(g_\varepsilon^{(j)}\left(s + s_{m_{k(\varepsilon)}}\right) + h_\varepsilon^{(j)}\left(s + s_{m_{k(\varepsilon)}}\right)\right)ds
\end{aligned}$$

As

$$\left|v_\varepsilon^{(j)}\left(t + s_{m_{k(\varepsilon)}}\right)\right| \leq \frac{1}{\operatorname{Re}\lambda}\left\|g_\varepsilon^{(j)}(s) + h_\varepsilon^{(j)}(s)\right\|_{L^\infty}$$

by the dominated convergence Theorem, we obtain

$$\widetilde{v}_\varepsilon(t) = \lim_{k\to+\infty} v_\varepsilon^{(j)}\left(t + s_{m_{k(\varepsilon)}}\right) = -\int\limits_t^{+\infty} e^{\lambda(t-s)}\left(\overline{g}_\varepsilon^{(j)}(s) + \overline{h}_\varepsilon^{(j)}(s)\right)ds \text{ exists for every } t \in \mathbb{R}$$

Also, according to Remark 2.2

$$\left|\widetilde{v}_\varepsilon\left(t - s_{m_{k(\varepsilon)}}\right)\right| \leq \frac{1}{\operatorname{Re}\lambda}\left\|\overline{g}_\varepsilon^{(j)}(s) + \overline{h}_\varepsilon^{(j)}(s)\right\|_{L^\infty} = \frac{1}{\operatorname{Re}\lambda}\left\|g_\varepsilon^{(j)}(s) + h_\varepsilon^{(j)}(s)\right\|_{L^\infty}$$

due to the dominated convergence Theorem, it follows

$$\begin{aligned}
\lim_{k\to+\infty} \widetilde{v}_\varepsilon^{(j)}\left(t - s_{m_{k(\varepsilon)}}\right) &= \lim_{k\to+\infty}\left(-\int\limits_t^{+\infty} e^{\lambda(t-s)}\left(\overline{g}_\varepsilon^{(j)}\left(s - s_{m_{k(\varepsilon)}}\right) + \overline{h}_\varepsilon^{(j)}\left(s - s_{m_{k(\varepsilon)}}\right)\right)ds\right) \\
&= -\int\limits_t^{+\infty} e^{\lambda(t-s)}\left(g_\varepsilon^{(j)}(s) + h_\varepsilon^{(j)}(s)\right)ds \\
&= v_\varepsilon^{(j)}(t) \quad \text{for every } t \in \mathbb{R}
\end{aligned}$$

It follows in view of Proposition 9 of [16], that $\forall\varepsilon \in I, v_\varepsilon \in \mathcal{B}_{aa}$. As $(v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{\mathcal{B}}$ then $(v_\varepsilon)_\varepsilon \in \mathcal{M}^M_{aa}$. We have the same reasoning for the cases $\operatorname{Re}\lambda < 0$ and for $\operatorname{Re}\lambda = 0, v_\varepsilon(t) = e^{i\operatorname{Im} t}C_\varepsilon + \int\limits_0^t e^{-i\operatorname{Im}(t-s)}(g_\varepsilon(s) + h_\varepsilon(s))\,ds$.

Since v_ε is bounded primitive of the almost automorphic function $e^{-i\,\mathrm{Im}(t-s)}\left(g_\varepsilon(s)+h_\varepsilon(s)\right)$ then by ([39], Theorem1), we obtain that v_ε is an almost automorphic function. Furthermore, By Proposition 2.3−(3) we get $\forall\varepsilon\in I, v_\varepsilon\in\mathcal{C}_{aa}\cap\mathcal{B}=\mathcal{B}_{aa}$. The fact that $(v_\varepsilon)_\varepsilon\in\mathcal{M}_{\mathcal{B}}^M$ gives that $(v_\varepsilon)_\varepsilon\in\mathcal{M}_{aa}^M$. □

Remark 5.3. *If we say that a bounded generalized ultradistribution $\widetilde{u}\in\left(\mathcal{G}_{\mathcal{B}}^M\right)^n$ is a solution of system 12 if it satisfies*

$$(\dot{u}_\varepsilon)_\varepsilon - A\,(u_\varepsilon)_\varepsilon - (f_\varepsilon)_\varepsilon \in \left(\mathcal{N}_{aa}^M\right)^n,$$

and as $\widetilde{u}\in\left(\mathcal{G}_{aa}^M\right)^n \Rightarrow \widetilde{u}\in\left(\mathcal{G}_{\mathcal{B}}^M\right)^n$ is obvious since $\mathcal{G}_{aa}^M\subset\mathcal{G}_{\mathcal{B}}^M$ due to Proposition 3.12-(3), then we have proved that the solution $\widetilde{u}$ of the system 11 is an almost automorphic Colombeau generalized ultradistribution if and only if it is a bounded generalized ultradistribution.

A primitive of $\widetilde{u}=[(u_\varepsilon)_\varepsilon]\in\mathcal{G}_{aa}^M$ is defined by the representative $(U_\varepsilon)_\varepsilon$ as a Colombeau generalized function, where

$$U_\varepsilon(x)=\left(\int_{x_0}^{x} u_\varepsilon(y)\,dy\right)_\varepsilon,\ x_0\in\mathbb{R}.$$

The following result is an extension to almost automorphic Colombeau generalized ultradistributions of the classical result of Bohl-Bohr on primitives.

Corollary 5.4. *A primitive of an almost automorphic Colombeau generalized ultradistribution is almost automorphic if and only if it is a bounded generalized ultradistribution.*

The existence of a bounded generalized solution of the system 11 is given by the following result.

Proposition 5.5. *If the matrix A has eigenvalues whose real parts are not zero, then there exists an almost automorphic Colombeau generalized ultradistribution solution $\widetilde{u}$ of the system* 11.

Proof. If $\mathrm{Re}\,\lambda\neq0$, the nth equation of the system 13 has a solution $v_{n,\varepsilon}$ defined by

1. $v_{n,\varepsilon}(t)=-\int_t^{+\infty} e^{\lambda_n(t-s)}\left(g_{n,\varepsilon}(s)+h_{n,\varepsilon}(s)\right)ds$, if $\mathrm{Re}\,\lambda_n>0$.
2. $v_{n,\varepsilon}(t)=\int_{-\infty}^{t} e^{\lambda_n(t-s)}\left(g_{n,\varepsilon}(s)+h_{n,\varepsilon}(s)\right)ds$, if $\mathrm{Re}\,\lambda_n<0$.

By replacing $v_{n,\varepsilon}$ in the $(n-1)$ th equation of the system 13, we obtain $v_{n-1\varepsilon}$ defined in the same way as 1 and 2. The same reasoning for the remaining components which gives that $\left((v_{1,\varepsilon})_\varepsilon,...,(v_{n,\varepsilon})_\varepsilon\right)$ is a solution of the system 13, actually this solution is a bounded generalized ultradistribution. Indeed, since $\mathrm{Re}\,\lambda_i\neq0, 1\leq i\leq n$, from 1 and 2, we have

$$\left\|v_{n,\varepsilon}\right\|_{L^\infty}\leq\frac{1}{|\mathrm{Re}\,\lambda_n|}\left\|g_{n,\varepsilon}+h_{n,\varepsilon}\right\|_{L^\infty}.$$

The $(n-1)$ th equation

$$\dot{v}_{n-1,\varepsilon}(t)=\lambda_{n-1}v_{n-1,\varepsilon}(t)+\left(b_{n-1,n}v_{n,\varepsilon}(t)+g_{n-1,\varepsilon}(t)+h_{n-1,\varepsilon}(t)\right)$$

so the following estimate $\forall\varepsilon\in I$, holds

$$\left\|v_{n-1,\varepsilon}\right\|_{L^\infty}\leq\left(\frac{|b_{n-1n}|}{|\mathrm{Re}\,\lambda_{n-1}|\,|\mathrm{Re}\,\lambda_n|}+\frac{1}{|\mathrm{Re}\,\lambda_{n-1}|}\right)\max\left(\left\|g_{n,\varepsilon}+h_{n,\varepsilon}\right\|_{L^\infty},\left\|g_{n-1,\varepsilon}+h_{n-1,\varepsilon}\right\|_{L^\infty}\right),$$

consequently, we obtain for $i = n, ..., 1$, that there exists $C > 0, \forall \varepsilon \in I$,

$$\left\|v_{i,\varepsilon}\right\|_{L^\infty} \leq C \max_{1\leq i\leq n} \left\|g_{i,\varepsilon} + h_{i,\varepsilon}\right\|_{L^\infty},$$

since the second $\left((g_{1,\varepsilon} + h_{1,\varepsilon})_\varepsilon, ..., (g_{n,\varepsilon} + h_{n,\varepsilon})_\varepsilon\right)$ defines a bounded generalized ultradistribution, then the solution $\left((v_{1,\varepsilon})_\varepsilon, ..., (v_{n,\varepsilon})_\varepsilon\right)$ of the system 13 is a bounded generalized ultradistribution. From the equality $\widetilde{v} = P^{-1}\widetilde{u}$, we have $\widetilde{u} \in \mathcal{G}^M_{\mathcal{B}} \Leftrightarrow \widetilde{u} \in \mathcal{G}^M_{aa}$ according to Propositions 5.2 and 5.3. □

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