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Time optimal controls for Hilfer fractional nonlocal evolution systems without compactness and Lipschitz condition

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Abstract. We explore the time optimal control problem for a Hilfer type fractional diffusion nonlocal control system. For this strategy, with the help of the established subordination principle and approximation theory, the abstract model adapted from the addressed diffusion system is tackled by the approximation solvability approach and the resolvent technique. We drop the compactness on semigroup and the Lipschitz restriction on the nonlinear term. We then employ a joint combination of the approach of formulating minimizing approximation sequences twice and the weak topology method to seek suitable trajectory-control dyads. Finally, the time optimal control problem for the diffusion system is solved by exploiting our mentioned abstract results.

1. Introduction

As is now well known, fractional theory has exhibited its potential superiority in characterizing many materials with long-memory properties and delineating massive systems [31]. As a consequence, a very large research efforts have been dedicated to this field and a series of effective findings have been received and reported (see [3]).

Fractional evolution equations are closely linked to semigroups or resolvents. The semigroup approach was firstly launched in [9] to address Caputo type fractional evolution systems. The solution operator (resolvent) technique introduced in [25] was presented to analyze a special Caputo type abstract model with the nonlinear term $J^{1-\beta}f(t, x(t))$. [16] developed the resolvent method to tackle Riemann-Liouville fractional evolution problems. The resolvent method was further developed in [21, 35] to deal with Hilfer type systems. For more recent inspiring results, one can refer to [4, 17, 19, 30].

Authors in [33] mentioned that the resolvent approach is convenient in dealing with fractional evolution models. Moreover, considering that the β-order γ-type resolvent introduced in [21] can cover the solution operator [25] and β-order resolvent [16] (see [35]), we need proceed with the investigation of the β-order γ-type resolvent. The theory of the subordination principle and approximation for resolvent is of vital importance to the control problem for evolution systems [34]. We thus look more closely at the theory.

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Nonlocal problems are important in representing large-scale models. Investigations on them have been extensively developed and many inspiring methods have been exhibited. The initial work for abstract models was displayed in [6] by Banach fixed point theorem. Fractional evolution nonlocal systems were tackled in [31] by the approach of Hausdorff measure of noncompactness. Under the compactness hypothesis of semigroups or resolvents, the approximation technique was exploited in [7, 15]. Recently, [5] and [28] developed the approximation solvability approach to analyze semilinear evolution systems, avoiding the compactness hypothesis on semigroups.

The time optimal controls for abstract systems have also received huge concern due to their potential application foreground, see [18, 22, 27, 35]. It is a pity that the existing literatures touched only a few aspects of them. They restricted their discussion to the models with compactness and Lipschitz hypotheses. New approaches need to be launched to the time optimal problems without these hypotheses.

Motivated by these aforementioned considerations, we need dedicate our effort to the study of the time optimal controls for the Hilfer type fractional model without compactness of semigroup and Lipschitz hypothesis on f. We enumerate the contributions of this work:

(*i*) We develop further the resolvent theory. We display the subordination principle and approximation theory of the resolvents.

(*ii*) With the aid of the established subordination principle and approximation theory, we develop the approximation solvability approach to address the Hilfer type models in the lack of compactness on semigroups and Lipschitz continuity of f.

(*iii*) By the weak topology theory, we design minimizing sequences twice to seek the most suitable trajectory-control dyad. We dispense with the compactness and Lipschitz restriction.

Let us now display the structure of the rest of this work. Section 2 includes our problem and some prerequisites required. We cope with the adapted abstract model in Section 3. Section 4 provides the approach to seek suitable trajectory-control dyads. We end the work with the addressed diffusion application.

2. Problem formulation and preliminaries

We are keen on the study of the following diffusion nonlocal control system with a Hilfer type fractional derivative operator $D^{\beta,\gamma}$:

$$\begin{cases} D^{\beta,\gamma}y(s,x) = \Delta y(s,x) + J^{\gamma(1-\beta)}(B(s)u(s,x) + f(s,x,y(s,x))), \text{ on } (0,b]\times\Omega, \\ y(s,x) = 0, \text{ on } (0,b)\times\partial\Omega, \\ \lim\limits_{s\to 0^+}\Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}y(s,x) = \sum\limits_{k=1}^{N} c_k s_k^{(1-\beta)(1-\gamma)} y(s_k,x), \ \ x\in\Omega. \end{cases} \tag{1}$$

Here $s_k \in (0,b]$, $c_k \in \mathbb{R}$, $k = 1,2,\cdots,N$, $\beta \in (0,1)$, $\gamma \in [0,1]$, $J^{\gamma(1-\beta)}$ is the fractional integral operator of $\gamma(1-\beta)$-order, $u \in L^2((0,b]\times\Omega)$, f is a function without Lipschitz restriction, B is a linear bounded mapping, Ω is a required bounded region in $\mathbb{R}^n$ ($n \geq 2$) with a Lipschitz boundary $\partial\Omega$.

A noteworthy fact is that this type of model (1) can serve as powerful tools for describing anamalous diffusion processes on fractals [8].

Designate

$$A = \Delta, \ D(A) = H_0^1(\Omega)\cap H^2(\Omega),$$

$$y(s)(x) = y(s,x), \ y(s_k)(x) = y(s_k,x), \ u(s)(x) = u(s,x),$$

$$f(s,y(s))(x) = f(s,x,y(s,x)), \ (\psi y)(x) = \sum_{k=1}^{N} c_k s_k^{(1-\beta)(1-\gamma)} y(s_k,x).$$

System (1) can be adapted to the abstract nonlocal control problem of the model

$$\begin{cases} D^{\beta,\gamma}y(s) = Ay(s) + J^{\gamma(1-\beta)}(B(s)u(s) + f(s,y(s))), \ s\in(0,b], \\ \lim\limits_{s\to 0^+}\Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}y(s) = \psi y. \end{cases} \tag{2}$$

In what follows, we will cope with model (2). For this strategy, let $\beta+\gamma(1-\beta)>\frac{1}{2}$. H and U are always separable Hilbert spaces. H_m is the m-dimensional subspace of H. The notation $\mathbb{P}_m : H \to H_m$ signifies the orthogonal projector from H to H_m. We designate a set $\mathscr{L}(H,U)$ as

$$\mathscr{L}(H,U)=\{f : H \to U |\ f \text{ is linear and bounded}\}.$$

To shorten symbol, we write $\mathscr{L}(H,H)$ as $\mathscr{L}(H)$. Let $\widetilde{y}(\cdot)=(\cdot)^{(1-\beta)(1-\gamma)}y(\cdot)$, $\widetilde{y}(0)=\lim_{\tau\to 0^+}\widetilde{y}(\tau)$, $J=[0,b]$ and $J'=(0,b]$. We can receive a Banach space $C_{\beta,\gamma}(J,H)$ designed by

$$C_{\beta,\gamma}(J,H)=\{y\in C(J',H)|\ \widetilde{y}\in C(J,H),\ 0<\beta<1,\ 0\leq\gamma\leq 1\}$$

with $\|y\|_{\beta,\gamma}=\sup_{\tau\in J}\|\widetilde{y}(\tau)\|$. For $R>0$, put

$$Q=\{y\in C_{\beta,\gamma}(J,H)|\ \|y\|_{\beta,\gamma}<R\} \text{ and } Q^{(m)}=Q\cap C_{\beta,\gamma}(J,H_m).$$

Additionally, we set $(g*h)(s)=\int_0^s g(s-\tau)h(\tau)\mathrm{d}\tau,\ s>0$.

Below, we begin by certain required notions.

Definition 2.1. *[24] Let $f\in L^1(J,H)$ and $\beta>0$. The β-order fractional integral operator J^β is designated as*

$$J^\beta f(s)=\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}f(\tau)\mathrm{d}\tau,\ s>0.$$

Definition 2.2. *[13] Let $0<\beta<1$ and $0\leq\gamma\leq 1$. The Hilfer type fractional derivative operator $D^{\beta,\gamma}$ is depicted as*

$$D^{\beta,\gamma}f(s)=J^{\gamma(1-\beta)}\frac{\mathrm{d}}{\mathrm{d}s}J^{(1-\beta)(1-\gamma)}f(s),\ s>0.$$

We then state the notion of β-order γ-type fractional resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$ and list some basic features of $\{R_{\beta,\gamma}(s)\}_{s>0}$.

Definition 2.3. *[21] By a β-order γ-type fractional resolvent, we mean a family $\{R_{\beta,\gamma}(s)\}_{s>0}\subseteq\mathscr{L}(H)$, which satisfies that for $x\in H$,*

(a) $R_{\beta,\gamma}(\cdot)x\in C(\mathbb{R}^+,H)$ and $\lim_{t\to 0^+}\Gamma(\beta+\gamma(1-\beta))t^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(t)x=x$;

(b) $R_{\beta,\gamma}(\tau)R_{\beta,\gamma}(s)=R_{\beta,\gamma}(s)R_{\beta,\gamma}(\tau)$, $s,\ \tau>0$;

(c) for $t,\ \tau>0$, $R_{\beta,\gamma}(\tau)J^\beta R_{\beta,\gamma}(t)-J^\beta R_{\beta,\gamma}(\tau)R_{\beta,\gamma}(t)=g_{\beta+\gamma(1-\beta)}(\tau)J^\beta R_{\beta,\gamma}(t)-g_{\beta+\gamma(1-\beta)}(t)J^\beta R_{\beta,\gamma}(\tau)$.

Here $g_{\beta+\gamma(1-\beta)}(\tau)=\frac{\tau^{\beta+\gamma(1-\beta)-1}}{\Gamma(\beta+\gamma(1-\beta))}$.

The generator of this resolvent is the operator A delineated by

$$Ax=\Gamma(2\beta+\gamma(1-\beta))\lim_{s\to 0^+}\frac{s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)x-\frac{x}{\Gamma(\beta+\gamma(1-\beta))}}{s^\beta},$$

$$D(A)=\left\{x\in H:\lim_{s\to 0^+}\frac{s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)x-\frac{x}{\Gamma(\beta+\gamma(1-\beta))}}{s^\beta}\ \text{ exists}\right\}.$$

Remark 2.4. *If $\gamma=0$, the resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$ is reduced to the β-order resolvent $\{T_\beta(s)\}_{s>0}$ introduced in [16]. In addition, due to (a) in Definition 2.3, we can easily receive that $M=\sup_{s\in J}\|s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)\|<\infty$.*

Lemma 2.5. *[21] For $\tau>0$, the resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$ possesses the following features:*

(a) $R_{\beta,\gamma}(\tau)D(A)\subseteq D(A)$ and $AR_{\beta,\gamma}(\tau)x=R_{\beta,\gamma}(\tau)Ax$ for $x\in D(A)$;

(b) $R_{\beta,\gamma}(\tau)x=g_{\beta+\gamma(1-\beta)}(\tau)x+J^\beta R_{\beta,\gamma}(\tau)Ax$ for $x\in D(A)$;

(c) $R_{\beta,\gamma}(\tau)x = g_{\beta+\gamma(1-\beta)}(\tau)x + AJ^{\beta}R_{\beta,\gamma}(\tau)x$ *for* $x \in V$;

(d) $\overline{D(A)} = V$.

We, in addition, propose the notion of $A \in C_{s_0}^{\beta,\gamma}(\overline{M},\omega)$ and display the equivalent statement adapted from [21].

Definition 2.6. *[35] Let* $\{R_{\beta,\gamma}(s)\}_{s>0}$ *be a resolvent. For sufficiently small* $s_0 > 0$, $\{R_{\beta,\gamma}(s)\}_{s>0}$ *is* $(\overline{M},\omega)$ *type for* $s \geq s_0$ *if there are two constants* $\omega > 0$ *and* $\overline{M} > 0$ *to guarantee that*

$$\|R_{\beta,\gamma}(s)\| \leq \overline{M}e^{\omega s},\ s \geq s_0. \tag{3}$$

For our comfort, we design the symbol $A \in C_{s_0}^{\beta,\gamma}(\overline{M},\omega)$ *to indicate that* $\{R_{\beta,\gamma}(s)\}_{s>0}$ *is a resolvent satisfying (3).*

Lemma 2.7. *[21, 35] If* $(\omega^{\beta},\infty) \subseteq \rho(A)$, $\omega > 0$, *and there exists a strongly continuous family* $\{R_{\beta,\gamma}(s)\}_{s>0} \subseteq \mathscr{L}(H)$, *which satisfies that for any* $x \in H$,

(a) $\lim_{s\to 0^+} \Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)x = x$;

(b) *for* $s \geq s_0$, $\|R_{\beta,\gamma}(s)\| \leq \overline{M}e^{\omega s}$;

(c) $R_{\beta,\gamma}(s)R_{\beta,\gamma}(\tau) = R_{\beta,\gamma}(\tau)R_{\beta,\gamma}(s)$ *for* $s,\tau > 0$;

(d) $R(\lambda^{\beta},A)x = (\lambda^{\beta}I - A)^{-1}x = \lambda^{\gamma(1-\beta)}\int_0^{\infty} e^{-\lambda s}R_{\beta,\gamma}(s)x\mathrm{d}s,\ \lambda > \omega$,

then $A \in C_{s_0}^{\beta,\gamma}(\overline{M},\omega)$.

Subsequently, we remind the features of the one-sided stable probability density $\varpi_{\beta}(s)$ [36] and the well-known Wright type function $\Psi_{\beta}(s)$ [20]:

$$\varpi_{\beta}(s) = \frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k s^{-(k+1)\beta-1}\frac{\Gamma((k+1)\beta+1)}{(k+1)!}\sin\left((k+1)\pi\beta\right),\ s \in \mathbb{R}^+,$$

$$\Psi_{\beta}(s) = \sum_{m=0}^{\infty}\frac{(-\tau)^m}{m!\Gamma(-\beta m+1-\beta)} = \frac{1}{\beta}s^{-1-\frac{1}{\beta}}\varpi_{\beta}(s^{-\frac{1}{\beta}}).$$

Lemma 2.8. *[3, 20, 36]* $\varpi_{\beta}(s)$ *and* $\Psi_{\beta}(s)$ *possess the following properties:*

(a) $\int_0^{\infty} e^{-\lambda s}\varpi_{\beta}(s)\mathrm{d}s = e^{-\lambda^{\beta}}$;

(b) $\Psi_{\beta}(s) \geq 0$ *for* $s > 0$;

(c) $\int_0^{\infty} s^k\Psi_{\beta}(s)\mathrm{d}s = \frac{\Gamma(1+k)}{\Gamma(1+\beta k)},\ k \in [0,1]$.

Next, with the aid of Lemma 2.7, we analyze and specify the subordination principle of the resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$.

Lemma 2.9. *If* $\{T(s)\}_{s\geq 0}$ *is an equicontinuous semigroup generated by* A *with* $\|T(s)\| \leq M$, $M > 0$, *then*

(a) $\{R_{\beta,\gamma}(s)\}_{s>0}$ *is a* β*-order* γ*-type resolvent, where*

$$R_{\beta,\gamma}(s) = J^{\gamma(1-\beta)}\left(s^{\beta-1}\int_0^{\infty}\beta\tau\Psi_{\beta}(\tau)T(s^{\beta}\tau)\mathrm{d}\tau\right).$$

(b) $\{s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)\}_{s>0}$ *is equicontinuous.*

Proof. (*a*) Firstly, for $s, \tau > 0$, because of the commutative property of $\{T(s)\}_{s\geq 0}$, we receive

$$R_{\beta,\gamma}(s)R_{\beta,\gamma}(\tau) = R_{\beta,\gamma}(\tau)R_{\beta,\gamma}(s).$$

We then assert that for $s \geq s_0$, $\{R_{\beta,\gamma}(s)\}_{s>0}$ is $(\overline{M}, \omega)$ type. On account of $\|T(\tau)\| \leq M$ and Lemma 2.8, we get

$$\begin{aligned}
\|R_{\beta,\gamma}(s)\| &\leq \left\| J^{\gamma(1-\beta)}\left(s^{\beta-1}\int_0^\infty \beta\tau\Psi_\beta(\tau)T(s^\beta\tau)\mathrm{d}\tau\right)\right\| \\
&\leq \left\| \frac{1}{\Gamma(\gamma(1-\beta))}\int_0^s (s-\tau)^{\gamma(1-\beta)-1}\left(\tau^{\beta-1}\int_0^\infty \beta\eta\Psi_\beta(\eta)T(\tau^\beta\eta)\mathrm{d}\eta\right)\mathrm{d}\tau\right\| \\
&\leq \frac{M}{\Gamma(\beta)\Gamma(\gamma(1-\beta))}\int_0^s (s-\tau)^{\gamma(1-\beta)-1}\tau^{\beta-1}\mathrm{d}\tau \\
&= \frac{Ms^{\beta+\gamma(1-\beta)-1}}{\Gamma(\beta+\gamma(1-\beta))} \leq \frac{Ms_0^{\beta+\gamma(1-\beta)-1}}{\Gamma(\beta+\gamma(1-\beta))}.
\end{aligned}$$

One thus can select $\overline{M} \geq 0$ and $\omega \geq 0$ to guarantee that

$$\|R_{\beta,\gamma}(s)\| \leq \overline{M}e^{\omega s},\ s \geq s_0.$$

Next, we examine that for $x \in H$,

$$(\lambda^\beta I - A)^{-1}x = \lambda^{\gamma(1-\beta)}\int_0^\infty e^{-\lambda s}R_{\beta,\gamma}(s)x\mathrm{d}s.$$

For $x \in H$, from Lemmas 2.7 and 2.8, it follows that

$$\begin{aligned}
&\lambda^{\gamma(1-\beta)}\int_0^\infty e^{-\lambda s}R_{\beta,\gamma}(s)x\mathrm{d}s \\
&= \int_0^\infty e^{-\lambda s}\left(s^{\beta-1}\int_0^\infty \tau^{-\frac{1}{\beta}}\varpi_\beta(\tau^{-\frac{1}{\beta}})T(s^\beta\tau)x\mathrm{d}\tau\right)\mathrm{d}s \\
&= \beta\int_0^\infty e^{-\lambda s}\left(\int_0^\infty \varpi_\beta(\theta)\frac{s^{\beta-1}}{\theta^\beta}T\left(\left(\frac{s}{\theta}\right)^\beta\right)x\mathrm{d}\theta\right)\mathrm{d}s \\
&= \beta\int_0^\infty\left(\int_0^\infty e^{-\lambda s}\varpi_\beta(\theta)\frac{s^{\beta-1}}{\theta^\beta}T\left(\left(\frac{s}{\theta}\right)^\beta\right)x\mathrm{d}s\right)\mathrm{d}\theta \\
&= \beta\int_0^\infty\left(\int_0^\infty e^{-\lambda t\theta}\varpi_\beta(\theta)t^{\beta-1}T\left(t^\beta\right)x\mathrm{d}t\right)\mathrm{d}\theta \\
&= \beta\int_0^\infty\left(\int_0^\infty e^{-\lambda t\theta}\varpi_\beta(\theta)\mathrm{d}\theta\right)t^{\beta-1}T\left(t^\beta\right)x\mathrm{d}t \\
&= \beta\int_0^\infty e^{-(\lambda t)^\beta}t^{\beta-1}T\left(t^\beta\right)x\mathrm{d}t \\
&= \int_0^\infty e^{-\lambda^\beta u}T(u)x\mathrm{d}u \\
&= (\lambda^\beta I - A)^{-1}x.
\end{aligned}$$

Furthermore, the argument used in Lemma 3.3 in [30] can give the continuity of $\{R_{\beta,\gamma}(s)x\}_{s>0}$.

In addition, thanks to the dominated convergence theorem, we arrive at

$$\Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}J^{\gamma(1-\beta)}\left(s^{\beta-1}\int_0^\infty \beta\tau\Psi_\beta(\tau)\left(T(s^\beta\tau)x - x\right)\mathrm{d}\tau\right)$$

$$
\begin{aligned}
&= \frac{\Gamma(\beta+\gamma(1-\beta))}{\Gamma(\gamma(1-\beta))}\int_0^1 (1-t)^{\gamma(1-\beta)-1}t^{\beta-1}\int_0^\infty \beta\tau\Psi_\beta(\tau)\left(T((st)^\beta\tau)x-x\right)\mathrm{d}\tau\mathrm{d}t \\
&\to 0,\ s\to 0,
\end{aligned}
$$

which leads to

$$\lim_{s\to 0^+}\Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)x = x.$$

Consequently, we receive a resolvent $\{R_{\beta,\gamma}(s)\}_{s>0}$.

(b) The equicontinuity can be drawn by the similar arguments in Theorem 3.1 in [29]. □

Remark 2.10. *If $\gamma = 0$, we receive the subordination principle [32] of the β-order resolvent $\{T_\beta(t)\}_{t>0}$ introduced in [16].*

We establish the Trotter-Kato type approximation theory of the resolvent $\{R_{\beta,\gamma}(t)\}_{t>0}$.

Lemma 2.11. *Assume that A_n and A respectively generate resolvents $\{R^n_{\beta,\gamma}(t)\}_{t>0}$ and $\{R_{\beta,\gamma}(t)\}_{t>0}$ on H. If $A_n \in C^{\beta,\gamma}_{s_0}(\overline{M},\omega)$ and $A \in C^{\beta,\gamma}_{s_0}(\overline{M},\omega)$, then the following declarations are equivalent:*

(a) $t^{(1-\beta)(1-\gamma)}R^n_{\beta,\gamma}(t)x \to t^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(t)x$ *for all $x\in H$.*

(b) $R(\lambda, A_n)x \to R(\lambda, A)x$ *for all $x \in H$ and $\lambda > \omega^\beta$.*

Proof. (a) ⇒ (b) For $x \in H$, $\lambda > \omega$, Lemma 2.7 implies that

$$\|R(\lambda^\beta, A_n)x - R(\lambda^\beta, A)x\| = \lambda^{\gamma(1-\beta)}\int_0^\infty e^{-\lambda s}s^{(1-\beta)(\gamma-1)}\|s^{(1-\beta)(1-\gamma)}R^n_{\beta,\gamma}(s)x - s^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(s)x\|\mathrm{d}s.$$

We hence conclude from the dominated convergence theorem that (b) holds.

(b) ⇒ (a) **Step 1**. For $x \in H$, we examine that $\int_0^t R^n_{\beta,\gamma}(s)x\mathrm{d}s \to \int_0^t R_{\beta,\gamma}(s)x\mathrm{d}s, n\to\infty$. For brief, put $f_n(t) = \int_0^t R^n_{\beta,\gamma}(s)x\mathrm{d}s$ and $f(t) = \lim_{n\to\infty}\int_0^t R^n_{\beta,\gamma}(s)x\mathrm{d}s$. When $t \le s_0$, Remark 2.4 yields $\|f_n(t)\| \le \frac{M\|x\|t^{\gamma(1-\beta)+\beta}}{\gamma(1-\beta)+\beta}$. When $t > s_0$, Definition 2.6 and Remark 2.4 force that

$$
\begin{aligned}
\|f_n(t)\| &\le \int_0^{s_0}\|R^n_{\beta,\gamma}(s)x\|\mathrm{d}s + \int_{s_0}^t \|R^n_{\beta,\gamma}(s)x\|\mathrm{d}s \\
&\le \frac{M\|x\|s_0^{\gamma(1-\beta)+\beta}}{\gamma(1-\beta)+\beta} + \frac{\overline{M}\|x\|e^{\omega t}}{\omega}.
\end{aligned}
$$

For any $t \ge 0$, we thereby can pick $M_1 > 0$ to ensure that

$$\left\|\int_0^t f_n(s)\mathrm{d}s\right\| \le M_1 e^{\omega t}.$$

Let $0 \le t < t+h$. If $0 \le t < t+h \le s_0$, we have

$$\|f_n(t+h) - f_n(t)\| \le \int_t^{t+h}\|R^n_{\beta,\gamma}(s)x\|\mathrm{d}s \le \frac{M\|x\|h^{\gamma(1-\beta)+\beta}}{\gamma(1-\beta)+\beta}.$$

For $t+h > t \ge s_0$, on account of Definition 2.6, we receive

$$\|f_n(t+h) - f_n(t)\| \le \int_t^{t+h}\|R^n_{\beta,\gamma}(s)x\|\mathrm{d}s \le \overline{M}\|x\|e^{\omega(t+h)}h.$$

Hence, $\{f_n\}$ is equicontinuous.

Hence, based upon (b), it may be easily concluded that

$$\int_0^\infty e^{-\lambda t} f_n(t)\mathrm{d}t \to \int_0^\infty e^{-\lambda t} f(t)\mathrm{d}t.$$

Thanks to Theorem 1.7.5 in [2], we thus assert that

$$\int_0^t R^n_{\beta,\gamma}(s)x\mathrm{d}s \to \int_0^t R_{\beta,\gamma}(s)x\mathrm{d}s,\ n\to\infty.$$

Step 2. We investigate that $(g_{1-\beta} * R^n_{\beta,\gamma})(t) \to (g_{1-\beta} * R_{\beta,\gamma})(t)$ strongly. For $x\in D(A)$, set $x_n = R(\lambda, A_n)(\lambda - A)x,\ \lambda > \omega^\beta$. In view of (b), one can deduce that $x_n \to x$ and $A_n x_n \to Ax$. Due to Lemma 2.5, we arrive at

$$(g_{1-\beta} * R^n_{\beta,\gamma})(t)x_n = g_{\gamma(1-\beta)+1}x_n + \int_0^t R^n_{\beta,\gamma}(s)A_n x_n\mathrm{d}s. \tag{4}$$

Due to step 1, we receive

$$\int_0^t R^n_{\beta,\gamma}(s)Ax\mathrm{d}s \to \int_0^t R_{\beta,\gamma}(s)Ax\mathrm{d}s.$$

In addition, it is a simple matter to deduce from Definition 2.6 and Remark 2.4 that

$$\left\|\int_0^t R^n_{\beta,\gamma}(s)A_n x_n\mathrm{d}s - \int_0^t R^n_{\beta,\gamma}(s)Ax\mathrm{d}s\right\| \to 0.$$

Hence, (4) gives

$$(g_{1-\beta} * R^n_{\beta,\gamma})(t)x_n \to (g_{1-\beta} * R_{\beta,\gamma})(t)x.$$

Therefore, $\overline{D(A)} = H$ can force that for $x\in H$,

$$\|(g_{1-\beta} * R^n_{\beta,\gamma}(t))x - (g_{1-\beta} * R_{\beta,\gamma}(t))x\| \to 0.$$

Step 3. For all $\beta\in(0,1)$, we check $(g_\beta * R^n_{\beta,\gamma})(t) \to (g_\beta * R_{\beta,\gamma})(t)$. If $\beta > \frac{1}{2}$, then $\beta > 1-\beta$. As such $(g_\beta * R^n_{\beta,\gamma})(t)x \to (g_\beta * R_{\beta,\gamma})(t)x$ for all $x\in H$. If $\beta > \frac{1}{3}$, then $2\beta > 1-\beta$. Based upon Lemma 2.5, we have

$$(g_\beta * R^n_{\beta,\gamma})(t)x_n = g_{2\beta+\gamma(1-\beta)}(t)x_n + (g_{2\beta} * R^n_{\beta,\gamma})(t)A_n x_n.$$

Moreover, a trivial verification gives

$$\|(g_{2\beta} * R^n_{\beta,\gamma})(t)A_n x_n - (g_{2\beta} * R^n_{\beta,\gamma})(t)Ax\| \to 0.$$

On the other hand, $2\beta > 1-\beta$ and step 2 force that

$$(g_{2\beta} * R^n_{\beta,\gamma})(t)Ax \to (g_{2\beta} * R_{\beta,\gamma})(t)Ax.$$

we thereby get

$$(g_\beta * R^n_{\beta,\gamma})(t)x_n \to (g_\beta * R_{\beta,\gamma})(t)x.$$

Hence, we can assert that $(g_\beta * R^n_{\beta,\gamma})(t)x \to (g_\beta * R_{\beta,\gamma})(t)x$ for all $x\in H$. Therefore, proceeding this fashion, we can receive that for all $\beta\in(0,1)$ and $t>0$,

$$(g_\beta * R^n_{\beta,\gamma})(t) \to (g_\beta * R_{\beta,\gamma})(t),\ \text{strongly}.$$

Step 4. We finally determine that (a) holds. According to $A_n x_n \to Ax$, an easy computation yields

$$\|(g_\beta * R^n_{\beta,\gamma})(t)A_n x_n - (g_\beta * R^n_{\beta,\gamma})(t)Ax\| \to 0.$$

By step 3, we see that

$$(g_\beta * R^n_{\beta,\gamma})(t)A_n x_n \to (g_\beta * R_{\beta,\gamma})(t)Ax.$$

Thereby, Lemma 2.5 and Remark 2.4 force that for every $x \in D(A)$ and $t > 0$,

$$R^n_{\beta,\gamma}(t)x \to R_{\beta,\gamma}(t)x.$$

Due to $\overline{D(A)} = H$, we thus achieve (a). □

Lemma 2.12. *Let D be a core of A and operators A_n and A respectively generate resolvents $\{R^n_{\beta,\gamma}(t)\}_{t>0}$ and $\{R_{\beta,\gamma}(t)\}_{t>0}$. If $A_n \in C^{\beta,\gamma}_{s_0}(\overline{M}, \omega)$, $A \in C^{\beta,\gamma}_{s_0}(\overline{M}, \omega)$ and $A_n x \to Ax$ as $n \to \infty$ for all $x \in D \subseteq D(A_n)$, then $t^{(1-\beta)(1-\gamma)}R^n_{\beta,\gamma}(t)x \to t^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(t)x$, for all $x \in H$, uniformly for $t \in [a, b] \subseteq [0, \infty)$.*

Proof. Combining the Trotter-Kato approximation theorem in [10] with $A_n x \to Ax$, we can arrive at $R(\lambda, A_n)x \to R(\lambda, A)x$ for all $x \in H$ and $\lambda > \omega^\beta$. Thus, by Lemma 2.11, we achieve the desired convergence. □

In the end, some crucial theorems are presented to prepare for the study of system (2).

By [28], the following weak compactness theorem in $C_{\beta,\gamma}(J, H)$ can be easily received:

Theorem 2.13. *Let $\widetilde{y}(\cdot) = (\cdot)^{(1-\beta)(1-\gamma)}y(\cdot)$ and B be a bounded subset of $C_{\beta,\gamma}(J, H)$. If, in addition, $\widetilde{B} = \{\widetilde{y} \in C(J, H) : y \in B\}$ is equicontinuous, then B is relatively weak sequentially compact.*

Theorem 2.14. *[1] Let E be a closed convex subset of H and $T : [0, 1] \times E \to H$ a compact map and $T(0, E) \subset \mathring{E}$. If, additionally, T admits a closed graph and for all $\lambda \in [0, 1)$, $T(\lambda, \cdot)$ is fixed point free on ∂E, then there admits $y \in E$ to guarantee that $T(1, y) = y$.*

3. Existence results

We here focus on exploring the nonlocal control system (2). By resorting to the resolvent approach and the approximation solvability technique, we can dispense with the Lipschitz restriction on f and the compactness of semigroup. To achieve our strategy now, we list the ensuing required conditions:

(HA) $\{T(t)\}_{t\geq 0}$ is an equicontinuous semigroup and $\|T(t)\| \leq M$.

(HB) $B \in L^\infty(J, \mathscr{L}(U, H))$.

(Hf) $f : J \times H \to H$ satisfies

(i) for every $z \in H$, $f(\cdot, z) : J \to H$ is measurable.

(ii) for a.e. $t \in J$, $f(t, \cdot) : H \to H$ is weak-to-weak continuous.

(iii) for a.e. $t \in J$ and every $y \in H$, $\|f(t, y)\| \leq \nu(t) + \rho t^{(1-\beta)(1-\gamma)}\|y\|$ with $\nu \in L^2(J, \mathbb{R}^+)$ and $\rho > 0$.

$(H\psi)$ $\psi \in \mathscr{L}(C_{\beta,\gamma}(J, H), H)$ and $\|\psi y\| \leq c\|y\|_{\beta,\gamma} + d$ for every $y \in C_{\beta,\gamma}(J, H)$ with $c > 0$ and $d > 0$.

Moreover, we frame the required admissible set U_{ad}. It is a convex closed bounded subset of $L^2(J, U)$.

Below, our analysis related to mild solutions begins by summarizing some important materials, the definition of mild solutions of (2) and some crucial lemmas.

Remark 3.1. *We make some notes here:*

(a) Due to Lemma 2.9, A can generate a β-order γ-type resolvent $\{R_{\beta,\gamma}(t)\}_{t>0}$ satisfying that $\{t^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(t)\}_{t>0}$ is equicontinuous. Moreover, Remark 2.4 forces that $\|t^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(t)\| \leq M$.

(b) By means of Lemma 2.9, $A^{(n)}$, the Yosida approximation of A, can also generate a β-order γ-type resolvent $\{R^n_{\beta,\gamma}(t)\}_{t>0}$ with $\|t^{(1-\beta)(1-\gamma)}R^n_{\beta,\gamma}(t)\| \leq M$.

(c) Put $A^{(n)}_m = \mathbb{P}_m A^{(n)} : H_m \to H_m$. We can suppose from the boundedness of $A^{(n)}_m$ and Lemma 2.9 that $A^{(n)}_m$ can also generate a β-order γ-type resolvent $\{R^{mn}_{\beta,\gamma}(s)\}_{s>0}$ on H_m with $\|s^{(1-\beta)(1-\gamma)}R^{mn}_{\beta,\gamma}(s)\| \leq M$.

(d) In virtue of the linearity boundedness of ψ, ψ is weakly continuous.

(e) $\mathbb{P}_m$ is weak-to-strong continuous.

Definition 3.2. *[21] By a mild solution to model (2), we understand the function $y \in C_{\beta,\gamma}(J, H)$ which satisfies*

$$y(s) = R_{\beta,\gamma}(s)\psi y + \int_0^s R_{\beta,\gamma}(s-\tau)(B(\tau)u(\tau) + f(\tau, y(\tau)))\mathrm{d}\tau,\ s \in J'.$$

Lemma 3.3. *Let $\{R^n_{\beta,\gamma}(s)\}_{s>0}$ and $\{R^{mn}_{\beta,\gamma}(s)\}_{s>0}$ be β-order γ-type resolvents generated by $A^{(n)}$ and $\mathbb{P}_m A^{(n)}$, respectively. Then for any $x \in H$ and $s \in J$,*

$$\|s^{(1-\beta)(1-\gamma)} R^{mn}_{\beta,\gamma}(s)\mathbb{P}_m x - s^{(1-\beta)(1-\gamma)} R^n_{\beta,\gamma}(s)x\| \to 0,\ m \to \infty.$$

Proof. For $x \in H$ with $\|x\| = 1$, Lemma 2.5 forces that

$$R^n_{\beta,\gamma}(t)x = g_{\beta+\gamma(1-\beta)}(t)x + A^{(n)}(g_\beta * R^n_{\beta,\tau})(t)x. \tag{5}$$

and

$$R^{mn}_{\beta,\gamma}(t)\mathbb{P}_m x = g_{\beta+\gamma(1-\beta)}(t)\mathbb{P}_m x + \mathbb{P}_m A^{(n)}(g_\beta * R^{mn}_{\beta,\gamma})(t)\mathbb{P}_m x. \tag{6}$$

For our comfort, delineate $u(t) = R^n_{\beta,\gamma}(t)x$, $u_m(t) = R^{mn}_{\beta,\gamma}(t)\mathbb{P}_m x$ and $z_m(t) = u(t) - u_m(t)$. Then combining (5) with (6), we arrive at

$$z_m(t) = g_{\beta+\gamma(1-\beta)}(t)(I - \mathbb{P}_m)x + A^{(n)}(g_\beta * z_m)(t) + (I - \mathbb{P}_m)A^{(n)}(g_\beta * u_m)(t). \tag{7}$$

For the resolvent $\{R^n_{\beta,\gamma}(t)\}_{t>0}$ and the β-order resolvent $\{T^n_\beta(t)\}_{t>0}$ (when $\gamma = 0$, see Remark 2.10) generated by A^n, based on Lemmas 2.7 and 2.9 and Remark 2.10, we arrive at

$$\int_0^\infty e^{-\lambda s}(g_\beta * R^n_{\beta,\gamma})(s)x\mathrm{d}s = \int_0^\infty e^{-\lambda s}(g_{\beta+\gamma(1-\beta)} * T^n_\beta)(s)x\mathrm{d}s. \tag{8}$$

We thus receive from (5), (7), (8) and Lemma 2.5 that

$$\begin{aligned} & g_{\beta+\gamma(1-\beta)} * z_m \\ = \ & (R^n_{\beta,\gamma} - A^{(n)} g_\beta * R^n_{\beta,\gamma}) * z_m \\ = \ & R^n_{\beta,\gamma} * (z_m - A^{(n)} g_\beta * z_m) \\ = \ & R^n_{\beta,\gamma} * g_{\beta+\gamma(1-\beta)}(I - \mathbb{P}_m)x + g_{\beta+\gamma(1-\beta)} * T^n_\beta * (I - \mathbb{P}_m)A^{(n)}u_m \\ = \ & g_{\beta+\gamma(1-\beta)} * \left(R^n_{\beta,\gamma}(I - \mathbb{P}_m)x + T^n_\beta * (I - \mathbb{P}_m)A^{(n)}u_m\right), \end{aligned}$$

which means that

$$z_m(t) = R^n_{\beta,\gamma}(t)(I - \mathbb{P}_m)x + \int_0^t T^n_\beta(t-s)(I - \mathbb{P}_m)A^{(n)}u_m(s)\mathrm{d}s.$$

We thus get

$$\|t^{(1-\beta)(1-\gamma)} z_m(t)\| \le \|I - \mathbb{P}_m\| \left(M + M^2 b^\beta \frac{\Gamma(\beta)\Gamma(\beta+\gamma(1-\beta))}{\Gamma(2\beta+\gamma(1-\beta))}\|A^{(n)}\|\right).$$

Thereby, we achieve the desired convergence result. □

Lemma 3.4. *Let $h \in L^2(J, H)$. The map $\Lambda : L^2(J, H) \to C_{\beta,\gamma}(J, H)$ described by $(\Lambda h)(\cdot) = (\cdot)^{(1-\beta)(1-\gamma)}(R_{\beta,\gamma} * h)(\cdot)$ is equicontinuous.*

Proof. For our confort, let $\widetilde{R_{\beta,\gamma}}(s) = s^{(1-\beta)(1-\gamma)} R_{\beta,\gamma}(s)$ and $\|z_n\|_{L^2} \le 1$. When $t_1, t_2 \in J$ with $0 < t_1 < t_2$, for $\eta \in (0, t_2)$, we have

$$\|(\Lambda z_n)(t_1) - (\Lambda z_n)(t_2)\|$$

$$
\begin{aligned}
&\leq \left(t_2^{(1-\beta)(1-\gamma)} - t_1^{(1-\beta)(1-\gamma)}\right)\left\|\int_0^{t_2} R_{\beta,\gamma}(t_2-\tau)z_n(\tau)\mathrm{d}\tau\right\| \\
&\quad + b^{(1-\beta)(1-\gamma)}\left\|\int_0^{t_2} R_{\beta,\gamma}(t_2-\tau)z_n(\tau)\mathrm{d}\tau - \int_0^{t_1} R_{\beta,\gamma}(t_1-\tau)z_n(\tau)\mathrm{d}\tau\right\| \\
&\leq \left(t_2^{(1-\beta)(1-\gamma)} - t_1^{(1-\beta)(1-\gamma)}\right)M\sqrt{\frac{b^{2(\beta+\gamma(1-\beta))-1}}{2(\beta+\gamma(1-\beta))-1}} \\
&\quad + b^{(1-\beta)(1-\gamma)}\left\|\int_0^{t_1-\eta}\left(\widetilde{R_{\beta,\gamma}}(t_2-\tau) - \widetilde{R_{\beta,\gamma}}(t_1-\tau)\right)(t_2-\tau)^{(\beta-1)(1-\gamma)}z_n(\tau)\mathrm{d}\tau\right\| \\
&\quad + b^{(1-\beta)(1-\gamma)}\left\|\int_{t_1-\eta}^{t_1}\left(\widetilde{R_{\beta,\gamma}}(t_2-\tau) - \widetilde{R_{\beta,\gamma}}(t_1-\tau)\right)(t_2-\tau)^{(\beta-1)(1-\gamma)}z_n(\tau)\mathrm{d}\tau\right\| \\
&\quad + b^{(1-\beta)(1-\gamma)}\left\|\int_0^{t_1}\widetilde{R_{\beta,\gamma}}(t_1-\tau)\left((t_2-\tau)^{(\beta-1)(1-\gamma)} - (t_1-\tau)^{(\beta-1)(1-\gamma)}\right)z_n(\tau)\mathrm{d}\tau\right\| \\
&\quad + b^{(1-\beta)(1-\gamma)}\left\|\int_{t_1}^{t_2} R_{\beta,\gamma}(t_2-\tau)z_n(\tau)\mathrm{d}\tau\right\| \\
&\leq \left(t_2^{(1-\beta)(1-\gamma)} - t_1^{(1-\beta)(1-\gamma)}\right)M\sqrt{\frac{b^{2(\beta+\gamma(1-\beta))-1}}{2(\beta+\gamma(1-\beta))-1}} \\
&\quad + \sup_{\tau\in[0,t_1-\eta]}\left\|\widetilde{R_{\beta,\gamma}}(t_2-\tau) - \widetilde{R_{\beta,\gamma}}(t_1-\tau)\right\|\sqrt{\frac{b}{2(\beta+\gamma(1-\beta))-1}} \\
&\quad + 2Mb^{(1-\beta)(1-\gamma)}\left(\int_{t_1-\eta}^{t_1}(t_2-\tau)^{2(\beta-1)(1-\gamma)}\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\quad + Mb^{(1-\beta)(1-\gamma)}\left(\int_0^{t_1}\left((t_2-\tau)^{(\beta-1)(1-\gamma)} - (t_1-\tau)^{(\beta-1)(1-\gamma)}\right)^2\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\quad + Mb^{(1-\beta)(1-\gamma)}\frac{(t_2-t_1)^{\beta+\gamma(1-\beta)-\frac{1}{2}}}{\sqrt{2(\beta+\gamma(1-\beta))-1}}.
\end{aligned}
$$

Thereby, from the absolute continuity of integration of $(t_2-\cdot)^{-2(1-\beta)(1-\gamma)}$, we receive

$$\lim_{t_2\to t_1}\|(\Lambda z_n)(t_1) - (\Lambda z_n)(t_2)\| = 0.$$

If $t_1 = 0$, we can immediately receive the above result. Hence, we achieve the equicontinuity. □

Theorem 3.5. *Let* (HA), (Hf), (HB) *and* $(H\psi)$ *be fulfilled. If, in addition,*

$$Mc + \frac{M\rho b}{\beta+\gamma(1-\beta)} < 1, \tag{9}$$

then for fixed $u \in U_{ad}$*, model (2) possesses mild solutions.*

Proof. **Step 1.** Let $R > 0$, $\lambda \in [0,1]$ and $q \in Q^{(m)}$, $m \in \mathbb{N}$. We delineate the ensuing linearized approximation auxiliary system which is indexed by q:

$$
\begin{cases}
D^{\beta,\gamma}y(s) = A_m^{(n)}y(s) + \lambda J^{\gamma(1-\beta)}\mathbb{P}_m(B(s)u(s) + f(s,q(s))),\ s\in J', \\
\lim\limits_{s\to 0^+}\Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}y(s) = \lambda\mathbb{P}_m\psi q.
\end{cases} \tag{10}
$$

Then a mild solution (see [21]) $y_m^n \in C_{\beta,\gamma}(J, H_m)$ is received and described by

$$y_m^n(t) = \lambda R_{\beta,\gamma}^{mn}(t)\mathbb{P}_m\psi q + \lambda \int_0^t R_{\beta,\gamma}^{mn}(t-\tau)\mathbb{P}_m(B(\tau)u(\tau) + f(\tau, q(\tau)))d\tau.$$

Moreover, we have

$$\|t^{(1-\beta)(1-\gamma)}y_m^n(t)\| \leq M\Big(cR + d + \frac{\rho Rb}{\beta+\gamma(1-\beta)}\Big) + M\sqrt{\frac{b}{2(\beta+\gamma(1-\beta))-1}}(\|\nu\|_{L^2} + \|Bu\|_{L^2}). \tag{11}$$

We thus can recognize the map $\Sigma : Q^{(m)} \times [0,1] \to C_{\beta,\gamma}(J, H_m)$ by

$$\Sigma(q,\lambda)(t) = \lambda R_{\beta,\gamma}^{mn}(t)\mathbb{P}_m\psi q + \lambda \int_0^t R_{\beta,\gamma}^{mn}(t-\tau)\mathbb{P}_m(B(\tau)u(\tau) + f(\tau, q(\tau)))d\tau.$$

Obviously, $\widetilde{\Sigma}(q,0) = 0 \in \mathring{Q}^{(m)}$. Below, we affirm that Σ fulfills the other conditions of Theorem 2.14.

We begin by confirming that Σ admits a closed graph. Let $\{q_k\} \subseteq Q^{(m)}$ with $q_k \to q_0$ and $\lambda_k \subseteq [0,1]$ with $\lambda_k \to \lambda_0$. We receive

$$\begin{aligned} & \Big\|t^{(1-\beta)(1-\gamma)}\Sigma(q_k,\lambda_k)(t) - t^{(1-\beta)(1-\gamma)}\Sigma(q_0,\lambda_0)(t)\Big\| \\ \leq\ & |\lambda_k - \lambda_0|\|t^{(1-\beta)(1-\gamma)}\Sigma(q_0,1)(t)\| + M\|\mathbb{P}_m\psi q_k - \mathbb{P}_m\psi q_0\| \\ & + b^{(1-\beta)(1-\gamma)}M\int_0^t (t-\tau)^{(\beta-1)(1-\gamma)}\|\mathbb{P}_m(f(\tau,q_k(\tau)) - f(\tau,q_0(\tau)))\|d\tau. \end{aligned}$$

Thus, combining with (Hf), $(H\psi)$, (11), Remark 3.1 and the dominated convergence theorem, we can affirm that Σ admits a closed graph.

We then explore the compactness of Σ. Set

$$\widetilde{\Sigma}(Q^{(m)} \times [0,1]) = \bigcup_{\lambda\in[0,1], q\in Q^{(m)}} \lambda(\cdot)^{(1-\beta)(1-\gamma)}\left\{R_{\beta,\gamma}^{mn}(\cdot)\mathbb{P}_m\psi q + \int_0^{\cdot} R_{\beta,\gamma}^{mn}(\cdot-\tau)\mathbb{P}_m(B(\tau)u(\tau)+f(\tau,q(\tau)))d\tau\right\}.$$

Since (11), Remark 3.1 and Lemma 3.4 can easily force the boundedness and equicontinuity of $\widetilde{\Sigma}(Q^{(m)}\times[0,1])$ on H_m, we can receive the compactness of Σ.

Subsequently, we check that $\Sigma(\cdot,\lambda)$ is fixed point free on $\partial Q^{(m)}$. Let $q = \Sigma(q,\lambda)$ and $\lambda \in (0,1)$. We arrive at

$$\|t^{(1-\beta)(1-\gamma)}q(t)\| \leq M\Big(c\|q\|_{\beta,\gamma} + d + \frac{\rho b\|q\|_{\beta,\gamma}}{\beta+\gamma(1-\beta)}\Big) + M\sqrt{\frac{b}{2(\beta+\gamma(1-\beta))-1}}(\|\nu\|_{L^2} + \|Bu\|_{L^2}).$$

Based on (9), we can pick R to guarantee that

$$R > \frac{Md + M\sqrt{\frac{b}{2(\beta+\gamma(1-\beta))-1}}\Big(\|\nu\|_{L^2} + \|Bu\|_{L^2}\Big)}{1 - Mc - M\frac{\rho b}{\beta+\gamma(1-\beta)}}.$$

We hence can confirm that $\|q\|_{\beta,\gamma} \neq R$. Thus, there is no $q \in \partial Q^{(m)}$ ensure that $(q,\lambda) \in \partial Q^{(m)} \times (0,1)$ with $q = \Sigma(q,\lambda)$.

Therefore, Theorem 2.14 forces that $q = \Sigma(q,1)$ holds at least a fixed point y_m^n.

Step 2. We design the auxiliary system of the model

$$\begin{cases} D^{\beta,\gamma}y(s) = A^{(n)}y(s) + J^{\gamma(1-\beta)}(B(s)u(s) + f(s,y(s))),\ s \in J', \\ \lim\limits_{s\to 0^+}\Gamma(\beta+\gamma(1-\beta))s^{(1-\beta)(1-\gamma)}y(s) = \psi y. \end{cases} \tag{12}$$

For the solution y_m^n of model (10), according to (11) and Lemma 3.4, $\{y_m^n\}_{m\geq 1}$ is equicontinuous and bounded. We thus can suppose from Theorem 2.13 that, up to subsequence, $y_m^n \rightharpoonup y^n$, $m \to \infty$, in $C_{\beta,\gamma}(J,H)$. The combination of Remark 3.1, $(H\psi)$ and Lemma 3.3 forces

$$\begin{aligned}
& \|t^{(1-\beta)(1-\gamma)}R^{mn}_{\beta,\gamma}(t)\mathbb{P}_m\psi y^n_m - t^{(1-\beta)(1-\gamma)}R^n_{\beta,\gamma}(t)\psi y^n\| \\
\leq\ & \|t^{(1-\beta)(1-\gamma)}R^{mn}_{\beta,\gamma}(t)\mathbb{P}_m\psi y^n_m - t^{(1-\beta)(1-\gamma)}R^{mn}_{\beta,\gamma}(t)\mathbb{P}_m\psi y^n\| \\
& + \|t^{(1-\beta)(1-\gamma)}R^{mn}_{\beta,\gamma}(t)\mathbb{P}_m\psi y^n - t^{(1-\beta)(1-\gamma)}R^n_{\beta,\gamma}(t)\psi y^n\| \\
\leq\ & M\|\mathbb{P}_m\psi y^n_m - \mathbb{P}_m\psi y^n\| + t^{(1-\beta)(1-\gamma)}\|R^{mn}_{\beta,\gamma}(t)\mathbb{P}_m\psi y^n - R^n_{\beta,\gamma}(t)\psi y^n\| \\
\to\ & 0,\ m\to\infty.
\end{aligned}$$

Similarly, (Hf) indicates that

$$(t-s)^{(1-\beta)(1-\gamma)}\|R^{mn}_{\beta,\gamma}(t-s)\mathbb{P}_m f(s,y^n_m(s)) - R^n_{\beta,\gamma}(t-s)f(s,y^n(s))\| \to 0,\ m\to\infty.$$

Moreover, Lemma 3.3 yields

$$(t-s)^{(1-\beta)(1-\gamma)}\|R^{mn}_{\beta,\gamma}(t-s)\mathbb{P}_m B(s)u(s) - R^n_{\beta,\gamma}(t-s)B(s)u(s)\| \to 0,\ m\to\infty.$$

Due to (Hf) and (HB), we arrive at

$$\begin{aligned}
& \|R^{mn}_{\beta,\gamma}(t-s)\mathbb{P}_m(B(s)u(s) + f(s,y^n_m(s))) - R^n_{\beta,\gamma}(t-s)(B(s)u(s) + f(s,y^n(s)))\| \\
\leq\ & 2M(t-s)^{(\beta-1)(1-\gamma)}(\nu(s) + \rho R + \|B(s)u(s)\|).
\end{aligned}$$

Hence, combing with the dominated convergence theorem and the uniqueness of the weak limit, we receive the following solution to (12):

$$y^n(t) = R^n_{\beta,\gamma}(t)\psi y^n + \int_0^t R^n_{\beta,\gamma}(t-\tau)(B(\tau)u(\tau) + f(\tau,y^n(\tau)))\mathrm{d}\tau.$$

Step 3. Finally, the task is now to confirm system (2) admits a solution.

From (11) and Lemma 3.4, one can easily receive the boundedness and equicontinuity of $\{y^n\}_{n\in\mathbb{N}}$. So, we can suppose, from Theorem 2.13, that, up to subsequence, $y^n \rightharpoonup y,\ n\to\infty$.

Let A^* be the adjoint of A and set $A^{(n)*} = nA^*(nI - A^*)^{-1}$. Then, the combination of the dual theorem of semigroup (see Theorem 3.7.1 in [26]), Remark 3.1 and Lemma 2.9 enables one to conclude that $A^{(n)*}$ and A^* can respectively generate resolvents $\{R^{n*}_{\beta,\gamma}(s)\}_{s>0}$ and $\{R^*_{\beta,\gamma}(s)\}_{s>0}$. We, in addition, can derive $\|R^{n*}_{\beta,\gamma}(s)z - R^*_{\beta,\gamma}(s)z\| \to 0$ for $z\in H$. We thereby arrive at

$$\begin{aligned}
& \langle z, R^n_{\beta,\gamma}(t)\psi y^n - R_{\beta,\gamma}(t)\psi y\rangle \\
=\ & \langle z, R^n_{\beta,\gamma}(t)(\psi y^n - \psi y)\rangle + \langle z, R^n_{\beta,\gamma}(t)\psi y - R_{\beta,\gamma}(t)\psi y\rangle \\
=\ & \langle R^{n*}_{\beta,\gamma}(t)z - R^*_{\beta,\gamma}(t)z, \psi y^n - \psi y\rangle + \langle R^*_{\beta,\gamma}(t)z, \psi y^n - \psi y\rangle + \langle z, R^n_{\beta,\gamma}(t)\psi y - R_{\beta,\gamma}(t)\psi y\rangle \\
\to\ & 0,
\end{aligned}$$

which gives $R^n_{\beta,\gamma}(t)\psi y^n \rightharpoonup R_{\beta,\gamma}(t)\psi y$. Similarly, we can receive

$$\int_0^t R^n_{\beta,\gamma}(t-\tau)(B(\tau)u(\tau) + f(\tau,y^n(\tau)))\mathrm{d}\tau \rightharpoonup \int_0^t R_{\beta,\gamma}(t-\tau)(B(\tau)u(\tau) + f(\tau,y(\tau)))\mathrm{d}\tau.$$

Thus, the uniqueness of the weak limit forces the mild solution of system (2), that is

$$y(t) = R_{\beta,\gamma}(t)\psi y + \int_0^t R_{\beta,\gamma}(t-\tau)(B(\tau)u(\tau) + f(\tau,y(\tau)))\mathrm{d}\tau.$$

□

Remark 3.6. *For now, we have received the existence result without the Lipschitz condition on f and the compactness restriction on semigroup or resolvent. But the uniqueness of solutions in Theorem 3.5 cannot be derived. To make our subsequent investigations comfort, designate*

$$S(u) = \{y \in C_{\beta,\gamma}(J, H) : y \text{ is the derived solution in Theorem 3.5}\}.$$

4. Time optimal controls

We here contemplate dropping the Lipschitz restriction on f and the compactness hypothesis on semigroup or resolvent, when considering the time optimal controls for model (2). We require a target set W to be a convex closed bounded subset in H. For our comfort, designate

$$\mathcal{A}_d^W = \{(y, u) \in S(u) \times U_{ad} : \text{ for some } t \in J,\ t^{(1-\beta)(1-\gamma)} y(t) \in W\},$$

$$U_0 = \{u \in U_{ad} : \text{ for some } y \in S(u),\ (y, u) \in \mathcal{A}_d^W\},$$

$$S_u^W = \{y \in S(u) : (y, u) \in \mathcal{A}_d^W\} \text{ for fixed } u \in U_0.$$

Let $\mathcal{A}_d^W \neq \emptyset$. We delineate the transition time as

$$t_{(y,u)} = \min\{t \in J : \text{ for fixed } (y, u) \in \mathcal{A}_d^W,\ t^{(1-\beta)(1-\gamma)} y(t) \in W\}.$$

Put $\bar{t} = \inf\limits_{(y,u)\in\mathcal{A}_d^W} t_{(y,u)}$. Our strategy now is to explore the ensuing time optimal control problem (TP):

Search for a suitable trajectory-control dyad $(\overline{y}, \overline{u})$ satisfying $t_{(\overline{y},\overline{u})} = \bar{t}$ in $\mathcal{A}_d^W$.

Theorem 4.1. *Let $\mathcal{A}_d^W \neq \emptyset$ and $Mc + \frac{M\rho b}{\beta+\gamma(1-\beta)} < 1$. If hypotheses (HA), (Hf), (HB) and (Hψ) hold, then problem (TP) possesses suitable trajectory-control dyads.*

Proof. **Step 1.** Based on $\mathcal{A}_d^W \neq \emptyset$, we receive $U_0 \neq \emptyset$ and $S_u^W \neq \emptyset$. Fix $u \in U_0$ and put $t_u = \inf\limits_{y\in S_u^W} t_{(y,u)}$. We begin by seeking some $\widehat{y} \in S_u^W$ to ensure that $t_u^{(1-\beta)(1-\gamma)}\widehat{y}(t_u) \in W$.

Since it is a trivial verification when $S(u)$ possesses only finite elements, one thus can take an approximation sequence $\{t_{(y_n,u)}\}_{n\geq 1}$, up to subsequence, in a way that $t_{(y_n,u)} \downarrow t_u,\ n \to \infty$ in J. For our comfort, set $t_n = t_{(y_n,u)}$ and choose some

$$R \geq \frac{Md + M\sqrt{\frac{b}{2(\beta+\gamma(1-\beta))-1}}(\|v\|_{L^2} + \|Bu\|_{L^2})}{1 - Mc - \frac{M\rho b}{\beta+\gamma(1-\beta)}}.$$

Based on $(y_n, u) \in \mathcal{A}_d^W$, we receive

$$y_n(t) = R_{\beta,\gamma}(t)\psi y_n + \int_0^t R_{\beta,\gamma}(t-\tau)\Big(B(\tau)u(\tau) + f(\tau, y_n(\tau))\Big)\mathrm{d}\tau.$$

The imposed hypotheses force

$$\|y_n\|_{\beta,\gamma} \leq Mc\|y_n\|_{\beta,\gamma} + Md + \frac{M\rho b\|y_n\|_{\beta,\gamma}}{\beta+\gamma(1-\beta)} + M\sqrt{\frac{b}{2(\beta+\gamma(1-\beta))-1}}(\|v\|_{L^2} + \|Bu\|_{L^2}).$$

This implies that $\|y_n\|_{\beta,\gamma} \leq R$. From Lemma 3.4, one, additionally, can derive the equicontinuity of $\{y_n\}_{n\geq 1}$. Then Theorem 2.13 can enable us to take a subsequence from $\{y_n\}_{n\geq 1}$, still written as itself, in a way that $y_n \rightharpoonup \widehat{y},\ n \to \infty$. Thereby, for $z \in H$, we arrive at

$$\langle z, R_{\beta,\gamma}(t)\psi y_n\rangle = \langle R^*_{\beta,\gamma}(t)z, \psi y_n\rangle \to \langle R^*_{\beta,\gamma}(t)z, \psi\widehat{y}\rangle,$$

which indicates that $R_{\beta,\gamma}(t)\psi y_n \rightharpoonup R_{\beta,\gamma}(t)\psi\widehat{y}$. Similar analysis can yield

$$\int_0^t R_{\beta,\gamma}(t-\tau)f(\tau,y_n(\tau))\mathrm{d}\tau \rightharpoonup \int_0^t R_{\beta,\gamma}(t-\tau)f(\tau,\widehat{y}(\tau))\mathrm{d}\tau.$$

Thereby, the uniqueness of the weak limit forces that

$$\widehat{y}(t) = R_{\beta,\gamma}(t)\psi\widehat{y} + \int_0^t R_{\beta,\gamma}(t-\tau)(B(\tau)u(\tau) + f(\tau,\widehat{y}(\tau)))\mathrm{d}\tau, \tag{13}$$

that is, $\widehat{y} \in S(u)$. Moreover, $(y_n, u) \in \mathcal{A}_d^W$ implies $t_n^{(1-\beta)(1-\gamma)}y_n(t_n) \in W$ and

$$y_n(t_n) = R_{\beta,\gamma}(t_n)\psi y_n + \int_0^{t_n} R_{\beta,\gamma}(t_n-\tau)\Big(B(\tau)u(\tau) + f(\tau,y_n(\tau))\Big)\mathrm{d}\tau.$$

From the hypotheses on W, we can derive that there exist some $\omega \in W$ in a way that, up to subsequence, $t_n^{(1-\beta)(1-\gamma)}y_n(t_n) \rightharpoonup \omega$. Due to $y_n \rightharpoonup \widehat{y}$, $n \to \infty$ and $(\cdot)^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(\cdot)x \in C(\mathbb{R},H)$, we receive that for $z \in H$,

$$\begin{aligned} & \langle z, t_n^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(t_n)\psi y_n\rangle = \langle t_n^{(1-\beta)(1-\gamma)}R^*_{\beta,\gamma}(t_n)z, \psi y_n\rangle \\ \to\ & \langle t_u^{(1-\beta)(1-\gamma)}R^*_{\beta,\gamma}(t_u)z, \psi\widehat{y}\rangle = \langle z, t_u^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(t_u)\psi\widehat{y}\rangle, \end{aligned}$$

which indicates

$$t_n^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(t_n)\psi y_n \rightharpoonup t_u^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(t_u)\psi\widehat{y}.$$

Similarly, we can get

$$\langle z, R_{\beta,\gamma}(t_n-\tau)f(\tau,y_n(\tau))\rangle \to \langle z, R_{\beta,\gamma}(t_u-\tau)f(\tau,\widehat{y}(\tau))\rangle.$$

Thus, the dominated convergence theorem forces that

$$\int_0^{t_u}\langle z, R_{\beta,\gamma}(t_n-\tau)f(\tau,y_n(\tau))\rangle\mathrm{d}\tau \to \int_0^{t_u}\langle z, R_{\beta,\gamma}(t_u-\tau)f(\tau,\widehat{y}(\tau))\rangle\mathrm{d}\tau.$$

In addition, we have

$$\begin{aligned} & \left\|\int_{t_u}^{t_n} R_{\beta,\gamma}(t_n-\tau)f(\tau,y_n(\tau))\mathrm{d}\tau\right\| \\ \leq\ & M\Big(\rho R\frac{(t_n-t_u)^{\beta+\gamma(1-\beta)}}{\beta+\gamma(1-\beta)} + \sqrt{\frac{(t_n-t_u)^{2(\beta+\gamma(1-\beta))-1}}{2(\beta+\gamma(1-\beta))-1}}\|v\|_{L^2}\Big) \\ \to\ & 0. \end{aligned}$$

So, we can immediately conclude that

$$\int_0^{t_n} R_{\beta,\gamma}(t_n-\tau)f(\tau,y_n(\tau))\mathrm{d}\tau \rightharpoonup \int_0^{t_u} R_{\beta,\gamma}(t_u-\tau)f(\tau,\widehat{y}(\tau))\mathrm{d}\tau.$$

The same reasoning can yield

$$\int_0^{t_n} R_{\beta,\gamma}(t_n-\tau)B(\tau)u(\tau)\mathrm{d}\tau \rightharpoonup \int_0^{t_u} R_{\beta,\gamma}(t_u-\tau)B(\tau)u(\tau)\mathrm{d}\tau.$$

Hence, by (13) and the uniqueness of the weak limit, we can easily conclude that

$$\omega = t_u^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(t_u)\psi\widehat{y}$$

$$
\begin{aligned}
&+t_u^{(1-\beta)(1-\gamma)}\int_0^{t_u} R_{\beta,\gamma}(t_u-\tau)(B(\tau)u(\tau)+f(\tau,\widehat{y}(\tau)))\mathrm{d}\tau\\
=\;& t_u^{(1-\beta)(1-\gamma)}\widehat{y}(t_u),
\end{aligned}
$$

which gives $t_u^{(1-\beta)(1-\gamma)}\widehat{y}(t_u)\in W$ and $\widehat{y}\in S_u^W$.

Step 2. Let $\bar{t}=\inf\limits_{u\in U_0} t_u$. We shall look for a control $\overline{u}\in U_0$ and a trajectory $\overline{y}\in S_{\overline{u}}^W$ to guarantee $\bar{t}^{(1-\beta)(1-\gamma)}\overline{y}(\bar{t})\in W$.

For the case that U_0 possesses only finite elements, it is trivial. Hence, an approximation sequence $\{t_{u_n}\}_{n\geq 1}$, up to subsequence, can be taken to ensure that $\lim\limits_{n\to\infty} t_{u_n}=\bar{t}$.

The discussion in step 1 enables us to choose a state $y_n\in S_{u_n}^W$ to guarantee that $t_{u_n}^{(1-\beta)(1-\gamma)}y_n(t_{u_n})\in W$ and

$$
y_n(t)=R_{\beta,\gamma}(t)\psi y_n+\int_0^t R_{\beta,\gamma}(t-\tau)\Big(B(\tau)u_n(\tau)+f(\tau,y_n(\tau))\Big)\mathrm{d}\tau.
$$

Thanks to the boundedness convexity closeness of U_{ad} and W, we can receive subsequences extracted from $\{u_n\}_{n\geq 1}$ and $\{t_{u_n}^{(1-\beta)(1-\gamma)}y_n(t_{u_n})\}_{n\geq 1}$, still written as them, that converge weakly to some $\overline{u}\in U_{ad}$ and some $\omega\in W$, respectively. Similar arguments in step 1 can give the boundedness and equicontinuity of $\{y_n\}_{n\geq 1}$. Due to Theorem 2.13, one may suppose that, up to subsequence, $y_n\rightharpoonup\overline{y}$ in $C_{\beta,\gamma}(J,H)$.

The same reasoning in step 2 can enable us to receive

$$
\begin{aligned}
\omega\;=\;& \bar{t}^{(1-\beta)(1-\gamma)}R_{\beta,\gamma}(\bar{t})\psi\overline{y}\\
&+\bar{t}^{(1-\beta)(1-\gamma)}\int_0^{\bar{t}} R_{\beta,\gamma}(\bar{t}-\tau)(B(\tau)\overline{u}(\tau)+f(\tau,\overline{y}(\tau)))\mathrm{d}\tau\\
=\;& \bar{t}^{(1-\beta)(1-\gamma)}\overline{y}(\bar{t}),
\end{aligned}
$$

which yields that $(\overline{y},\overline{u})$ is our suitable trajectory-control dyad. □

Remark 4.2. *Up to now, the suitable state-control pairs have been received by resorting to the weak topology approach and the technique of designing minimizing approximation sequences twice. We have dispensed with the compactness condition or Lipschitz restriction imposed in the existing findings.*

5. An application

As an application of our theoretical findings, we now return back to the diffusion model (1).

Let $H=L^2(\Omega)$. Then A can generate a contractive equicontinuous semigroup $\{T(s)\}_{s\geq 0}$ (see [23]). Lemma 2.9 forces that a resolvent $\{R_{\beta,\gamma}(t)\}_{t>0}$ can also be generated by A. Set $c=\sum\limits_{k=1}^{N}c_k$. Then $(H\psi)$ is fulfilled. Let conditions (Hf), (HB) and (9) be satisfied. Then, all hypotheses in Theorem 4.1 hold. Hence, problem (1) admits suitable trajectory-control dyads.

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