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Lie higher derivations on triangular algebras without assuming unity

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Abstract. In this paper, we focus on the structure of Lie higher derivations on triangular algebras $\mathcal{T}$ without assuming unity. We prove that Lie higher derivation on every triangular algebra can be decomposed into a sum of a higher derivation, an extreme Lie higher derivation, and a central mapping vanishing on commutators $[x, y]$. As by-products, we use it on some typical algebras: upper triangular matrix algebras over faithful algebras and semiprime algebras, respectively.

1. Introduction

Let $\mathcal{R}$ be a commutative ring with identity and $\mathcal{A}$ be an algebra over $\mathcal{R}$. $C(\mathcal{A})$ be the center of $\mathcal{A}$. Let us denote the Lie product of arbitrary elements $x, y \in \mathcal{R}$ by $[x, y] = xy - yx$. This paper is devoted to the treatment of Lie higher derivations of algebras $\mathcal{A}$. Let's introduce some necessary mappings. Let $n \in \mathcal{N}$ be the set of all non-negative integers and $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ be a family of $\mathcal{R}$-linear mappings on $\mathcal{A}$ such that $\delta_0 = id_{\mathcal{A}}$. Δ is called:

(a) a higher derivation if

$$\delta_n(xy) = \sum_{i+j=n} \delta_i(x)\delta_j(y)$$

for all $x, y \in \mathcal{A}$ and for each $n \in \mathcal{N}$;

(b) a Lie higher derivation if

$$\delta_n([x, y]) = \sum_{i+j=n} [\delta_i(x), \delta_j(y)]$$

for all $x, y \in \mathcal{A}$ and for each $n \in \mathcal{N}$.

It is obvious that Lie higher derivations and higher derivations are usual Lie derivations and derivations for $n = 1$ respectively. Lie derivations (resp. derivations) are an active subject of research in algebras which may not be associative or commutative see[6, 10, 11, 16]. It is easy to verify that every higher derivation is a Lie higher derivation. But, the converse statement is in general not true. Higher derivations play an

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important role in the algebraic theory and are also an active subject of research in algebras which may not be associative or commutative [1, 7–9, 17, 18].

In this paper, we mainly study the structure of Lie higher derivations on triangular algebras that do not assume to contain unit elements. Many mappings related to Lie higher derivation on triangular algebras have been extensively studied. Under certain conditions, Lie derivation was studied by Cheung[6] and proved to have a standard form. Subsequently, the result of Cheung[6] was extended to Lie higher derivations by Li and Shen [9], Li and Shen's results[9] also extend the results of Lie higher derivations on nest algebras studied by Qi and Hou [15]. Moafian and Ebrahimi Vishki [12] given the structure of Lie higher derivations on triangular algebras by the entries of matrices and obtained a similar conclusion as shown by Li and Shen [9]. It should be noted that the unit elements of triangular algebra play an important role in the research process of the above article[6, 9, 12, 15].

It should be noted that the unit elements of triangular algebra play an important role in the research process of the above article. So an interesting question is proposed: how to completely characterize the structural form of Lie higher derivations on triangular algebra without assuming unity? This is the main purpose of this paper. Note that some mappings on triangular algebras without assuming unit elements have attracted the attention of many scholars. Lie centralizer, Lie derivations, and generalized Lie 2-derivations at zero products on triangular algebras without assuming unity was worked by Ghahramani and collaborators[2–4] The origin of this problem comes from the study of Lie derivations by [16]. Specifically, under suitable conditions, he proved that Lie derivations L on triangular algebras have the following form:

$$L = \delta + \sigma + \tau,$$

where δ is a derivation of $\mathcal{T}$, σ is an extreme Lie derivation defined in Definition refxxsec2.2 of $\mathcal{T}$, and τ is a linear mapping into the ordinary center $C_1(\mathcal{T})$ defined in Part 2 of $\mathcal{T}$ and $\tau([\mathcal{T},\mathcal{T}]) = 0$. Inspired by the article[16], the focus of this article is to characterize the structural form of Lie higher derivation on triangular algebras. We provide necessary and sufficient conditions for each higher derivations on a triangular algebra $\left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ without assuming unital (Theorem 3.1), on this basis, we provide sufficient conditions for each Lie higher derivations on a triangular algebra $\left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ without assuming unital have has a decomposition form (Theorem 3.2). And then we apply this result to describe the Lie higher derivations on upper triangular matrix algebras $T_n(\mathcal{A})$ over a faithful algebra $\mathcal{A}$ (see Corollary 3.6) and a semiprime algebra $\mathcal{A}$ (see Corollary 3.7) respectively.

This paper assumes that triangular algebra does not contain identity element, so the property of the identity element is not used in the whole proof process. Based on this, under certain conditions, this paper proves that each Lie higher derivation has a completely new decomposition form(see Theorem 3.2). This is the first feature of this article. In addition, the proof process in this paper is also suitable for the case of triangular algebra containing identity elements. Only by modifying the proof process appropriately, the conclusion of Li and Shen's results[9] can be obtained, which is the second feature of this paper.

2. Triangular algebras without assuming unity

In this section, we will introduce the concept of triangular algebra and introduce some basic knowledge related to the topic. Let's start this section with a definition. Wang[16] described a generalization of faithful bimodule as follows.

Definition 2.1. *[16, Theorem 2.1] Let A and B be algebras. Let M be an (A, B)-bimodule: if $aAMB = 0$(resp.$AMBb$=0), then $a = 0$(resp.$b = 0$), it is called **left strong faithful**(resp. **right strong faithful**). If M satisfies the definition of both left strong faithful and right strong faithful, it is said to **strong faithful**.*

If algebras A and B both contain identity elements, then left strong faithful, right strong faithful and strong faithful evolve into strong faithful, strong faithful and faithful in papers[5, 19], respectively. Thus it can be known that every faithful (A, B)-bimodul is equivalent to strong faithful bimodule Many examples of faithful bimodule are constructed by Wang [16, Example 2.1, Example 2.2] which is not strong faithful.

For algebras A and B, and their (A, B)-bimodule M, we define an algebra

$$\mathcal{T} = \begin{bmatrix} A & M \\ 0 & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \middle\| \ \forall\, a \in A,\ m \in M,\ b \in B \right\} \tag{2.1}$$

according to the usual matrix addition and multiplication operations. Note that $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ is unital if and only if both A and B are unital. If algebra $\mathcal{T}$ contains an unit elements, it evolves into a triangular algebra studied by many scholars[6, 9, 12, 15].

Starting from this point, we assume that triangular algebras $\mathcal{T}$ do not contain identity elements. At this point, it should be noted that triangular algebra $\mathcal{T}$ without assuming unity still has the form of equation (2.1).

With the help of [16, Proposition 2.1], the center $C(\mathcal{T})$ of algebra $\mathcal{T}$ is characterized as follows

$$C(\mathcal{T}) = \left\{ \begin{bmatrix} a & m_0 \\ 0 & b \end{bmatrix} \middle\| \ am = mb,\ \forall\, m \in M \text{ and } Am_0 = 0 = m_0 B \right\}. \tag{2.2}$$

Based on the structural form of triangular algebra, we define two natural $\mathcal{R}$-linear projections $\pi_A : \mathcal{T} \to A$ and $\pi_B : \mathcal{T} \to B$ by

$$\pi_A : \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \longmapsto a \quad \text{and} \quad \pi_B : \begin{bmatrix} a & m \\ 0 & b \end{bmatrix} \longmapsto b.$$

It is obvious that algebras $\pi_A\,(C(\mathcal{T}))$ and $\pi_B(C(\mathcal{T}))$ are subalgebras $C(A)$ and $C(B)$, respectively. Furthermore, there exists a unique algebraic isomorphism $\tau\colon \pi_A(C(\mathcal{T})) \longrightarrow \pi_B(C(\mathcal{T}))$ such that $am = m\tau(a)$ for all $a \in \pi_A(C(\mathcal{T}))$ and for all $m \in M$.

In light of (2.2) and the definitions of strong faithful bimodule, the definitions of ordinary centers and extreme centers was defined by Wang[16, Proposition 2.1] as follows respectively :

$$C_1(\mathcal{T}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle\| \ am = mb,\ \forall\, m \in M \right\}.$$

and

$$C_2(\mathcal{T}) = \left\{ \begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix} \middle\| \ Am_0 = 0 = m_0 B,\ \forall\, m \in M \right\}.$$

The sets $C_1(\mathcal{T})$ and $C_2(\mathcal{T})$ is said to be **ordinary centres** and **extreme centres** respectively. According to [16, Proposition 2.1], both $C_1(\mathcal{T})$ and $C_2(\mathcal{T})$ are subalgebras of $C(\mathcal{T})$ satisfying relation

$$C(\mathcal{T}) = C_1(\mathcal{T}) \bigoplus C_2(\mathcal{T}).$$

Moreover, $\pi_A\,(C(\mathcal{T})) = \pi_A\,(C_1(\mathcal{T})) \subseteq C(A)$ and $\pi_B\,(C(\mathcal{T})) = \pi_B\,(C_1(\mathcal{T})) \subseteq C(B)$. Notice that if the algebra $\mathcal{T}$ has the identity element, then $C_2(\mathcal{T}) = 0$ and $C(\mathcal{T}) = C_1(\mathcal{T})$.

Based on the above definition of center, we introduce a new concept: extreme Lie higher derivation, which extends the definition of extreme Lie derivation in the article[16, Definition 2.2].

Definition 2.2. *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$. A Lie higher derivation of $\mathcal{T}$ is said to be an extreme Lie higher derivation if it is a higher derivation modulo $C_2(\mathcal{T})$ and is not a nonzero higher derivation. In other words, extreme Lie higher derivation $\Sigma = \{\sigma_n\}_{n \in N}$ satisfy the following relationship:*

(a) σ_n is a nonzero mapping on $\mathcal{T}$ and $\sigma_0 = Id_{\mathcal{T}}$ is an identity mapping on $\mathcal{T}$,
(b) $\sigma_n(xy) - \sum_{i+j=n}(\delta_i(x)\sigma_j(y) + \sigma_i(x)\delta_j(y)) \in C_2(\mathcal{T})$,
(c) $\sigma_n([x, y]) = \sum_{i+j=n}([\delta_i(x), \sigma_j(y)] + [\sigma_i(x), \delta_j(y)])$.

It should be noted that if algebra A contains a unit element, each extreme Lie higher derivation degenerates to zero. During the research, I was surprised to find that there are non-zero extreme Lie higher derivations in many algebras, such as [6, Example 8] and [16, Example 2.3].

What is studied in the literature[9, 13, 14] is indeed the structure of Lie type derivatives on triangular algebras. But they used the identity element properties of triangular algebras. Comparing those literatures[9, 13, 14] on Lie higher derivations, we characterize the structure of Lie higher derivations on triangular algebras without assuming that triangular algebras contain unit elements, and obtain a new decomposition structure (see Theorem 3.2) of Lie higher derivation. From the proof of Theorem 3.2, it can be seen that this structure generalizes Wang's results[16, Theorem 3.2]. In the research process, we just do not assume that there is a unit element. When this assumption is removed, the triangular algebra will evolve into a triangular algebra with an unit element. Therefore, the strong faithful bimodule used in theorem 3.2 will evolve into a faithful bimodule, and the extreme center will evolve into zero. Therefore, Theorem 3.2 generalized some existing conclusions. It should be noted that, in order to obtain Theorem 3.2, we first gave the equivalent characterization of higher derivations on triangular algebras(see Theorem 3.1). In the research process of Theorem 3.1, we also did not assume that triangular algebras contains identity elements. In other words, we use Theorem 3.1 to describe the new structural characterization of Lie higher derivations on triangular algebras without assuming unity (see Theorem 3.2).

3. Lie higher derivations of triangular algebras

In this section we characterize Lie higher derivations on triangular algebras without assuming unity. For this purpose, We first give a description of higher derivations of triangular algebras $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ without assuming unity and with also M strong faithful. Next, we show that a sequence $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ of linear mappings $\delta_n : \mathcal{T} \to \mathcal{T}$ on triangular algebra without assuming unity is an equivalent characterization of higher derivation.

Theorem 3.1. *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ with M strong faithful. A sequence $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ of linear mappings $\delta_n : \mathcal{T} \to \mathcal{T}$ is a higher derivation if and only if for each $n \in \mathcal{N}$, δ_n can be written as*

$$\delta_n\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big) = \begin{bmatrix} p_A^{(n)}(a) & r_1^{(n)}(a) - r_2^{(n)}(b) + f^{(n)}(m) \\ 0 & p_B^{(n)}(b) \end{bmatrix},$$

where, for arbitrary $n \in \mathcal{N}$,

$$p_A^{(n)} : A \to A,\ p_B^{(n)} : B \to B;$$
$$r_1^{(n)} : A \to M,\ r_2^{(n)} : B \to M,\ f^{(n)} : M \to M$$

are all $\mathcal{R}$-linear mappings satisfying the following conditions:

(i) $f^{(n)}(am) = \sum_{i+j=n} p_A^{(i)}(a) f^{(j)}(m)$;

(ii) $f^{(n)}(mb) = \sum_{i+j=n} f^{(i)}(m) p_B^{(j)}(b)$;

(iii) $\sum_{i+j=n} p_A^{(i)}(a) r_2^{(j)}(b) = \sum_{i+j=n} r_1^{(i)}(a) p_B^{(j)}(b)$,

(iv) $r_1^{(n)}(aa') = \sum_{i+j=n} p_A^{(i)}(a) r_1^{(j)}(a')$, *and* $r_2^{(n)}(bb') = \sum_{i+j=n} r_2^{(i)}(b) p_B^{(j)}(b')$

for all $a, a' \in A, b, b' \in B$.

Proof. Assume that a squence $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ of linear mappings $\delta_n : \mathcal{T} \to \mathcal{T}$ is a higher derivation on triangular algebras $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$. We shall use an induction method for n. For $n = 1$, $\delta_1 : \mathcal{T} \to \mathcal{T}$ is a derivation and the result follows from [16, Theorem 3.1].

Suppose that δ_n is a higher derivation on $\mathcal{T}$. Writer δ_n as

$$\delta_n(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}) = \begin{bmatrix} p_A^{(n)}(a) + q_B^{(n)}(b) + k_1^{(n)}(m) & r_1^{(n)}(a) - r_2^{(n)}(b) + f^{(n)}(m) \\ 0 & p_B^{(n)}(b) + q_A^{(n)}(a) + k_2^{(n)}(m) \end{bmatrix} \tag{3.1}$$

for all $a \in A, b \in B$ and $m \in M$, where, for arbitrary $n \in \mathcal{N}$,

$$\begin{aligned} &p_A^{(n)} : A \to A,\ q_B^{(n)} : B \to A,\ k_1^{(n)} : M \to A;\\ &r_1^{(n)} : A \to M,\ r_2^{(n)} : B \to M,\ f^{(n)} : M \to M;\\ &p_B^{(n)} : B \to B,\ q_A^{(n)} : A \to A,\ k_2^{(n)} : M \to M \end{aligned}$$

are all $\mathcal{R}$-linear mappings. It is important to note that linear maps incorporate the following conditions: for $n = 0$, we know that $\delta_n(0) = Id_{\mathcal{T}}$ is an identity mapping, this is,

$$\begin{aligned} &p_A^{(0)}(a) = a,\ k_2^{(0)}(m) = m,\ r_2^{(0)}(b) = b,\\ &q_B^{(0)}(b) = 0,\ k_1^{(0)}(m) = 0,\ r_1^{(0)}(a) = 0, f^{(n)}(a) = 0,\ p_B^{(0)}(a) = q_A^{(0)}(a) = 0. \end{aligned}$$

Since

$$\begin{aligned} 0 = \delta_n\Big(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\Big) &= \sum_{i+j=n} \delta_i\Big(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\Big)\delta_j\Big(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\Big)\\ = \sum_{i+j=n} &\begin{bmatrix} q_B^{(i)}(b) & -r_2^{(i)}(b) \\ 0 & p_B^{(i)}(b) \end{bmatrix}\begin{bmatrix} p_A^{(j)}(a) & r_1^{(j)}(a) \\ 0 & q_A^{(j)}(a) \end{bmatrix},\\ = \sum_{i+j=n} &\begin{bmatrix} q_B^{(i)}(b)p_A^{(j)}(a) & q_B^{(i)}(b)r_1^{(j)}(a) - r_2^{(i)}(b)q_A^{(j)}(a) \\ 0 & p_B^{(i)}(b)q_A^{(j)}(a) \end{bmatrix} \end{aligned}$$

we arrive at

$$\sum_{i+j=n} q_B^{(i)}(b)p_A^{(j)}(a) = 0 \text{ and } \sum_{i+j=n} p_B^{(i)}(b)q_A^{(j)}(a) = 0 \tag{3.2}$$

for all $a \in A, b \in B$.

Let's check $q_B^{(n)}(b) = 0 = q_A^{(n)}(a)$ by complete induction on n. When $n = 1$, by the definition of Lie higher derivations, $\delta_1 : \mathcal{T} \to \mathcal{T}$ is a Lie derivation and hence it follows from [16, Theorem 3.1], it is clear that $q_B^{(1)}(b) = 0 = q_A^{(1)}(a)$ for all $a \in A, b \in B$.

Suppose that

$$q_B^{(i)}(b) = 0 = q_A^{(j)}(a)$$

for all $1 \leq i, j \leq n-1$ and all $a \in A, b \in B$. We therefore get from (3.2)

$$\begin{aligned} 0 = \sum_{i+j=n} q_B^{(i)}(b)p_A^{(j)}(a) &= \sum_{i+j=n, i\neq 0} q_B^{(i)}(b)p_A^{(j)}(a) + q_B^{(n)}(b)a\\ &= q_B^{(n)}(b)a \end{aligned}$$

and

$$\begin{aligned} 0 = \sum_{i+j=n} p_B^{(i)}(b)q_A^{(j)}(a) &= \sum_{i+j=n, j\neq n} p_B^{(i)}(b)q_A^{(j)}(a) + bq_A^{(n)}(a)\\ &= bq_A^{(n)}(a), \end{aligned}$$

i.e., for arbitrary $a \in A, b \in B$ we have $q_B^{(n)}(b)a = 0 = bq_A^{(n)}(a)$. And then $q_B^{(n)}(b)AMB = 0$ and $AMBq_A^{(n)}(a) = 0$. Hence $q_B^{(n)} = 0 = q_A^{(n)}$ as M is strong faithful.

On one hand, according to (3.1), we have

$$\begin{aligned} \delta_n\Big(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}\Big) &= \delta_n\Big(\begin{bmatrix} aa' & 0 \\ 0 & bb' \end{bmatrix}\Big)\\ &= \delta_n\Big(\begin{bmatrix} p_A^{(n)}(aa') & r_1^{(n)}(aa') - r_2^{(n)}(bb') \\ 0 & p_B^{(n)}(bb') \end{bmatrix}\Big) \end{aligned}$$

for all $a, a' \in A, b, b' \in B$.

On the other hand, in light of (3.1), we have

$$\begin{aligned}
&\delta_n\Big(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}\Big)\\
&= \Sigma_{i+j=n}\delta_i\Big(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\Big)\delta_j\Big(\begin{bmatrix} a' & 0 \\ 0 & b' \end{bmatrix}\Big)\\
&= \sum_{i+j=n}\Big(\begin{bmatrix} p_A^{(i)}(a) & r_1^{(i)}(a) - r_2^{(i)}(b) \\ 0 & p_B^{(i)}(b) \end{bmatrix}\Big)\Big(\begin{bmatrix} p_A^{(j)}(a') & r_1^{(j)}(a') - r_2^{(j)}(b') \\ 0 & p_B^{(j)}(b') \end{bmatrix}\Big)\\
&= \sum_{i+j=n}\Big(\begin{bmatrix} p_A^{(i)}(a)p_A^{(j)}(a') & p_A^{(i)}(a)(r_1^{(j)}(a') - r_2^{(j)}(b')) + (r_1^{(i)}(a) - r_2^{(i)}(b))p_B^{(j)}(b') \\ 0 & p_B^{(i)}(b)p_B^{(j)}(b') \end{bmatrix}\Big)
\end{aligned}$$

for all $a, a' \in A, b, b' \in B$.

Thus, we arrive at

$$p_A^{(n)}(aa') = \sum_{i+j=n} p_A^{(i)}(a)p_A^{(j)}(a'),\ \ p_B^{(n)}(bb') = \sum_{i+j=n} p_B^{(i)}(b)p_B^{(j)}(b')$$

and

$$r_1^{(n)}(aa') - r_2^{(n)}(bb') = \sum_{i+j=n}(p_A^{(i)}(a)(r_1^{(j)}(a') - r_2^{(j)}(b')) + (r_1^{(i)}(a) - r_2^{(i)}(b))p_B^{(j)}(b')) \tag{3.3}$$

for all $a, a' \in A, b, b' \in B$. This implies that both a sequence $\mathcal{P}_{\mathcal{A}} = \{p_A^{(n)}\}_{n\in\mathcal{N}}$ of linear mapping $p_A^{(n)} : A \to A$ and a sequence $\mathcal{P}_{\mathcal{B}} = \{p_B^{(n)}\}_{n\in\mathcal{N}}$ of linear mapping $p_B^{(n)} : B \to B$ be a higher derivation on A and B respectively.

Let $a' = 0 = b$ in (3.3), we get

$$\sum_{i+j=n} p_A^{(i)}(a)r_2^{(j)}(b') = \sum_{i+j=n} r_1^{(i)}(a)p_B^{(j)}(b') \tag{3.4}$$

for all $a \in A, b' \in B$.

Let $b' = 0 = b$ and $a' = 0 = a$ in (3.3), respectively, we obtain

$$r_1^{(n)}(aa') = \Sigma_{i+j=n}p_A^{(i)}(a)r_1^{(j)}(a') \text{ and } r_2^{(n)}(bb') = \sum_{i+j=n} r_2^{(i)}(b)p_B^{(j)}(b')$$

for all $a, a' \in A$ and $b, b' \in B$.

Since

$$\begin{aligned}
0 =\delta_n\Big(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\Big) &= \sum_{i+j=n}\delta_i\Big(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\Big)\delta_j\Big(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\Big)\\
&= \sum_{i+j=n}\Big(\begin{bmatrix} k_1^{(i)}(m) & f^{(i)}(m) \\ 0 & k_2^{(i)}(m) \end{bmatrix}\Big)\Big(\begin{bmatrix} p_A^{(j)}(a) & r_1^{(j)}(a) \\ 0 & 0 \end{bmatrix}\Big)\\
&= \sum_{i+j=n}\Big(\begin{bmatrix} k_1^{(i)}(m)p_A^{(j)}(a) & k_1^{(i)}(m)r_1^{(j)}(a) \\ 0 & 0 \end{bmatrix}\Big)
\end{aligned}$$

for all $a \in A, m \in M$. Thus we obtain that $\sum_{i+j=n} k_1^{(i)}(m)p_A^{(j)}(a) = 0$ and $\Sigma_{i+j=n}k_1^{(i)}(m)r_1^{(j)}(a) = 0$ for all $a \in A, m \in M$.

In order to prove $k_1^{(n)} = 0$ for arbitrary $n \in \mathcal{N}$, we use mathematical induction for n. According to the proof of [16, Theorem 3.1], we obtain that $k_1^{(1)} = 0$ for $n = 1$. Suppose that

$$k_1^{(i)} = 0 \ \text{ for all } \ 1 \le i \le n - 1.$$

We therefore obtain
$$\begin{aligned}0 &= \sum_{i+j=n} k_1^{(i)}(m)p_A^{(j)}(a) = \sum_{i+j=n, j\neq 0} k_1^{(i)}(m)p_A^{(j)}(a) + k_1^{(n)}(m)a \\ &= k_1^{(n)}(m)a\end{aligned}$$

for all $a \in A, m \in M$. Therefore, we obtain $k_1^{(n)}(m)a = 0$ and so $k_1^{(n)}(m)AMB = 0$. Since M is strong faithful, we obtain $k_1^{(n)} = 0$. For the same reason, we have

$$\begin{aligned}0 =&\delta_n\Big(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\Big) \\ &= \sum_{i+j=n} \delta_i\Big(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\Big)\delta_j\Big(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\Big) \\ &= \sum_{i+j=n} \Big(\begin{bmatrix} 0 & -r_2^{(i)}(b) \\ 0 & p_B^{(i)}(b) \end{bmatrix}\Big)\Big(\begin{bmatrix} 0 & f^{(j)}(m) \\ 0 & k_2^{(j)}(m)) \end{bmatrix}\Big) \\ &= \sum_{i+j=n} \Big(\begin{bmatrix} 0 & -r_2^{(i)}(b)k_2^{(j)}(m)) \\ 0 & p_B^{(i)}(b)k_2^{(j)}(m) \end{bmatrix}\Big),\end{aligned}$$

for all $b \in B, m \in M$. Thus we arrive at
$$\sum_{i+j=n} p_B^{(i)}(b)k_2^{(j)}(m) = 0 \tag{3.5}$$

for all $b \in B, m \in M$. In order to prove $k_2^{(n)} = 0$ for arbitrary $n \in \mathcal{N}$, we use mathematical induction for n. According to the proof of [16, Theorem 3.1], we obtain that $k_2^{(1)} = 0$ for $n = 1$. Suppose that
$$k_2^{(i)} = 0 \ \text{ for all } \ 1 \leq i \leq n-1.$$

With the help of (3.5), we therefore obtain
$$\begin{aligned}0 &= \sum_{i+j=n} p_B^{(i)}(b)k_2^{(j)}(m) = \sum_{i+j=n, i\neq 0} p_B^{(i)}(b)k_2^{(j)}(m) + bk_2^{(n)}(m) \\ &= bk_2^{(n)}(m)\end{aligned}$$

for all $b \in B, m \in M$. Therefore, we obtain $bk_2^{(n)}(m) = 0$ and so $AMBk_2^{(n)}(m) = 0$. Since M is strong faithful, we obtain $k_2^{(n)} = 0$.

On the one hand, we have
$$\begin{aligned}&\delta_n\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\Big) \\ &= \sum_{i+j=n} \delta_i\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\Big)\delta_j\Big(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\Big) \\ &= \sum_{i+j=n} \Big(\begin{bmatrix} p_A^{(i)}(a) & r_1^{(j)}(a) \\ & 0 \end{bmatrix}\Big)\Big(\begin{bmatrix} 0 & f^{(j)}(m) \\ & 0 \end{bmatrix}\Big) \\ &= \sum_{i+j=n} \Big(\begin{bmatrix} 0 & p_A^{(i)}(a)f^{(j)}(m) \\ & 0 \end{bmatrix}\Big),\end{aligned}$$

for all $b \in B, m \in M$. On the other hand, we have
$$\delta_n\Big(\begin{bmatrix} a & 0 \\ & 0 \end{bmatrix}\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\Big) = \delta_n\Big(\begin{bmatrix} 0 & am \\ & 0 \end{bmatrix}\Big) = \begin{bmatrix} 0 & f^{(n)}(am) \\ & 0 \end{bmatrix}$$

for all $a \in A, m \in M$. We have

$$f^{(n)}(am) = \sum_{i+j=n} p_A^{(i)}(a) f^{(j)}(m)$$

for all $a \in A, m \in M$. In an analogous manner, by computing $\delta_n\Big(\begin{bmatrix} 0 & m \\ & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ & b \end{bmatrix}\Big)$, we can obtain

$$f^{(n)}(mb) = \sum_{i+j=n} f^{(i)}(m) p_B^{(j)}(b)$$

for all $b \in B, m \in M$.

Conversely, suppose that a sequence $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ of linear mapping $\delta_n : \mathcal{T} \to \mathcal{T}$ is of the form

$$\delta_n(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}) = \begin{bmatrix} p_A^{(n)}(a) & r_1^{(n)}(a) - r_2^{(n)}(b) + f^{(n)}(m) \\ 0 & p_B^{(n)}(b) \end{bmatrix}$$

with (i), (ii), (iii) and (iv) satisfied.

We first show that a sequence $\mathcal{P}_{\mathcal{A}} = \{p_A^{(n)}\}_{n\in\mathcal{N}}$ of linear mapping $p_A^{(n)} : A \to A$ and a sequence $\mathcal{P}_{\mathcal{B}} = \{p_B^{(n)}\}_{n\in\mathcal{N}}$ of linear mapping $p_B^{(n)} : B \to B$ is a higher derivation on A and B respectively. By assumption (i) we have

$$f^{(n)}(aa'm) = \sum_{i+j=n} p_A^{(i)}(aa') f^{(j)}(m)$$

for all $a, a' \in A, m \in M$. On the other hand, we have

$$\begin{aligned} f^{(n)}(aa'm) &= \sum_{i+j=n} p_A^{(i)}(a) f^{(j)}(a'm) \\ &= \sum_{i+j=n} p_A^{(i)}(a) (\sum_{j_1+j_2=j} p_A^{(j_1)}(a') f^{(j_2)}(m)) \\ &= \sum_{i+j=n} \sum_{j_1+j_2=j} p_A^{(i)}(a) p_A^{(j_1)}(a') f^{(j_2)}(m) \\ &= \sum_{i+j_1+j_2=n} p_A^{(i)}(a) p_A^{(j_1)}(a') f^{(j_2)}(m) \\ &= \sum_{i_1+j_1+j=n} p_A^{(i_1)}(a) p_A^{(j_1)}(a') f^{(j)}(m) \\ &= \sum_{i+j=n} \Big(\sum_{i_1+j_1=i} p_A^{(i_1)}(a) p_A^{(j_1)}(a') \Big) f^{(j)}(m) \end{aligned}$$

for all $a, a' \in A, m \in M$.

On comparing the above two relations, we see that

$$\sum_{i+j=n} (p_A^{(i)}(aa') - \sum_{i_1+j_1=i} p_A^{(i_1)}(a) p_A^{(j_1)}(a')) f^{(j)}(m) = 0 \tag{3.6}$$

for all $a, a' \in A, m \in M$. Next, we use mathematical induction to prove that a sequence $\mathcal{P}_{\mathcal{A}} = \{p_A^{(n)}\}_{n\in\mathcal{N}}$ of linear mapping $p_A^{(n)} : A \to A$ is a higher derivation on A. For this purpose, By the proof of [16, Theorem 3.1] we know that for all $a, a' \in A$,

$$p_A^{(1)}(aa') = p_A^{(1)}(a)a' + ap_A^{(1)}(a')$$

whenever $n = 1$. Suppose that for all $a, a' \in A$,

$$p_A^{(k)}(aa') - \sum_{i_1+j_1=k} p_A^{(i_1)}(a) p_A^{(j_1)}(a') = 0$$

whenever $1 \le k \le n-1$. Taking into accounts (3.6), we therefore arrive at

$$(p_A^{(n)}(aa') - \sum_{i_1+j_1=n} p_A^{(i_1)}(a)p_A^{(j_1)}(a'))m = 0$$

for all $a, a' \in A, m \in M$. Since M is faithful as a left A-module, we immediately obtain

$$p_A^{(n)}(aa') = \sum_{i_1+j_1=n} p_A^{(i_1)}(a)p_A^{(j_1)}(a')$$

and so a sequence $\mathcal{P}_{\mathcal{A}} = \{p_A^{(n)}\}_{n\in\mathcal{N}}$ of linear mapping $p_A^{(n)} : A \to A$ is a higher derivation on A for all $a, a' \in A$. Adopt the same discussion as relations $P_A^{(n)}$ and the proof of [16, Theorem 3.1], we can prove that a sequence $\mathcal{P}_{\mathcal{B}} = \{p_B^{(n)}\}_{n\in\mathcal{N}}$ of linear mappings $P_B^{(n)} : B \to B$ is a higher derivation on B. On the one hand, in view of the relation (iii) and (3.4) we have

$$\begin{aligned}
&\delta_n\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\begin{bmatrix} a' & m' \\ & b' \end{bmatrix}\Big) = \delta_n\Big(\begin{bmatrix} aa' & am' + mb' \\ & bb' \end{bmatrix}\Big) \\
&= \begin{bmatrix} p_A^{(n)}(aa') & r_1^{(n)}(aa') - r_2^{(n)}(bb') + f^{(n)}(am' + mb') \\ 0 & p_B^{(n)}(bb') \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i+j=n} p_A^{(i)}(a)p_A^{(j)}(a') & \sum_{i+j=n} H(i,j) \\ 0 & \sum_{i+j=n} p_B^{(i)}(b)p_B^{(j)}(b') \end{bmatrix} \\
&= \sum_{i+j=n} \begin{bmatrix} p_A^{(i)}(a)p_A^{(j)}(a') & H(i,j) \\ 0 & p_B^{(i)}(b)p_B^{(j)}(b') \end{bmatrix} \\
&= \sum_{i+j=n} \begin{bmatrix} p_A^{(i)}(a)p_A^{(j)}(a') & H(i,j) + r_1^{(i)}(a)p_B^{(j)}(b') - p_A^{(i)}(a)r_2^{(j)}(b') \\ 0 & p_B^{(i)}(b)p_B^{(j)}(b') \end{bmatrix} \\
&= \sum_{i+j=n} \begin{bmatrix} p_A^{(i)}(a) & r_1^{(i)}(a) - r_2^{(i)}(b) + f^{(i)}(a) \\ 0 & p_B^{(i)}(b) \end{bmatrix}\begin{bmatrix} p_A^{(j)}(a') & r_1^{(j)}(a') - r_2^{(j)}(b') + f^{(j)}(m') \\ 0 & p_B^{(j)}(b') \end{bmatrix} \\
&= \sum_{i+j=n} \delta_i\Big(\begin{bmatrix} a & m \\ & b \end{bmatrix}\Big)\delta_j\Big(\begin{bmatrix} a' & m' \\ & b' \end{bmatrix}\Big),
\end{aligned}$$

where

$$H(i,j) = p_A^{(i)}(a)r_1^{(j)}(a') - r_2^{(i)}(b)p_B^{(j)}(b') + p_A^{(i)}(a)f^{(j)}(m') + f^{(i)}(m)p_A^{(j)}(a')$$

for all $a, a' \in A, b, b' \in B, m, m' \in M$. Therefore, a sequence $\delta = \{\delta_n\}_{n\in\mathcal{N}}$ of linear mappings $\delta_n : \mathcal{T} \to \mathcal{T}$ is a higher derivation. □

We now give the main result of the paper.

Theorem 3.2. *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a triangular algebra. If the following statements hold true:*

(i) $\mathcal{Z}(\mathcal{A}) = \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T}))$ and $\mathcal{Z}(\mathcal{B}) = \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T}))$;

(ii) M is strong faithful;

(ii) For any $m_0 \in M$, we have that $Am_0 = 0$ if and only if $m_0B = 0$.

Let a sequence $\mathfrak{L} = \{L_n\}_{n\in\mathcal{N}}$ of linear mappings $L_n : \mathcal{T} \to \mathcal{T}$ is a Lie higher derivation. Then there exists a higher derivation $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ of linear mapping $\delta_n : \mathcal{T} \to \mathcal{T}$, extreme Lie higher derivation $\Sigma = \{\sigma_n\}_{n\in\mathcal{N}}$ of linear mapping $\sigma_n : \mathcal{T} \to \mathcal{T}$ and an $\mathcal{R}$-linear mapping $\tau_n : \mathcal{T} \to C_1(\mathcal{T})$ vanishing on commutators $[x, y]$ such that

$$L_n = \delta_n + \sigma_n + \tau_n$$

for all $n \in \mathcal{N}$.

Proof. Suppose that a sequence $\mathfrak{L} = \{L_n\}_{n\in\mathcal{N}}$ of linear mappings $L_n : \mathcal{T} \to \mathcal{T}$ is a Lie higher derivation. Writer L_n as

$$L_n\left(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\right) = \begin{bmatrix} g_A^{(n)}(a) + h_B^{(n)}(b) + k_1^{(n)}(m) & r_1^{(n)}(a) - r_2^{(n)}(b) + f^{(n)}(m) \\ 0 & g_B^{(n)}(b) + h_A^{(n)}(a) + k_2^{(n)}(m) \end{bmatrix} \tag{3.7}$$

for all $a \in A, b \in B$ and $m \in M$, where $L_0 = Id_{\mathcal{T}}$ is an identity map on algebra $\mathcal{T}$.

Since

$$\begin{aligned} 0 &= L_n\left(\left[\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right]\right) \\ &= \sum_{i+j=n} \left[L_i\left(\begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}\right), L_j\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}\right)\right] = \sum_{i+j=n} \left[\begin{bmatrix} h_B^{(i)}(b) & -r_2^{(i)}(b) \\ 0 & g_B^{(i)}(b) \end{bmatrix}, \begin{bmatrix} g_A^{(j)}(a) & r_1^{(j)}(a) \\ 0 & h_A^{(j)}(a) \end{bmatrix}\right] \\ &= \sum_{i+j=n} \begin{bmatrix} h_B^{(i)}(b)g_A^{(j)}(a) - g_A^{(j)}(a)h_B^{(i)}(b) & h_B^{(i)}(b)r_1^{(j)}(a) - r_2^{(i)}(b)h_A^{(j)}(a) + g_A^{(j)}(a)r_2^{(i)}(b) - r_1^{(j)}(a)g_B^{(i)}(b) \\ 0 & g_B^{(i)}(b)h_A^{(j)}(a) - h_A^{(j)}(a)g_B^{(i)}(b) \end{bmatrix} \end{aligned}$$

for all $a \in A, b \in B$. Thus we arrive at

$$\begin{aligned} &\sum_{i+j=n} (h_B^{(i)}(b)g_A^{(j)}(a) - g_A^{(j)}(a)h_B^{(i)}(b)) = 0, \\ &\sum_{i+j=n} (g_B^{(i)}(b)h_A^{(j)}(a) - h_A^{(j)}(a)g_B^{(i)}(b)) = 0, \\ &\text{and} \\ &\sum_{i+j=n} (h_B^{(i)}(b)r_1^{(j)}(a) - r_2^{(i)}(b)h_A^{(j)}(a) + g_A^{(j)}(a)r_2^{(i)}(b) - r_1^{(j)}(a)g_B^{(i)}(b)) = 0 \end{aligned} \tag{3.8}$$

for all $a \in A, b \in B$. Let's check

$$h_B^{(n)}(b) \in C(A) \text{ and } h_A^{(n)}(a) \in C(B) \tag{3.9}$$

by complete induction on n for all $a \in A, b \in B$. When $n = 1$, due to the proof of [16, Theorem 3.2], it is clear that

$$h_B^{(1)}(b) \in C(A) \text{ and } h_A^{(1)}(a) \in C(B)$$

for all $a \in A, b \in B$.

Suppose that

$$h_B^{(i)}(b) \in C(A) \text{ and } h_A^{(j)}(a) \in C(B)$$

for all $1 \le i, j \le n-1$. With the help of (3.8), we therefor get

$$0 = \sum_{i+j=n} (h_B^{(i)}(b)g_A^{(j)}(a) - g_A^{(j)}(a)h_B^{(i)}(b)) = h_B^{(n)}(b)a - ah_B^{(n)}(b)$$

and

$$0 = \sum_{i+j=n} (g_B^{(i)}(b)h_A^{(j)}(a) - h_A^{(j)}(a)g_B^{(i)}(b)) = bh_A^{(j)}(a) - h_A^{(j)}(a)b$$

for all $a \in A, b \in B$. And then $h_B^{(n)}(b) \in \mathcal{Z}(A)$ and $h_A^{(n)}(a) \in \mathcal{Z}(B)$ for all $a \in A, b \in B$. Taking into accounts (3.9), the third equation in (3.8) can be rewritten as

$$\sum_{i+j=n} ((g_A^{(j)}(a) - \varphi^{-1}(h_A^{(j)}(a)))r_2^{(i)}(b) - r_1^{(j)}(a)(g_B^{(i)}(b) - \varphi(h_B^{(i)}(b)))) = 0 \tag{3.10}$$

for all $a \in A, b \in B$. Let us define $p_A^{(j)}(a) = g_A^{(j)}(a) - \varphi^{-1}(h_A^{(j)}(a))$ and $p_B^{(i)}(b) = g_B^{(i)}(b) - \varphi(h_B^{(i)}(b))$ for all $1 \leq i, j \leq n$. Thus the equation (3.10) can be rewritten as

$$\sum_{i+j=n} (p_A^{(i)}(a) r_2^{(j)}(b) - r_1^{(i)}(a) p_B^{(j)}(b)) = 0 \tag{3.11}$$

On the one hand, in view of (3.7), we have

$$\begin{aligned}
L_n\left(\left[\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right]\right) &= \sum_{i+j=n}\left[L_i\left(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\right), L_i\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right)\right] \\
&= \sum_{i+j=n}\left[\begin{bmatrix} k_1^{(i)}(m) & f^{(i)}(m) \\ 0 & k_2^{(i)}(m) \end{bmatrix}, \begin{bmatrix} g_A^{(j)}(a) + h_B^{(j)}(b) & r_1^{(j)}(a) - r_2^{(j)}(b) \\ 0 & g_B^{(j)}(b) + h_A^{(j)}(a) \end{bmatrix}\right] \\
&= \sum_{i+j=n}\begin{bmatrix} [k_1^{(i)}(m), g_A^{(j)}(a) + h_B^{(j)}(b)] & U \\ 0 & [k_2^{(i)}(m), g_B^{(j)}(b) + h_A^{(j)}(a)] \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
U = &(k_1^{(i)}(m)(r_1^{(j)}(a) - r_2^{(j)}(b)) + f^{(i)}(m)(g_B^{(j)}(b) + h_A^{(j)}(a)) \\
&- (g_A^{(j)}(a) + h_B^{(j)}(b)) f^{(i)}(m) - (r_1^{(j)}(a) - r_2^{(j)}(b)) k_2^{(i)}(m))
\end{aligned}$$

for all $a \in A, b \in B$ and $m \in M$.

On the other hand,

$$L_n\left(\left[\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right]\right) = L_n\left(\begin{bmatrix} 0 & mb - am \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} k_1^{(n)}(mb - am) & f^{(n)}(mb - am) \\ 0 & k_2^{(n)}(mb - am) \end{bmatrix}$$

for all $a \in A, b \in B$ and $m \in M$.

On comparing the above two relations, we see that

$$k_1^{(n)}(mb - am) = \sum_{i+j=n} [k_1^{(i)}(m), g_A^{(j)}(a) + h_B^{(j)}(b)],\ k_2^{(n)}(mb - am) = \sum_{i+j=n} [k_2^{(i)}(m), g_B^{(j)}(b) + h_A^{(j)}(a)]$$

and

$$\begin{aligned}
f^{(n)}(mb - am) = \sum_{i+j=n} &((k_1^{(i)}(m)(r_1^{(j)}(a) - r_2^{(j)}(b)) + f^{(i)}(m)(g_B^{(j)}(b) + h_A^{(j)}(a)) \\
&- (g_A^{(j)}(a) + h_B^{(j)}(b)) f^{(i)}(m) - (r_1^{(j)}(a) - r_2^{(j)}(b)) k_2^{(i)}(m)))
\end{aligned} \tag{3.12}$$

for all $a \in A, b \in B$ and $m \in M$. Let us set $a = 0$ in (3.12), we have

$$\begin{aligned}
&k_1^{(n)}(mb) = \sum_{i+j=n} (k_1^{(i)}(m) h_B^{(j)}(b) - h_B^{(j)}(b) k_1^{(i)}(m)), \\
&k_2^{(n)}(mb) = \sum_{i+j=n} (k_2^{(i)}(m) g_B^{(j)}(b) - g_B^{(j)}(b) k_2^{(i)}(m)), \\
&\text{and} \\
&f^{(n)}(mb) = \sum_{i+j=n} (f^{(i)}(m) g_B^{(j)}(b) - k_1^{(i)}(m) r_2^{(j)}(b) - h_B^{(j)}(b) f^{(i)}(m) + r_2^{(j)}(b) k_2^{(i)}(m))
\end{aligned} \tag{3.13}$$

for all $b \in B, m \in M$.

By an analogous manner, taking $b = 0$ in (3.12), we arrive at

$$\begin{aligned} &k_1^{(n)}(am) = \sum_{i+j=n}(g_A^{(j)}(a)k_1^{(i)}(m) - k_1^{(i)}(m)g_A^{(j)}(a)),\\ &k_2^{(n)}(am) = \sum_{i+j=n}(h_A^{(j)}(a)k_2^{(i)}(m) - k_2^{(i)}(m)h_A^{(j)}(a)),\\ &\text{and}\\ &f^{(n)}(am) = \sum_{i+j=n}(g_A^{(j)}(a)f^{(i)}(m) + r_1^{(j)}(a)k_2^{(i)}(m) - k_1^{(i)}(m)r_1^{(j)}(a) - f^{(i)}(m)h_A^{(j)}(a)) \end{aligned} \tag{3.14}$$

for all $b \in B, m \in M$.

Note that

$$\begin{aligned} 0 &= L_n\Big(\Big[\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix}\Big]\Big)\\ &= \sum_{i+j=n}\Big[L_i\Big(\begin{bmatrix} 0 & m \\ 0 & 0 \end{bmatrix}\Big), L_j\Big(\begin{bmatrix} 0 & m' \\ 0 & 0 \end{bmatrix}\Big)\Big]\\ &= \sum_{i+j=n}\Big[\begin{bmatrix} k_1^{(i)}(m) & f^{(i)}(m) \\ 0 & k_2^{(i)}(m) \end{bmatrix}, \begin{bmatrix} k_1^{(j)}(m') & f^{(j)}(m') \\ 0 & k_2^{(j)}(m') \end{bmatrix}\Big]\\ &= \sum_{i+j=n}\begin{bmatrix} [k_1^{(i)}(m), k_1^{(j)}(m')] & k_1^{(i)}(m)f^{(j)}(m') + f^{(i)}(m)k_2^{(j)}(m') - k_1^{(j)}(m')f^{(i)}(m) - f^{(j)}(m')k_2^{(i)}(m) \\ 0 & [k_2^{(i)}(m), k_2^{(j)}(m')] \end{bmatrix}, \end{aligned}$$

It follows that

$$\begin{aligned} &\sum_{i+j=n}(k_1^{(i)}(m)f^{(j)}(m') + f^{(i)}(m)k_2^{(j)}(m') - k_1^{(j)}(m')f^{(i)}(m) - f^{(j)}(m')k_2^{(i)}(m))\\ &= \sum_{i+j=n,\ j\neq 0 \text{ or } i\neq 0}(k_1^{(i)}(m)f^{(j)}(m') + f^{(i)}(m)k_2^{(j)}(m') - k_1^{(j)}(m')f^{(i)}(m) - f^{(j)}(m')k_2^{(i)}(m))\\ &\quad + k_1^{(n)}(m)m' - m'k_2^{(n)}(m) - k_1^{(n)}(m')m + mk_2^{(n)}(m')\\ &= 0 \end{aligned} \tag{3.15}$$

for all $m, m' \in M$.

By the proof of [16, Theorem 3.2], we know that

$$k_1^{(1)}(mb) = 0, k_2^{(1)}(am) = 0, \text{ and } k_1^{(1)}(m) \oplus k_2^{(1)}(m) \in C_1(\mathcal{T})$$

whenever $n = 1$ for all $a \in A, m \in M, b \in B$. Suppose that

$$k_1^{(x_1)}(mb) = 0, k_2^{(x_2)}(am) = 0, \text{ and } k_1^{(x_3)}(m) \oplus k_2^{(x_3)}(m) \in C_1(\mathcal{T})$$

whenever $1 \leq x_1, x_2, x_3 \leq n-1$ for all $a \in A, m \in M, b \in B$. On comparing the equation (3.13) and (3.14), and using complete mathematical induction, we therefore arrive at

$$k_1^{(n)}(mb) = \sum_{i+j=n}(k_1^{(i)}(m)h_B^{(j)}(b) - h_B^{(j)}(b)k_1^{(i)}(m)) = 0 \text{ and } k_2^{(n)}(am) = \sum_{i+j=n}(g_B^{(j)}(b)k_2^{(i)}(m) - k_2^{(i)}(m)g_B^{(j)}(b)) = 0$$

for all $a \in A, m \in M, b \in B$. Hence

$$k_1^{(n)}(am) = 0 = k_2^{(n)}(mb) \tag{3.16}$$

for all $a \in A, m \in M, b \in B$. At the same time, taking into (3.15) and inductive hypothesis $k_1^{(x_3)}(m) \oplus k_2^{(x_3)}(m) \in C_1(\mathcal{T})$, we can obtain

$$k_1^{(n)}(m)m' - m'k_2^{(n)}(m) - k_1^{(n)}(m')m + mk_2^{(n)}(m') = 0 \tag{3.17}$$

for all $m, m' \in M$. Substituting m' with $am'b$ in (3.17) and using (3.16), we obtain

$$k_1^{(n)}(m)am'b - am'bk_2^{(n)}(m) = 0 \tag{3.18}$$

Replacing a by aa' in (3.18) and then subtracting the left multiplication of (3.18) by a, we arrive at

$$(k_1^{(n)}(m)a' - a'k_1^{(n)}(m))AMB = 0$$

for all $a' \in A, m \in M$. Since M is strong faithful, we get

$$k_1^{(n)}(m)a' = a'k_1^{(n)}(m)$$

i.e., $k_1^{(n)}(m) \in C(A)$ for all $a' \in A, m \in M$. With the help of (3.18), we have

$$am'b(\varphi(k_1^{(n)}(m)) - k_2^{(n)}(m)) = 0$$

for all $a \in A, b \in B, m' \in M$. Thus

$$AMB(\varphi(k_1^{(n)}(m)) - k_2^{(n)}(m)) = 0$$

for all $m \in M$. Since M is strong faithful, we get

$$\varphi(k_1^{(n)}(m)) - k_2^{(n)}(m) = 0$$

for all $m \in M$. Hence

$$k_1^{(n)}(m) \oplus k_2^{(n)}(m) \in C_1(\mathcal{T}) \tag{3.19}$$

for all $m \in M$. In view of the relation (3.13), (3.14), (3.19) and the relations $h_B^{(n)}(b) \in C(A)$ and $h_A^{(n)}(a) \in C(B)$, we have

$$\begin{aligned} f^{(n)}(mb) &= \sum_{i+j=n} (f^{(i)}(m)g_B^{(j)}(b) - k_1^{(i)}(m)r_2^{(j)}(b) - h_B^{(j)}(b)f^{(i)}(m) + r_2^{(j)}(b)k_2^{(i)}(m)) \\ &= \sum_{i+j=n} (f^{(i)}(m)(g_B^{(j)}(b) - \varphi(h_B^{(j)}(b))) - (k_1^{(i)}(m) - \varphi^{-1}(k_2^{(i)}(m)))r_2^{(j)}(b)) \\ &= \sum_{i+j=n} f^{(i)}(m)(g_B^{(j)}(b) - \varphi(h_B^{(j)}(b))) \end{aligned}$$

and

$$\begin{aligned} f^{(n)}(am) &= \sum_{i+j=n} (g_A^{(j)}(a)f^{(i)}(m) + r_1^{(j)}(a)k_2^{(i)}(m) - k_1^{(i)}(m)r_1^{(j)}(a) - f^{(i)}(m)h_A^{(j)}(a)) \\ &= \sum_{i+j=n} ((g_A^{(j)}(a) - \varphi^{-1}(h_A^{(j)}(a)))f^{(i)}(m) + r_1^{(j)}(a)k_2^{(i)}(m) - k_1^{(i)}(m)r_1^{(j)}(a)) \\ &= \sum_{i+j=n} (g_A^{(j)}(a) - \varphi^{-1}(h_A^{(j)}(a)))f^{(i)}(m) \end{aligned}$$

for all $a \in A, b \in B, m \in M$. Hence we obtain

$$\begin{aligned} f^{(n)}(mb) &= \sum_{i+j=n} f^{(i)}(m)(g_B^{(j)}(b) - \varphi(h_B^{(j)}(b))) \\ f^{(n)}(am) &= \sum_{i+j=n} (g_A^{(j)}(a) - \varphi^{-1}(h_A^{(j)}(a)))f^{(i)}(m) \end{aligned} \tag{3.20}$$

for all $a \in A, b \in B, m \in M$.

On the basis of equation (3.20), we propose a claim.

Claim 1: A sequence $\mathcal{P}_{\mathcal{A}} = \{p_A^{(n)}\}_{n\in\mathcal{N}}$ of linear $p_A^{(n)} : A \to A$ and a sequence $\mathcal{P}_{\mathcal{B}} = \{p_B^{(n)}\}_{n\in\mathcal{N}}$ of linear $p_B^{(n)} : B \to B$ is a higher derivation on A and B respectively.

Indeed, we prove this assertion by full mathematical induction for n. For $n = 1$, with the help of the mapping "δ" in [16, Theorem 3.2], we known that $p_A^{(1)} : A \to A$ and $p_B^{(1)} : B \to B$ be a derivation on A and B respectively. On the basis of "n=1", we can prove this conclusion by using complete mathematical induction for n and combining with equation (3.6).

Let's check

$$r_1^{(n)}(aa') - \sum_{i+j=n} p_A^{(i)}(a)r_1^{(j)}(a') \in C_2(\mathcal{T}) \tag{3.21}$$

and

$$r_2^{(n)}(bb') - \sum_{i+j=n} r_2^{(i)}(b)p_B^{(j)}(b') \in C_2(\mathcal{T}) \tag{3.22}$$

for all $a, a' \in A$ and $b, b' \in B$ by complete induction on n.

Let's prove the relationship ((3.21)) by mathematical induction for n. When $n = 1$, due to the proof of [16, Theorem 3.2], it is clear that

$$r_1^{(1)}(aa') - ar_1^{(1)}(a') = r_1^{(1)}(aa') - ar_1^{(1)}(a') - p_A^{(1)}(a)r_1^{(0)}(a') \in C_2(A)$$

with the help of the definition of Lie higher derivation for all $a \in A, b \in B$.

Suppose that

$$r_1^{(k)}(aa') - \sum_{i+j=k} p_A^{(i)}(a)r_1^{(j)}(a') \in C_2(\mathcal{T})$$

for all $1 \leq k \leq n-1$, i.e.,

$$(r_1^{(k)}(aa') - \sum_{i+j=k} p_A^{(i)}(a)r_1^{(j)}(a'))B = 0 = A(r_1^{(k)}(aa') - \sum_{i+j=k} p_A^{(i)}(a)r_1^{(j)}(a')).$$

With the help of (3.11) and replacing a by aa', we have

$$\begin{aligned}
\sum_{i+j=n} r_1^{(i)}(aa')p_B^{(j)}(b) &= \sum_{i+j=n} p_A^{(i)}(aa')r_2^{(j)}(b) \\
&= \sum_{i+j=n} (\sum_{i_1+i_2=i} p_A^{(i_1)}(a)p_A^{(i_2)}(a'))r_2^{(j)}(b) \\
&= \sum_{i+j=n} \sum_{i_1+i_2=i} p_A^{(i_1)}(a)p_A^{(i_2)}(a')r_2^{(j)}(b) \\
&= \sum_{i_1+i_2+j=n} p_A^{(i_1)}(a)p_A^{(i_2)}(a')r_2^{(j)}(b) \\
&= \sum_{i_1+s=n} p_A^{(i_1)}(a)(\sum_{i_2+j=s} p_A^{(i_2)}(a')r_2^{(j)}(b)) \\
&= \sum_{i_1+s=n} p_A^{(i_1)}(a)(\sum_{i_2+j=s} r_1^{(i_2)}(a')p_A^{(j)}(b)) \\
&= \sum_{i_1+i_2+j=n} p_A^{(i_1)}(a)r_1^{(i_2)}(a'))p_A^{(j)}(b) \\
&= \sum_{i+j=n} (\sum_{i_1+i_2=i} p_A^{(i_1)}(a)r_1^{(i_2)}(a'))p_A^{(j)}(b)
\end{aligned}$$

for all $aa' \in A$ and $b \in B$. Furthermore, we have

$$\begin{aligned} 0 &= \sum_{i+j=n}(r_1^{(i)}(aa') - \sum_{i_1+i_2=i} p_A^{(i_1)}(a)r_1^{(i_2)}(a'))p_B^{(j)}(b) \\ &= \sum_{i+j=n,\ j\neq 0}(r_1^{(i)}(aa') - \sum_{i_1+i_2=i} p_A^{(i_1)}(a)r_1^{(i_2)}(a'))p_B^{(j)}(b) \\ &\quad + (r_1^{(n)}(aa') - \sum_{i_1+i_2=n} p_A^{(i_1)}(a)r_1^{(i_2)}(a'))b \end{aligned}$$

for all $aa' \in A$ and $b \in B$. Taking into accounts the above equation and inductive hypothesis, we have

$$(r_1^{(n)}(aa') - \sum_{i+j=n} p_A^{(i)}(a)r_1^{(j)}(a'))B = 0$$

for all $a, a' \in A$. The condition (iii) implies that

$$A(r_1^{(n)}(aa') - \sum_{i+j=n} p_A^{(i)}(a)r_1^{(j)}(a')) = 0$$

for all $aa' \in A$. In view of the extreme centers $C_2(\mathcal{A})$ we see that

$$\begin{bmatrix} 0 & r_1^{(n)}(aa') - \sum_{i+j=n} p_A^{(i)}(a)r_1^{(j)}(a') \\ 0 & 0 \end{bmatrix} \in C_2(\mathcal{T})$$

for all $a, a' \in A$. Similarly, with the help of [16, Theorem 3.2] and (3.11), We can conclude that (3.22) is true, i.e.,

$$r_2^{(1)}(bb') - r_2^{(1)}(b)b' \in C_2(A)$$

for all $b, b' \in B$. In other words,

$$\begin{bmatrix} 0 & r_2^{(n)}(bb') - \sum_{i+j=n} r_2^{(i)}(b')p_B^{(j)}(b) \\ 0 & 0 \end{bmatrix} \in C_2(\mathcal{T})$$

for all $b, b' \in B$.

We define a mapping $\tau_n : \mathcal{T} \to \mathcal{T}$ in the following way:

$$\tau_n\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big) = \begin{bmatrix} \varphi^{-1}(h_A^{(n)}(a)) + h_B^{(n)}(b) + k_1^{(n)}(m) & 0 \\ 0 & \varphi(h_B^{(n)}(b)) + h_A^{(n)}(a) + k_2^{(n)}(m) \end{bmatrix}$$

for all $a \in A, b \in B, m \in M$. We note that $\tau_n : \mathcal{T} \to C_1(\mathcal{T})$ for arbitrary $n \in \mathcal{N}$.

In order to obtain the conclusion of this Theorem, we divide its proof into two different cases.

Case 1: Suppose first that

$$r_1^{(n)}(aa') - \sum_{i+j=n} p_A^{(i)}(a)r_1^{(j)}(a') - r_2^{(n)}(bb') + \sum_{i+j=n} r_2^{(i)}(b)p_B^{(j)}(b') = 0$$

for all $a, a\prime \in A, b, b' \in B$. This implies that

$$r_1^{(n)}(aa') - \sum_{i+j=n} p_A^{(i)}(a)r_1^{(j)}(a') = 0 \text{ and } r_2^{(n)}(bb') - \sum_{i+j=n} r_2^{(i)}(b)p_B^{(j)}(b') = 0 \tag{3.23}$$

for all $a, a\prime \in A, b, b' \in B$.

We define two mappings here:

(1) Defining the first mapping $\delta_n : \mathcal{T} \to \mathcal{T}$ as defined as

$$\delta_n\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big) = \begin{bmatrix} g_A^{(n)}(a) - \varphi^{-1}(h_A^{(n)}(a)) & r_1^{(n)}(a) - r_2^{(n)}(b) + f_1^{(n)}(m) \\ 0 & g_B^{(n)}(b) - \varphi(h_B^{(n)}(b)) \end{bmatrix} = \begin{bmatrix} p_A^{(n)}(a) & r_1^{(n)}(a) - r_2^{(n)}(b) + f_1^{(n)}(m) \\ 0 & p_B^{(n)}(b) \end{bmatrix}$$

for all $a \in A, b \in B, m \in M$. Using (3.20) − (3.22), we get from Theorem 3.1 that a sequence $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ of linear mappings $\delta_n : \mathcal{T} \to \mathcal{T}$ is a higher derivation; therefore, we obtain that a sequence $\mathcal{P}_{\mathcal{A}} = \{p_A^{(n)} = g_A^{(n)}(a) - \varphi^{-1}(h_A^{(n)}(a))\}_{n\in\mathcal{N}}$ of linear mapping $p_A^{(n)} : A \to A$ and a sequence $\mathcal{P}_{\mathcal{B}} = \{p_B^{(n)}(b) = g_B^{(n)}(b) - \varphi(h_B^{(n)}(b))\}_{n\in\mathcal{N}}$ of linear mapping $p_B^{(n)} : B \to B$ be a higher derivation on A and B respectively.

(2) Let's define the second mapping $\sigma_n : \mathcal{T} \to \mathcal{T}$ as follows

$$\sigma_n\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for all $a \in A, b \in B, m \in M$.

It is clear that $L_n = \delta_n + \sigma_n + \tau_n$. Using (3.19) and (3.20), we obtain from Theorem 3.1 that $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ is a higher derivation on $\mathcal{T}$. We easily check that $\tau_n([x, y]) = 0$ for all $x, y \in \mathcal{T}$.

Case 2: Suppose next that

$$r_1^{(n)}(a_0a_0') - \sum_{i+j=n} p_A^{(i)}(a_0)r_1^{(j)}(a_0') - r_2^{(n)}(b_0b_0') + \sum_{i+j=n} r_2^{(i)}(b_0)p_B^{(j)}(b_0') \neq 0 \tag{3.24}$$

for some $a_0, a_0' \in A, b_0, b_0' \in B$. We define two mappings here:

(1) Defining the second mapping $\delta_n : \mathcal{T} \to \mathcal{T}$ as defined as

$$\delta_n\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big) = \begin{bmatrix} p_A^{(n)}(a) & f_1^{(n)}(m) \\ 0 & p_B^{(n)}(b) \end{bmatrix}$$

for all $a \in A, b \in B, m \in M$. Using (3.20), we get from Theorem 3.1 that a sequence $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ of linear mappings $\delta_n : \mathcal{T} \to \mathcal{T}$ is a higher derivation; therefore, we obtain that a sequence $\mathcal{P}_{\mathcal{A}} = \{p_A^{(n)} = g_A^{(n)}(a) - \varphi^{-1}(h_A^{(n)}(a))\}_{n\in\mathcal{N}}$ of linear mapping $p_A^{(n)} : A \to A$ and a sequence $\mathcal{P}_{\mathcal{B}} = \{p_B^{(n)}(b) = g_B^{(n)}(b) - \varphi(h_B^{(n)}(b))\}_{n\in\mathcal{N}}$ of linear mapping $p_B^{(n)} : B \to B$ be a higher derivation on A and B respectively.

(2) Let's define the third mapping $\sigma_n : \mathcal{T} \to \mathcal{T}$ as follows

$$\sigma_n\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\Big) = \begin{bmatrix} 0 & r_1^{(n)}(a) - r_2^{(n)}(b) \\ 0 & 0 \end{bmatrix}$$

for all $a \in A, b \in B, m \in M$.

It is clear that $L_n = \delta_n + \sigma_n + \tau_n$. Using (3.20) we obtain from Theorem 3.1 that δ_n is a higher derivation on $\mathcal{T}$.

We propose the second claim.

Claim 2: With the notations as above, we have that a sequence $\Sigma = \{\sigma_n\}_{n\in\mathcal{N}}$ of the mappings $\sigma_n : \mathcal{T} \to \mathcal{T}$ and τ_n satisfy the following relations:

(a) σ_n is a nonzero mapping on $\mathcal{T}$,

(b) $\sigma_n(xy) - \sum_{i+j=n}(\delta_i(x)\sigma_j(y) + \sigma_i(x)\delta_j(y)) \in C_2(\mathcal{T})$,

(c) $\sigma_n([x,y]) = \sum_{i+j=n}([\delta_i(x),\sigma_j(y)] + [\sigma_i(x),\delta_j(y)])$,

(d) $\tau_n([x,y]) = 0$

for all $x, y \in \mathcal{T}$.

Indeed, we first prove conclusion (*b*).

According to (3.21) and (3.22), there exists $m_0, m_0' \in M$ satisfying the following relation

$$\begin{aligned} r_1^{(n)}(aa') &= \sum_{i+j=n} p_A^{(i)}(a)r_1^{(j)}(a') \in C_2(\mathcal{T}) + m_0 \ \text{ and} \\ r_2^{(n)}(bb') &= \sum_{i+j=n} r_2^{(i)}(b)p_B^{(j)}(b') \in C_2(\mathcal{T}) + m_0' \end{aligned} \tag{3.25}$$

for all $a, a' \in A$ and $b, b' \in B$ and some $m_0, m_0' \in M$.

For arbitrary $x = a + m + b$ and $y = a' + m' + b'$, using (3.11) and (3.25) we have

$$\begin{aligned} \sigma_n(xy) &= \sigma_n\Big(\begin{bmatrix} a & m \\ 0 & b \end{bmatrix}\begin{bmatrix} a' & m' \\ 0 & b' \end{bmatrix}\Big) \\ &= \sigma_n\Big(\begin{bmatrix} aa' & am' + mb' \\ 0 & bb' \end{bmatrix}\Big) = \begin{bmatrix} 0 & r_1^{(n)}(aa') - r_2^{(n)}(bb') \\ 0 & 0 \end{bmatrix} \\ &= \sum_{i+j=n}\begin{bmatrix} 0 & p_A^{(i)}(a)r_1^{(j)}(a') - r_2^{(i)}(b)p_B^{(j)}(b') \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & m_0 - m_0' \\ 0 & 0 \end{bmatrix} \\ &= \sum_{i+j=n}\begin{bmatrix} 0 & p_A^{(i)}(a)r_1^{(j)}(a') - p_A^{(i)}(a)r_2^{(j)}(b)) + r_1^{(i)}(a)p_B^{(j)}(b) - r_2^{(i)}(b)p_B^{(j)}(b') \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & m_0 - m_0' \\ 0 & 0 \end{bmatrix} \\ &= \sum_{i+j=n}\begin{bmatrix} p_A^{(i)}(a) & r_1^{(i)}(a) - r_2^{(i)}(b) \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & r_1^{(j)}(a') + r_2^{(j)}(b')) \\ 0 & p_B^{(j)}(b') \end{bmatrix} + \begin{bmatrix} 0 & m_0 - m_0' \\ 0 & 0 \end{bmatrix} \\ &= \sum_{i+j=n}(\delta_i(x)\sigma_j(y) + \sigma_i(x)\delta_j(y)) + \begin{bmatrix} 0 & m_0 - m_0' \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Therefore, we have

$$\sigma_n(xy) - \sum_{i+j=n}(\delta_i(x)\sigma_j(y) + \sigma_i(x)\delta_j(y)) \in C_2(\mathcal{T})$$

for all $x, y \in \mathcal{T}$.

In view of (3.23), and the last relation we see that

$$\begin{aligned} \sigma_n\Big(\begin{bmatrix} a_0 & m_0 \\ 0 & b_0 \end{bmatrix}\begin{bmatrix} a_0' & m_0' \\ 0 & b_0' \end{bmatrix}\Big) - \sum_{i+j=n}\Big(\delta_i\Big(\begin{bmatrix} a_0 & m_0 \\ 0 & b_0 \end{bmatrix}\Big)\sigma_j\Big(\begin{bmatrix} a_0' & m_0' \\ 0 & b_0' \end{bmatrix}\Big) \\ - \sigma_i\Big(\begin{bmatrix} a_0 & m_0 \\ 0 & b_0 \end{bmatrix}\Big)\delta_j\Big(\begin{bmatrix} a_0' & m_0' \\ 0 & b_0' \end{bmatrix}\Big)\Big) \neq 0. \end{aligned}$$

We see that σ_n is not a nonzero mapping.

In order to prove conclusions (*c*) and (*d*), set

$$\mu_n(x,y) = \sigma_n([x,y]) - \sum_{i+j=n}([\delta_i(x),\sigma_j(y)] + [\sigma_i(x),\delta_j(y)])$$

for all $x, y \in \mathcal{T}$. Note $\mu_n(x, y) \in Z_2(\mathcal{T})$. Since $L_n = \delta_n + \sigma_n + \tau_n$ is a Lie higher derivation we have

$$\begin{aligned} L_n([x, y]) &= \sum_{i+j=n} [L_i(x), L_i(y)] \\ &= \sum_{i+j=n} [\delta_i(x) + \sigma_i(x) + \tau_i(x), \delta_j(y) + \sigma_j(y) + \tau_j(y)] \\ &= \sum_{i+j=n} [\delta_i(x) + \sigma_i(x), \delta_j(y) + \sigma_j(y)] \\ &= \sum_{i+j=n} ([\delta_i(x), \delta_j(y)] + [\delta_i(x), \sigma_j(y)] + [\sigma_i(x), \delta_j(y)]) \\ &= L_n([x, y]) - \sigma_n([x, y]) + \mu_n(x, y) \end{aligned}$$

for all $x, y \in \mathcal{T}$. This implies that

$$\sigma_n([x, y]) + \mu_n(x, y) = 0$$

for all $x, y \in \mathcal{T}$. Since $C_1(\mathcal{T}) \bigcap C_2(\mathcal{T}) = 0$, we get that

$$\tau_n([x, y]) = 0 = \mu_n(x, y)$$

for all $x, y \in \mathcal{T}$. The proof of the result is now complete. □

Remark 3.3. *When $n = 1$, a sequence $\Sigma = \{\sigma_n\}_{n\in\mathcal{N}}$ of linear mapping $\sigma_n : \mathcal{T} \to \mathcal{T}$ will degenerate into a extreme Lie derivation introduced by Wang [16, Definition 2.2].*

As a consequence we have the following results by Wang(see [16, Theorem 3.2]) and Cheung(see [5, Theorem 11]).

Corollary 3.4. *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a triangular algebra. If the following statements hold true:*

(i) $C(e\mathcal{A}e) = C(\mathcal{A})e$ and $C(f\mathcal{A}f) = C(\mathcal{A})f$
(ii) M is strong faithful;
(ii) For any $m_0 \in M$, we have that $Am_0 = 0$ if and only if $m_0B = 0$.

Let a linear mappings $L : \mathcal{T} \to \mathcal{T}$ is a Lie derivation. Then there exists a derivation δ, extreme Lie derivation σ and an $\mathcal{R}$-linear mapping $\tau : \mathcal{T} \to C(\mathcal{T})$ vanishing on commutators $[x, y]$ such that

$$L = \delta + \sigma + \tau$$

for all $x, y \in \mathcal{T}$.

Corollary 3.5. *Let $\mathcal{T} = \left[\begin{smallmatrix} A & M \\ O & B \end{smallmatrix}\right]$ be a triangular algebra. If the following statements hold true:*

(i) $\mathcal{Z}(\mathcal{A}) = \pi_{\mathcal{A}}(\mathcal{Z}(\mathcal{T}))$ and $\mathcal{Z}(\mathcal{B}) = \pi_{\mathcal{B}}(\mathcal{Z}(\mathcal{T}))$;
(ii) M is faithful;
(ii) Both A and B are unital.

Let a linear mappings $L : \mathcal{T} \to \mathcal{T}$ is a Lie derivation. Then there exists a derivation δ and an $\mathcal{R}$-linear mapping $\tau : \mathcal{T} \to C(\mathcal{T})$ vanishing on commutators $[x, y]$ such that

$$L = \delta + \tau$$

for all $x, y \in \mathcal{T}$.

In addition to the above corollaries, Theorem 3.2 has important applications in other algebras, such as upper triangular matrix algebras $T_n(\mathcal{A})(n \geq 2)$ defined on faithful algebras $\mathcal{A}$ (see [16, Theorem 4.1] for definition). With the help of [16, Theorem 4.1], $T_n(\mathcal{A})$ can be viewed as the triangular algebra

$$\begin{bmatrix} \mathcal{A} & \mathcal{A}^{n-1} \\ 0 & T_{n-1}(\mathcal{A}) \end{bmatrix} := \begin{bmatrix} A & M \\ 0 & B \end{bmatrix}.$$

Through the medium of [16, Theorem 4.1], the upper triangular matrix algebra $T_n(\mathcal{A})(n \geq 2)$ coincides with the conditions of Theorem 3.2.

Corollary 3.6. *Let $T_m(\mathcal{A})$ be an upper triangular matrix algebras over a faithful alebra $\mathcal{A}$, where $m \geq 2$. Let a sequence $\mathfrak{L} = \{L_n\}_{n\in\mathcal{N}}$ of linear mappings $\delta_n : T_m(\mathcal{A}) \to T_m(\mathcal{A})$ is a Lie higher derivation. Then there exists a higher derivation $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ and an $\mathcal{R}$-linear mapping $\tau_n : T_m(\mathcal{A}) \to C(T_m(\mathcal{A}))$ vanishing on commutators $[x, y]$ such that*

$$L_n = \delta_n + \tau_n$$

for all $x, y \in T_m(\mathcal{A})$.

At the same instant, noting that the unital algebra and semi-prime algebras are both faithful algebras[16, Theorem 4.1], so the following corollary follows from Theorem 3.2 and Corollary 3.6:

Corollary 3.7. *Let $T_m(\mathcal{A})$ be an upper triangular matrix algebras over a semiprime algebra $\mathcal{A}$, where $m \geq 2$. Let a sequence $\mathfrak{L} = \{L_n\}_{n\in\mathcal{N}}$ of linear mappings $\delta_n : T_m(\mathcal{A}) \to T_m(\mathcal{A})$ is a Lie higher derivation. Then there exists a higher derivation $\Delta = \{\delta_n\}_{n\in\mathcal{N}}$ and an $\mathcal{R}$-linear mapping $\tau_n : T_m(\mathcal{A}) \to C(T_m(\mathcal{A}))$ vanishing on commutators $[x, y]$ such that*

$$L_n = \delta_n + \tau_n$$

for all $x, y \in T_m(\mathcal{A})$.

Proof. With the help of [16, 4.An application], the definition conditions of each semiprime algebra coincide with those of the faithful algebra. Therefore, Corollary 3.6 tells us that Corollary 3.7 is true □

In view of [19, Example 2.5], there are many classical examples of semi-prime algebras, such as that double affine Hecke algebras, graded Hecke algebras, rational Cherednik algebras, Iwasawa algebras, algebras of bounded linear operators, von Neumann algebras and so on. We apply the Corollary 3.7 to the above algebra, which leads to many conclusions similar to Corollary 3.7.

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