Filomat 38:3 (2024), 939–956
https://doi.org/10.2298/FIL2403939H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

The MPWG inverse of third-order F-square tensors based on the T-product

Mengyu He[a], Xiaoji Liu[b], Hongwei Jin[a,*]

[a] *School of Mathematics and Physics, Guangxi Minzu University, Nanning 530006, China*
[b] *School of Education, Guangxi Vocational Normal University, Nanning 530006, China*

Abstract. We define the T-MPWG inverse of third-order F-square tensors by using the T-core EP decomposition of tensors via the T-product. Then, we present some characterizations and properties of the T-MPWG inverse. Moreover, the Cayley-Hamilton theorem of the third-order tensors is extended to T-MPWG inverses. Examples are also given to illustrate these results.

1. Introduction

A tensor $\mathcal{A}$ can be regarded as a multidimensional array of data, which takes the form:

$$\mathcal{A} = (a_{i_1 i_2 \cdots i_N}) \in \mathbb{C}^{I_1 \times I_2 \times \cdots \times I_N}.$$

The order of a tensor is the number of dimensions. For the given tensor $\mathcal{A}$ the order is N. In general, a vector is a first-order tensor, and a matrix is considered a second-order tensor. Products of tensors include Einstein products and T-products, etc.

Kilmer and Martin proposed the tensor T-product and used the discrete Fourier transform to transform the tensor multiplication into the matrix multiplication for calculation in [6]. Jin, Bai, Benitez and Liu defined the Moore-penrose inverse of tensors and derived an application to linear models in [4]. Miao, Qi and Wei introduced T-Drazin inverse and its properties when an F-square tensor was not invertible with T-product in [8]. Zhang introduced the weak group inverse, core inverse and core-EP inverse of tensors based on the T-product in [17].

In [10], Wang, Liu and Jin defined the MP weak group inverse of a complex square matrix A with $\mathrm{Ind}(A) = k$, denoted as $A^{\dagger,WG}$. The MPWG inverse $A^{\dagger,WG}$ of A is the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying $XAX = X, AX = A^D C$ and $XA = A^\dagger A^D A^2$. Moreover, it was proved that

$$A^{\dagger,WG} = A^\dagger A^{\textcircled{w}} A$$

2020 *Mathematics Subject Classification*. 15A09, 15A24, 15A57
Keywords. T-MPWG inverse; T-product; T-core EP decomposition; Cayley-Hamilton theorem.
Received: 02 July 2023; Revised: 20 July 2023; Accepted: 02 August 2023
Communicated by Dijana Mosić
Research supported by the National Natural Science Foundation of China (No. 12061015), the Guangxi Natural Science Foundation (No. 2018GXNSFDA281023) and the Special Fund for Science and Technological Bases and Talents of Guangxi (No. GUIKE AD21220024).
* Corresponding author: Hongwei Jin
Email addresses: hmy04280611@163.com (Mengyu He), xiaojiliu72@126.com (Xiaoji Liu), jhw_math@126.com (Hongwei Jin)

where C is the weak core part of A with $C = AA^{ⓦ}A$. $A^{\dagger}$ and $A^{ⓦ}$ represent the Moore-Penrose inverse and weak group inverse of A respectively.

In [10], Wang, Chen and Yan gave the polynomial equations of the core-EP inverse matrix $A^{\textcircled{\dagger}}$ on complex field by using the classical Cayley-Hamliton theorem. Furthermore, some properties of the characteristic polynomials of $A^{\textcircled{\dagger}}$ were derived. Liu and Wang also gave the Cayley-Hamliton theorem of the weak group inverse $A^{ⓦ}$ on complex field by using the core-EP decomposition in [7].

The work is organized as follows. In section 2, we provide some preliminaries. We introduce basic definitions and properties of tensors firstly Then, we show the definitions of the T-Moore-Penrose inverse, T-core EP inverse and T-weak group inverse. In section 3, we defined the MPWG inverse of the third-order tensors based on T-product. Then, we prove that the MPWG inverse of an arbitrary tensor $\mathcal{A}$ exists and is unique by using the technique of discrete Fourier transform. Then, we give some properties of the T-MPWG inverse and some new representations by using the T-core EP decomposition. In section 4, we discuss the relationships between the T-MPWG inverse and other known generalized inverses of tensors. Furthermore, we present the limit expression of the MPWG inverse of the third-order tensors. Supplementary example is given to illustrate the relationships. In section 5, we extend the Cayley-Hamliton theorem of the third-order tensors to the T-MPWG inverse, and give some examples to illustrate.

2. Preliminaries

In this section, we mainly introduce the definitions, properties and operation rules of the third-order tensors based on the T-product.

Let $\mathcal{A} \in \mathbb{C}^{m\times n\times p}$ be a third-order tensor, we denote its frontal faces as $A^{(k)} \in \mathbb{C}^{m\times n}$, $k = 1, \cdots, p$. The operations bcirc, unfold and fold are defined as follows [6]:

$$\text{bcirc}(\mathcal{A}) := \begin{bmatrix} A^{(1)} & A^{(p)} & A^{(p-1)} & \cdots & A^{(2)} \\ A^{(2)} & A^{(1)} & A^{(p)} & \cdots & A^{(3)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ A^{(p)} & A^{(p-1)} & A^{(p-2)} & \cdots & A^{(1)} \end{bmatrix}, \quad \text{unfold}(\mathcal{A}) := \begin{bmatrix} A^{(1)} \\ A^{(2)} \\ \vdots \\ A^{(p)} \end{bmatrix},$$

and fold (unfold($\mathcal{A}$)) := $\mathcal{A}$, which means that fold is inverse operator of unfold. We can also define the corresponding operation $\text{bcirc}^{-1} : \mathbb{C}^{mp\times np} \to \mathbb{C}^{m\times n\times p}$, which is the inverse operator of bcirc, such that $\text{bcirc}^{-1}(\text{bcirc}(\mathcal{A})) = \mathcal{A}$.

On the basis of the above operators, the conjugate transpose of $\mathcal{A}$ is introduced in [8]. The conjugate transpose $\mathcal{A}^*$ is obtained by conjugate transposing each of the frontal slices and then reversing the order of transposed frontal slices 2 through n:

$$\mathcal{A}^* = \text{fold}\left(\begin{bmatrix} (A^{(1)})^* \\ (A^{(p)})^* \\ (A^{(p-1)})^* \\ \vdots \\ (A^{(2)})^* \end{bmatrix}\right).$$

Definition 2.1. *[8] Let $\mathcal{A} \in \mathbb{C}^{m\times n\times p}$ be the third-order tensor.*
(i) *The T-range space of $\mathcal{A}$:*

$$\mathcal{R}(\mathcal{A}) := \text{Ran}\left(\left(F_p^H \otimes I_n\right)\text{bcirc}\left(\mathcal{A}\right)\left(F_p \otimes I_n\right)\right)$$

where Ran *means the range space;*
(ii) *The T-null space of $\mathcal{A}$:*

$$\mathcal{N}(\mathcal{A}) := \text{Null}\left(\left(F_p^H \otimes I_n\right)\text{bcirc}\left(\mathcal{A}\right)\left(F_p \otimes I_n\right)\right)$$

where Null *represents the null space.*

The following definitions are introduced in [8]:

Definition 2.2. *[8] Let $\mathcal{A} \in \mathbb{C}^{m\times n\times p}$, $\mathcal{B} \in \mathbb{C}^{n\times s\times p}$ be two complex tensors. Then, the T-product $\mathcal{A} * \mathcal{B}$ is an $m \times s \times p$ complex tensor defined by*

$$\mathcal{A} * \mathcal{B} := \text{fold}\,(\text{bcirc}(\mathcal{A})\text{unfold}(\mathcal{B})).$$

Definition 2.3. *[8] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$. If there exists a tensor $\mathcal{X} \in \mathbb{C}^{n\times n\times p}$ such that*

$$(1)\ \mathcal{A}{*}\mathcal{X}{*}\mathcal{A} = \mathcal{A},\ (2)\ \mathcal{X}{*}\mathcal{A}{*}\mathcal{X} = \mathcal{X},\ (3)\ (\mathcal{A}{*}\mathcal{X})^* = \mathcal{A}{*}\mathcal{X},\ (4)\ (\mathcal{X}{*}\mathcal{A})^* = \mathcal{X}{*}\mathcal{A}.$$

Then $\mathcal{X}$ is called the Moore-Penrose inverse of the tensor $\mathcal{A}$ and is denoted by $\mathcal{A}^\dagger$.

The cyclic matrix can be transformed into diagonal shape by the discrete Fourier transform. For the cyclic matrix bcirc($\mathcal{A}$), in [1, 3], the authors used the discrete Fourier transform to transform it into diagonal shape: let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$, then

$$\text{bcirc}(\mathcal{A}) = \left(F_p^H \otimes I_n\right)\text{Diag}\left(A_1,\cdots,A_p\right)\left(F_p \otimes I_n\right),$$

where $A_i \in \mathbb{C}^{n\times n}$, $(i = 1, \cdots, p)$. On the basis of block diagonal shape, T-rank and T-index was introduced in [8]:

Definition 2.4. *[8] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$.*
(i) *Let* $\text{rank}_T(\mathcal{A})$ *be the rank of the tensor $\mathcal{A}$:*

$$\text{rank}_T(\mathcal{A}) = \text{rank}\,(\text{bcirc}\,(\mathcal{A})) = \textstyle\sum_{i=1}^p (\text{rank}\,(A_i)),$$

where $\text{rank}\,(A_i)$ *represents the rank of the matrix A_i, $i = 1, \ldots, p$.*
(ii) *Let* $\text{Ind}_T(\mathcal{A})$ *be the index of the tensor $\mathcal{A}$:*

$$\text{Ind}_T(\mathcal{A}) = \text{Ind}\,(\text{bcirc}\,(\mathcal{A})) = \max_{1\le i\le p} (\text{Ind}\,(A_i)),$$

where $\text{Ind}\,(A_i)$ *is the smallest positive integer satisfying* $\text{rank}(A_i^k) = \text{rank}(A_i^{k+1})$. *Obviously,* $\text{Ind}_T(\mathcal{A}) = 1 \Leftrightarrow \text{Ind}\,(A_i) = 1$ *for any $i = 1, \ldots, p$* $\Leftrightarrow \text{rank}\,(\text{bcirc}\,(\mathcal{A})) = \text{rank}\left(\text{bcirc}\,(\mathcal{A})^2\right)$.

Definition 2.5. *[8] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ and* $\text{Ind}_T(\mathcal{A}) = k$. *If there exists a tensor $\mathcal{X} \in \mathbb{C}^{n\times n\times p}$ such that*

$$(1^k)\ \mathcal{A}^{k+1}{*}\mathcal{X} = \mathcal{A}^k,\ (2)\ \mathcal{X}{*}\mathcal{A}{*}\mathcal{X} = \mathcal{X},\ (5)\ \mathcal{A}{*}\mathcal{X} = \mathcal{X}{*}\mathcal{A}.$$

Then $\mathcal{X}$ is called the Drazin inverse of the tensor $\mathcal{A}$, and is denoted by $\mathcal{A}^D$. In particular, when $k = 1$, $\mathcal{X}$ is called the group inverse of the tensor $\mathcal{A}$ and is denoted by $\mathcal{A}^\#$.

Definition 2.6. *[9] Let $\mathcal{A} \in \mathbb{C}^{m\times n\times p}$. Then*
(i) *the tensor $\mathcal{A}$ is called EP if $\mathcal{A}{*}\mathcal{A}^\dagger = \mathcal{A}^\dagger{*}\mathcal{A}$;*
(ii) *the tensor $\mathcal{A}$ is idempotent if $\mathcal{A}^2 = \mathcal{A}$;*
(iii) *the tensor $\mathcal{A}$ is tripotent if $\mathcal{A}^3 = \mathcal{A}$;*
(iv) *the tensor $\mathcal{A}$ is called Hermitian idempotent if $\mathcal{A}^2 = \mathcal{A} = \mathcal{A}^*$;*
(v) *the tensor $\mathcal{A}$ is unitary if $\mathcal{A}^*{*}\mathcal{A} = \mathcal{A}{*}\mathcal{A}^* = \mathcal{I}$.*

Lemma 2.7. *[9] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ and* $\text{Ind}_T(\mathcal{A}) = k$. *If there exists a tensor $\mathcal{X} \in \mathbb{C}^{n\times n\times p}$ satisfying*

$$(1)\ \mathcal{X}{*}\mathcal{A}{*}\mathcal{X} = \mathcal{X},\ (2)\ \mathcal{R}(\mathcal{A}^k) = \mathcal{R}(\mathcal{X}).$$

Then $\mathcal{X}$ is called the core-EP inverse of tensor $\mathcal{A}$, and it is denoted as $\mathcal{A}^{\oplus}$. It's also expressed as

$$\mathcal{X} = (\mathcal{A})^k{*}(\mathcal{A}^*)^k{*}\left((\mathcal{A}^*)^k{*}\mathcal{A}^{k+1}\right)^\dagger{*}(\mathcal{A}^*)^k.$$

Lemma 2.8. *[8] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ and* $\text{Ind}_T(\mathcal{A}) = k$. *If there exists a tensor $\mathcal{X} \in \mathbb{C}^{n\times n\times p}$ satisfying*

$$(1)\ \mathcal{A}{*}\mathcal{X}^2 = \mathcal{X},\quad (2)\ \mathcal{A}{*}\mathcal{X} = \mathcal{A}^{\oplus}{*}\mathcal{A}.$$

Then $\mathcal{X}$ is called the weak group inverse of $\mathcal{A}$ and is denoted as $\mathcal{A}^{ⓦ}$. It's also expressed as $\mathcal{X} = (\mathcal{A}^{\oplus})^2{*}\mathcal{A}$. In particular, when $\text{Ind}_T(\mathcal{A}) = 1$, $\mathcal{X} = \mathcal{A}^\#$.

3. T-MPWG inverse

In this section, we introduce the MPWG inverse of the third-order tensors based on the T-product, and give some characterizations and properties of it.

Lemma 3.1. *[11, 12] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$. Then the following statements about $\mathcal{A}^{\textcircled{w}}$ hold:*
(i) *$\mathcal{A}^{\textcircled{w}}$ is an outer inverse of $\mathcal{A}$, that is, $\mathcal{A}^{\textcircled{w}} * \mathcal{A} * \mathcal{A}^{\textcircled{w}} = \mathcal{A}^{\textcircled{w}}$,*
(ii) *$\mathcal{R}(\mathcal{A}^{\textcircled{w}}) = \mathcal{R}\left(\mathcal{A}^k\right)$,*
(iii) *$\mathcal{A}^{\textcircled{w}} * \mathcal{A}^k = \mathcal{A}^{k+1}$,*
(iv) *$\mathcal{A} * \mathcal{A}^{\textcircled{w}} = \mathcal{A}^k * \mathcal{B}$, for some tensor $\mathcal{B}$,*
(v) *$\mathcal{A}^{\textcircled{w}} = \mathcal{A}^k * \mathcal{Z}$, for some tensor $\mathcal{Z}$.*

Theorem 3.2. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ and $\mathrm{Ind}_T(\mathcal{A}) = k$. Then the following system of equations*

$$(1)\ \mathcal{X} * \mathcal{A} * \mathcal{X} = \mathcal{X}, \quad (2)\ \mathcal{A} * \mathcal{X} = \mathcal{A}^D * \mathcal{C}, \quad (3)\ \mathcal{X} * \mathcal{A} = \mathcal{A}^\dagger * \mathcal{A}^{\textcircled{w}} * \mathcal{A}^2$$

is consistent and its unique solution is the tensor $\mathcal{X} = \mathcal{A}^\dagger * \mathcal{A}^D * \mathcal{C}$, where $\mathcal{C} = \mathcal{A} * \mathcal{A}^{\textcircled{w}} * \mathcal{A}$ is the weak core part of tensor $\mathcal{A}$.

Proof. Let

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}))) = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{bmatrix},$$

then

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^D))) = \begin{bmatrix} A_1^D & & \\ & \ddots & \\ & & A_p^D \end{bmatrix},$$

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^\dagger))) = \begin{bmatrix} A_1^\dagger & & \\ & \ddots & \\ & & A_p^\dagger \end{bmatrix},$$

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^{\textcircled{\dagger}}))) = \begin{bmatrix} A_1^{\textcircled{\dagger}} & & \\ & \ddots & \\ & & A_p^{\textcircled{\dagger}} \end{bmatrix},$$

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^{\textcircled{w}}))) = \begin{bmatrix} A_1^{\textcircled{w}} & & \\ & \ddots & \\ & & A_p^{\textcircled{w}} \end{bmatrix}.$$

Let

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{X}))) = \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_p \end{bmatrix}.$$

We will check $\mathcal{X} = \mathcal{A}^\dagger * \mathcal{A}^D * \mathcal{C}$ satisfies the there equations in the system.
Notice that

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{X}))) = \mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^\dagger * \mathcal{A}^D * \mathcal{C})))$$

i.e.

$$\begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_p \end{bmatrix} = \begin{bmatrix} A_1^{\dagger}A_1^{D}C_1 & & \\ & \ddots & \\ & & A_p^{\dagger}A_p^{D}C_p \end{bmatrix}, X_i = A_i^{\dagger}A_i^{D}C_i, (i = 1, \cdots, p)$$

where $\mathcal{C} = \mathcal{A}*\mathcal{A}^{\circledW}*\mathcal{A}$, then

$$\text{DFT}(\text{Circ}(\text{Unfold}(\mathcal{C}))) = \text{DFT}(\text{Circ}(\text{Unfold}(\mathcal{A}*\mathcal{A}^{\circledW}*\mathcal{A}))),$$

i.e.

$$\begin{bmatrix} C_1 & & \\ & \ddots & \\ & & C_p \end{bmatrix} = \begin{bmatrix} A_1A_1^{\circledW}A_1 & & \\ & \ddots & \\ & & A_pA_p^{\circledW}A_p \end{bmatrix}, C_i = A_iA_i^{\circledW}A_i, (i = 1, \cdots, p).$$

By $AA^D = A^DA$ and $CA^DC = AA^{\circledW}AA^DAA^{\circledW}A = AA^{\circledW}AA^{\circledW}A = AA^{\circledW}A = A$, we can get

$$\begin{aligned}
\text{DFT}(\text{Circ}(\text{Unfold}(\mathcal{X}*\mathcal{A}*\mathcal{X}))) &= \begin{bmatrix} X_1A_1X_1 & & \\ & \ddots & \\ & & X_pA_pX_p \end{bmatrix} \\
&= \begin{bmatrix} (A_1^{\dagger}A_1^{D}C_1)A_1(A_1^{\dagger}A_1^{D}C_1) & & \\ & \ddots & \\ & & (A_p^{\dagger}A_p^{D}C_p)A_p(A_p^{\dagger}A_p^{D}C_p) \end{bmatrix} \\
&= \begin{bmatrix} A_1^{\dagger}A_1^{D}C_1A_1A_1^{\dagger}A_1^{D}A_1A_1^{\circledW}A_1 & & \\ & \ddots & \\ & & A_p^{\dagger}A_p^{D}C_pA_pA_p^{\dagger}A_p^{D}A_pA_p^{\circledW}A_p \end{bmatrix} \\
&= \begin{bmatrix} A_1^{\dagger}A_1^{D}C_1A_1^{D}C_1 & & \\ & \ddots & \\ & & A_p^{\dagger}A_p^{D}C_pA_pC_p \end{bmatrix} \\
&= \begin{bmatrix} A_1^{\dagger}A_1^{D}C_1 & & \\ & \ddots & \\ & & A_p^{\dagger}A_p^{D}C_p \end{bmatrix} \\
&= \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_p \end{bmatrix} = \text{DFT}(\text{Circ}(\text{Unfold}(\mathcal{X}))).
\end{aligned}$$

Therefore, $\mathcal{X}*\mathcal{A}*\mathcal{X} = \mathcal{X}$.
On the other hand,

$$\begin{aligned}
\text{DFT}(\text{Circ}(\text{Unfold}(\mathcal{A}*\mathcal{X}))) &= \begin{bmatrix} A_1X_1 & & \\ & \ddots & \\ & & A_pX_p \end{bmatrix} \\
&= \begin{bmatrix} A_1A_1^{\dagger}A_1^{D}C_1 & & \\ & \ddots & \\ & & A_pA_p^{\dagger}A_p^{D}C_p \end{bmatrix} \\
&= \begin{bmatrix} A_1A_1^{\dagger}A_1^{D}A_1A_1^{\circledW}A_1 & & \\ & \ddots & \\ & & A_pA_p^{\dagger}A_p^{D}A_pA_p^{\circledW}A_p \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} A_1A_1^{\dagger}A_1A_1^{D}A_1^{\textcircled{w}}A_1 & & \\ & \ddots & \\ & & A_pA_p^{\dagger}A_pA_p^{D}A_p^{\textcircled{w}}A_p \end{bmatrix} \\
&= \begin{bmatrix} A_1A_1^{D}A_1^{\textcircled{w}}A_1 & & \\ & \ddots & \\ & & A_pA_p^{D}A_p^{\textcircled{w}}A_p \end{bmatrix} \\
&= \begin{bmatrix} A_1^{D}C_1 & & \\ & \ddots & \\ & & A_p^{D}C_p \end{bmatrix} = \mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^{D}*\mathcal{C}))).
\end{aligned}
$$

Therefore, $\mathcal{A}*\mathcal{X} = \mathcal{A}^{D}*\mathcal{C}$.

From (v) in Lemma 3.1, for some tensor $\mathcal{Z}$, we have $\mathcal{A}^{\textcircled{w}} = \mathcal{A}^{k}*\mathcal{Z}$, $A^{\textcircled{w}} = A^{k}Z$, and because $A^{k+1}A^{D} = A^{k}$, then

$$
\begin{aligned}
\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{X}*\mathcal{A}))) &= \begin{bmatrix} X_1A_1 & & \\ & \ddots & \\ & & X_pA_p \end{bmatrix} \\
&= \begin{bmatrix} A_1^{\dagger}A_1^{D}C_1A_1 & & \\ & \ddots & \\ & & A_p^{\dagger}A_p^{D}C_pA_p \end{bmatrix} \\
&= \begin{bmatrix} A_1^{\dagger}A_1^{D}A_1A_1^{\textcircled{w}}A_1^{2} & & \\ & \ddots & \\ & & A_pA_p^{\dagger}A_p^{D}A_pA_p^{\textcircled{w}}A_p^{2} \end{bmatrix} \\
&= \begin{bmatrix} A_1^{\dagger}A_1^{D}A_1A_1^{k}Z_1A_1^{2} & & \\ & \ddots & \\ & & A_pA_p^{\dagger}A_p^{D}A_pA_p^{k}Z_pA_p^{2} \end{bmatrix} \\
&= \begin{bmatrix} A_1^{\dagger}A_1^{k}Z_1A_1^{2} & & \\ & \ddots & \\ & & A_pA_p^{\dagger}A_p^{k}Z_pA_p^{2} \end{bmatrix} \\
&= \begin{bmatrix} A_1^{\dagger}A_1^{\textcircled{w}}A_1^{2} & & \\ & \ddots & \\ & & A_pA_p^{\dagger}A_p^{\textcircled{w}}A_p^{2} \end{bmatrix} = \mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^{\dagger}*\mathcal{A}^{\textcircled{w}}*\mathcal{C}))).
\end{aligned}
$$

Therefore, $\mathcal{X}*\mathcal{A} = \mathcal{A}^{\dagger}*\mathcal{A}^{\textcircled{w}}*\mathcal{A}^{2}$.

Above all, $\mathcal{X}$ satisfies the three equations.

For the uniqueness, we assume that $\mathcal{X}_1$ and $\mathcal{X}_2$ are two solutions of the system. From

$$\mathcal{A}*\mathcal{X}_1 = \mathcal{A}^{D}*\mathcal{C} = \mathcal{A}*\mathcal{X}_2,\ \mathcal{X}_1*\mathcal{A} = \mathcal{A}^{\dagger}*\mathcal{A}^{\textcircled{w}}*\mathcal{A}^{2} = \mathcal{X}_2*\mathcal{A},$$

we have

$$\mathcal{X}_1 = (\mathcal{X}_1*\mathcal{A})*\mathcal{X}_1 = (\mathcal{X}_2*\mathcal{A})*\mathcal{X}_1 = \mathcal{X}_2*(\mathcal{A}*\mathcal{X}_1) = \mathcal{X}_2*\mathcal{A}*\mathcal{X}_2 = \mathcal{X}_2.$$

The uniqueness is proved. □

Definition 3.3. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ with $\mathrm{Ind}_T(\mathcal{A}) = k$, and $\mathcal{C}$ be the weak core part of $\mathcal{A}$. The MPWG inverse of tensor $\mathcal{A}$, denoted as $\mathcal{A}^{\dagger,WG}$, is defined to be the solution of the system in Theorem* 3.2.

Theorem 3.4. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ with $\mathrm{Ind}_T(\mathcal{A}) = k$. Then*

$$\mathcal{A}^{\dagger,WG} = \mathcal{A}^{\dagger} * \mathcal{A}^{\textcircled{w}} * \mathcal{A}.$$

Proof. Applying (v) in Lemma 3.1 and Theorem 3.2, we obtain

$$\begin{aligned}\mathcal{A}^{\dagger,WG} &= \mathcal{A}^{\dagger} * \mathcal{A}^{D} * \mathcal{C} = \mathcal{A}^{\dagger} * \mathcal{A}^{D} * \mathcal{A} * \mathcal{A}^{\textcircled{w}} * \mathcal{A} \\ &= \mathcal{A}^{\dagger} * \mathcal{A}^{D} * \mathcal{A} * \mathcal{A}^{\textcircled{w}} * \mathcal{A} \\ &= \mathcal{A}^{\dagger} * \mathcal{A}^{D} * \mathcal{A} * \mathcal{A}^{k} * \mathcal{Z} * \mathcal{A} \\ &= \mathcal{A}^{\dagger} * \mathcal{A}^{k} * \mathcal{Z} * \mathcal{A} \\ &= \mathcal{A}^{\dagger} * \mathcal{A}^{\textcircled{w}} * \mathcal{A}\end{aligned}$$

□

Let $\mathcal{A} \in \mathbb{C}^{n \times n \times p}$ be a third-order tensor, and $A^{(k)}$ represents the *kth* frontal slice of tensor $\mathcal{A}$. For the simplicity of the discussion, let

$$\mathcal{A} = \left[A^{(1)} | A^{(2)} | \cdots | A^{(p)}\right].$$

Example 3.5. *Let*

$$\mathcal{A} \in \mathbb{C}^{2\times2\times2}, \ \mathcal{A} = \left[A^{(1)} | A^{(2)}\right],$$

where

$$A^{(1)} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \ A^{(2)} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Applying the discrete Fourier transform, we obtain

$$\text{bcirc}(\mathcal{A}) = \left(F_2^H \otimes I_2\right) \text{Diag}\,(A_1, A_2)\,(F_2 \otimes I_2),$$

where

$$A_1 = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

From the above, we can get $\text{Ind}(A_1) = 2, \text{Ind}(A_2) = 1$, *so* $\text{Ind}_T(\mathcal{A}) = 2$. *Besides,*

$$\mathcal{A}^{\dagger} = \left[\begin{array}{cc|cc} \frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{array}\right], \ \mathcal{A}^{*} = \left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right],$$

$$\mathcal{A}^{2} = \left[\begin{array}{cc|cc} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \ \mathcal{A}^{3} = \left[\begin{array}{cc|cc} 4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$(\mathcal{A}^{*})^2 = \mathcal{A}^{*} * \mathcal{A}^{*} = \text{fold}(\text{bcirc}(\mathcal{A}^{*})\text{unfold}(\mathcal{A}^{*})) = \left[\begin{array}{cc|cc} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$(\mathcal{A}^{*})^2 * \mathcal{A}^{3} = \left[\begin{array}{cc|cc} 16 & 0 & -16 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \ \left((\mathcal{A}^{*})^2) * \mathcal{A}^{3}\right)^{\dagger} = \left[\begin{array}{cc|cc} \frac{1}{64} & 0 & -\frac{1}{64} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$
\begin{aligned}
\mathcal{A}^2*((\mathcal{A}^*)^2*\mathcal{A}^3)^\dagger &= \text{fold}\left(\text{bcirc}(\mathcal{A}^2)\text{unfold}((\mathcal{A}^*)^2*(\mathcal{A}^3))^\dagger\right)\\
&= \text{fold}\left(\begin{bmatrix} 2 & 0 & -2 & 0\\ 0 & 0 & 0 & 0\\ -2 & 0 & 2 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} \frac{1}{64} & 0\\ 0 & 0\\ -\frac{1}{64} & 0\\ 0 & 0\end{bmatrix}\right) = \left[\begin{array}{cc|cc} \frac{1}{16} & 0 & -\frac{1}{16} & 0\\ 0 & 0 & 0 & 0\end{array}\right],\\
\mathcal{A}^{\textcircled{\#}} = \mathcal{A}^2*((\mathcal{A}^*)^2*\mathcal{A}^3)^\dagger*(\mathcal{A}^*)^2 &= \text{fold}\left(\text{bcirc}(\mathcal{A}^2*((\mathcal{A}^*)^2*\mathcal{A}^3)^\dagger)\text{unfold}((\mathcal{A}^*)^2\right)\\
&= \text{fold}\left(\begin{bmatrix} \frac{1}{16} & 0 & -\frac{1}{16} & 0\\ 0 & 0 & 0 & 0\\ -\frac{1}{16} & 0 & \frac{1}{16} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} 2 & 0\\ 0 & 0\\ -2 & 0\\ 0 & 0\end{bmatrix}\right) = \left[\begin{array}{cc|cc} \frac{1}{4} & 0 & -\frac{1}{4} & 0\\ 0 & 0 & 0 & 0\end{array}\right],\\
\mathcal{A}^{\textcircled{w}} = (\mathcal{A}^{\textcircled{\#}})^2*\mathcal{A} &= \text{fold}\left(\text{bcirc}((\mathcal{A}^{\textcircled{\#}})^2)\text{unfold}(\mathcal{A})\right)\\
&= \text{fold}\left(\begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0\\ 0 & 0 & 0 & 0\\ -\frac{1}{2} & 0 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} 1 & 1\\ 0 & 0\\ -1 & 1\\ 0 & 0\end{bmatrix}\right) = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right],\\
\mathcal{A}^{\textcircled{w}}*\mathcal{A} &= \text{fold}\left(\text{bcirc}(\mathcal{A}^{\textcircled{w}})\text{unfold}(\mathcal{A})\right)\\
&= \text{fold}\left(\begin{bmatrix} 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix} 1 & 1\\ 0 & 0\\ -1 & 1\\ 0 & 0\end{bmatrix}\right) = \left[\begin{array}{cc|cc} 2 & 0 & -2 & 0\\ 0 & 0 & 0 & 0\end{array}\right],\\
\mathcal{A}^{\dagger,WG} &= \mathcal{A}^\dagger*\mathcal{A}^{\textcircled{w}}*\mathcal{A} = \text{fold}\left(\text{bcirc}(\mathcal{A}^\dagger)\text{unfold}(\mathcal{A}^{\textcircled{w}}*\mathcal{A})\right)\\
&= \text{fold}\left(\begin{bmatrix} \frac{1}{4} & 0 & -\frac{1}{4} & 0\\ \frac{1}{4} & 0 & \frac{1}{4} & 0\\ -\frac{1}{4} & 0 & \frac{1}{4} & 0\\ \frac{1}{4} & 0 & \frac{1}{4} & 0\end{bmatrix}\begin{bmatrix} 2 & 0\\ 0 & 0\\ -2 & 0\\ 0 & 0\end{bmatrix}\right) = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right].
\end{aligned}
$$

In the following, we will give some properties of the T-MPWG inverse of the tensor $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$.

Lemma 3.6. *[5] Let $\mathcal{X} \in \mathbb{C}^{n\times n\times p}, \mathcal{B} \in \mathbb{C}^{n\times n\times p}$, and $\mathcal{C} \in \mathbb{C}^{n\times n\times p}$ be given tensors. Then*
(i) *$\mathcal{R}(\mathcal{X}) \subseteq \mathcal{R}(\mathcal{B})$ if and only if there exists $\mathcal{U} \in \mathbb{C}^{n\times n\times p}$ such that $\mathcal{X} = \mathcal{B}*\mathcal{U}$.*
(ii) *$\mathcal{N}(\mathcal{C}) \subseteq \mathcal{N}(\mathcal{X})$ if and only if there exists $\mathcal{V} \in \mathbb{C}^{n\times n\times p}$ such that $\mathcal{X} = \mathcal{V}*\mathcal{C}$.*
(iii) *$\mathcal{R}(\mathcal{X}) \subseteq \mathcal{R}(\mathcal{B})$ and $\mathcal{N}(\mathcal{C}) \subseteq \mathcal{N}(\mathcal{X})$ if and only if there exists $\mathcal{U} \in \mathbb{C}^{n\times n\times p}$ such that $\mathcal{X} = \mathcal{B}*\mathcal{V}*\mathcal{C}$.*

On the basis of the T-range space $\mathcal{R}(\mathcal{A})$ and T-null space $\mathcal{N}(\mathcal{A})$ of $\mathcal{A}$, we introduce the orthogonal complement space of the T-range space:

Definition 3.7. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$, $\mathcal{R}(\mathcal{A})^\perp$ represents the orthogonal complement space of $\mathcal{R}(\mathcal{A})$, that is, each tensor $\mathcal{X} \in \mathbb{C}^{n\times n\times p}$ can be uniquely represented as*

$$\mathcal{X} = \mathcal{Y} + \mathcal{Z}, \mathcal{Y} \in \mathcal{R}(\mathcal{A}), \mathcal{Z} \in \mathcal{R}(\mathcal{A})^\perp.$$

Lemma 3.8. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$. Then,*

$$\mathcal{N}(\mathcal{A}) = \mathcal{R}(\mathcal{A}^*)^\perp,\ \mathcal{N}(\mathcal{A}^*) = \mathcal{R}(\mathcal{A})^\perp.$$

Proof. For any $\mathcal{Z} \in \mathcal{R}(\mathcal{A}^*)^\perp$, $\mathcal{Y} \in \mathcal{R}(\mathcal{A}^*)$, $\mathcal{Y} = \mathcal{A}^**\mathcal{X}$, where $\mathcal{X}$ is arbitrary. Then,

$$\mathcal{Y}^**\mathcal{Z} = \mathcal{O} \Longleftrightarrow (\mathcal{A}^**\mathcal{X})^**\mathcal{Y} = \mathcal{X}^**\mathcal{A}*\mathcal{Z} = \mathcal{O},$$

so $\mathcal{A}*\mathcal{Z} = \mathcal{O} \Longleftrightarrow \mathcal{Z} \in \mathcal{N}(\mathcal{A})$. Therefore, $\mathcal{N}(\mathcal{A}) = \mathcal{R}(\mathcal{A}^*)^\perp$. $\mathcal{N}(\mathcal{A}^*) = \mathcal{R}(\mathcal{A})^\perp$ is similarly proved. $\square$

Theorem 3.9. *Let* $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ *with* $\mathrm{Ind}_T(\mathcal{A}) = k$. *Then the following conditions are equivalent:*
(i) $\mathcal{X} = \mathcal{A}^{\dagger,WG} = \mathcal{A}^{\dagger}*\mathcal{A}^{Ⓦ}*\mathcal{A}$,
(ii) $\mathcal{X} = \mathcal{X}*\mathcal{A}^{D}*\mathcal{C}$, $\mathcal{X}*\mathcal{A}^{k} = \mathcal{A}^{\dagger}*\mathcal{A}^{k}$,
(iii) $\mathcal{A}^{\dagger}*\mathcal{A}*\mathcal{X} = \mathcal{X}$, $\mathcal{A}*\mathcal{X} = \mathcal{A}^{Ⓦ}*\mathcal{A}$,
(iv) $\mathcal{X}*\mathcal{A}^{\dagger} = \mathcal{A}^{\dagger}*\mathcal{A}^{k}$, $\mathcal{X}*\mathcal{A}^{Ⓦ}*\mathcal{A} = \mathcal{X}$,
(v) $\mathcal{X} = \mathcal{X}*\mathcal{A}^{D}*\mathcal{C}$, $\mathcal{R}(\mathcal{X}) = \mathcal{R}\left(\mathcal{A}^{\dagger}\mathcal{A}^{k}\right)$,
(vi) $\mathcal{A}*\mathcal{X} = \mathcal{A}^{D}*\mathcal{C}$, $\mathcal{R}(\mathcal{X}) = \mathcal{R}(\mathcal{A}^{*})$.

Proof. That (i) implies all other items (ii)-(vi) can be checked directly.
(ii) ⇒ (i) From (v) in Lemma 3.1, since $\mathcal{A}^{Ⓦ} = \mathcal{A}^{k}*\mathcal{Z}$, for some third-order tensor $\mathcal{Z}$, it follows that

$$\mathcal{X} = \mathcal{X}*\mathcal{A}^{D}*\mathcal{C} = \mathcal{X}*\mathcal{A}^{D}*\mathcal{A}*\mathcal{A}^{Ⓦ}*\mathcal{A} = \mathcal{X}*\mathcal{A}^{D}*\mathcal{A}*\mathcal{A}^{k}*\mathcal{Z}*\mathcal{A} = \mathcal{X}*\mathcal{A}^{k}*\mathcal{Z}*\mathcal{A} = \mathcal{A}^{\dagger}*\mathcal{A}^{k}*\mathcal{Z}*\mathcal{A} = \mathcal{A}^{\dagger}*\mathcal{A}^{Ⓦ}*\mathcal{A}.$$

(iii) ⇒ (i) It's obvious that $\mathcal{X} = \mathcal{A}^{\dagger}*\mathcal{A}*\mathcal{X} = \mathcal{A}^{\dagger}*\mathcal{A}^{Ⓦ}*\mathcal{A}$.
(iv) ⇒ (i) According to $\mathcal{X}*\mathcal{A}^{k} = \mathcal{A}^{\dagger}*\mathcal{A}^{k}$, we obtain

$$\mathcal{X} = \mathcal{X}*\mathcal{A}^{Ⓦ}*\mathcal{A} = \mathcal{X}*\mathcal{A}^{k}*\mathcal{Z}*\mathcal{A} = \mathcal{A}^{\dagger}*\mathcal{A}^{k}*\mathcal{Z}*\mathcal{A} = \mathcal{A}^{\dagger}*\mathcal{A}^{Ⓦ}*\mathcal{A}.$$

(v) ⇒ (ii) Since $\mathcal{A}*\mathcal{X}$ is idempotent, it follows that

$$\mathcal{A}*\mathcal{X} - (\mathcal{A}*\mathcal{X})^2 = (\mathcal{A} - \mathcal{A}*\mathcal{X}*\mathcal{A})*\mathcal{X} = 0,$$

so $\mathcal{R}\left(\mathcal{A}^{\dagger}\mathcal{A}^{k}\right) = \mathcal{R}(\mathcal{X}) \subseteq \mathcal{N}(\mathcal{A} - \mathcal{A}*\mathcal{X}*\mathcal{A})$. We have $(\mathcal{A} - \mathcal{A}*\mathcal{X}*\mathcal{A})*\mathcal{A}^{\dagger}*\mathcal{A}^{k} = 0$. That is,

$$\mathcal{A}*\mathcal{A}^{\dagger}*\mathcal{A}^{k} - \mathcal{A}*\mathcal{X}*\mathcal{A}*\mathcal{A}^{\dagger}*\mathcal{A}^{k} = 0 \Rightarrow \mathcal{A}^{k} = \mathcal{A}*\mathcal{X}*\mathcal{A}^{k}.$$

Multiplying the last equality by $\mathcal{A}^{\dagger}$ from the left side, we get $\mathcal{A}^{\dagger}*\mathcal{A}^{k} = \mathcal{A}^{\dagger}*\mathcal{A}*\mathcal{X}*\mathcal{A}^{k}$.

From $(\mathcal{I} - \mathcal{A}^{\dagger}*\mathcal{A})*\mathcal{A}^{\dagger}*\mathcal{A}^{k} = 0$, we have $\mathcal{R}(\mathcal{X}) = \mathcal{R}\left(\mathcal{A}^{\dagger}*\mathcal{A}^{k}\right) \subseteq \mathcal{N}(\mathcal{I} - \mathcal{A}^{\dagger}*\mathcal{A})$. Then $(\mathcal{I} - \mathcal{A}^{\dagger}*\mathcal{A})*\mathcal{X} = 0$. i.e. $\mathcal{X} = \mathcal{A}^{\dagger}*\mathcal{A}*\mathcal{X}$. Hence, $\mathcal{X}*\mathcal{A}^{k} = \mathcal{A}^{\dagger}*\mathcal{A}*\mathcal{X}*\mathcal{A}^{k} = \mathcal{A}^{\dagger}*\mathcal{A}^{k}$.

Since $\mathcal{R}(\mathcal{I} - \mathcal{A}^{\dagger}*\mathcal{A}) \subseteq \mathcal{N}((\mathcal{A}^{k})^{*}*\mathcal{A}^{2}) = \mathcal{N}(\mathcal{X})$, we have that $\mathcal{X} = \mathcal{A}^{\dagger}*\mathcal{A}*\mathcal{X}$, and hence, $\mathcal{X}*\mathcal{A}^{k} = \mathcal{A}^{\dagger}*\mathcal{A}^{k}$.
(vi) ⇒ (i) Let $\mathcal{X} = \mathcal{A}^{\dagger,WG}$. From its definition, we have $\mathcal{A}*\mathcal{X} = \mathcal{A}^{D}*\mathcal{C}$. Then,

$$\mathcal{A}^{\dagger}*\mathcal{A}*\mathcal{A}^{\dagger,WG} = \mathcal{A}^{\dagger}*\mathcal{A}*\mathcal{A}^{\dagger}*\mathcal{A}^{Ⓦ}*\mathcal{A} = \mathcal{A}^{\dagger}*\mathcal{A}^{Ⓦ}*\mathcal{A} = \mathcal{A}^{\dagger,WG}.$$

We obtain that $\mathcal{R}(\mathcal{X}) \subseteq \mathcal{R}(\mathcal{A}^{\dagger}*\mathcal{A}) = \mathcal{R}(\mathcal{A}^{*})$.

In order to show that $\mathcal{X}$ is the unique solution to the system, we assume that both $\mathcal{X}_1$ and $\mathcal{X}_2$ satisfy the equations. Then,

$$\mathcal{A}*\mathcal{X}_1 = \mathcal{A}^{D}*\mathcal{C} = \mathcal{A}*\mathcal{X}_2,\ \mathcal{R}(\mathcal{X}_1) \subseteq \mathcal{R}(\mathcal{A}^{*}),\ \mathcal{R}(\mathcal{X}_2) \subseteq \mathcal{R}(\mathcal{A}^{*}),$$

So we get that $\mathcal{R}(\mathcal{X}_1 - \mathcal{X}_2) \subseteq \mathcal{R}(\mathcal{A}^{*})$.

Since $\mathcal{A}*(\mathcal{X}_1 - \mathcal{X}_2) = 0$, we obtain $\mathcal{R}(\mathcal{X}_1 - \mathcal{X}_2) \subseteq \mathcal{N}(\mathcal{A}) = \mathcal{R}(\mathcal{A}^{*})^{\perp}$, Therefore, $\mathcal{R}(\mathcal{X}_1 - \mathcal{X}_2) \subseteq (\mathcal{R}(\mathcal{A}^{*})^{\perp}) \cap \mathcal{R}(\mathcal{A}^{*}) = 0$. Thus, $\mathcal{X}_1 = \mathcal{X}_2$. □

Theorem 3.10. *Let* $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ *with* $\mathrm{Ind}_T(\mathcal{A}) = k$. *Then*
(i) $\mathcal{A}^{\dagger,WG} = \mathcal{A}^{\dagger}*(\mathcal{A}*\mathcal{A}^{\textcircled{\#}}*\mathcal{A})^{\#}*\mathcal{A}$,
(ii) $\mathcal{A}^{\dagger,WG} = \mathcal{A}^{\dagger}*(\mathcal{A}^{\textcircled{\#}})^{2}*\mathcal{A}^{2} = \mathcal{A}^{\dagger}*(\mathcal{A}^{2})^{\textcircled{\#}}*\mathcal{A}^{2}$,
(iii) $\mathcal{A}^{\dagger,WG} = \mathcal{A}^{\dagger}*\mathcal{A}^{k}*(\mathcal{A}^{k+2})^{\textcircled{\#}}*\mathcal{A}^{2}$,
(iv) $\mathcal{A}^{\dagger,WG} = \mathcal{A}^{\dagger}*(\mathcal{A}^{k+2}*(\mathcal{A}^{k})^{\dagger})^{\dagger}*\mathcal{A}^{2}$.

Proof. Let

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}))) = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{bmatrix},$$

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^{\oplus}))) = \begin{bmatrix} A_1^{\oplus} & & \\ & \ddots & \\ & & A_p^{\oplus} \end{bmatrix},$$

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^{W}))) = \begin{bmatrix} A_1^{W} & & \\ & \ddots & \\ & & A_p^{W} \end{bmatrix},$$

then

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}((\mathcal{A}*\mathcal{A}^{W}*\mathcal{A})^{\#}))) = \begin{bmatrix} (A_1A_1^{\oplus}A_1)^{\#} & & \\ & \ddots & \\ & & (A_pA_p^{\oplus}A_p)^{\#} \end{bmatrix}.$$

From Theorem 3.8 and Theorem 3.9 in reference [12], we have $A^{W} = (AA^{\oplus}A)^{\#} = (A^{\oplus})^2A = (A^2)^{\oplus}A = A^k(A^{k+2})^{\oplus}A = (A^2P_{A^k})^{\dagger}A$, i.e.

$$A_i^{W} = (A_iA_i^{\oplus}A_i)^{\#} = (A_i^{\oplus})^2A_i = (A_i^2)^{\oplus}A_i = A_i^k(A_i^{k+2})^{\oplus}A_i = (A_i^2P_{A_i^k})^{\dagger}A_i.$$

Then,

$$\mathcal{A}^{W} = (\mathcal{A}*\mathcal{A}^{W}*\mathcal{A})^{\#} = (\mathcal{A}^{\oplus})^2*\mathcal{A} = (\mathcal{A}^2)^{\oplus}*\mathcal{A} = \mathcal{A}^k*(\mathcal{A}^{k+2})^{\oplus}*\mathcal{A} = (\mathcal{A}^2*\mathcal{A}^k*(\mathcal{A}^k)^{\dagger})^{\dagger}*\mathcal{A},$$

pre-multiplying the last equality by $\mathcal{A}^{\dagger}$ and post-multiplying by $\mathcal{A}$, we obtain $\mathcal{A}^{\dagger,WG} = \mathcal{A}^{\dagger}*\mathcal{A}^{W}*\mathcal{A}$.

So (i)-(iv) are established. □

On the basis of the core-EP decomposition which was introduced in [11], Cong and Ma introduced the core-EP decomposition of the third-order tensors $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ based on the T-product in [2]:

Lemma 3.11. *[2] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ with $\mathrm{Ind}_T(\mathcal{A}) = k$, $rank_T(\mathcal{A}^k) = p$. Then $\mathcal{A}$ can be decomposed as $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$,*

$$(1)\ rank_T(\mathcal{A}_1^2) = rank_T(\mathcal{A}_1);\quad (2)\ \mathcal{A}_2^k = \mathcal{O};\quad (3)\ \mathcal{A}_1{}^{*}*\mathcal{A}_2 = \mathcal{A}_2*\mathcal{A}_1 = \mathcal{O},$$

where $\mathcal{A}_1$ is the core part with $\mathrm{Ind}_T(\mathcal{A}_1) = 1$, $\mathcal{A}_2$ is EP. There exists a unitary tensor $\mathcal{U} \in \mathbb{C}^{n\times n\times p}$ such that

$$\mathcal{A} = \mathcal{U}*\begin{bmatrix} \mathcal{T} & \mathcal{S} \\ \mathcal{O} & \mathcal{N} \end{bmatrix}*\mathcal{U}^*,\ \mathcal{A}_1 = \mathcal{U}*\begin{bmatrix} \mathcal{T} & \mathcal{S} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}*\mathcal{U}^*,\ \mathcal{A}_2 = \mathcal{U}*\begin{bmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{N} \end{bmatrix}*\mathcal{U}^*,$$

where $\mathcal{T} \in \mathbb{C}^{r\times r\times p}$ is singular, $\mathcal{S} \in \mathbb{C}^{r\times(n-r)\times p}$, $\mathcal{N} \in \mathbb{C}^{(n-r)\times(n-r)\times p}$ is nilpotent of index k, i.e. $\mathcal{N}^k = 0$.

Lemma 3.12. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$. Its T-core EP decomposition as above, then*

(i) $\mathcal{A}^{\dagger} = \mathcal{U}*\begin{bmatrix} \mathcal{T}^{*}*\Delta & -\mathcal{T}*\Delta*\mathcal{S}*\mathcal{N}^{\dagger} \\ (\mathcal{I}_{n-r} - \mathcal{N}^{\dagger}*\mathcal{N})*\mathcal{S}^{*}*\Delta & \mathcal{N}^{\dagger} - (\mathcal{I}_{n-r} - \mathcal{N}^{\dagger}*\mathcal{N})*\mathcal{S}^{*}*\Delta*\mathcal{S}*\mathcal{N}^{\dagger} \end{bmatrix}*\mathcal{U}^*,$

(ii) $\mathcal{A}^{D} = \mathcal{U}*\begin{bmatrix} \mathcal{T}^{-1} & (\mathcal{T}^{k+1})^{-1}*\widetilde{\mathcal{T}} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}*\mathcal{U}^*,$

(iii) $\mathcal{A}^{\oplus} = \mathcal{U}*\begin{bmatrix} \mathcal{T}^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}*\mathcal{U}^*,$

(iv) $\mathcal{A}^{W} = \mathcal{U}*\begin{bmatrix} \mathcal{T}^{-1} & \mathcal{T}^{-2}*\mathcal{S} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}*\mathcal{U}^*,$

which can be expressed as the core-EP decomposition form, where
$\Delta = [\mathcal{T}^{*}*\mathcal{T} + (\mathcal{I}_{n-r} - \mathcal{N}^{\dagger}*\mathcal{N})*\mathcal{S}^{*}]^{-1}$, $\widetilde{\mathcal{T}} = \sum_{j=0}^{k-1}\mathcal{T}^{j}*\mathcal{S}*\mathcal{N}^{k-1-j}$, *and*

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{I}_{n-r}))) = \begin{bmatrix} I_1 & & \\ & \ddots & \\ & & I_p \end{bmatrix},\ I_i = I_{n-r},\ (i = 1,\cdots,p).$$

Proof.

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^{\dagger}))) = \begin{bmatrix} A_1^{\dagger} & & \\ & \ddots & \\ & & A_p^{\dagger} \end{bmatrix}$$

$$= \begin{bmatrix} U_1 \begin{bmatrix} E_1 & F_1 \\ G_1 & H_1 \end{bmatrix} U_1^* & & \\ & \ddots & \\ & & U_p \begin{bmatrix} E_p & F_p \\ G_p & H_p \end{bmatrix} U_p^* \end{bmatrix}$$

$$= \begin{bmatrix} U_1 & & \\ & \ddots & \\ & & U_p \end{bmatrix} \begin{bmatrix} \begin{bmatrix} E_1 & F_1 \\ G_1 & H_1 \end{bmatrix} & & \\ & \ddots & \\ & & \begin{bmatrix} E_p & F_p \\ G_p & H_p \end{bmatrix} \end{bmatrix} \begin{bmatrix} U_1^* & & \\ & & \\ & & U_p^* \end{bmatrix}$$

$$= \mathrm{DFT}\left(\mathrm{Circ}\left(\mathrm{Unfold}\left(\mathcal{U} * \begin{bmatrix} \mathcal{E} & \mathcal{F} \\ \mathcal{G} & \mathcal{H} \end{bmatrix} * \mathcal{U}^*\right)\right)\right).$$

According to reference [11], we obtain

$$A^{\dagger} = U \begin{bmatrix} E & F \\ G & H \end{bmatrix} U^* = U \begin{bmatrix} T^* \triangle & -T \triangle S N^{\dagger} \\ (I_{n-r} - N^{\dagger} N) S^* \triangle & N^{\dagger} - (I_{n-r} - N^{\dagger} N) S^* \triangle S N^{\dagger} \end{bmatrix} U^*,$$

so

$$\mathcal{A}^{\dagger} = \mathcal{U} * \begin{bmatrix} \mathcal{T}^* * \Delta & -\mathcal{T} * \Delta * \mathcal{S} * \mathcal{N}^{\dagger} \\ (\mathcal{I}_{n-r} - \mathcal{N}^{\dagger} * \mathcal{N}) * \mathcal{S}^* * \Delta & \mathcal{N}^{\dagger} - (\mathcal{I}_{n-r} - \mathcal{N}^{\dagger} * \mathcal{N}) * \mathcal{S}^* * \Delta * \mathcal{S} * \mathcal{N}^{\dagger} \end{bmatrix} * \mathcal{U}^*,$$

where $\Delta = [\mathcal{T}^* * \mathcal{T} + (\mathcal{I}_{n-r} - \mathcal{N}^{\dagger} * \mathcal{N}) * \mathcal{S}^*]^{-1}$.

Similarly,

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^{D}))) = \begin{bmatrix} A_1^{D} & & \\ & \ddots & \\ & & A_p^{D} \end{bmatrix}$$

$$= \mathrm{DFT}\left(\mathrm{Circ}\left(\mathrm{Unfold}\left(\mathcal{U} * \begin{bmatrix} \mathcal{T}^{-1} & (\mathcal{T}^{k+1})^{-1} * \widetilde{\mathcal{T}} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} * \mathcal{U}^*\right)\right)\right)$$

so

$$\mathcal{A}^{D} = \mathcal{U} * \begin{bmatrix} \mathcal{T}^{-1} & (\mathcal{T}^{k+1})^{-1} * \widetilde{\mathcal{T}} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} * \mathcal{U}^*,$$

where $\widetilde{\mathcal{T}} = \sum_{j=0}^{k-1} \mathcal{T}^{j} * \mathcal{S} * \mathcal{N}^{k-1-j}$. Furthermore,

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^{\oplus}))) = \begin{bmatrix} A_1^{\oplus} & & \\ & \ddots & \\ & & A_p^{\oplus} \end{bmatrix} = \mathrm{DFT}\left(\mathrm{Circ}\left(\mathrm{Unfold}\left(\mathcal{U} * \begin{bmatrix} \mathcal{T}^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} * \mathcal{U}^*\right)\right)\right)$$

so

$$\mathcal{A}^{\oplus} = \mathcal{U} * \begin{bmatrix} \mathcal{T}^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} * \mathcal{U}^*.$$

On the other hand,

$$\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{A}^{\circledw}))) = \begin{bmatrix} A_1^{\circledw} & & \\ & \ddots & \\ & & A_p^{\circledw} \end{bmatrix} = \mathrm{DFT}\left(\mathrm{Circ}\left(\mathrm{Unfold}\left(\mathcal{U}*\begin{bmatrix} \mathcal{T}^{-1} & \mathcal{T}^{-2}*\mathcal{S} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}*\mathcal{U}^*\right)\right)\right).$$

Therefore,

$$\mathcal{A}^{\circledw} = \mathcal{U}*\begin{bmatrix} \mathcal{T}^{-1} & \mathcal{T}^{-2}*\mathcal{S} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}*\mathcal{U}^*.$$

□

According to the decomposition of $\mathcal{A}^\dagger$ and $\mathcal{A}^{\circledw}$, we can easily get the following two inferences.

Corollary 3.13. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$. Then we have*

$$\mathcal{A}^{\dagger,WG} = \mathcal{A}^\dagger*\mathcal{A}^{\circledw}*\mathcal{A} = \mathcal{U}*\begin{bmatrix} \mathcal{T}^**\Delta & -\mathcal{T}^**\Delta*(\mathcal{T}^{-1}*\mathcal{S}+\mathcal{T}^{-2}*\mathcal{S}*\mathcal{N}) \\ (\mathcal{I}_{n-r}-\mathcal{N}^\dagger*\mathcal{N})*\mathcal{S}^**\Delta & (\mathcal{I}_{n-r}-\mathcal{N}^\dagger*\mathcal{N})*\mathcal{S}^**\Delta*(\mathcal{T}^{-1}*\mathcal{S}+\mathcal{T}^{-2}*\mathcal{S}*\mathcal{N}) \end{bmatrix}*\mathcal{U}^*,$$

where $\Delta = [\mathcal{T}^**\mathcal{T} + (\mathcal{I}_{n-r}-\mathcal{N}^\dagger*\mathcal{N})*\mathcal{S}^*]^{-1}$, $\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{I}_{n-r}))) = \begin{bmatrix} I_1 & & \\ & \ddots & \\ & & I_p \end{bmatrix}$, $I_i = I_{n-r}, (i = 1, \cdots, p)$.

Proof. Since $\mathcal{A}^{\dagger,WG} = \mathcal{A}^\dagger*\mathcal{A}^{\circledw}*\mathcal{A}$, then

$$\begin{aligned}
\mathcal{A}^{\dagger,WG} &= \mathrm{bcirc}^{-1}\left(\mathrm{bcirc}(\mathcal{A}^{\dagger,WG})\right) = \mathrm{bcirc}^{-1}\left(\mathrm{bcirc}(\mathcal{A}^\dagger*\mathcal{A}^{\circledw}*\mathcal{A})\right) \\
&= \mathrm{bcirc}^{-1}\left((F_p^H \otimes I_n)\begin{bmatrix} A_1^\dagger A_1^{\circledw} A_1 & & \\ & \ddots & \\ & & A_p^\dagger A_p^{\circledw} A_p \end{bmatrix}(F_p \otimes I_n)\right) \\
&= \mathcal{U}*\begin{bmatrix} \mathcal{T}^**\Delta & -\mathcal{T}^**\Delta*(\mathcal{T}^{-1}*\mathcal{S}+\mathcal{T}^{-2}*\mathcal{S}*\mathcal{N}) \\ (\mathcal{I}_{n-r}-\mathcal{N}^\dagger*\mathcal{N})*\mathcal{S}^**\Delta & (\mathcal{I}_{n-r}-\mathcal{N}^\dagger*\mathcal{N})*\mathcal{S}^**\Delta*(\mathcal{T}^{-1}*\mathcal{S}+\mathcal{T}^{-2}*\mathcal{S}*\mathcal{N}) \end{bmatrix}*\mathcal{U}^*
\end{aligned}$$

where $\Delta = [\mathcal{T}^**\mathcal{T} + (\mathcal{I}_{n-r}-\mathcal{N}^\dagger*\mathcal{N})*\mathcal{S}^*]^{-1}$. □

Remark 3.14. *Using the T-core EP decomposition, we can get that*

$$\mathcal{A}*\mathcal{A}^{\circledw}*\mathcal{A}^\dagger = \mathcal{U}*\begin{bmatrix} \mathcal{T}^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}*\mathcal{U}^* = \mathcal{A}^{\oplus}.$$

4. Relationships with other generalized inverses of tensors

In this section, we discuss the equivalence between the T-MPWG inverse and other known generalized inverses of tensors by using the T-core EP decomposition.

Theorem 4.1. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ be a complex tensor with* $\mathrm{Ind}_T(\mathcal{A}) = k$. *Then*
(i) $\mathcal{A}^{\dagger,WG} = \mathcal{A} \Leftrightarrow \mathcal{T}^2 = \mathcal{I}_r, \mathcal{S} = \mathcal{O}$ *and* $\mathcal{N} = \mathcal{O}$;
(ii) $\mathcal{A}^{\dagger,WG} = \mathcal{A}^* \Leftrightarrow \mathcal{T}*\mathcal{T}^* = \mathcal{I}_r, \mathcal{S} = \mathcal{O}$ *and* $\mathcal{N} = \mathcal{O}$;
(iii) $\mathcal{A}^{\dagger,WG} = \mathcal{P}_{\mathcal{A}} = \mathcal{A}*\mathcal{A}^\dagger \Leftrightarrow \mathcal{A}$ *is orthogonal and idempotent;*
(iv) $\mathcal{A}^{\dagger,WG} = \mathcal{Q}_{\mathcal{A}} = \mathcal{A}^\dagger*\mathcal{A} \Leftrightarrow \mathcal{T} = \mathcal{I}_r$ *and* $\mathcal{N} = \mathcal{O}$,

where $\mathrm{DFT}(\mathrm{Circ}(\mathrm{Unfold}(\mathcal{I}_r))) = \begin{bmatrix} I_1 & & \\ & \ddots & \\ & & I_p \end{bmatrix}$, $I_i = I_r, (i = 1, \cdots, p)$.

Proof. Let

$$\mathcal{A}^{\dagger,WG} = \mathcal{U}*\begin{bmatrix} \mathcal{T}^{*}*\triangle & -\mathcal{T}^{*}*\triangle*(\mathcal{T}^{-1}*\mathcal{S}+\mathcal{T}^{-2}*\mathcal{S}*\mathcal{N}) \\ (\mathcal{I}_{n-r}-\mathcal{N}^{\dagger}*\mathcal{N})*\mathcal{S}^{*}*\triangle & (\mathcal{I}_{n-r}-\mathcal{N}^{\dagger}*\mathcal{N})*\mathcal{S}^{*}*\triangle*(\mathcal{T}^{-1}*\mathcal{S}+\mathcal{T}^{-2}*\mathcal{S}*\mathcal{N}) \end{bmatrix}*\mathcal{U}^{*} = \mathcal{U}*\begin{bmatrix} \mathcal{G}_1 & \mathcal{G}_2 \\ \mathcal{G}_3 & \mathcal{G}_4 \end{bmatrix}*\mathcal{U}^{*}.$$

(i) $\mathcal{A}^{\dagger,WG} = \mathcal{A} \Leftrightarrow \begin{bmatrix} \mathcal{G}_1 & \mathcal{G}_2 \\ \mathcal{G}_3 & \mathcal{G}_4 \end{bmatrix} = \begin{bmatrix} \mathcal{T} & \mathcal{S} \\ \mathcal{O} & \mathcal{N} \end{bmatrix}$
$\Leftrightarrow \mathcal{T}^{*}*\triangle = \mathcal{T}, \mathcal{S} = \mathcal{S}*\mathcal{N}^{\dagger}*\mathcal{N}, \mathcal{N} = \mathcal{O}$ and $\mathcal{T}*(\mathcal{T}^{-1}*\mathcal{S}+\mathcal{T}^{-2}*\mathcal{S}*\mathcal{N}) = \mathcal{S}$.
$\Leftrightarrow \mathcal{T}^2 = \mathcal{I}_r, \mathcal{S} = \mathcal{O}$ and $\mathcal{N} = \mathcal{O}$.
(ii) $\mathcal{A}^{\dagger,WG} = \mathcal{A}^{*} \Leftrightarrow \begin{bmatrix} \mathcal{G}_1 & \mathcal{G}_2 \\ \mathcal{G}_3 & \mathcal{G}_4 \end{bmatrix} = \begin{bmatrix} \mathcal{T}^{*} & \mathcal{O} \\ \mathcal{S}^{*} & \mathcal{N}^{*} \end{bmatrix}$
$\Leftrightarrow \mathcal{T}^{*}*\triangle = \mathcal{T}^{*}, \mathcal{T}^{*}*(\mathcal{T}^{-1}*\mathcal{S}+\mathcal{T}^{-2}*\mathcal{S}*\mathcal{N}) = \mathcal{O},$
$(\mathcal{I}_{n-r}-\mathcal{N}^{\dagger}*\mathcal{N})*\mathcal{S}^{*}*\triangle = \mathcal{S}^{*}$ and $\mathcal{S}^{*}*(\mathcal{T}^{-1}*\mathcal{S}+\mathcal{T}^{-2}*\mathcal{S}*\mathcal{N}) = \mathcal{N}^{*}$.
$\Leftrightarrow \triangle = \mathcal{I}, \mathcal{T}^{-1}*\mathcal{S}+\mathcal{T}^{-2}*\mathcal{S}*\mathcal{N} = \mathcal{O}, \mathcal{S}*\mathcal{N}^{\dagger}*\mathcal{N} = \mathcal{O}, \mathcal{N}^{*} = \mathcal{O}$
$\Leftrightarrow \mathcal{T}*\mathcal{T}^{*} = \mathcal{I}_r, \mathcal{S} = \mathcal{O}$ and $\mathcal{N} = \mathcal{O}$.
(iii) $\mathcal{A}^{\dagger,WG} = \mathcal{P}_{\mathcal{A}} \Leftrightarrow \mathcal{A}^{\dagger,WG} = \mathcal{A}*\mathcal{A}^{\dagger}$
$\Leftrightarrow \mathcal{T}^{*}*\triangle = \mathcal{I}_r, \mathcal{T}^{-1}*\mathcal{S}+\mathcal{T}^{-2}*\mathcal{S}*\mathcal{N} = \mathcal{O},$
$(\mathcal{I}_{n-r}-\mathcal{N}^{\dagger}*\mathcal{N})*\mathcal{S}^{*}*\triangle = \mathcal{O}$ and $\mathcal{O} = \mathcal{N}*\mathcal{N}^{\dagger}$.
From the reference [15], $\mathcal{A}$ is orthogonal and idempotent.
(iv) $\mathcal{A}^{\dagger,WG} = \mathcal{Q}_{\mathcal{A}} \Leftrightarrow \mathcal{A}^{\dagger,WG} = \mathcal{A}^{\dagger}*\mathcal{A}$
$\Leftrightarrow \begin{bmatrix} \mathcal{G}_1 & \mathcal{G}_2 \\ \mathcal{G}_3 & \mathcal{G}_4 \end{bmatrix} = \begin{bmatrix} \mathcal{T}^{*}*\triangle*\mathcal{T} & \mathcal{T}^{*}*\triangle*\mathcal{S}-\mathcal{T}^{*}*\triangle*\mathcal{S}*\mathcal{N}^{\dagger}*\mathcal{N} \\ (\mathcal{I}_{n-r}-\mathcal{N}^{\dagger}*\mathcal{N})*\mathcal{S}^{*}*\triangle*\mathcal{T} & \mathcal{N}^{\dagger}*\mathcal{N}+(\mathcal{I}_{n-r}-\mathcal{N}^{\dagger}*\mathcal{N})*\mathcal{S}^{*}*\triangle*\mathcal{S}*(\mathcal{I}_{n-r}-\mathcal{N}^{\dagger}*\mathcal{N}) \end{bmatrix}$
$\Leftrightarrow \mathcal{T} = \mathcal{I}_r$ and $\mathcal{N} = \mathcal{O}$. □

Remark 4.2. *When the tensor $\mathcal{A}$ is EP, i.e. $\mathcal{A}*\mathcal{A}^{\dagger} = \mathcal{A}^{\dagger}*\mathcal{A}$, we have that*

$$\mathcal{A}^{\dagger,WG} = \mathcal{A}^{\dagger} = \mathcal{A}^{\#} = \mathcal{A}^{\textcircled{\#}} = \mathcal{A}^{\textcircled{w}} = \mathcal{A}^{\textcircled{\dagger}}.$$

In [16], Yuan and Zuo pointed out the limit expressions for some important generalized inverses. We extend it to tensors and obtain the limit expressions of the third-order tensors based on the T-product.

Lemma 4.3. *[16] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ with $\mathrm{Ind}_T(\mathcal{A}) = k$. There exists $\mathcal{A}^{(2)}$ satisfying $\mathcal{A}^{(2)}*\mathcal{A}*\mathcal{A}^{(2)} = \mathcal{A}^{(2)}$, then*

$$\mathcal{A}^{(2)}_{\mathcal{R}(\mathcal{X}*\mathcal{Y}),\mathcal{N}(\mathcal{X}*\mathcal{Y})} = \lim_{z\to 0}\mathcal{X}*(z*\mathcal{I}+\mathcal{Y}*\mathcal{A}*\mathcal{X})^{-1}*\mathcal{Y}.$$

In [14],Wang and Liu pointed out the limit expressions of the MP inverse of the third-order tensors based on the T-product.

Theorem 4.4. *[14] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ with $\mathrm{Ind}_T(\mathcal{A}) = k$. Then*

$$\mathcal{A}^{\dagger} = \lim_{z\to 0}\mathcal{A}^{*}*(z*\mathcal{I}+\mathcal{A}*\mathcal{A}^{*})^{-1}.$$

Applying Theorem 4.4 and the relationship between the T-MPWG inverse and the T-MP inverse $\mathcal{A}^{\dagger,WG} = \mathcal{A}^{\dagger}*\mathcal{A}^{\textcircled{w}}*\mathcal{A}$, we obtain the limit expression of the MPWG inverse of the tensor.

Lemma 4.5. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ with $\mathrm{Ind}_T(\mathcal{A}) = k$. Then*

$$\mathcal{A}^{\dagger,WG} = \mathcal{A}^{(2)}_{\mathcal{R}(\mathcal{A}^{\dagger}*\mathcal{A}^{k}),\mathcal{N}((\mathcal{A}^{k})^{*}*\mathcal{A}^{2})}.$$

Proof. By the definition of MPWG inverse of matrix, X_i is an outer inverse of A_i, i.e.$X_iA_iX_i = X_i$ $(i = 1, \cdots, p)$. From $X_i = A_i^{\dagger}A_i^{\textcircled{w}}A_i$, we have

$$\begin{bmatrix} X_1A_1X_1 & & \\ & \ddots & \\ & & X_pA_pX_p \end{bmatrix} = \begin{bmatrix} X_1 & & \\ & \ddots & \\ & & X_p \end{bmatrix} = \begin{bmatrix} A_1^{\dagger}A_1^{\textcircled{w}}A_1 & & \\ & \ddots & \\ & & A_p^{\dagger}A_p^{\textcircled{w}}A_p \end{bmatrix}.$$

Hence,

$$\text{DFT}\,(\text{Circ}(\text{Unfold}(\mathcal{X}*\mathcal{A}*\mathcal{X}))) = \text{DFT}\,(\text{Circ}(\text{Unfold}(\mathcal{X}))).$$

Therefore, $\mathcal{A}^{\dagger,WG} = \mathcal{A}^{\dagger,WG}*\mathcal{A}*\mathcal{A}^{\dagger,WG}$.

Using the T-core EP decomposition, we have $\mathcal{A}*\mathcal{A}^{\dagger,WG} = \mathcal{A}^{\circledW}*\mathcal{A}$. Then

$$\mathcal{N}\,(\mathcal{A}^{\circledW}*\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}^{\dagger}*\mathcal{A}^{\circledW}*\mathcal{A}) = \mathcal{N}(\mathcal{A}^{\dagger,WG}) \subseteq \mathcal{N}(\mathcal{A}*\mathcal{A}^{\dagger,WG}) = \mathcal{N}(A^{\circledW}*\mathcal{A}),$$

$$\mathcal{N}\,(\mathcal{A}^{\circledW}*\mathcal{A}) \subseteq \mathcal{N}(\mathcal{A}*\mathcal{A}^{\circledW}*\mathcal{A}) = \mathcal{N}(\mathcal{A}^{\oplus}*\mathcal{A}^2) \subseteq \mathcal{N}(\mathcal{A}^{\oplus})^2*\mathcal{A}^2) = \mathcal{N}(A^{\circledW}*\mathcal{A}),$$

so $\mathcal{N}(\mathcal{A}^{\dagger,WG}) = \mathcal{N}(A^{\circledW}*\mathcal{A}) = \mathcal{N}(\mathcal{A}^{\oplus})$. Hence,

$$\mathcal{X} \in \mathcal{N}(\mathcal{A}^{\dagger,WG}) \Leftrightarrow \mathcal{A}^2*\mathcal{X} \in \mathcal{N}(\mathcal{A}^{\oplus}) = \mathcal{N}((\mathcal{A}^k)^*),$$

$$\mathcal{X} \in \mathcal{N}(\mathcal{A}^{\dagger,WG}) \Leftrightarrow \mathcal{X} \in \mathcal{N}((\mathcal{A}^k)^**\mathcal{A}^2).$$

So we have

$$\mathcal{R}(\mathcal{A}^{\dagger}*\mathcal{A}^k) = \mathcal{R}(\mathcal{A}^{\dagger,WG}*\mathcal{A}^k) \subseteq \mathcal{R}(\mathcal{A}^{\dagger,WG}) = \mathcal{R}(\mathcal{A}^{\dagger}*\mathcal{A}^{\circledW}*\mathcal{A}) = \mathcal{R}(\mathcal{A}^{\dagger}*\mathcal{A}^k*\mathcal{Z}*\mathcal{A}) \subseteq \mathcal{R}(\mathcal{A}^{\dagger}*\mathcal{A}^k).$$

Therefore,

$$\mathcal{A}^{\dagger,WG} = \mathcal{A}^{(2)}_{\mathcal{R}(\mathcal{A}^{\dagger}*\mathcal{A}^k),\mathcal{N}((\mathcal{A}^k)^**\mathcal{A}^2)}.$$

□

From Lemma 4.4, we can get

$$\mathcal{A}^{\dagger,WG} = \mathcal{A}^{(2)}_{\mathcal{R}(\mathcal{A}^{\dagger}*\mathcal{A}^k),\mathcal{N}((\mathcal{A}^k)^**\mathcal{A}^2)} = \mathcal{A}^{(2)}_{\mathcal{R}(\mathcal{A}^{\dagger}*\mathcal{A}^k*(\mathcal{A}^k)^**\mathcal{A}^2),\mathcal{N}((\mathcal{A}^k)^**\mathcal{A}^2*\mathcal{A}^{\dagger}*\mathcal{A}^k)}.$$

Theorem 4.6. *Let* $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ *with* $\text{Ind}_T(\mathcal{A}) = k$. *Then*

$$\mathcal{A}^{\dagger,WG} = \lim_{z\to 0}\mathcal{A}^{\dagger}*\left(z*\mathcal{I} + \mathcal{A}^k*(\mathcal{A}^k)^**\mathcal{A}^3*\mathcal{A}^{\dagger}\right)^{-1}*\mathcal{A}^k*(\mathcal{A}^k)^**\mathcal{A}^2.$$

Proof. Let $\mathcal{X} = \mathcal{A}^{\dagger}, \mathcal{Y} = \mathcal{A}^k*(\mathcal{A}^k)^**\mathcal{A}^2$. Then from Lemma 4.4, we can get

$$\begin{aligned}\mathcal{A}^{\dagger,WG} &= \lim_{z\to 0}\mathcal{X}*(z*\mathcal{I} + \mathcal{Y}*\mathcal{A}*\mathcal{X})^{-1}*\mathcal{Y}\\ &= \lim_{z\to 0}\mathcal{A}^{\dagger}*\left(z*\mathcal{I} + \mathcal{A}^k*(\mathcal{A}^k)^**\mathcal{A}^2*\mathcal{A}*\mathcal{A}^{\dagger}\right)^{-1}*\mathcal{A}^k*(\mathcal{A}^k)^**\mathcal{A}^2\\ &= \lim_{z\to 0}\mathcal{A}^{\dagger}*\left(z*\mathcal{I} + \mathcal{A}^k*(\mathcal{A}^k)^**\mathcal{A}^3*\mathcal{A}^{\dagger}\right)^{-1}*\mathcal{A}^k*(\mathcal{A}^k)^**\mathcal{A}^2.\end{aligned}$$

□

An example is given to illustrate the theorem:

Example 4.7. *Consider the tensor*

$$\mathcal{A} \in \mathbb{C}^{2\times 2\times 2},\ \mathcal{A} = \left[A^{(1)}|A^{(2)}\right],$$

in Example 3.1, where

$$A^{(1)} = \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix},\ A^{(2)} = \begin{bmatrix}-1 & 1\\ 0 & 0\end{bmatrix},$$

$$A_1 = \begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix},\ A_2 = \begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix}.$$

Since $\text{Ind}_T(\mathcal{A}) = 2$,

$$\mathcal{A}^\dagger = \left[\begin{array}{cc|cc} \frac{1}{4} & 0 & -\frac{1}{4} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 \end{array}\right], \mathcal{A}^* = \left[\begin{array}{cc|cc} -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right], \mathcal{A}^2 = \left[\begin{array}{cc|cc} 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$\mathcal{A}^3 = \left[\begin{array}{cc|cc} 4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \mathcal{A}^4 = \mathcal{A}^2 * (\mathcal{A}^2)^* = \left[\begin{array}{cc|cc} 8 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$\mathcal{A}^2 * (\mathcal{A}^2)^* * \mathcal{A}^3 = \left[\begin{array}{cc|cc} 64 & 0 & -64 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \mathcal{A}^{\dagger,WG} = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right],$$

$$\begin{aligned} \mathcal{A}^2 * (\mathcal{A}^2)^* * \mathcal{A}^3 * \mathcal{A}^\dagger &= \text{fold}\left(\text{bcirc}(\mathcal{A}^2 * (\mathcal{A}^2)^* * \mathcal{A}^3)\text{unfold}(\mathcal{A}^\dagger)\right) \\ &= \text{fold}\left(\begin{bmatrix} 64 & 0 & -64 & 0 \\ 0 & 0 & 0 & 0 \\ -64 & 0 & 64 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 0 \\ -\frac{1}{4} & 0 \\ 0 & 0 \end{bmatrix}\right) = \left[\begin{array}{cc|cc} 32 & 0 & -32 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]. \end{aligned}$$

And

$$z * \mathcal{I} + \mathcal{A}^2 * (\mathcal{A}^2)^* * \mathcal{A}^3 * \mathcal{A}^\dagger = \left[\begin{array}{cc|cc} z & 0 & 0 & 0 \\ 0 & z & 0 & 0 \end{array}\right] + \left[\begin{array}{cc|cc} 32 & 0 & -32 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cc|cc} z+32 & 0 & -32 & 0 \\ 0 & z & 0 & 0 \end{array}\right],$$

$$(z * \mathcal{I} + \mathcal{A}^2 * (\mathcal{A}^2)^* * \mathcal{A}^3 * \mathcal{A}^\dagger)^{-1} = \left[\begin{array}{cc|cc} \frac{z+32}{z(z+64)} & 0 & \frac{32}{z(z+64)} & 0 \\ 0 & \frac{1}{z} & 0 & 0 \end{array}\right],$$

$$\begin{aligned} &(z * \mathcal{I} + \mathcal{A}^2 * (\mathcal{A}^2)^* * \mathcal{A}^3 * \mathcal{A}^\dagger)^{-1} * \mathcal{A}^2 * (\mathcal{A}^2)^* * \mathcal{A}^2 = \text{fold}\left(\text{bcirc}((z * \mathcal{I} + \mathcal{A}^2 * (\mathcal{A}^2)^* * \mathcal{A}^3 * \mathcal{A}^\dagger)^{-1})\text{unfold}(\mathcal{A}^2 * (\mathcal{A}^2)^* * \mathcal{A}^2)\right) \\ &= \text{fold}\left(\begin{bmatrix} \frac{z+32}{z(z+64)} & 0 & \frac{32}{z(z+64)} & 0 \\ 0 & \frac{1}{z} & 0 & 0 \\ \frac{32}{z(z+64)} & 0 & \frac{z+32}{z(z+64)} & 0 \\ 0 & 0 & 0 & \frac{1}{z} \end{bmatrix}\begin{bmatrix} 64 & 0 \\ 0 & 0 \\ -64 & 0 \\ 0 & 0 \end{bmatrix}\right) = \left[\begin{array}{cc|cc} \frac{64}{z+64} & 0 & -\frac{64}{z+64} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right], \end{aligned}$$

Therefore,

$$\lim_{z\to 0} \mathcal{A}^\dagger * \left(z * \mathcal{I} + \mathcal{A}^k * (\mathcal{A}^k)^* * \mathcal{A}^3 * \mathcal{A}^\dagger\right)^{-1} * \mathcal{A}^k * (\mathcal{A}^k)^* * \mathcal{A}^2 = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \mathcal{A}^{\dagger,WG}.$$

5. Cayley-Hamilton theorem of the T-MPWG inverse

In this section, we extend the Cayley-Hamilton theorem of the third-order tensors to the T-MPWG inverse. If bcirc($\mathcal{A}$) can be Fourier block diagonalized as:

$$\text{bcirc}(\mathcal{A}) = \left(F_p^H \otimes I_n\right)\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_p \end{bmatrix}\left(F_p \otimes I_n\right).$$

$P_{A_i}(x)$ is the characteristic polynomial of the matrix A_i,

$$P_{A_i}(x) = det(sI_n - A_i) = x^n + a_{i,n-1}x^{n-1} + \cdots + a_{i,1}x + a_{i,0},$$

where $a_{i,0} = det(A_i), i = 1, \cdots, p$.

Firstly, we introduce the concept of the T-characteristic polynomial and Cayley-Hamilton theorem for the tensor $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$.

Definition 5.1. *[8] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ be a complex tensor. Then the T-characteristic polynomial $P_T(x)$ of tensor $\mathcal{A}$ has the expression:*

$$P_T(x) := \mathrm{LCM}\left(P_{A_1}(x), P_{A_2}(x), \cdots, P_{A_p}(x)\right)$$

where LCM *means the least common multipiler.*

According to the above definition, let the T-characteristic polynomial of $\mathcal{A}$ be of order t.

$$P_T(x) = x^t + b_{t-1}x^{t-1} + \cdots + b_1 x + b_0.$$

Assuming that $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ is singular, then there exists at least one matrix A_i $(i = 1, \cdots, p)$ in $P_{A_i}(x)$ is singular, i.e., there is at least one $det(A_i) = 0$. Therefore, $b_0 = 0$.

Theorem 5.2. *[8] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ be a complex tensor, and $P_T(x)$ be the T-characteristic polynomial of $\mathcal{A}$. Then $\mathcal{A}$ satisfies the T-characteristic polynomial $P_T(x)$, which $P_T(\mathcal{A}) = \mathcal{O}$.*

Lemma 5.3. *[13] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$. The T-core EP decomposition is as*

$$\mathcal{A} = \mathcal{U} * \begin{bmatrix} \mathcal{T} & \mathcal{S} \\ \mathcal{O} & \mathcal{N} \end{bmatrix} * \mathcal{U}^*,$$

then

$$P_T(\mathcal{A}^{\oplus}) = a_1(\mathcal{A}^{\oplus})^n + a_2(\mathcal{A}^{\oplus})^{n-1} + \cdots + a_{n-1}(\mathcal{A}^{\oplus})^2 + \mathcal{A}^{\oplus} = \mathcal{O}.$$

Lemma 5.4. *[7] Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$. The T-core EP decomposition is as*

$$\mathcal{A} = \mathcal{U} * \begin{bmatrix} \mathcal{T} & \mathcal{S} \\ \mathcal{O} & \mathcal{N} \end{bmatrix} * \mathcal{U}^*,$$

then

$$P_T(\mathcal{A}^{Ⓦ}) = a_1(\mathcal{A}^{Ⓦ})^n + a_2(\mathcal{A}^{Ⓦ})^{n-1} + \cdots + a_{n-1}(\mathcal{A}^{Ⓦ})^2 + \mathcal{A}^{Ⓦ} = \mathcal{O}.$$

Theorem 5.5. *Let $\mathcal{A} \in \mathbb{C}^{n\times n\times p}$ with $\mathrm{Ind}_T(\mathcal{A}) = k$. The characteristic polynomial of the matrix A_i is $P_{A_i}(s) = det(sI_n - A_i) = s^n + a_{i,n-1}s^{n-1} + \cdots + a_{i,1}s$, then*

$$P_T(\mathcal{A}^{\dagger,WG}) = a_1(\mathcal{A}^{\dagger,WG})^n + a_2(\mathcal{A}^{\dagger,WG})^{n-1} + \cdots + a_{n-1}(\mathcal{A}^{\dagger,WG})^2 + \mathcal{A}^{\dagger,WG} = \mathcal{O}.$$

$\mathcal{A}^{\dagger,WG} \in \mathbb{C}^{n\times n\times p}$ is the T-MPWG inverse of $\mathcal{A}$.

Proof. Since $P_T(\mathcal{A}^{\dagger,WG})$ is a tensor on $\mathbb{C}^{n\times n\times p}$, we apply bcirc to it:

$$\begin{aligned}
\mathrm{bcirc}\left(P_T(\mathcal{A}^{\dagger,WG})\right) &= P_T\left(\mathrm{bcirc}(\mathcal{A}^{\dagger,WG})\right) \\
&= P_T\left((F_p^H \otimes I_n)\begin{bmatrix} A_1^{\dagger,WG} & & & \\ & A_2^{\dagger,WG} & & \\ & & \ddots & \\ & & & A_p^{\dagger,WG} \end{bmatrix}(F_p \otimes I_n)\right) \\
&= (F_p^H \otimes I_n)\begin{bmatrix} P_T(A_1^{\dagger,WG}) & & & \\ & P_T(A_2^{\dagger,WG}) & & \\ & & \ddots & \\ & & & P_T(A_p^{\dagger,WG}) \end{bmatrix}(F_p \otimes I_n) \\
&= (F_p^H \otimes I_n)\begin{bmatrix} O & & & \\ & O & & \\ & & \ddots & \\ & & & O \end{bmatrix}(F_p \otimes I_n) = O.
\end{aligned}$$

According to the literature [8], we obtain

$$P_T(\mathcal{A}^{\dagger,WG}) = a_1(\mathcal{A}^{\dagger,WG})^n + a_2(\mathcal{A}^{\dagger,WG})^{n-1} + \cdots + a_{n-1}(\mathcal{A}^{\dagger,WG})^2 + \mathcal{A}^{\dagger,WG} = \mathcal{O}.$$

□

Here is an example to illustrate:

Example 5.6. *Consider tensor*

$$\mathcal{A} \in \mathbb{C}^{2\times2\times2}, \mathcal{A} = \left[A^{(1)}|A^{(2)}\right],$$

in Example 3.1, where

$$A^{(1)} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A^{(2)} = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix},$$

and

$$\mathcal{A}^{\dagger,WG} = \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Applying the discrete Fourier transform

$$\text{bcirc}(\mathcal{A}^{\dagger,WG}) = \left(F_2^H \otimes I_2\right) \text{Diag}\left(A_1^{\dagger,WG}, A_2^{\dagger,WG}\right)(F_2 \otimes I_2),$$

we obtain

$$A_1^{\dagger,WG} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2^{\dagger,WG} = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Thus

$$P_{A_1^{\dagger,WG}}(x) = \left|xE - A_1^{\dagger,WG}\right| = \begin{vmatrix} x-1 & -1 \\ 0 & x \end{vmatrix} = x^2 - x,$$

$$P_{A_2^{\dagger,WG}}(x) = \left|xE - A_2^{\dagger,WG}\right| = \begin{vmatrix} x+1 & 1 \\ 0 & x \end{vmatrix} = x^2 + x,$$

so

$$P_{A_1^{\dagger,WG}}\left(A_1^{\dagger,WG}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^2 - \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = O,$$

$$P_{A_2^{\dagger,WG}}\left(A_2^{\dagger,WG}\right) = \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}^2 - \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} = O.$$

Therefore

$$\begin{aligned}
\text{bcirc}\left(P_T(\mathcal{A}^{\dagger,WG})\right) &= P_T\left(\text{bcirc}(\mathcal{A}^{\dagger,WG})\right) \\
&= P_T\left((F_2^H \otimes I_2)\begin{bmatrix} A_1^{\dagger,WG} & \\ & A_2^{\dagger,WG} \end{bmatrix}(F_2 \otimes I_2)\right) \\
&= (F_2^H \otimes I_2)\begin{bmatrix} P_T(A_1^{\dagger,WG}) & \\ & P_T(A_2^{\dagger,WG}) \end{bmatrix}(F_2 \otimes I_2) \\
&= (F_2^H \otimes I_2)\begin{bmatrix} O & \\ & O \end{bmatrix}(F_2 \otimes I_2) = O.
\end{aligned}$$

Hence, $P_T(\mathcal{A}^{\dagger,WG}) = a_1(\mathcal{A}^{\dagger,WG})^2 + \mathcal{A}^{\dagger,WG} = \mathcal{O}.$

Acknowledgement

The authors would like to thank the referee for their valuable comments, which have significantly improved the paper.

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