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J-spaces and C-normal spaces: An algebraic perspective

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Abstract. In this article, algebraic characterisations of J-spaces and C-normal spaces are exhibited. The concept of a Z-connected ideal in $C(X)$ is presented and characterised using certain connected subsets of X. We define the class of JC-spaces and characterise its members via Z-connected ideals. Two more classes of ideals in $C(X)$, namely the *coz*-free and F-free ideals, are instituted. These types of ideals are used to establish conditions under which a given space is a strong J-space. We introduce the notion of a J-lattice and show that the lattice, $\mathrm{CL}(X)$, of closed subsets of X is a J-lattice if and only if X is a J-space. A pointfree topology exposition of J-lattices is also presented, with more attention to complete Boolean algebras.

1. Introduction

Michael formally introduced J-spaces in 2000, where he observed that one characterisation of these spaces is reminiscent of one aspect of the *Jordan Curve Theorem* (see [9]). We recall that a space is called a J-space if every binary closed cover with a compact intersection has a property that one member of the cover must be compact. These spaces have since been studied by a few authors (for example, see Gao [5], Manoussos [8], Mthethwa and Taherifar [11]). We underscore that spaces satisfying conditions which are stronger than those defining J-spaces had already been examined by Nowiński's in his 1972 paper without a name (see [12]), and Michael refers to such spaces as "strong J-spaces". A space X is called strong J-space if for every compact $K \subseteq X$ there is another compact $L \subseteq X$ which contains K such that $X \setminus L$ is connected.

Herein, we introduce *JC-spaces*, the class of topological spaces obtained by substituting compactness with connectedness in the definition of a J-space. Connected spaces are JC-spaces. The fact that the converse of the latter is not true is witnessed through the observation that if a space X can be expressed as a union of two of its disjoint connected subsets, then it is a JC-space. In particular, we show that the remainder of the real line in its Stone-Čech compactification is a disconnected JC-space.

Recall from [11] that a space X is *C-normal* if two disjoint closed connected subsets can be separated by two disjoint open sets. This paper presents algebraic characterizations of J-spaces, JC-spaces, and C-normal spaces. We are interested in characterizing these phenomena via some ideals in $C(X)$.

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Let I be an ideal of $C(X)$ and $A \subseteq X$. Then the set $I_{C(A)}$ consisting of all restrictions of functions in I from X to A will be of utmost importance. We note that $I_{C(A)}$ is not necessarily an ideal in $C(X)$; it is an ideal precisely when A is a C-embedded subspace. An ideal in $C(X)$ shall be called a Z-connected ideal if $I_{C(A)}$ is contained in no principal generated by a proper idempotent. An ideal I in $C(X)$ is called R-connected if whenever J, K are two orthogonal ideals in $C(X)$ and $I + J + K$ is free, then either $I + J$ or $I + K$ is free. We show that a space is C-normal if and only if any two Z-connected ideals whose sum is free have a product that is not R-connected. Every Z-connected ideal in $C(X)$ is R-connected. The converse of the latter is not true. We also introduce two other classes of ideals in $C(X)$, namely the F-free and the *coz*-free ideals. Several properties and characterizations of F-free and *coz*-free ideals are established. More specifically, we show that a space X is compact if and only if every ideal in $C(X)$ is F-free. As one of our main results, we show that a space X is a J-space if and only if any two orthogonal ideals in $C(X)$ whose sum is F-free have a property that at least one of them is F-free. A space X is a *strong J-space* (see [9]) if for every compact $K \subseteq X$ there is another compact $L \subseteq X$ which contains K such that $X \setminus L$ is connected. We show that a space X is a strong J-space if and only if whenever I is a closed fixed F-free ideal in $C(X)$, there exists an F-free ideal J in $C(X)$ such that $J \subseteq I$ and J is a *coz*-free ideal. Again, in [9], a space X is a *weak J-space* if, whenever $\{A, B, K\}$ is a closed covering of X with K compact and $A \cap B = \emptyset$, then A or B is compact. We show that a space X is a weak J-space if and only if for every three orthogonal ideals I, J and L in $C(X)$ such that $I + J$ is free, and L is F-free, either I or J is F-free.

As usual, we denote the *support* of a real-valued function f on X by $\mathrm{cl}(X \setminus Z(f))$; where $Z(f)$ is the zero-set of f. Taking a cue from [6], we denote the family of all functions in $C(X)$ having compact support by $C_K(X)$. Amongst other things, we show that $C_K(X)$ is F-free if and only if X is a non-compact locally compact space. Other general results which appear to be of independent interest are distributed throughout the paper, one of which is that a space X is connected if and only if for every two orthogonal ideals I, J in $C(X)$ with $I + J$ free, either I or J is free.

We close the paper with a lattice-theoretic slant on J-spaces by introducing J-lattices. A glance at the bounded distributive lattice of closed subsets of a space delivers a result which asserts that such a lattice is a J-lattice if and only if the space is a J-space. Conditions under which the coframe of sublocales of a given frame is a J-lattice are given.

We would like to refer the reader to [4] and [6] for basic concepts on topology and rings of continuous maps that we use without defining them in this paper. Before we kick off, let us mention that, for convenience, we shall impose the blanket assumption of Hausdorffness and complete regularity on all our space, although some of our results are valid without these assumptions.

2. An algebraic characterisation of C-normal spaces

Before we proceed to our first result, a word on notation is in order: for a subset A of a space X and an ideal I in $C(X)$, the restriction of I to $C(A)$ is denoted by $I_{C(A)}$, and it is given by:

$$I_{C(A)} = \{f|_A : f \in I\}.$$

We note that, in general, $I_{C(A)}$ need not be an ideal of $C(A)$. We shall show that, for every ideal I of $C(X)$, $I_{C(A)}$ is an ideal of $C(A)$ precisely when A is a C-embedded subspace of X; viz., for every $g \in C(A)$ there is some $f \in C(X)$ such that $g = f|_A$.

Proposition 2.1. *Let A be a subspace of a space X. Then for every ideal I in $C(X)$, $I_{C(A)}$ is an ideal of $C(A)$ if and only if A is a C-embedded subspace of X.*

Proof. ($\Longrightarrow$) Let $g \in C(A)$ and $I = C(X)$. Since $1|_A \in I_{C(A)}$, by the hypothesis, $g.1 = g \in I_{C(A)}$. So there exists $f \in C(X)$ such that $g = f|_A$.

($\Longleftarrow$) Define $\phi : C(X) \to C(A)$ by $\phi(f) = f|_A$. Since for two elements $f, g \in C(X)$, we have $f|_A + g|_A = (f + g)|_A$ and $(fg)|_A = f|_A.g|_A$, ϕ is a ring homomorphism. Now, A is C-embedded in X, so ϕ is onto. Thus, for each ideal I in $C(X)$, $\phi(I) = I_{C(A)}$ is an ideal of $C(A)$. □

Let us recall that an idempotent of a ring is *proper* if it is not equal to zero or one. As usual, (f) shall denote the principal ideal generated by $f \in C(X)$ and for an ideal I in $C(X)$, we write $Z[I] = \{Z(f) : f \in I\}$. An ideal I in $C(X)$ is *fixed* if $\bigcap Z[I]$ is nonempty and *free* if $\bigcap Z[I]$ is empty.

Definition 2.2. Let I be an ideal in $C(X)$ and $A = \bigcap Z[I]$. We say that I is *Z-connected* if $I_{C(A)}$ is contained in no principal ideal (f) for any proper idempotent f in $C(A)$.

In [9], a subset A of a topological space X is called *relatively connected* in X if no open $U \supseteq A$ in X has a disjoint, open cover $\{U_1, U_2\}$ with $U_1 \cap A \neq \emptyset$ and $U_2 \cap A \neq \emptyset$. Relatively connected subsets were further studied in [11]. In the following definition, we introduce another class of ideals in $C(X)$, which we shall utilize to characterize relatively connected subsets and, subsequently, C-normal spaces.

Definition 2.3. An ideal I in $C(X)$ is called *R-connected* if whenever J, K are two orthogonal ideals in $C(X)$ and $I + J + K$ is free, then either $I + J$ or $I + K$ is free.

Lemma 2.4. *The following statements hold.*

1. *An ideal I in $C(X)$ is Z-connected if and only if $\bigcap Z[I]$ is a connected subset of X.*
2. *An ideal I in $C(X)$ is R-connected if and only if $\bigcap Z[I]$ is a relatively connected subset of X.*
3. *Every Z-connected ideal in $C(X)$ is an R-connected. The converse is not true.*

Proof. (1) Suppose I is a Z-connected ideal in $C(X)$. Put $A = \bigcap Z[I]$. Let G be a non-empty clopen subset in A. Then $G = Z(f)$ for some proper idempotent $f \in C(A)$. Thus for each $g \in I$, we have $Z(f) \subseteq Z(g|_A)$. This implies $Z(f) \subseteq \operatorname{int}_A Z(g|_A)$. By [6, 1D.1], $g|_A$ is a multiple of f. Thus $I_{C(A)} \subseteq (f)$, which is a contradiction since I is a Z-connected. Conversely, if $A = \bigcap Z[I]$ is connected, then the only idempotents in $C(A)$ are 0 and 1, by [6, 1B.4]. This implies that $I_{C(A)}$ is not contained in a principal ideal generated by a proper idempotent. So I is a Z-connected ideal.

(2) Suppose I is an R-connected ideal and let $A = \bigcap Z[I]$. To show that A is relatively connected in X, we use the equivalence of (1) and (5) in [11, Lemma 2.1]. So, consider two closed sets C, D in X such that $X = C \cup D$ and $C \cap D \subseteq X \setminus A$. Since X is a completely regular space, there are ideals J, K in $C(X)$ such that $C = \bigcap Z[J]$ and $D = \bigcap Z[K]$. The equality $X = C \cup D$ implies $JK = 0$, and $C \cap D \subseteq X \setminus A$ implies $\bigcap Z[I] \cap \bigcap Z[J] \cap \bigcap Z[K] = \emptyset$, hence $\bigcap Z[I + J + K] = \emptyset$. Thus, we have that K, J are orthogonal and $I + J + K$ is free. Hence $I + J$ or $I + K$ is free, by the hypothesis. Let us assume that $I + J$ is free. Then $A \cap C = \bigcap Z[I + J] = \emptyset$, i.e., $C \subseteq X \setminus A$. Hence $A = \bigcap Z[I]$ is relatively connected, by [11, Lemmma 2.1]. For the converse, suppose $\bigcap Z[I]$ is relatively connected in X and let $I + J + K$ be a free ideal where J, K are orthogonal ideals in $C(X)$. Then we have,

$$\bigcap Z[I] \cap \bigcap Z[J] \cap \bigcap Z[K] = \emptyset \quad \text{and} \quad \bigcap Z[K] \cup \bigcap Z[J] = X.$$

Therefore $\bigcap Z[J] \cap \bigcap Z[K] \subseteq X \setminus (\bigcap Z[I])$. By the hypothesis and [11, Lemma 2.1],

$$\bigcap Z[J] \subseteq X \setminus (\bigcap Z[I]) \quad \text{or} \quad \bigcap Z[K] \subseteq X \setminus (\bigcap Z[I]).$$

This implies that

$$\bigcap Z[I + J] = \bigcap Z[J] \cap \bigcap Z[I] = \emptyset \quad \text{or} \quad \bigcap Z[I + K] = \bigcap Z[K] \cap \bigcap Z[I] = \emptyset.$$

Hence either $I + J$ or $I + K$ is free.

(3) Since every connected subset is relatively connected in X, the result follows from (1) and (2). The converse is not true since not every relatively connected subset is connected (e.g., see [9, Example 9.8]). □

An easy direct consequence of part (1) of the above lemma is that the zero ideal in $C(X)$ is Z-connected if and only if X is a connected space.

The product of two Z-connected (resp., R-connected) ideals need not be a Z-connected (resp., R-connected) ideal. Consider two principal ideals $I = (f)$ and $J = (g)$, where $f, g \in C(\mathbb{R})$, $Z(f) = [0, 1]$ and $Z(g) = [2, 3]$. Then by Lemma 2.4, I, J are two Z-connected ideals but $IJ = (fg)$ is not a Z-connected (resp., R-connected) ideal. However, we have the following result:

Proposition 2.5. *If I, J are Z-connected (resp., R-connected) ideals where $I + J$ is a fixed ideal, then IJ is a Z-connected (resp., R-connected) ideal.*

Proof. Suppose I, J are Z-connected (resp., R-connected) ideals in $C(X)$. By Lemma 2.4, $\bigcap Z[I]$ and $\bigcap Z[J]$ are connected (resp., relatively connected). If $I + J$ is a fixed ideal, then $(\bigcap Z[I]) \cap (\bigcap Z[J]) = \bigcap Z[I + J] \neq \emptyset$. Thus $\bigcap Z[IJ] = (\bigcap Z[I]) \cup (\bigcap Z[J])$ is connected (resp., relatively connected). Hence IJ is a Z-connected (resp., R-connected) ideal, by Lemma 2.4. □

Recall from [11] that a space X is *C-normal* if any two disjoint closed connected subsets can be separated by two disjoint open sets. We characterise this class of spaces using Z-connected ideals whose sum is free.

Theorem 2.6. *The following statements are equivalent.*

1. *A space X is C-normal.*
2. *For any two Z-connected ideals I, J in $C(X)$ where $I + J$ is free, there are two orthogonal ideals I_1, J_1 in $C(X)$ such that $I + I_1$ and $J + J_1$ are free.*
3. *For any two Z-connected ideals I, J such that $I + J$ is free, IJ is never R-connected.*

Proof. (1) $\Longrightarrow$ (2) Let I, J be two Z-connected ideals in $C(X)$ such that $I + J$ is a free ideal. Note that the sets $A = \bigcap Z[I]$ and $B = \bigcap Z[J]$ are closed in X, and by Lemma 2.4, they are also connected subsets of X. But $I + J$ is free, so $A \cap B = \emptyset$. By the hypothesis, there are two disjoint open sets U, V in X such that $A \subseteq U$ and $B \subseteq V$. By complete regularity of X, we can find two subsets $\{f_\alpha : \alpha \in \Lambda\}$ and $\{g_\alpha : \alpha \in \Gamma\}$ of $C(X)$ such that $U = \bigcup_{\alpha\in\Lambda}(X \setminus Z(f_\alpha))$ and $V = \bigcup_{\alpha\in\Gamma}(X \setminus Z(g_\alpha))$. Let $I_1 =< f_\alpha : \alpha \in \Lambda >$ and $J_1 =< g_\alpha : \alpha \in \Gamma >$. Then $I_1J_1 = 0$, since $U \cap V = \emptyset$. Now, $A \subseteq U$ implies $\bigcap Z[I + I_1] = (\bigcap Z[I]) \cap (\bigcap Z[I_1]) = \emptyset$, and $B \subseteq V$ implies $\bigcap Z[J + J_1] = (\bigcap Z[J]) \cap (\bigcap Z[J_1]) = \emptyset$. Thus $I + I_1$ and $J + J_1$ are free ideals in $C(X)$.

(2) $\Longrightarrow$ (1) Let A, B be two disjoint closed connected subsets of X. Since X is a completely regular space, by [6, Theorem 3.2], there are two ideals I, J in $C(X)$ such that $A = \bigcap Z[I]$ and $B = \bigcap Z[J]$. From the equality $A \cap B = \emptyset$, we get $\bigcap Z[I + J] = (\bigcap Z[I]) \cap (\bigcap Z[J]) = \emptyset$. Thus, $I + J$ is a free ideal. Lemma 2.4, together with the fact that A and B are connected, implies that I and J are Z-connected in $C(X)$. By the hypothesis, there are two orthogonal ideals I_1, J_1 in $C(X)$ such that $I + I_1$ and $J + J_1$ are free. Let $U = \bigcup_{f\in I_1}(X \setminus Z(f))$ and $V = \bigcup_{g\in J_1}(X \setminus Z(g))$. Since $I_1J_1 = 0$, then $U \cap V = \emptyset$. But $I + I_1$ and $J + J_1$ are free, so $A \cap (\bigcap Z[I_1]) = \emptyset$ and $B \cap (\bigcap Z[J_1]) = \emptyset$. Thus $A \subseteq U$ and $B \subseteq V$.

(1) $\Longrightarrow$ (3) Let I, J be two Z-connected ideals in $C(X)$ such that $I + J$ is free. Then $\bigcap Z[I]$ and $\bigcap Z[J]$ are connected subsets of X, and $\bigcap Z[I] \cap \bigcap Z[J] = \bigcap Z[I+J] = \emptyset$. By [11, Proposition 2.8], $\bigcap Z[IJ] = \bigcap Z[I] \cup \bigcap Z[J]$ is not relatively connected. Hence IJ is not an R-connected ideal, by Lemma 2.4.

(3) $\Longrightarrow$ (1) Let A, B be two disjoint closed connected subsets of X. Then, by [6, Theorem 3.2], there are two ideals I, J in $C(X)$ such that $A = \bigcap Z[I]$ and $B = \bigcap Z[J]$. Hence $\bigcap Z[I + J] = \bigcap Z[I] \cap \bigcap Z[J] = A \cap B = \emptyset$, i.e., $I + J$ is free. By the hypothesis, IJ is not an R-connected ideal. Hence $A \cup B = \bigcap Z[IJ]$ is not relatively connected by Lemma 2.4. This implies that X is C-normal, by [11, Proposition 2.8]. □

3. *JC*-spaces

Given two orthogonal ideals I and J in $C(X)$, such that $I + J$ is Z-connected, under what conditions is one of these ideals Z-connected? To answer this question, we introduce a new class of topological spaces containing the class of connected spaces as a proper subclass.

Definition 3.1. We say that a space X is a *JC-space* if whenever $X = A \cup B$, where A, B are closed subsets of X and $A \cap B$ is connected, then A or B is connected.

Proposition 3.2. *Every connected space is a JC-space.*

Proof. Suppose X is connected and let $X = A \cup B$, where A, B are two closed subsets of X with $A \cap B$ connected. Let us proceed by contradiction and suppose that, say, A is disconnected. Then we can write $A = A_1 \cup A_2$ for some disjoint closed sets A_1, A_2 in A. Since A is closed and $A \cap B$ is connected in X, then $A \cap B$ is connected in A, and whence $A \cap B \subseteq A_1$ or $A \cap B \subseteq A_2$. Suppose $A \cap B \subseteq A_2$. Then $X = A_1 \cup (A_2 \cup B)$ and $A_1 \cap (A_2 \cup B) = \emptyset$. But A_1 and $A_2 \cup B$ are closed subsets of X, which contradicts the connectedness of X. Hence, A is connected. Similarly, we can show that B is connected. □

From the proof of the previous proposition, we see that if a connected space X is the union of two of its closed subsets with a connected intersection, then both of these two closed subsets are connected. This is not true for disconnected spaces:

Example 3.3. Consider the rational numbers $\mathbb{Q}$ as a subspace of the real line $\mathbb{R}$ with the standard topology. Put $A = (-\infty, 0] \cap \mathbb{Q}$ and $B = [0, \infty) \cap \mathbb{Q}$. Then A and B are closed in $\mathbb{Q}$ with a connected intersection $A \cap B = \{0\}$, and $\mathbb{Q} = A \cup B$. However, neither A nor B is connected in $\mathbb{Q}$. Thus, $\mathbb{Q}$ is not a JC-space.

The class of JC-spaces ought to be bigger than that of connected spaces. We shall use the following result to produce an example of a disconnected JC-space.

Theorem 3.4. *If a space X is a union of two disjoint connected subsets, then X is a JC-space.*

Proof. Suppose $X = U_1 \cup U_2$ with $U_1 \cap U_2 = \emptyset$ where U_1, U_2 are connected subset of X. To show that X is a JC-space, let $X = A \cup B$, where A, B are closed subsets of X such that $A \cap B$ is connected. We need to show that either A or B is connected, so suppose $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ for some disjoint closed subsets A_1, A_2 contained in A and disjoint closed subsets B_1, B_2 contained in B. Then we have $X = (A_1 \cup A_2) \cup (B_1 \cup B_2)$. Since $A \cap B$ is connected in X, it is connected in A and B. Hence, $A \cap B \subseteq A_1$ or $A \cap B \subseteq A_2$, and similarly, $A \cap B \subseteq B_1$ or $A \cap B \subseteq B_2$. We assume $A \cap B \subseteq A_1$ and $A \cap B \subseteq B_1$. Now, $X = (A_1 \cup B_1) \cup (A_2 \cup B_2)$, where $A_1 \cup B_1$ and $A_2 \cup B_2$ are two disjoint closed subsets of X. Since U_1, U_2 are connected subsets of X, we must have $U_1 \subseteq A_1 \cup B_1$ or $U_1 \subseteq A_2 \cup B_2$, and similarly, $U_2 \subseteq A_1 \cup B_1$ or $U_2 \subseteq A_2 \cup B_2$. Note that if we assume, without loss of generality, that both U_1, U_2 are contained in $A_1 \cup B_1$, we must have $A_2 \cup B_2 = \emptyset$. This implies that $A_2 = \emptyset$ and $B_2 = \emptyset$, and whence both A, B are connected. So we may assume that $U_1 \subseteq A_1 \cup B_1$ and $U_2 \subseteq A_2 \cup B_2$. From this, we get $U_1 = A_1 \cup B_1$ and $U_2 = A_2 \cup B_2$, since U_1, U_2 are also disjoint. But U_2 is connected, so the equality $U_2 = A_2 \cup B_2$ implies that $A_2 = \emptyset$ or $B_2 = \emptyset$. Therefore A is connected, or B is connected. □

In the following example, the real line is considered with the standard topology:

Example 3.5. (1) Consider the space $X = (-\infty, 0) \cup (0, \infty)$ as a subspace of $\mathbb{R}$. Then X is a disconnected JC-space, by Theorem 3.4.

(2) A disjoint union of three connected spaces need not be a JC-space. Consider $X = (-3, 0) \cup (0, 1) \cup (1, 2)$ as a subspace of $\mathbb{R}$. Then X is not a JC-space since $A = (-3, 0) \cup (0, 1)$ and $B = (0, 1) \cup (1, 2)$ are two closed subsets of X and $A \cap B = (0, 1)$ is connected, but neither A nor B is connected.

(3) Let $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the subspaces of $\mathbb{R}$ consisting of nonnegative and nonpositive real numbers respectively. Consider $\beta\mathbb{R}$, the Stone-Čech compactification of $\mathbb{R}$. By [6, 6.10.b], $\mathrm{cl}_{\beta\mathbb{R}} \mathbb{R}^+ \setminus \mathbb{R}^+$ and $\mathrm{cl}_{\beta\mathbb{R}} \mathbb{R}^- \setminus \mathbb{R}^-$ are disjoint connected subsets of $\beta\mathbb{R}$ such that $\beta\mathbb{R} \setminus \mathbb{R} = (\mathrm{cl}_{\beta\mathbb{R}} \mathbb{R}^+ \setminus \mathbb{R}^+) \cup (\mathrm{cl}_{\beta\mathbb{R}} \mathbb{R}^- \setminus \mathbb{R}^-)$. Hence, by Theorem 3.4, $\beta\mathbb{R} \setminus \mathbb{R}$ is a disconnected JC-space.

Recall from [4] that a continuous mapping $f : X \longrightarrow Y$ is *monotone* if all fibers $f^{-1}(y)$ are connected.

Theorem 3.6. *Let $f : X \to Y$ be a monotone closed mapping from X onto Y. Then X is a JC-space if and only if Y is a JC-space.*

Proof. ($\Longrightarrow$) Let X be a JC-space and $\{A, B\}$ be a closed cover for Y with $A \cap B$ connected. Then $\{f^{-1}(A), f^{-1}(B)\}$ is a closed cover for X and $f^{-1}(A) \cap f^{-1}(B)$ is connected because $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B)$ and $f^{-1}(A \cap B)$ is connected, by [4, Theorem 6.1.29]. Thus $f^{-1}(A)$ or $f^{-1}(B)$ is connected. Hence $A = f(f^{-1}(A))$ or $B = f(f^{-1}(B))$ is connected. Thus Y is a JC-space.

($\Longleftarrow$) Let $\{H, K\}$ be a closed cover for X and $H \cap K$ be connected. Then $\{f(H), f(K)\}$ is a closed cover for Y, and $f(H) \cap f(K) = f(H \cap K)$ by [9, Lemma 5.5], hence $f(H) \cap f(K)$ is connected in Y. Thus $f(H)$ or $f(K)$ is connected in Y. Hence $f^{-1}(f(H))$ or $f^{-1}(f(K))$ is connected in X, by [4, Theorem 6.1.29]. To see that H or K is connected, let $\{H_1, H_2\}$ and $\{K_1, K_2\}$ be two disjoint closed cover for H and K, respectively. Then we have

$$X = f^{-1}(f(H)) \cup f^{-1}(f(K)) = H \cup K = (H_1 \cup H_2) \cup (K_1 \cup K_2).$$

Since $H \cap K$ is a connected subset of H and K, then $H \cap K \subseteq H_1$ or $H \cap K \subseteq H_2$ and similarly, $H \cap K \subseteq K_1$ or $H \cap K \subseteq K_2$. Without loss of generality, suppose $H \cap K \subseteq H_1$ and $H \cap K \subseteq K_1$. Then $H \cap K \subseteq H_1 \cup K_1$ and $\{H_1 \cup K_1, H_2 \cup K_2\}$ is a disjoint closed cover for X. By connectedness of $f^{-1}(f(H))$, we have $H_1 \cup H_2 = H \subseteq f^{-1}(f(H)) \subseteq H_1 \cup K_1$ or $H_1 \cup H_2 = H \subseteq f^{-1}(f(H)) \subseteq H_2 \cup K_2$. Thus $H_2 \subseteq K_1$ or $H_1 \subseteq K_2$. Hence $H_2 \subseteq K_1 \cap (H_2 \cup K_2) = \emptyset$ or $H_1 \subseteq (H_1 \cup K_1) \cap K_2 = \emptyset$. Thus H is connected. □

Corollary 3.7. *Let Z be a connected space and Y be any space. Then $Z \times Y$ is a JC-space if and only if Y is a JC-space.*

Proof. By Theorem 3.6, with $X = Z \times Y$ and f being the projection map $f = \pi_Y : X \to Y$. □

Corollary 3.8. *If $f : X \to Y$ is a monotone quotient mapping from X onto Y and X is a JC-space, then Y is a JC-space.*

Proof. The argument is similar to the one provided in the forward direction of Theorem 3.6, with [4, Theorem 6.1.28] being used instead of [4, Theorem 6.1.29]. □

We conclude this section by addressing the question that was discussed in the introduction of this section:

Theorem 3.9. *A space X is a JC-space if and only if for each two orthogonal ideals I, J in $C(X)$ such that $I + J$ is Z-connected, either I or J is Z-connected.*

Proof. ($\Longrightarrow$) Let X be a JC-space and I, J be two orthogonal ideals in $C(X)$ such that $I + J$ is Z-connected. Then we have $X = \bigcap Z[I] \cup \bigcap Z[J]$, and since $\bigcap Z[I] \cap \bigcap Z[J] = \bigcap Z[I + J]$, we also have that $\bigcap Z[I] \cap \bigcap Z[J]$ is connected. By the hypothesis, $\bigcap Z[I]$ or $\bigcap Z[J]$ is connected. Hence I or J is Z-connected, by Lemma 2.4.

($\Longleftarrow$) Let $X = A \cup B$, where A, B are two closed subsets of X such that $A \cap B$ is connected. Since X is a completely regular space, there are two ideals I, J in $C(X)$ such that $A = \bigcap Z[I]$ and $B = \bigcap Z[J]$. The equality $X = A \cup B$ implies $IJ = 0$ and the connectedness of $A \cap B$ implies that $\bigcap Z[I+J] = \bigcap Z[I] \cap \bigcap Z[J]$ is connected and hence $I + J$ is a Z-connected ideal in $C(X)$. Therefore, by the hypothesis, I or J is Z-connected and hence A or B is connected, by Lemma 2.4. □

4. *F*-free ideals and *J*-spaces

We begin this section with the following definitions. The ideal generated by a subset H of functions in $C(X)$ is denoted by $< H >$.

Definition 4.1. (1) Let $A = \bigcup_{f \in I}(X \setminus Z(f))$. The ideal I in $C(X)$ is called a *coz*-free ideal if there is no proper idempotent $e \in C(A)$ such that $< I|_{C(A)} > + < e >$ is a free ideal in $C(A)$.

(2) An ideal I in $C(X)$ is called an *F-free* ideal if whenever $I + J$ is free for some ideal J in $C(X)$, then there exists a finite subset H of J such that $I+ < H >$ is a free ideal.

Since for every finite subset $\{f_1, f_2, ..., f_n\}$ of $C(X)$, we have $\bigcap_{i=1}^{n} Z(f_i) = Z(f)$, where $f = f_1^2 + f_2^2 + ... + f_n^2$. We observe that an ideal I of $C(X)$ is an F-free ideal if and only if whenever $I + J$ is free for some ideal J of $C(X)$, then there exists an element $f \in J$ such that $I+ < f >$ is free.

For trivial reasons, the zero ideal is *coz*-free, and every free ideal in $C(X)$ is an F-free ideal. Recall that for any $A \subseteq X$, $M_A = \{f \in C(X) : A \subseteq Z(f)\}$ is an ideal of $C(X)$. For every finite subset H of X, the ideal M_H is an F-free ideal, and M_H is not a free ideal. For another example of *coz*-free ideals, let x be an isolated point in a space X. Then for the ideal $M_{X\setminus\{x\}}$, we have $\bigcup_{f\in M_{X\setminus\{x\}}}(X \setminus Z(f)) = \{x\}$. The next result not only tells us that $M_{X\setminus\{x\}}$ is a *coz*-free ideal, but it is also instrumental for the remainder of this paper.

Lemma 4.2. *The following statements hold.*

1. *An ideal I in $C(X)$ is coz-free if and only if $\bigcup_{f\in I}(X \setminus Z(f))$ is a connected subset of X.*
2. *An ideal I in $C(X)$ is an F-free ideal if and only if $\bigcap Z[I]$ is compact.*

Proof. (1) For the forward direction, let $A = \bigcup_{f\in I}(X \setminus Z(f))$ and g be a proper idempotent in $C(A)$ such that $Z(g) \subseteq A$. Then $Z(g) \cap (\bigcap_{f\in I} Z(f)) = \emptyset$. This implies $Z(g) \cap (\bigcap_{f\in I} Z(f|_A)) = \emptyset$. Thus, $\bigcap Z[< g > + < I_{C(A)} >] = Z(g) \cap (\bigcap_{h\in I|_{C(A)}} Z(h)) = \emptyset$. Hence, $< g > + < I_{C(A)} >$ is a free ideal in $C(A)$, which is a contradiction since I is *coz*-free. Conversely, if the subset $A = \bigcup_{f\in I}(X \setminus Z(f))$ is connected, then the only idempotents of $C(A)$ are 0 and 1, by [6, 1.B.4]. Hence I is a *coz*-free ideal.

(2) Let I be an F-free ideal in $C(X)$ and $\bigcap Z[I] \subseteq \bigcup_{\alpha\in\Gamma}(X \setminus Z(f_\alpha))$, where each $f_\alpha \in C(X)$ and Γ is any index set. Put $J =< f_\alpha : \alpha \in \Gamma >$. Then $\bigcap Z[I + J] = (\bigcap Z[I]) \cap (\bigcap Z[J]) = \emptyset$. This shows that $I + J$ is a free ideal. By the hypothesis, there exists a finite subset $H = \{g_1, g_2, ..., g_n\}$ of J such that $I+ < H >$ is free. Thus $(\bigcap Z[I]) \cap (\bigcap_{i=1}^n Z(g_i)) = \emptyset$. For each $1 \le i \le n$ there are finitely many elements of Γ, say $\alpha_1^i, \alpha_2^i, ..., \alpha_{m_i}^i$, such that $Z(f_{\alpha_1^i}) \cap Z(f_{\alpha_2^i}) \cap ... \cap Z(f_{\alpha_{m_i}^i}) \subseteq Z(g_i)$. This implies that $(\bigcap Z[I]) \cap (\bigcap_{i=1}^n Z(f_{\alpha_{m_i}^i}) = \emptyset$, and therefore $\bigcap Z[I] \subseteq \bigcup_{i=1}^n (X \setminus Z(f_{\alpha_{m_i}^i}))$. For the converse, suppose that $I + J$ is free for some ideal J in $C(X)$. Then $(\bigcap Z[I]) \cap (\bigcap Z[J]) = \bigcap Z[I + J] = \emptyset$. This implies that $\bigcap Z[I] \subseteq \bigcup_{f\in J}(X \setminus Z(f))$. By the hypothesis, there exists a finite subset $H = \{f_1, f_2, ..., f_n\}$ of J such that $\bigcap Z[I] \subseteq \bigcup_{i=1}^n (X \setminus Z(f_i))$. This implies

$$(\bigcap Z[I]) \cap (\bigcap_{i=1}^n Z(f_i)) = \bigcap Z[I+ < H >] = \emptyset.$$

That is, $I+ < H >$ is a free ideal. □

The set $L = \{I : I \text{ is an } F\text{-free ideal of } C(X)\}$ partially ordered by set-inclusion, equipped with the following operations:

$$I \vee J = I + J \quad \text{and} \quad I \wedge J = I \cap J,$$

is a lattice. This, and other properties of F-free ideals, are given in the result below:

Proposition 4.3. *The following statements hold for the ring $C(X)$.*

1. *The sum of two F-free ideals is an F-free ideal.*
2. *The product of two F-free ideals is an F-free ideal.*
3. *The intersection of two F-free ideals is an F-free ideal.*
4. *An ideal I is an F-free ideal if and only if $\sqrt{I}$ is an F-free ideal.*
5. *If $I \subseteq J$ and I is an F-free ideal, then so is J.*

Proof. (1) Let I, J be two F-free ideals in $C(X)$. Then $\bigcap Z[I]$ and $\bigcap Z[J]$ are two compact subsets of X, by Lemma 4.2. So $\bigcap Z[I + J] = (\bigcap Z[I]) \cap (\bigcap Z[J])$ is a compact subset in X. Thus, again by Lemma 4.2, $I + J$ is an F-free ideal.

(2) For two ideals I, J in $C(X)$ we have

$$\bigcap Z[IJ] = (\bigcap Z[I]) \cup (\bigcap Z[J]).$$

If I, J are F-free, then both $\bigcap Z[I]$ and $\bigcap Z[J]$ are compact, and so $\bigcap Z[IJ]$ is compact, hence IJ is an F-free ideal, by Lemma 4.2.

(3) Let I, J be two F-free ideals in $C(X)$. We always have

$$\bigcap Z[I \cap J] = (\bigcap Z[I]) \cup (\bigcap Z[J]).$$

By an argument similar to the one provided in part (2), we are done.

(4) This follows by Lemma 4.2 and by the fact that $\bigcap Z[I] = \bigcap Z[\sqrt{I}]$.

(5) $I \subseteq J$ implies $\bigcap Z[J] \subseteq \bigcap Z[I]$. Since I is F-free, $\bigcap Z[I]$ is compact by Lemma 4.2, so $\bigcap Z[J]$ is also compact. This implies that J is F-free. □

Corollary 4.4. *The following statements are equivalent.*

1. *Every ideal in $C(X)$ is an F-free ideal.*
2. *Any intersection of F-free ideals in $C(X)$ is an F-free ideal.*
3. *The space X is compact.*

Proof. (1) $\Longrightarrow$ (2) This is trivial.

(2) $\Longrightarrow$ (3) Notice that every maximal ideal in $C(X)$ is an F-free ideal. Since the zero ideal in $C(X)$ is the intersection of all maximal ideals, it follows that the zero ideal is an F-free ideal by the hypothesis. Hence $X = \bigcap Z[0]$ is compact, by Lemma 4.2.

(3) $\Longrightarrow$(1) The compactness of X and the fact that $X = \bigcap Z[0]$ implies that the zero ideal is F-free, by Lemma 4.2. Now let I be an ideal in $C(X)$. Since $0 \subseteq I$, I is an F-free ideal, by part (5) of Proposition 4.3. □

Proposition 4.5. *The following statements hold.*

(1) If I, J are two coz-free ideals in $C(X)$ and $IJ \neq 0$, then $I + J$ is a coz-free ideal.

(2) If I, J are two ideals in $C(X)$ with $I + J$ is coz-free, then $IJ \neq 0$.

(3) Every ideal in $C(X)$ is coz-free if and only if every open set in X is connected and hence X is a connected space.

Proof. (1) Let I, J be *coz*-free ideals and $IJ \neq 0$. Then $\bigcup_{f\in I}(X \setminus Z(f))$ and $\bigcup_{g\in J}(X \setminus Z(g))$ be two connected subsets of X. $IJ \neq 0$ implies that:

$$\bigcup_{f\in I}(X \setminus Z(f)) \cap \bigcup_{g\in J}(X \setminus Z(g)) \neq \emptyset.$$

Thus $\bigcup_{f\in I}(X \setminus Z(f)) \cup \bigcup_{g\in J}(X \setminus Z(g)) = \bigcup_{h\in I+J}(X \setminus Z(h))$ is connected. This shows that $I + J$ is a *coz*-free ideal.

(2) If $IJ = 0$, then $\bigcup_{f\in I}(X \setminus Z(f)) \cap \bigcup_{g\in J}(X \setminus Z(g)) = \emptyset$. This shows that $\bigcup_{h\in I+J}(X \setminus Z(h))$ is a union of two disjoint open sets, i.e., this is a disconnected subset. Hence $I + J$ is not a *coz*-free ideal, which is a contradiction.

(3) If A is an open subset of X, then there is an ideal I of $C(X)$ such that $A = \bigcup_{f\in I}(X \setminus Z(f))$. By the hypothesis, A is connected. This implies X is connected. Conversely, let I be an ideal of $C(X)$. Then $\bigcup_{f\in I}(X \setminus Z(f))$ is connected, by the hypothesis. Hence I is a *coz*-free ideal. □

The connectedness of X in the above result does not imply that every ideal of $C(X)$ is a *coz*-free ideal. For, consider $X = \mathbb{R}$ with standard topology and $f(x) = |x|$. Then we have $X \setminus Z(f) = \mathbb{R} \setminus \{0\}$ which is disconnected. Hence the ideal $I =< f >$ is not a *coz*-free ideal.

Using parts (1) and (2) of Proposition 4.5 to obtain the next result.

Corollary 4.6. *The sum of two coz-free ideals in $C(X)$ is coz-free if and only if they are not orthogonal.*

Proposition 4.7. *The following statements are equivalent.*

1. *The ideal $C_K(X)$ is an F-free ideal.*
2. *The space X is locally compact.*
3. *The ideal $C_K(X)$ is a free ideal.*

Proof. (1) $\Longrightarrow$ (2) By [1, Lemma 2.1],

$$\bigcup_{f\in C_K(X)}(X\setminus Z(f))=X_L.$$

Thus, $\bigcap Z[C_K(X)]=X\setminus X_L$. By the hypothesis and Lemma 4.2, we have that $X\setminus X_L$ is compact. Thus we must have $X=X_L$, i.e., X is locally compact.

(2) $\Longrightarrow$ (3) If X is compact, then $C_K(X)=C(X)$, hence $C_K(X)$ is free. If X is a non-compact locally compact space, then $C_K(X)$ is free, by [6, 4D.3].

(3) $\Longrightarrow$ (1) This is trivial. □

Our next goal is to provide an algebraic characterization of J-spaces (resp., strong J-spaces) via F-free ideals. But before we do this, let us remind the reader that an ideal I in $C(X)$ is called *closed fixed* ideal if $I=\bigcap_{I\subseteq M_p}M_p$. Also, for a subset A of X, $M_A=\bigcap_{p\in A}M_p=\{f\in C(X):A\subseteq Z(f)\}$. If I is a closed fixed ideal and $A=\bigcap Z[I]$, we have that $I=M_A$. Let us also point out that, for each closed subset A of X, M_A is a closed fixed ideal in $C(X)$ since $M_A=\bigcap_{p\in A}M_p$.

Theorem 4.8. *The following statements hold.*

1. *A space X is a J-space if and only if for any two orthogonal ideals I,J in $C(X)$ where $I+J$ is F-free, either I or J is F-free.*
2. *A space X is a strong J-space if and only if whenever I is a closed fixed F-free ideal in $C(X)$, there exists an F-free ideal J in $C(X)$ such that $J\subseteq I$ and J is a coz-free ideal.*

Proof. (1) Suppose that X is a J-space. Let I,J be ideals in $C(X)$ such that $IJ=0$ and $I+J$ be F-free. Put $A=\bigcap Z[I]$ and $B=\bigcap Z[J]$. So, A and B are closed. Now, $IJ=0$ implies that $A\cup B=X$ and $A\cap B=\bigcap Z[I+J]$. Since $I+J$ is F-free, then $A\cap B$ is compact, by Lemma 4.2. By the hypothesis, A or B is compact. Again by Lemma 4.2, I or J is F-free. For the reverse direction, let $X=A\cup B$, where A,B are closed, and $A\cap B$ is compact. Since X is a completely regular space, there are two ideals I,J in $C(X)$ such that $A=\bigcap Z[I]$ and $B=\bigcap Z[J]$. The equality $A\cup B=X$ implies that $\bigcap Z[IJ]=X$, hence $IJ=0$. By compactness of $A\cap B$, we have that $\bigcap Z[I+J]$ is compact. Thus $I+J$ is F-free. Hence, by the hypothesis, I or J is F-free. Thus, A or B is compact, by Lemma 4.2.

(2) Suppose X is a strong J-space. Let I be a closed fixed and F-free ideal in $C(X)$. Then $A=\bigcap Z[I]$ is a compact subset of X, by Lemma 4.2. We also have $I=M_A$. Now, by the hypothesis, there exists a compact subset L of X such that $A\subseteq L$ and $X\setminus L$ is connected. Since X is a Hausdorff space, L is a closed subset of X. Now, observe that $L=\bigcap Z[M_L]$, therefore $M_L\subseteq M_A=I$. Let $J=M_L$. The compactness of $\bigcap Z[J]$ implies that J is an F-free ideal. Also, $X\setminus L=\bigcup_{g\in J}(X\setminus Z(g))$. Hence $\bigcup_{g\in J}(X\setminus Z(g))$ is connected, and therefore, the ideal J is *coz*-free, by Lemma 4.2. Conversely, let A be a compact subset of X. Then A is a closed subset of Hausdorff space X, hence $A=\bigcap Z[M_A]$. We know that M_A is a closed fixed ideal which is also an F-free ideal, by Lemma 4.2. So, by the hypothesis, there exists an F-free ideal J such that $J\subseteq M_A$ and J is *coz*-free. Let $B=\bigcap Z[J]$. Then B is compact, by Lemma 4.2. Moreover, $A=\bigcap Z[M_A]\subseteq\bigcap Z[J]=B$. Now, $X\setminus B=\bigcup_{g\in J}(X\setminus Z(g))$. But J is a *coz*-free ideal, therefore $X\setminus B$ is a connected subset of X, by Lemma 4.2. Thus X is a strong J-space. □

Recall from [9] that a space X is a *weak J-space* if, whenever $\{A,B,K\}$ is a closed covering of X with K compact and $A\cap B=\emptyset$, then A or B is compact. Like J-spaces and strong J-spaces, weak J-spaces can be characterised via F-free ideals. The details of the latter are provided in part (1) of the next result, while part (2) contains an independent result that characterises connectedness in terms of free ideals:

Theorem 4.9. *The following statements hold.*

1. *A space X is a weak J-space if and only if for each three orthogonal ideals I,J and L in $C(X)$ with $I+J$ free and L is F-free, either I or J is F-free.*

2. *A space X is connected if and only if for each two orthogonal ideals I, J in $C(X)$ with $I + J$ free, either I or J is free.*

Proof. (1) Suppose X is a weak J-space. Let I, J and L be three ideals in $C(X)$ such that $IJL = 0$, $I + J$ is free and L is F-free. We have that $X = \bigcap Z[IJL] = \bigcap Z[I] \cup \bigcap Z[J] \cup \bigcap Z[L]$, $\bigcap Z[I] \cap \bigcap Z[J] = \emptyset$ and $\bigcap Z[L]$ is compact (the latter follow by Lemma 4.2). Therefore $\bigcap Z[I]$ or $\bigcap Z[J]$ is compact, by the hypothesis. Thus I or J is F-compact, by Lemma 4.2. For the converse, let $X = A \cup B \cup K$, where A, B are two disjoint closed subsets of X, and K is compact (K is also closed since X is Hausdorff). By complete regularity of X, then there are three ideals I, J and L in $C(X)$ such that $A = \bigcap Z[I]$, $B = \bigcap Z[J]$ and $K = \bigcap Z[L]$. Since A and B are disjoint, we have $\bigcap Z[I + J] = \emptyset$ and hence $I + J$ is a free ideal. The compactness of K implies that L is an F-free ideal, by Lemma 4.2. By the hypothesis, either I or J is an F-free ideal. Hence A or B is compact, again by Lemma 4.2.

(2) Let X be connected and I, J be two orthogonal ideals in $C(X)$ such that $I+J$ is free. Then $\bigcap Z[I] \cap \bigcap Z[J] = \bigcap Z[I + J] = \emptyset$. The equality $IJ = 0$ implies that $\bigcap Z[I] \cup \bigcap Z[J] = X$. By the hypothesis, we must have $\bigcap Z[I] = \emptyset$ or $\bigcap Z[J] = \emptyset$, i.e., I or J is free. Conversely, suppose that $X = A \cup B$, where A, B are two disjoint closed subsets of X. By complete regularity of X, there are two ideals I, J in $C(X)$ such that $A = \bigcap Z[I]$ and $B = \bigcap Z[J]$. Thus $X = \bigcap Z[IJ]$. We now have, $IJ = 0$, and $\bigcap Z[I + J] = A \cap B = \emptyset$. By the hypothesis, either I or J is free. That is, $A = \emptyset$ or $B = \emptyset$. So, X is a connected space. □

5. *J*-lattices

In this section, all the considered lattices are complete and therefore bounded. We shall denote the top element of a lattice by 1 and the bottom element by 0.

Definition 5.1. Let L be a complete lattice. An element $a \in L$ is called *F-compact* if whenever $a \wedge (\bigwedge S) = 0$ for some $S \subseteq L$, then there exists a finite $F \subseteq S$ such that $a \wedge (\bigwedge F) = 0$.

Recall that an element a in a lattice L is *compact* if whenever $a \leq \bigvee A$ for some $A \subseteq L$, we can find a finite $A_0 \subseteq A$ such that $a \leq \bigvee A_0$. A complement of a, is an element b with the property that $a \vee b = 1$ and $a \wedge b = 0$. Recall that a *Boolean algebra*, B, is a distributive lattice where all elements have complements. Complements are not unique in general. However, in a Boolean algebra, they are. A *frame* L is a complete lattice which satisfies the infinite distributive law:

$$x \wedge \bigvee S = \bigvee \{x \wedge s : s \in S\}$$

for every $x \in L$ and every $S \subseteq L$. In a frame L, an element need not have a complement. A *coframe* L is a complete lattice satisfying infinite distributive law:

$$x \vee \bigwedge S = \bigwedge \{x \vee s : s \in S,\}$$

for every $x \in L$ and every $S \subseteq L$. For each $a \in L$ we have the *pseudocomplement* a^* of a, given by $a^* = \bigvee \{x \in L : x \wedge a = 0\}$. Thus, $a \wedge y = 0 \Longleftrightarrow a \leq y^*$. In any distributive lattice (in particular, frames), each complement is a pseudocomplement and, therefore, uniquely determined. A *Heyting algebra* is a bounded meet-semilattice equipped with a binary relation " $\rightarrow$ " (called the *Heyting operation*) satisfying:

$$c \leq a \rightarrow b \quad \text{if and only if} \quad c \wedge a \leq b.$$

Any Boolean algebra is a Heyting algebra. Frames are precisely the complete Heyting algebras. In any Heyting algebra L, we always have the *first De Morgan law:*

$$(\bigvee_{s \in S} s)^* = \bigwedge_{s \in S} s^*$$

whenever $\bigvee_{s\in S} s$ exists for $S \subseteq L$. For a Boolean algebra B, one also has the *second De Morgan law*:

$$(\bigwedge_{s\in S} s)^* = \bigvee_{s\in S} s^*$$

whenever $\bigvee_{s\in S} s$ exists for $S \subseteq B$. Moreover, in a Boolean algebra B, $a^{**} = a$ for all $a \in B$. For the proofs of all the facts mentioned above, see, for example, [13, Appendix I]

Proposition 5.2. *In a Boolean algebra B, an element is compact if and only if it is F-compact.*

Proof. ($\Longrightarrow$) If $a \in B$ is compact and $a \wedge (\bigwedge S) = 0$ for some $S \subseteq B$, then $a \leq (\bigwedge S)^* = (\bigwedge_{s\in S} s)^* = \bigvee_{s\in S} s^*$. By compactness of a, there exists some finite $S_0 \subseteq S$ such that $a \leq \bigvee_{s\in S_0} s^* = (\bigwedge_{s\in S_0} s)^*$. Thus, $a \wedge (\bigwedge_{s\in S_0} s) = 0$.

($\Longleftarrow$) Let $a \in B$ be F-compact. Suppose $a \leq \bigvee S$ for some $S \subseteq B$. Then $a \wedge (\bigvee S)^* = 0$. That is, $a \wedge (\bigwedge_{s\in S} s^*) = 0$, and by F-compactness of a, we have $a \wedge (\bigwedge_{s\in S_0} s^*) = 0$ for some finite $S_0 \subseteq S$. Hence, $a \leq (\bigwedge_{s\in S_0} s^*)^* = \bigvee_{s\in S_0} s^{**} = \bigvee_{s\in S_0} s$. □

A Boolean frame is a frame that is also a Boolean algebra. We immediately have:

Corollary 5.3. *In a Boolean frame L, an element is compact if and only if it is F-compact.*

Remark 5.4. Note that in both directions of the proof on the previous proposition, we used the fact that our ambient lattice is a Boolean one. The fact that, in general, elements of a lattice may not all be complemented alerts us that F-compactness and compactness of elements may be distinct notions in non-Boolean lattices.

Example 5.5. (1) Let X be a set. Then the power set $\mathcal{P}(X)$ ordered by set inclusion, having joins as unions and meets as intersections, is a Boolean algebra (the complement of an element of $\mathcal{P}(X)$ is precisely the set-theoretic complement). So, F-compact elements of $\mathcal{P}(X)$ are compact.

(2) Let L be a frame. The *Boolenisation* of L is the set $\mathfrak{B}(L) = \{a \in L : a^{**} = a\}$. It is a Boolean algebra, with the same meets as in L, and the joins are given by $a \sqcup b = (a^* \wedge b^*)^*$. Thus, F-compact elements are compact in this lattice.

(3) Let L be a frame. A *sublocale* of a frame L is a subset S of L such that S is closed under arbitrary meets, and for each $x \in L$ and each $s \in S$, $x \to s \in S$. The lattice $\mathcal{S}(L)$ of all sublocales of L is a coframe under inclusion (see [13, Theorem III.3.2.1]). Here, meets are precisely the intersections, $\mathrm{O} = \{1\}$ is the bottom element, and L is the top element of $\mathcal{S}(L)$. The joins are defined by the formula:

$$\bigvee_{i\in I} S_i = \{\bigwedge A : A \subseteq \bigcup_{i\in I} S_i\}$$

for any $\{S_i\}_{i\in I} \subseteq \mathcal{S}(L)$. In general, $\mathcal{S}(L)$ is not a Bolean algebra. Hence, its compact elements need not be F-compact.

Let $\mathrm{CL}(X)$ be the lattice of closed subsets of topological spaces X, ordered by set-inclusion. Then $\mathrm{CL}(X)$ is a bounded distributive lattice with X as its largest element and $\emptyset$ as its smallest element. It may be needless to mention here, but that $\mathrm{CL}(X)$ is a coframe. Moreover, we have the following quick result:

Proposition 5.6. *Let X be a topological space. Then $A \in \mathrm{CL}(X)$ is F-compact if and only if A is a compact subset of X.*

Proof. Let $C = \{C_i : i \in I\}$ be an arbitrary collection of closed subsets of X. Note that $A \in \mathrm{CL}(X)$ is F-compact if and only if $A \cap (\bigcap C) = \emptyset$ implies that $A \cap (\bigcap_{i=1}^{n} C_i) = \emptyset$ for some finite collection $\{C_i\}_{1=1}^n \subseteq C$. This is equivalent to saying that, $A \in \mathrm{CL}(X)$ is F-compact if and only if $A \subseteq \bigcup(X \setminus C_i)$ implies that $A \subseteq \bigcup(X \setminus C_i)$ for some finite $\{C_i\}_{1=1}^n \subseteq C$. Thus, $A \in \mathrm{CL}(X)$ is F-compact if and only if A is a compact subset of X. □

Let us return to J-spaces. Note that a pointfree enunciation of the concept of a J-space is (see [10]):

*A frame L is a J-*frame *if and only whenever $L = S \vee T$ where S and T are closed sublocales of L with $S \cap T$ compact, then S or T is compact.*

One immediately has that:

A Hausdorff space X is a J-space if and only if $\mathfrak{O}X$ is a J-frame.

Note that the joins and meets in the pointfree enunciation of a J-space are taken in the lattice $\mathcal{S}(L)$. To attain some level of generality, let us re-state this definition for any lattice, modulo replacing "compact" with "F-compact":

Definition 5.7. A lattice L is called a *J-lattice* if whenever $a \vee b = 1$ in L and $a \wedge b$ is F-compact, then a or b is F-compact.

Proposition 5.8. CL(X) *is a J-lattice if and only if X is a J-space.*

Proof. ($\Longrightarrow$) Suppose CL(X) is a J-lattice. Proposition 5.6 says that F-compact elements in CL(X) are closed compact subsets in X. Let $X = A \cup B$, where A, B are closed and $A \cap B$ is compact. Then $A \cap B$ is an F-compact element in CL(X). So by the hypothesis, A or B is F-compact. This implies A or B is a compact subset of X.

($\Longleftarrow$) Let X be a J-space and suppose that $A \cup B = X$ in CL(X) and $A \cap B$ is F-compact. So A, B are closed, and since $A \cap B$ is F-compact, $A \cap B$ is compact by Proposition 5.6. So by the hypothesis, A or B is a compact subset of X. This implies A or B is F-compact in CL(X). Thus, CL(X) is a J-lattice. □

Example 5.9. Let X be a J-space (for example, the set of all nonnegative real numbers $\mathbb{R}^+$ with the standard topology, see [9]). Then CL(X) is a J-lattice

A space X is said to be *scattered* if for every non-empty closed set A there is an isolated point $a \in A$, and an open $U \ni a$ such that $U \cap A = \{a\}$. One speaks of a frame L as being *scattered* if $\mathcal{S}(L)$ is also a frame. In general (even when $L = \mathfrak{O}X$), the coframe $\mathcal{S}(L)$ is not a Boolean algebra. However, if X is a T_D-space, then X is scattered if and only if $\mathcal{S}(\mathfrak{O}X)$ is a Boolean algebra (see [3, Theorem 2.4.2]). The localic counterpart of the latter was observed in [3], that is, a frame L is scattered if and only if $\mathcal{S}(L)$ is a Boolean algebra.

Proposition 5.10. *Let L be a scattered frame. Then, L is a J-frame if and only if $\mathcal{S}(L)$ is a J-lattice.*

Proof. ($\Longrightarrow$) Suppose L is a J-frame. Let $S, T \in \mathcal{S}(L)$ and suppose that $S \vee T = L$ where $S \cap T$ is F-compact. By scatteredness of L, we have that $\mathcal{S}(L)$ is a Boolean algebra. Therefore $S \cap T$ is compact, by Proposition 5.2. Since L is a J-frame, then S or T is compact. Thus, by Proposition 5.2 again, S or T is an F-compact element of $\mathcal{S}(L)$.

($\Longleftarrow$) Follows similarly. □

Not all scattered spaces are Hausdorff (the Sierpiński space is a typical counter-example). The scattered Hausdorff spaces (for example, all countable compact Hausdorff spaces, see [7, Corollary 10]) are useful in our setting. The fact that any compact Hausdorff space is a J-space (see [9]) is important for our next example:

Example 5.11. Consider the frame $\mathfrak{O}X$, where X is any scattered Hausdorff space which is a J-space (e.g., the countable compact Hausdorff spaces). Since any Hausdorff space is a T_D-space, the scatteredness of X implies that $\mathcal{S}(\mathfrak{O}X)$ is a Boolean algebra. This implies that $\mathfrak{O}X$ is a scattered frame. Since X is a J-space, then $\mathfrak{O}X$ is also a J-frame. Hence, $\mathcal{S}(\mathfrak{O}X)$ is a J-lattice by the previous proposition.

Remark 5.12. (1) It is noteworthy that all examples of J-lattices provided herein arise from J-spaces. It would be good to get some examples of J-lattices that do not arise this way.

(2) In view of Proposition 5.2, compactness and F-compactness may not coincide for lattices which are not Boolean. Hence, exploring examples and properties of non-Boolean lattices with the "J-lattice property" may be an interesting avenue. The dual lattice of a J-lattice may also be worth studying for independent interest.

We intend to pursue (1) and (2) of this remark elsewhere.

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