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On class of fractional impulsive hybrid integro-differential equation

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Abstract. In this work, a class of a impulsive hybrid fractional integro-differential equation with hybrid boundary conditions is studied by the generalization of Dhage's fixed point theorem by three operators. This study ends with on example illustrating the theoretical findings.

1. Introduction

During the last three decades fractional calculus and its applications become diversified more and has materialize as a significant tool for the comprehensive applications in mathematical modeling of nonlinear systems. The nonlocal nature of fractional order operators accounts the hereditary properties involved in various systems in terms of fractional differential operator. For further reference see [12, 14, 19, 20]. and the references cited therein.

The definitions like Riemann-Liouville (1832), Grunwald-Letnikov(1867), Hadamard (1891,[21]) and Caputo(1997) are used to model problems in engineering and applied sciences and the formulations are used to model the physical systems and has given more accurate results. In 1891, Hadamard introduced the new derivative. For more details one can refer [6, 8, 10, 18] and the references cited therein.

The impulsive differential equations served as the foundations of micro world of biology, which has to be led to a reconsideration of nature. It is also important for a variety of applications in bio-informatics and practical utilizations in biotechnologies [7, 15]. In addition to the great importance of studying the existence of solutions to fractional differential equations using the many theories of the fixed point, several studies have been conducted over the years to investigate how stability concepts such as the Mittag-Leffler function, exponential, and Lyapunov stability apply to various types of dynamic systems. Ulam and Hyers, on the other hand, identified previously unknown types of stability known as Ulam-stability [25]. This example is not exclusive, many similar works can be found in [26–29].

In 2015, Surang. Sitho et al.[16], discussed the following boundary value problem:

$$\begin{cases} D^{\tau}\left(\frac{\rho(\hat{\iota})-\sum_{i=1}^{m} I^{\beta_i}\varphi_i(\hat{\iota},\rho(t))}{\psi(\hat{\iota},\rho(t))}\right) = \varpi(\hat{\iota},\rho(\hat{\iota})), \hat{\iota}\in J=[0,T], \quad 0<\tau\le 1, \\ \rho(0)=0, \end{cases}$$

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where D^{τ} denotes the Riemann-Liouville fractional derivative of order τ, $0 < \tau \leq 1$, I^{ς} is the Riemann-Liouville fractional integral of order $\varsigma > 0$, $\varsigma \in \{\beta_1, \beta_2, \ldots, \beta_m\}$, $\psi \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $\varpi \in C(J \times \mathbb{R}, \mathbb{R})$, with $\varphi_i \in C(J \times \mathbb{R}, \mathbb{R})$ with $\varphi_i(0,0) = 0, i = 1, 2, \ldots, m$.
Benchohra and al.[11] are discussed the following boundary value problems for differential equations with fractional order

$$\begin{cases} {}^cD^{\alpha} y(t) = f(t, y(t)), \quad for \quad each \quad t \in J = [0, T], \quad 0 < \alpha < 1, \\ ay(0) + by(T) = c, \end{cases}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, is a continuous function, a, b, c are real constants with $a + b \neq 0$.
Motivated by some recent studies on impulsive hybrid fractional integro-differential equations , we consider the following value problem :

$$\begin{cases} D^{\hat{\iota}}\left(\frac{\vartheta(\hat{\varkappa}) - I^{\hat{\kappa}}\xi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}\right) = \varpi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\kappa}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\kappa}_k}\vartheta(\hat{\varkappa})), \hat{\varkappa} \in J = [0, T], \quad 1 < \hat{\iota} \leq 2, \\ \vartheta(\hat{\varkappa}_i^+) = \vartheta(\hat{\varkappa}_i^-) + I_i(\vartheta(\hat{\varkappa}_i^-)), \qquad \hat{\varkappa}_i \in (0, 1), i = 1, 2, \ldots, n, \\ \frac{\vartheta(0)}{\varphi(0, \vartheta(0), I^{\hat{\iota}_1}\vartheta(0), \ldots, I^{\hat{\iota}_n}\vartheta(0))} = \vartheta_0, \qquad \frac{\vartheta(T)}{\varphi(T, \vartheta(T), I^{\hat{\iota}_1}\vartheta(T), \ldots, I^{\hat{\iota}_n}\vartheta(T))} = \vartheta_T, \end{cases} \tag{1}$$

where $\hat{\iota}_1, \ldots, \hat{\iota}_n > 0$, $\hat{\kappa}_1, \ldots, \hat{\kappa}_n > 0$, $\vartheta \in \mathbb{R}$, $D^{\hat{\iota}}$ denotes Caputo fractional derivative of order $\hat{\iota}$. $I^{\hat{\kappa}}$ is the Riemann-Liouville fractional integral of order $\hat{\kappa} > 0$. $\varphi : J \times \mathbb{R}^n \longrightarrow \mathbb{R} \setminus \{0\}$, $\xi : J \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous with $\xi(0, \vartheta(0), I^{\hat{\iota}_1}\vartheta(0), \ldots, I^{\hat{\iota}_n}\vartheta(0)) = 0$ and $\varpi \in C(J \times \mathbb{R}^k, \mathbb{R})$ is a function via some properties.
The problem (1) considered here is general in the sense that it includes the following three well-known classes of initial value problems of fractional differential equations.
By a solution of the peoblem (1) we mean a function $\vartheta \in C(\mathcal{J}, \mathbb{R})$ such that
(H_0) (i) The function $\hat{\varkappa} \mapsto \frac{\vartheta}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}$ is increasing in $\mathbb{R}$ for every $\hat{\varkappa} \in J$, and
(ii) ϑ satisfies the equations in (1).
This paper is arranged as follows. In Section 2, we recall some concepts and some fractional calculation law and establish preparation results. In Section 3, we study the existence solution of the initial value problem (1), based on the Dhage fixed point theorem. In section 4, example is provided to further clarify of the study's finding. In section 5, a conclusion and a future work are introduced.

2. Preliminaries

Recalling some preliminary facts, some basic definitions and properties of the fractional calculus .
Throughout this paper denotes $J_0 = [0, \hat{\varkappa}_1]$, $J_1 = (\hat{\varkappa}_1, \hat{\varkappa}_2]$, $\ldots$, $J_{n-1} = (\hat{\varkappa}_{n-1}, \hat{\varkappa}_n]$, $J_n = (\hat{\varkappa}_n, T], n \in \mathbb{N}, n > 1$.
For $\hat{\varkappa}_i \in (0, 1)$ such that $\hat{\varkappa}_1 < \hat{\varkappa}_2 < \ldots < \hat{\varkappa}_n$, we define the following spaces:
$J' = J \setminus \{\hat{\varkappa}_1, \hat{\varkappa}_2, ..., \hat{\varkappa}_n\}$,

$$\hat{\mathfrak{X}} = \{\vartheta \in C(J, \mathbb{R}) : \vartheta \in C(J') \text{ and left } \vartheta(\hat{\varkappa}_i^+) \text{ and right limit } \vartheta(\hat{\varkappa}_i^-)) \text{ exist and } \vartheta(\hat{\varkappa}_i^-) = \vartheta(\hat{\varkappa}_i), 1 \leq i \leq n\}.$$

Then the space $(\hat{\mathfrak{X}}, \|.\|)$ endowed with the norm:

$$\|\vartheta\| = \sup\{|\vartheta(\hat{\varkappa})|, \hat{\varkappa} \in J\}.$$

Clearly $\hat{\mathcal{X}}$ is a Banach algebra with respect to above supremum norm .

Definition 2.1. *[10] The R-L fractional integral of the function $\eta \in L^1([a, b], \mathbb{R}^+)$, of order $\hat{\iota} \in \mathbb{R}^+$ is defined by*

$$I_a^{\hat{\iota}}\eta(\hat{\varkappa}) = \int_a^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\alpha)}\eta(s)ds,$$

where Γ is the gamma function.

Definition 2.2. *[10] For a function η given on the interval $[a, b]$, the Riemann-Liouville fractional-order derivative of η, is defined by*

$$({}^{c}D^{\hat{\iota}}_{a^+}\eta)(\hat{\varkappa}) = \frac{1}{\Gamma(n-\kappa)}\left(\frac{d}{dt}\right)^n \int_a^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{n-\hat{\iota}-1}}{\Gamma(\hat{\iota})}\eta(s)ds,$$

where $n = [\hat{\iota}] + 1$ and $[\hat{\iota}]$ denotes the integer part of $\hat{\iota}$.

Definition 2.3. *[10] For a function η given on the interval $[a, b]$, the Caputo fractional-order derivative of η, is defined by*

$$({}^{c}D^{\hat{\iota}}_{a^+}\eta)(\hat{\varkappa}) = \frac{1}{\Gamma(n-\hat{\iota})}\int_a^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{n-\hat{\iota}-1}}{\Gamma(\kappa)}\eta^{(n)}(s)ds,$$

where $n = [\hat{\iota}] + 1$ and $[\hat{\iota}]$ denotes the integer part of $\hat{\iota}$.

Lemma 2.4. *[10] Let $\hat{\iota} > 0$ and $\vartheta \in C(0, T) \cap L(0, T)$. Then the fractional differential equation*

$$D^{\hat{\iota}}\vartheta(\hat{\varkappa}) = 0,$$

has a unique solution

$$\vartheta(\hat{\varkappa}) = \tau_1\hat{\varkappa}^{\hat{\iota}-1} + \tau_2\hat{\varkappa}^{\hat{\iota}-2} + \ldots + \tau_n\hat{\varkappa}^{\hat{\iota}-n},$$

where $\tau_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $n - 1 < \hat{\iota} < n$.

Lemma 2.5. *Let $\hat{\iota} > 0$. Then for $\vartheta \in C(0, T) \cap L(0, T)$ we have*

$$I^{\hat{\iota}}D^{\hat{\iota}}\vartheta(\hat{\varkappa}) = \vartheta(\hat{\varkappa}) + c_0 + c_1t + \ldots + c_{n-1}\hat{\varkappa}^{n-1},$$

fore some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n - 1$. Where $n = [\hat{\iota}] + 1$.

Lemma 2.6. *[9] For any $\hat{\mathfrak{S}}$ nonempty, closed convex and bounded subset of a Banach algebra $\hat{\mathfrak{X}}$ and for any operators $\hat{\mathfrak{A}}, \hat{\mathfrak{C}} : \hat{\mathfrak{X}} \longrightarrow \hat{\mathfrak{X}}$ and $\hat{\mathfrak{B}} : \hat{\mathfrak{S}} \longrightarrow \hat{\mathfrak{X}}$ such that:*

(*i*) $\hat{\mathfrak{A}}$ *and* $\hat{\mathfrak{C}}$ *are Lipschitzian with Lipschitz constants $\hat{\tau}$ and ρ, respectively,*

(*ii*) $\hat{\mathfrak{B}}$ *is compact and continuous,*

(*iii*) $\vartheta = \hat{\mathfrak{A}}\vartheta\hat{\mathfrak{B}}\eta + \hat{\mathfrak{C}}\vartheta \Longrightarrow \vartheta \in \hat{\mathfrak{S}}$ *for all $\eta \in \hat{\mathfrak{S}}$,*

(*iv*) $\hat{\tau}\hat{\mathfrak{W}} + \rho < 1$, *where $\hat{\mathfrak{W}} = \|\hat{\mathfrak{F}}(\hat{\mathfrak{S}})\|$.*

Then the equation $\vartheta = \hat{\mathfrak{A}}\vartheta\hat{\mathfrak{B}}\vartheta + \hat{\mathfrak{C}}\vartheta$ has a solution.

Lemma 2.7. *:[13] Let $\hat{\iota} \in (0, 1)$ and $\eta : [0, T] \longrightarrow \mathbb{R}$ be continuous. A function $\vartheta \in \hat{\mathfrak{C}}([0, T], \mathbb{R})$ is a solution of the fractional integral equation*

$$\vartheta(\hat{\varkappa}) = \vartheta_0 - \int_0^a \frac{(\hat{\varkappa}-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\eta(s)ds + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa}-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\eta(s)ds,$$

if and only if ϑ is a solution of the following problem:

$$\begin{cases} D^{\hat{\iota}}\vartheta(\hat{\varkappa}) = \eta(\hat{\varkappa}), \hat{\varkappa} \in [0, T] \\ \vartheta(a) = \vartheta_0, \quad a > 0. \end{cases} \tag{2}$$

For brevity let us take,

$$\eta = \frac{1}{T}\vartheta_T - \frac{1}{T}\vartheta_0 - \frac{1}{T\Gamma(\hat{\iota})}\int_0^T (T-s)^{\hat{\iota}-1}\chi(s)ds - \frac{d}{T},$$

$$d = \frac{I^{\beta}\varpi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}{\varphi(T, \vartheta(T), I^{\hat{\iota}_1}\vartheta(T), \ldots, I^{\hat{\iota}_n}\vartheta(T))},$$

$$\gamma = \sum_{i=1}^{n} \frac{I_i(\vartheta(\hat{\varkappa}_i^-) - I^{\hat{\kappa}}\xi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i), I^{\hat{\iota}_i}\vartheta(\hat{\varkappa}_i), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_i))}{\varphi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_i), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_i))},$$

$$\delta = \sum_{i=1}^{n} \frac{\int_0^{\hat{\varkappa}_i} \frac{(\hat{\varkappa}_i - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s))ds}{\varphi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_i), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_i))}.$$

Lemma 2.8. *Then, for any $\chi \in L^1(J, \mathbb{R})$, the function $\vartheta \in \hat{\mathfrak{C}}(J, \mathbb{R})$ is a solution of the*

$$\begin{cases} D^{\hat{\iota}}\left(\frac{\vartheta(\hat{\varkappa}) - I^{\hat{\kappa}}\xi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}\right) = \chi(\hat{\varkappa}), \quad a.e. \quad \hat{\varkappa} \in J = [0, T], \quad 1 < \hat{\iota} \le 2, \\ \vartheta(\hat{\varkappa}_i^+) = \vartheta(\hat{\varkappa}_i^-) + I_i(\vartheta(\hat{\varkappa}_i^-)), \quad \hat{\varkappa}_i \in (0, 1), i = 1, 2, \ldots, n, \\ \frac{\vartheta(0)}{\varphi(0, \vartheta(0), I^{\hat{\iota}_1}\vartheta(0), \ldots, I^{\hat{\iota}_n}\vartheta(0))} = \vartheta_0, \qquad \frac{\vartheta(T)}{\varphi(T, \vartheta(T), I^{\hat{\iota}_1}\vartheta(T), \ldots, I^{\hat{\iota}_n}\vartheta(T))} = \vartheta_T, \end{cases} \tag{3}$$

if and only if ϑ satisfies the hybrid integral equation

$$\begin{aligned} \vartheta(\hat{\varkappa}) &= \varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))\Big[\gamma + \delta + \eta\varkappa_1 + \vartheta_0 \\ &+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds\Big] + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s))ds, \quad \hat{\varkappa} \in [0, T]. \end{aligned} \tag{4}$$

Proof. We assume that ϑ is a solution of the problem (3).
If $\hat{\varkappa} \in [\hat{\varkappa}_0, \hat{\varkappa}_1[$, By definition, $\left(\frac{\vartheta(\hat{\varkappa}) - I^{\hat{\kappa}}\xi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}\right)$ is continuous. Applying $I^{\hat{\iota}}$ of the order $\hat{\iota}$ on both sides of (3), we can obtain,

$$\frac{\vartheta(\hat{\varkappa}) - I^{\hat{\kappa}}\xi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))} = I^{\hat{\iota}}\chi(\hat{\varkappa}) - c_0 - c_1\hat{\varkappa},$$

so we get

$$\frac{\vartheta(\hat{\varkappa})}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))} = I^{\hat{\iota}}\chi(\hat{\varkappa}) - c_0 - c_1\vartheta + \frac{I^{\hat{\kappa}}\xi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}.$$

Substituting $\vartheta = 0$ we have

$$c_0 = -\frac{\vartheta(0)}{\varphi(0, \vartheta(0), I^{\hat{\iota}_1}\vartheta(0), \ldots, I^{\vartheta_n}\vartheta(0))} = -\vartheta_0.$$

And substituting $\vartheta = T$ we have

$$\frac{\vartheta(T)}{\varphi(T, \vartheta(T), I^{\alpha_1}\vartheta(T), \ldots, I^{\hat{\iota}_n}\vartheta(T))} = I^{\hat{\iota}}\chi(T) + \vartheta_0 - c_1 T + d.$$

Then

$$c_1 = \frac{1}{T}(\vartheta_0 + I^{\hat{\iota}}\chi(T) - \vartheta_T + d).$$

In consequence, we have

$$\begin{aligned}\vartheta(\hat{\varkappa}) &= \Big(\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))\Big)\Big(\frac{1}{\Gamma(\hat{\iota})}\int_0^{\hat{\varkappa}}(\hat{\varkappa}-s)^{\hat{\iota}-1}\chi(s)ds+(1-\frac{\hat{\varkappa}}{T})\vartheta_0+\frac{\hat{\varkappa}}{T}\vartheta_T\\ &- \frac{\hat{\varkappa}}{T\Gamma(\hat{\iota})}\int_0^T(T-s)^{\hat{\iota}-1}\chi(s)ds-\frac{\hat{\varkappa}d}{T}\Big)+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s,\vartheta(s),I^{\hat{\iota}_1}\vartheta(s),\dots,I^{\hat{\iota}_n}\vartheta(s))ds.\end{aligned}$$

If $\hat{\varkappa}\in[\hat{\varkappa}_1,\hat{\varkappa}_2[$, then

$$D^{\hat{\iota}}\Bigg(\frac{\vartheta(\hat{\varkappa})-I^{\hat{\kappa}}\xi(\hat{\varkappa},\vartheta(\hat{\varkappa}),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}\Bigg)=\chi(\hat{\varkappa}),\quad \hat{\varkappa}\in[\hat{\varkappa}_1,\hat{\varkappa}_2[, \tag{5}$$

$$\vartheta(\hat{\varkappa}_1^+)=\vartheta(\hat{\varkappa}_1^-)+I_1(\vartheta(\hat{\varkappa}_1^-)). \tag{6}$$

According to Lemma 2.7 and the continuity of $\hat{\varkappa}\longrightarrow\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))$, we have

$$\begin{aligned}\frac{\vartheta(\hat{\varkappa})-I^{\hat{\kappa}}\xi(\hat{\varkappa},\vartheta(\hat{\varkappa}),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))} &= \frac{(\vartheta(\hat{\varkappa}_1^+)-I^{\hat{\kappa}}\xi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))}{\varphi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))}\\ &-\int_0^{\hat{\varkappa}_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds.\end{aligned}$$

Since

$$\begin{aligned}\frac{\vartheta(\hat{\varkappa})-I^{\hat{\kappa}}\xi(\hat{\varkappa},\vartheta(\hat{\varkappa}),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))} &= \frac{(\vartheta(\hat{\varkappa}_1^-)+I_1(\vartheta(\hat{\varkappa}_1^-))-I^{\hat{\kappa}}\xi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1)}{\varphi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))}\\ &-\int_0^{\hat{\varkappa}_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds,\end{aligned}$$

according to

$$\begin{aligned}\vartheta(\hat{\varkappa}^-) &= \Big(\varphi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))\Big)\Big(\frac{1}{\Gamma(\hat{\iota})}\int_0^{\hat{\varkappa}_1}(\hat{\varkappa}_1-s)^{\hat{\iota}-1}\chi(s)ds+(1-\frac{\hat{\varkappa}_1}{T})\vartheta_0+\frac{\hat{\varkappa}_1}{T}\vartheta_T\\ &-\frac{\hat{\varkappa}_1}{T\Gamma(\hat{\iota})}\int_0^T(T-s)^{\hat{\iota}-1}\chi(s)ds-\frac{\hat{\varkappa}_1 d}{T}\Big)+\int_0^{\hat{\varkappa}_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s,\vartheta(s),I^{\hat{\iota}_1}\vartheta(s),\dots,I^{\hat{\iota}_n}\vartheta(s))ds\Big).\end{aligned}$$

Then we get

$$\begin{aligned}\vartheta(\hat{\varkappa}) &= \varphi(\hat{\varkappa},\vartheta(\hat{\varkappa}),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))\Big[\frac{1}{\Gamma(\hat{\iota})}\int_0^{\hat{\varkappa}_1}(\hat{\varkappa}_1-s)^{\hat{\iota}-1}\chi(s)ds\\ &+\int_0^{\hat{\varkappa}_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s,\vartheta(s),I^{\hat{\iota}_1}\vartheta(s),\dots,I^{\hat{\iota}_n}\vartheta(s))ds+\frac{I_1(\vartheta(\hat{\varkappa}_1^-))-I^{\hat{\kappa}}\xi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))}{\varphi(\hat{\varkappa}_1,\vartheta(\hat{\varkappa}_1),I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1),\dots,I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))}\\ &-\int_0^{\hat{\varkappa}_1}\frac{(\hat{\varkappa}_1-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds+\eta\varkappa_1+\vartheta_0\Big]\\ &+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s,\vartheta(s),I^{\hat{\iota}_1}\vartheta(s),\dots,I^{\hat{\iota}_n}\vartheta(s))ds,\end{aligned}$$

so, one has

$$\vartheta(\hat{\varkappa}) = \varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))\Big[\int_0^{\hat{\varkappa}_1} \frac{(\hat{\varkappa}_1 - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s))ds$$
$$+ \frac{I_1(\vartheta(\hat{\varkappa}_1^-)) - I^{\hat{\kappa}}\xi(\hat{\varkappa}_1, \vartheta(\hat{\varkappa}_1), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))}{\varphi(\hat{\varkappa}_1, \vartheta(\hat{\varkappa}_1), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))} + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds + \eta\varkappa_1 + \vartheta_0\Big]$$
$$+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s))ds.$$

If $\hat{\varkappa} \in [\hat{\varkappa}_2, \hat{\varkappa}_3[$, we have

$$\frac{\vartheta(\hat{\varkappa}) - I^{\hat{\kappa}}\xi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))} = \frac{(\vartheta(\hat{\varkappa}_2^+) - I^{\hat{\kappa}}\xi(\hat{\varkappa}_2, \vartheta(\hat{\varkappa}_2), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_2), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))}{\varphi(\hat{\varkappa}_2, \vartheta(\hat{\varkappa}_2), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_2), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_2))}$$
$$- \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds.$$

Since

$$\frac{\vartheta(\hat{\varkappa}) - I^{\hat{\kappa}}\xi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))} = \frac{(\vartheta(\hat{\varkappa}_2^-) + I_2(\vartheta(\hat{\varkappa}_2^-)) - I^{\hat{\kappa}}\xi(\hat{\varkappa}_2, \vartheta(\hat{\varkappa}_2), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_2), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_2)}{\varphi(\hat{\varkappa}_2, \vartheta(\hat{\varkappa}_2), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_2), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_2))}$$
$$- \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds.$$

For

$$\vartheta(\hat{\varkappa}_2^-) = \varphi(\hat{\varkappa}_2, \vartheta(\hat{\varkappa}_2), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_2), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_2))\Big[\frac{\int_0^{\hat{\varkappa}_1} \frac{(\hat{\varkappa}_1 - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s))ds}{\varphi(\hat{\varkappa}_1, \vartheta(\hat{\varkappa}_1), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))}$$
$$+ \frac{I_1(\vartheta(\hat{\varkappa}_1^-)) - I^{\hat{\kappa}}\xi(\hat{\varkappa}_1, \vartheta(\hat{\varkappa}_1), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))}{\varphi(\hat{\varkappa}_1, \vartheta(\hat{\varkappa}_1), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))} + \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds + \eta\varkappa_1 + \vartheta_0\Big]$$
$$+ \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s). \ldots, I^{\hat{\iota}_n}\vartheta(s))ds.$$

Therefore, we obtain

$$\vartheta(\hat{\varkappa}) = \varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))\Big[\frac{I_1(\vartheta(\hat{\varkappa}_1^-) - I^{\hat{\kappa}}\xi(\hat{\varkappa}_1, \vartheta(\hat{\varkappa}_1), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))}{\varphi(\hat{\varkappa}_1, \vartheta(\hat{\varkappa}_1), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))}$$
$$+ \frac{(I_2\vartheta(\hat{\varkappa}_2^-) - I^{\hat{\kappa}}\xi(\hat{\varkappa}_2, \vartheta(\hat{\varkappa}_2), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_2), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_2)}{\varphi(\hat{\varkappa}_2, \vartheta(\hat{\varkappa}_2), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_2), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_2))} + \frac{\int_0^{\hat{\varkappa}_1} \frac{(\hat{\varkappa}_1 - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s))ds}{\varphi(\hat{\varkappa}_1, \vartheta(\hat{\varkappa}_1), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_1), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_1))}$$
$$+ \frac{\int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s))ds}{\varphi(\hat{\varkappa}_2, \vartheta(\hat{\varkappa}_2), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_2), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_2))} + \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds + \eta\varkappa_1 + \vartheta_0$$
$$+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds - \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds\Big] + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s))ds.$$

Consequently, we get

$$\vartheta(\hat{\varkappa}) = \varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))\Big[\sum_{i=1}^{2} \frac{I_i(\vartheta(\hat{\varkappa}_i^-) - I^{\hat{\kappa}}\xi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i), I^{\hat{\iota}_i}\vartheta(\hat{\varkappa}_i), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_i))}{\varphi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_i), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_i))}$$
$$+ \sum_{i=1}^{2} \frac{\int_0^{\hat{\varkappa}_i} \frac{(\hat{\varkappa}_i - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \dots, I^{\hat{\iota}_n}\vartheta(s))ds}{\varphi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_i), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_i))} + \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds + \eta\varkappa_1 + \vartheta_0$$
$$+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds - \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds\Big] + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \dots, I^{\hat{\iota}_n}\vartheta(s))ds.$$

By using the same method, for $\hat{\varkappa} \in [\hat{\varkappa}_i, \hat{\varkappa}_{i+1}[(i = 3, 4, ..., n)$, one has

$$\vartheta(\hat{\varkappa}) = \varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))\Big[\sum_{i=1}^{n} \frac{I_i(\vartheta(\hat{\varkappa}_i^-) - I^{\hat{\kappa}}\xi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i), I^{\hat{\iota}_i}\vartheta(\hat{\varkappa}_i), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_i))}{\varphi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_i), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_i))}$$
$$+ \sum_{i=1}^{n} \frac{\int_0^{\hat{\varkappa}_i} \frac{(\hat{\varkappa}_i - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \dots, I^{\hat{\iota}_n}\vartheta(s))ds}{\varphi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_i), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_i))} + \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds + \eta\varkappa_1 + \vartheta_0$$
$$+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds - \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds\Big] + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \dots, I^{\hat{\iota}_n}\vartheta(s))ds.$$

Conversely, assume that ϑ satisfies (20). If $\hat{\varkappa} \in [\hat{\varkappa}_0, \hat{\varkappa}_1[$, then we have

$$\begin{aligned}\vartheta(\hat{\varkappa}) &= \Big(\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))\Big)\Big(\frac{1}{\Gamma(\hat{\iota})}\int_0^{\hat{\varkappa}} (\hat{\varkappa} - s)^{\hat{\iota}-1}\chi(s)ds + (1 - \frac{\hat{\varkappa}}{T})\vartheta_0 + \frac{\hat{\varkappa}}{T}\vartheta_T \\ &- \frac{\hat{\varkappa}}{T\Gamma(\hat{\iota})}\int_0^{T} (T - s)^{\hat{\iota}-1}\chi(s)ds - \frac{\hat{\varkappa}d}{T}\Big) + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \dots, I^{\hat{\iota}_n}\vartheta(s))ds.\end{aligned} \quad (7)$$

Then, we multiplied by $\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))$ and applying $D^{\hat{\iota}}$ on both sides of (8), we get first equation in (3).

Again, substituting $\hat{\varkappa} = 0$ and $\hat{\varkappa} = T$ in (8), and $\vartheta \longrightarrow \dfrac{\vartheta}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}$, is increasing in $\mathbb{R}$ for $\hat{\varkappa} \in [\hat{\varkappa}_0, \hat{\varkappa}_1[$, the map $\vartheta \longrightarrow \dfrac{\vartheta}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}$, is injective in $\mathbb{R}$. Then we get

$$\frac{\vartheta(0)}{\varphi(0, \vartheta(0), I^{\hat{\iota}_1}\vartheta(0), \dots, I^{\hat{\iota}_n}\vartheta(0))} = \vartheta_0, \qquad \frac{\vartheta(T)}{\varphi(T, \vartheta(T), I^{\hat{\iota}_1}\vartheta(T), \dots, I^{\hat{\iota}_n}\vartheta(T))} = \vartheta_T,$$

If $\hat{\varkappa} \in [\hat{\varkappa}_1, \hat{\varkappa}_2[$, then we have

$$\vartheta(\hat{\varkappa}) = \varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))\Big[\sum_{i=1}^{2} \frac{I_i(\vartheta(\hat{\varkappa}_i^-) - I^{\hat{\kappa}}\xi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i), I^{\hat{\iota}_i}\vartheta(\hat{\varkappa}_i), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_i))}{\varphi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_i), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_i))}$$
$$+ \sum_{i=1}^{2} \frac{\int_0^{\hat{\varkappa}_i} \frac{(\hat{\varkappa}_i - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \dots, I^{\hat{\iota}_n}\vartheta(s))ds}{\varphi(\hat{\varkappa}_i, \vartheta(\hat{\varkappa}_i), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}_i), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}_i))} + \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds + \eta\varkappa_1 + \vartheta_0$$
$$+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds - \int_0^{\hat{\varkappa}_2} \frac{(\hat{\varkappa}_2 - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\chi(s)ds\Big] + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \dots, I^{\hat{\iota}_n}\vartheta(s))ds. \quad (8)$$

Then, we dividing by $\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))$ and applying $D^{\hat{\iota}}$ on both sides of (8), we get equation (5) . Again by (H_0), substituting $\hat{\varkappa} = \hat{\varkappa}_1$ in (8) and taking the limit of (8), then (8) minus (8) gives (6).

Similarly, for $\hat{\varkappa} \in [\hat{\varkappa}_i, \hat{\varkappa}_{i+1}[(i = 2, 3, ..., n)$, we get

$$D^{\hat{\iota}}\left(\frac{\vartheta(\hat{\varkappa}) - I^{\hat{\kappa}}\xi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}{\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \dots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))}\right) = \chi(\hat{\varkappa}), \tag{9}$$

$$\vartheta(\hat{\varkappa}_1^+) = \vartheta(\hat{\varkappa}_1^-) + I_1(\vartheta(\hat{\varkappa}_1^-)). \tag{10}$$

□

This completes the proof.

3. Main Result

In this section, we prove the existence of a mild solution for problem (1) by Lemma 2.6. For this study, we need the following assumptions. Assume that :

(H_1) The functions $\varphi : J\mathbb{R}^{n+1} \longrightarrow \times\mathbb{R} \setminus \{0\}$, $\xi : J \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$, $\varpi : J \times \mathbb{R}^{k+1} \longrightarrow \mathbb{R}$ are be a Carathéodory functions, $\xi(0, \vartheta(0), I^{\hat{\iota}_1}\vartheta(0), \dots, I^{\hat{\iota}_n}\vartheta(0))) = 0$ and there exist
$p, m : J \longrightarrow (0, \infty)$ with bound $\|p\|$ and $\|m\|$ respectively, such that

$$|\varphi(\hat{\varkappa}, \nu_1, \nu_2, \dots, \nu_{n+1}) - \varphi(\hat{\varkappa}, \omega_1, \omega_2, \dots, \omega_{n+1})| \leq p(\hat{\varkappa}) \sum_{i=1}^{n+1} |\nu_i - \omega_i|, \tag{11}$$

and

$$|\xi(\hat{\varkappa}, \nu_1, \nu_2, \dots, \nu_{n+1}) - \xi(\hat{\varkappa}, \omega_1, \omega_2, \dots, \omega_{n+1})| \leq m(\hat{\varkappa}) \sum_{i=1}^{n+1} |\nu_i - \omega_i|, \tag{12}$$

for $\hat{\varkappa} \in J$ and $(\omega_1, \omega_2, \dots, \omega_{n+1}), (\nu_1, \nu_2, \dots, \nu_{n+1}) \in \mathbb{R}^{n+1}$.

(H_2) There exists a function $h \in L^1(J, \mathbb{R})$ such that

$$|\varpi(\hat{\varkappa}, \omega_1, \omega_2, \dots, \omega_k)| \leq h(\hat{\varkappa}), (\hat{\varkappa}, \omega_1, x_2, \dots, \omega_k) \in J \times \mathbb{R}^k. \tag{13}$$

(H_3) There exists $r > 0$ such that

$$r \geq \frac{F_0\Big(\mid \gamma + \delta + \eta\varkappa_1 + \vartheta_0 \mid + \frac{\|h\|_{L^1} T^{\hat{\iota}}}{\Gamma(\hat{\iota}+1)}\Big) + \frac{T^{\hat{\kappa}}}{\Gamma(\hat{\kappa}+1)} k_0}{1 - \Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1+1)} + \dots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n+1)}\Big)\Big[\|p\|\Big(\mid \gamma + \delta + \eta\varkappa_1 + \vartheta_0 \mid \frac{\|h\|_{L^1} T^{\hat{\iota}}}{\Gamma(\hat{\iota}+1)}\Big) + \|m\| \frac{T^{\hat{\kappa}}}{\Gamma(\hat{\kappa}+1)}\Big]}. \tag{14}$$

where $F_0 = \sup_{\hat{\varkappa} \in J} |\varpi(\hat{\varkappa}, 0, \underbrace{0, 0 \dots, 0}_{n})|$ and $K_0 = \sup_{\hat{\varkappa} \in J} |\xi(\hat{\varkappa}, 0, \underbrace{0, 0 \dots, 0}_{n})|$.

Theorem 3.1. *Assume that the conditions $(H_1) - (H_3)$ hold. Then the initial value problem (1) has at least one solution on J provided that*

$$\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} + \dots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n + 1)}\Big)\Big[\|p\|\Big(\mid \gamma + \delta + \eta\varkappa_1 + \vartheta_0 \mid \frac{\|h\|_{L^1} T^{\hat{\iota}}}{\Gamma(\hat{\iota} + 1)}\Big) + \|m\| \frac{T^{\hat{\kappa}}}{\Gamma(\hat{\kappa} + 1)}\Big] < 1. \tag{15}$$

Proof. we define a subset $\hat{\mathfrak{S}}$ of $\hat{\mathfrak{X}}$ as

$$\hat{\mathfrak{S}} = \{\vartheta \in \hat{\mathfrak{X}} : \|\vartheta\| \leq r\},$$

where r satisfies inequality (14).
Clearly $\hat{\mathfrak{S}}$ is closed, convex, and bounded subset of the Banach space $\hat{\mathfrak{X}}$. Now we define three operators, $\hat{\mathfrak{A}} : \hat{\mathfrak{X}} \longrightarrow \hat{\mathfrak{X}}$ by:

$$\hat{\mathfrak{A}}\vartheta(\hat{\varkappa}) = \varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa})), \quad \hat{\varkappa} \in J. \tag{16}$$

$\hat{\mathfrak{B}} : \hat{\mathfrak{S}} \longrightarrow \hat{\mathfrak{X}}$ by:

$$\hat{\mathfrak{B}}\vartheta(\hat{\varkappa}) = \gamma + \delta + \eta\varkappa_1 + \vartheta_0 + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})} \varpi(s, \vartheta(s), I^{\hat{\kappa}_1}\vartheta(s), \ldots, I^{\hat{\kappa}_k}\vartheta(s))ds, \quad \hat{\varkappa} \in J, \tag{17}$$

and $\hat{\mathfrak{C}} : \hat{\mathfrak{X}} \longrightarrow \hat{\mathfrak{X}}$ by:

$$\hat{\mathfrak{C}}\vartheta(\hat{\varkappa}) = \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})} \xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s))ds, \quad \hat{\varkappa} \in J. \tag{18}$$

We shall prove that the operators $\hat{\mathfrak{A}}$, $\hat{\mathfrak{B}}$, and $\hat{\mathfrak{C}}$ satisfy all the the hypothesis of Lemma 2.6.
Claim 1. We will prove that $\hat{\mathfrak{A}}$ and $\hat{\mathfrak{C}}$ are Lipschitzian on $\hat{\mathfrak{X}}$, that is, the assumption (i) of Lemma 2.6 holds.
Let $\vartheta, \upsilon \in \hat{\mathfrak{X}}$. Then by (H_1), for $\hat{\varkappa} \in J$ we have

$$\begin{aligned}
|\hat{\mathfrak{A}}\vartheta(\hat{\varkappa}) - \hat{\mathfrak{A}}\upsilon(\hat{\varkappa})| &= |\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa})) - \varphi(\hat{\varkappa}, \upsilon(\hat{\varkappa}), I^{\hat{\iota}_1}\upsilon(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\upsilon(\hat{\varkappa}))| \\
&\leq \sup_{\hat{\varkappa} \in J}(|p||\vartheta(\hat{\varkappa}) - \upsilon(\hat{\varkappa})|)\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n + 1)}\Big) \\
&\leq \|p\|\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n + 1)}\Big)\|\vartheta - \upsilon\|,
\end{aligned}$$

for all $\hat{\varkappa} \in J$.
This implies that

$$\|\hat{\mathfrak{A}}\vartheta - \hat{\mathfrak{A}}\upsilon\| \leq \|p\|\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n + 1)}\Big)\|\vartheta - \upsilon\|,$$

for all $\vartheta, \upsilon \in \hat{\mathfrak{E}}$.
So $\hat{\mathfrak{A}}$ is a Lipschitz on $\hat{\mathfrak{X}}$ with Lipschitz constant $\|p\|\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1+1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n+1)}\Big)$.
Analogously, for any $\vartheta, \upsilon \in \hat{\mathfrak{X}}$, we have

$$\begin{aligned}
|\hat{\mathfrak{C}}\vartheta(\hat{\varkappa}) - \hat{\mathfrak{C}}\upsilon(\hat{\varkappa})| &= \Big|\int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}[\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s)) - \xi(s, \upsilon(s), I^{\hat{\iota}_1}\upsilon(s), \ldots, I^{\hat{\iota}_n}\upsilon(s))]ds\Big| \\
&\leq \sup_{\hat{\varkappa} \in J}(|m||\vartheta(\hat{\varkappa}) - \upsilon(\hat{\varkappa})|)\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n + 1)}\Big)\int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})}ds \\
&\leq \|m\|\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n + 1)}\Big)\frac{T^{\hat{\kappa}}}{\Gamma(\hat{\kappa} + 1)}\|\vartheta - \upsilon\|,
\end{aligned}$$

for all $\hat{\varkappa} \in J$.
This implies that

$$\|\hat{\mathfrak{C}}\vartheta - \hat{\mathfrak{C}}\upsilon\| \leq \|m\|\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n + 1)}\Big)\frac{T^{\hat{\kappa}}}{\Gamma(\hat{\kappa} + 1)}\|\vartheta - \upsilon\|.$$

So, $\hat{\mathfrak{C}}$ is a Lipschitzian on $\hat{\mathfrak{X}}$ with Lipschitz constant $\|m\|\Big(1+\frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1+1)}+\ldots+\frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n+1)}\Big)\frac{T^{\hat{\kappa}}}{\Gamma(\hat{\kappa}+1)}$.
Claim 2. We prove that the operator $\hat{\mathfrak{B}}$ is completely continuous on $\hat{\mathfrak{S}}$. We first prove that the operator $\hat{\mathfrak{B}}$ is continuous on $\hat{\mathfrak{S}}$.
Let $\{\vartheta_n\}$ be a sequence in $\hat{\mathfrak{S}}$ converging to a point $\vartheta\in\hat{\mathfrak{S}}$. Then by the Lebesgue dominated convergence theorem, for all $\hat{\varkappa}\in J$, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\Big(\gamma+\delta+\eta\varkappa_1+\vartheta_0+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\varpi(s,\vartheta_n(s),I^{\hat{\kappa}_1}\vartheta_n(s),\ldots,I^{\hat{\kappa}_k}\vartheta_n(s))ds\Big)\\
&=\gamma+\delta+\eta\varkappa_1+\vartheta_0+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\lim_{n\to\infty}\Big(\varpi(s,\vartheta_n(s),I^{\hat{\kappa}_1}\vartheta_n(s),\ldots,I^{\hat{\kappa}_k}\vartheta_n(s))ds\Big)\\
&=\gamma+\delta+\eta\varkappa_1+\vartheta_0+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\varpi(s,\vartheta(s),I^{\hat{\kappa}_1}\vartheta(s),\ldots,I^{\hat{\kappa}_k}\vartheta(s))ds.
\end{aligned}$$

In consequence, we have

$$\lim_{n\to\infty}\hat{\mathfrak{B}}\vartheta_n=\hat{\mathfrak{B}}\vartheta.$$

This shows that $\hat{\mathfrak{B}}$ is continuous on $\hat{\mathfrak{S}}$.
It is sufficient to show that the set $\hat{\mathfrak{B}}(\hat{\mathfrak{S}})$ is a uniformly bounded in $\hat{\mathfrak{S}}$. For any $\vartheta\in\hat{\mathfrak{S}}$, we have

$$\begin{aligned}
|\hat{\mathfrak{B}}\vartheta(\hat{\varkappa})|&=\Big|\gamma+\delta+\eta\varkappa_1+\vartheta_0+\int_0^{\hat{\varkappa}}\frac{(\hat{\varkappa}-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\varpi(s,\vartheta(s),I^{\hat{\kappa}_1}\vartheta(s),\ldots,I^{\hat{\kappa}_k}\vartheta(s))ds\Big|\\
&\le|\gamma|+|\delta|+|\eta||\varkappa_1|+|\vartheta_0|+\Big(\|h\|\frac{T^{\hat{\iota}}}{\Gamma(\hat{\iota}+1)}\Big)=K_1,
\end{aligned}$$

for all $\hat{\varkappa}\in J$. Taking supremum over the interval J, the above inequality becomes, $\|\hat{\mathfrak{B}}\vartheta\|\le K_1$ for all $\vartheta\in\hat{\mathfrak{S}}$.
This shows that $\hat{\mathfrak{B}}(\hat{\mathfrak{S}})$ is uniformly bounded on $\hat{\mathfrak{S}}$.
Next we show that $\hat{\mathfrak{S}}$ is an equicontinuous set in $\hat{\mathfrak{X}}$. We take, $\tau_1,\tau_2\in J$ with $\tau_1<\tau_2$ and and $\vartheta\in\hat{\mathfrak{S}}$.
Then we have

$$\begin{aligned}
|\hat{\mathfrak{B}}\vartheta(\tau_2)-\hat{\mathfrak{B}}\vartheta(\tau_1)|&=\Big|\int_0^{\tau_2}\frac{(\tau_2-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\varpi(s,\vartheta(s),I^{\hat{\kappa}_1}\vartheta(s),\ldots,I^{\hat{\kappa}_k}\vartheta(s))ds\\
&-\int_0^{\tau_1}\frac{(\tau_1-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}\varpi(s,x(s),I^{\hat{\kappa}_1}\vartheta(s),\ldots,I^{\hat{\kappa}_k}\vartheta(s))ds+\Big[(\gamma+\delta+\eta\varkappa_1+\vartheta_0)-(\gamma+\delta+\eta\varkappa_1+\vartheta_0)\Big]\\
&\le\int_0^{\tau_1}\frac{|(\tau_2-s)^{\hat{\iota}-1}-(\tau_1-s)^{\hat{\iota}-1}|}{\Gamma(\hat{\iota})}|\varpi(s,x(s),I^{\hat{\kappa}_1}\vartheta(s),\ldots,I^{\hat{\kappa}_k}\vartheta(s))|ds\\
&+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})}|\varpi(s,x(s),I^{\hat{\kappa}_1}\vartheta(s),\ldots,I^{\hat{\kappa}_k}\vartheta(s))|ds\\
&\le\int_0^{\tau_1}\frac{|(\tau_2-s)^{\hat{\iota}-1}-(\tau_1-s)^{\hat{\iota}-1}|}{\Gamma(\hat{\iota})}\|h\|_{L^1}ds+\int_{\tau_1}^{\tau_2}\frac{(\tau_2-s)^{\alpha-1}}{\Gamma(\hat{\iota})}\|h\|_{L^1}ds.
\end{aligned}$$

Thus, we have that $|\hat{\mathfrak{B}}\vartheta(\tau_2)-\hat{\mathfrak{B}}\vartheta(\tau_1)|\longrightarrow 0$ as $\tau_2\longrightarrow\tau_1$
which is independent of $\vartheta\in\hat{\mathfrak{S}}$.
which is independent of $\vartheta\in\hat{\mathfrak{S}}$.Thus, $\hat{\mathfrak{B}}(\hat{\mathfrak{S}})$ is equicontinuous. So $\hat{\mathfrak{B}}$ is relatively compact on $\hat{\mathfrak{S}}$. Hence, by the Arzelá-Ascoli theorem, $\hat{\mathfrak{B}}$ is compact on $\hat{\mathfrak{S}}$.
Claim 3. The hypothesis (*iii*) of Lemma 2.6 is satisfied.

Let $\vartheta \in \hat{\mathfrak{X}} \in \hat{\mathfrak{S}}$ be arbitrary elements such that $\vartheta = \hat{\mathfrak{A}}\vartheta\hat{\mathfrak{B}}\vartheta + \hat{\mathfrak{C}}\vartheta$. Then we have

$$\vartheta(\hat{\varkappa}) = \varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))\Big[\gamma + \delta + \eta\varkappa_1 + \vartheta_0$$
$$+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})} \varpi(s, \vartheta(s), I^{\hat{\kappa}_1}\vartheta(s), \ldots, I^{\hat{\kappa}_k}\vartheta(s))ds\Big] \tag{19}$$
$$+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})} \xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s))ds, \quad \hat{\varkappa} \in [0, T]. \tag{20}$$

$$\begin{aligned}
|\vartheta(\hat{\varkappa})| &\leq |\hat{\mathfrak{A}}\vartheta\hat{\mathfrak{B}}\vartheta + \hat{\mathfrak{C}}\vartheta| \\
&\leq |\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa}))|\Big[\ |\ \gamma + \delta + \eta\varkappa_1 + \vartheta_0\ | \\
&+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})} \Big|\varpi(s, \vartheta(s), I^{\hat{\kappa}_1}\vartheta(s), \ldots, I^{\hat{\kappa}_k}\vartheta(s))ds\Big|ds + \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})} \xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s))ds\Big|ds \\
&\leq (|\varphi(\hat{\varkappa}, \vartheta(\hat{\varkappa}), I^{\hat{\iota}_1}\vartheta(\hat{\varkappa}), \ldots, I^{\hat{\iota}_n}\vartheta(\hat{\varkappa})) - \varphi(\hat{\varkappa}, 0, \ldots, 0)| + |\varphi(\hat{\varkappa}, 0, \ldots, 0)|)\Big(\ |\ \gamma + \delta + \eta\varkappa_1 + \vartheta_0\ | \\
&+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\iota}-1}}{\Gamma(\hat{\iota})} |h(s)|ds\Big) \\
&+ \int_0^{\hat{\varkappa}} \frac{(\hat{\varkappa} - s)^{\hat{\kappa}-1}}{\Gamma(\hat{\kappa})} \Big[|\xi(s, \vartheta(s), I^{\hat{\iota}_1}\vartheta(s), \ldots, I^{\hat{\iota}_n}\vartheta(s)) - \xi(s, 0, \ldots, 0)| + |\xi(s, 0, \ldots, 0)|\Big]ds \\
&\leq \Big[r\|p\|\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n + 1)}\Big) + F_0\Big]\Big(\ |\ \gamma + \delta + \eta\varkappa_1 + \vartheta_0\ | + \frac{\|h\|_{L^1} T^{\hat{\iota}}}{\Gamma(\alpha + 1)}\Big) \\
&+ \frac{r\|m\|T^{\hat{\kappa}}}{\Gamma(\hat{\kappa} + 1)}\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\alpha_n + 1)}\Big) + \frac{T^{\hat{\kappa}}}{\Gamma(\hat{\kappa} + 1)} k_0.
\end{aligned}$$

We deduce that

$$\begin{aligned}
\|\vartheta\| &\leq \Big[r\|p\|\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n + 1)}\Big) + F_0\Big]\Big(\ |\ \gamma + \delta + \eta\varkappa_1 + \vartheta_0\ | + \frac{\|h\|_{L^1} T^{\hat{\iota}}}{\Gamma(\alpha + 1)}\Big) \\
&+ \frac{r\|m\|T^{\hat{\kappa}}}{\Gamma(\hat{\kappa} + 1)}\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\alpha_n + 1)}\Big) + \frac{T^{\hat{\kappa}}}{\Gamma(\hat{\kappa} + 1)} k_0,
\end{aligned} \tag{21}$$

that is, $\vartheta \in \hat{\mathfrak{S}}$.
Claim 4. Finally we show that $\hat{\tau}\mathcal{W} + \rho < 1$, that is, (iv) of Lemma 2.6 holds.
Since

$$\begin{aligned}
\mathcal{W} = \|\hat{\mathfrak{B}}(\hat{\mathfrak{S}})\| &= \sup_{\vartheta \in \hat{\mathfrak{S}}}\{\sup_{\hat{\varkappa} \in J} |\hat{\mathfrak{B}}\vartheta(\hat{\varkappa})|\} \\
&\leq \Big(\ |\ \gamma + \delta + \eta\varkappa_1 + \vartheta_0\ | + \frac{\|h\|_{L^1} T^{\hat{\iota}}}{\Gamma(\hat{\iota} + 1)}\Big),
\end{aligned} \tag{22}$$

and by theorem 3.1 we have

$$\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1 + 1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n + 1)}\Big)\Big[\|p\|\mathcal{W} + \|m\|\frac{T^{\hat{\kappa}}}{\Gamma(\hat{\kappa} + 1)}\Big] < 1,$$

with $\hat{\nu} = \Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1+1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n+1)}\Big)\|p\|$ and $\rho = \|m\|\frac{T^{\hat{\kappa}}}{\Gamma(\hat{\kappa}+1)}\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1+1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n+1)}\Big)$.
Thus all the conditions of Lemma 2.6 are satisfied and hence the operator equation $\vartheta = \hat{\mathfrak{A}}\vartheta\hat{\mathfrak{B}}\vartheta + \hat{\mathfrak{C}}\vartheta$ has a solution in $\hat{\mathfrak{S}}$. In consequence, problem (1) has a solution on J. This completes the proof. □

4. Exemple

In this section we give an example to illustrate the usefulness of our main results. Let us consider the following fractional boundary value problem:

$$\begin{cases} D^{\frac{1}{2}}\left(\frac{\vartheta(\hat{\varkappa})-I^{\frac{1}{2}}\left[\frac{2\hat{\varkappa}e^{-3\hat{\varkappa}}}{15(3+\hat{\varkappa})}\left(\sin\vartheta(\hat{\varkappa})+\frac{\vartheta(\hat{\varkappa})+9I^{\sqrt{2}}|\vartheta(\hat{\varkappa})|}{I^{\sqrt{2}}|\vartheta(\hat{\varkappa})|+5}\right)\right]}{\frac{(\hat{\varkappa}+1)^2}{100}\left(\sin\vartheta(\hat{\varkappa})+\frac{|I^{\sqrt{2}}\vartheta(\hat{\varkappa})|}{1+|I^{\sqrt{2}}\vartheta(\hat{\varkappa})|}+3\right)}\right) = \hat{\varkappa}^2\sin\vartheta(\hat{\varkappa})+\cos(I^{\frac{1}{4}}\vartheta(\hat{\varkappa}))+1, \hat{\varkappa}\in J=[0,1], \\ \vartheta(\hat{\varkappa}_1^+) = \vartheta(\hat{\varkappa}_1^-)+(-2\vartheta(\hat{\varkappa}_1^-)), \hat{\varkappa}_1 \neq 0,1, \\ \frac{\vartheta(0)}{\varphi(0,\vartheta(0),0)} = \frac{\pi}{2}, \qquad \frac{\vartheta(1)}{\varphi(1,\vartheta(1),I^{\alpha_1}\vartheta(1))} = 0, \end{cases} \tag{23}$$

Put $\hat{\iota} = \frac{1}{2}$, $\hat{\iota}_1 = \sqrt{2}, \hat{\kappa} = \frac{1}{2}, \hat{\kappa} = \frac{1}{4}, T = 1$, $n = k = 1, \varphi(\hat{\varkappa},\vartheta,\nu) = \frac{(\hat{\varkappa}+1)^2}{100}\left(\sin\nu(\hat{\varkappa}) + \frac{|\vartheta|}{1+|\vartheta|} + 3\right)$, $\varpi(\hat{\varkappa},\vartheta,\nu) = \hat{\varkappa}^2\sin\vartheta(\hat{\varkappa}) + \cos(I^{\frac{1}{4}}\vartheta(\hat{\varkappa})) + 1$, $\xi(\hat{\varkappa},\vartheta,\nu) = \frac{2\hat{\varkappa}e^{-3\hat{\varkappa}}}{15(3+\hat{\varkappa})}(\sin\nu(\hat{\varkappa}) + \frac{\vartheta^2(\hat{\varkappa})+9|\vartheta(\hat{\varkappa})|}{|\vartheta(\hat{\varkappa})|+5})$, $m(\hat{\varkappa}) = \frac{2\hat{\varkappa}}{15(3+\hat{\varkappa})}$, and $p(\hat{\varkappa}) = \frac{(\hat{\varkappa}+1)^2}{100}$ for $\hat{\varkappa} \in [0,1]$.
Note that, $\|\varpi(\hat{\varkappa},\vartheta,\nu)\| \leq \hat{\varkappa}^2 + 2$, and

$$|\varphi(\hat{\varkappa},\vartheta,\nu) - \varpi(\hat{\varkappa},\vartheta',\nu')| \leq \frac{(\hat{\varkappa}+1)^2}{100}(|\vartheta-\vartheta'| + |\nu-\nu'|),$$

and

$$|\xi(\hat{\varkappa},\vartheta,\nu) - \xi(\hat{\varkappa},\vartheta',\nu')| \leq \Big(\frac{2\hat{\varkappa}}{15(3+\hat{\varkappa})}\Big)(|\vartheta-\vartheta'| + |\nu-\nu'|).$$

We have

$$\Big(1 + \frac{T^{\hat{\iota}_1}}{\Gamma(\hat{\iota}_1+1)} + \ldots + \frac{T^{\hat{\iota}_n}}{\Gamma(\hat{\iota}_n+1)}\Big)\Big[\|p\|\Big(\mid \gamma + \delta + \eta\varkappa_1 + \vartheta_0 \mid \frac{\|h\|_{L^1}T^{\hat{\iota}}}{\Gamma(\hat{\iota}+1)}\Big) + \|m\|\frac{T^{\hat{\kappa}}}{\Gamma(\hat{\kappa}+1)}\Big] \simeq 0.18957628293 < 1.$$

By using the theorem 3.1, the problem (23) has a solution.

5. Conclusion

Most natural phenomena are treated using different types of fractional differential equations. This diversity in this type of equation helps us to scrutinize the integration of many phenomena in various fields. This helps us in creating programs that enable us to consume rational materials.In our contribution, we presented the existence of the integral solution of a impulsive hybrid differential equation in the the frame of the fractional derivative. The main results are obtained by using on the generalization of Dhage's fixed point theorem by three operators.For future work, we study the coupled for this problem.

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