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Interpolation formulas for 1-harmonic functions on the unit circle

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Abstract. A generalization of the deeply investigated harmonic functions, known as α-harmonic functions, have recently gained considerable attention. Similarly to the harmonic functions, an α-harmonic function u on the unit disc $\mathbb{D}$ is uniquely determined by its values on the boundary of the disc $\partial\mathbb{D}$. In fact, for any $z \in \mathbb{D}$, the value of $u(z)$ can be given as a contour integral over $\partial\mathbb{D}$ with a modified Poisson kernel. However, this integral can be difficult to evaluate, or the values on the boundary are known only empirically. In such cases, approximating $u(z)$ with an interpolatory formula, as a weighted sum of values of u at n nodes on $\partial\mathbb{D}$, can be an attractive alternative. The nodes and weights are to be chosen so that the degree d of exactness of the formula is maximized. In other words, the formula should be exact for all basis functions for α-harmonic functions of degree up to d, with d as large as possible. In the case of harmonic functions, it is known that there is an interpolation formula of degree of exactness as large as $d = n - 1$. The objective of this paper are formulas of this type for α-harmonic functions. We will prove that, given n, in this case the degree of exactness cannot be $n - 1$, but there is a unique interpolation formula of degree $n - 2$. Finally, we will prove convergence of such formulas to $u(z)$ as $n \to \infty$.

1. Introduction

For a complex-valued function u defined in a region D in the complex plane, two differential operators are commonly used:

$$\partial_z(u) = \frac{1}{2}(u_x - iu_y) \quad \text{and} \quad \bar{\partial}_z(u) = \frac{1}{2}(u_x + iu_y), \qquad \text{where} \quad z = x + iy.$$

In what follows, D will be the unit disc $\mathbb{D}$: $|z| \leqslant 1$.

The standard Laplace operator is $\Delta = \partial_z\bar{\partial}_z$. A function $u : \mathbb{D} \to \mathbb{C}$ is *harmonic* if it satisfies the Laplace equation $\Delta u = 0$. The functions $\operatorname{Re} z^k$ and $\operatorname{Im} z^k$ are harmonic and form a basis for *harmonic polynomials*, which are dense in the space of harmonic functions.

The Dirichlet boundary problem for harmonic functions is the problem of determining a harmonic function u if its values on the boundary $\partial\mathbb{D}$ are known:

$$u(z) = f(z) \quad \text{for} \quad z \in \partial\mathbb{D} \qquad \text{and} \qquad \Delta u = 0. \tag{1}$$

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It is not required that u be defined on $\partial\mathbb{D}$. Thus, f is in general a distribution on $\partial\mathbb{D}$ and the boundary condition actually means $\lim_{r\to 1-} u(re^{i\theta}) = f(e^{i\theta})$.

The solution to the boundary problem (1) is then given by the Poisson integral

$$u(z) = \mathcal{P}[f](z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\frac{1-|z|^2}{|z-e^{i\theta}|^2}d\theta \quad \text{for} \quad z \in \mathbb{D}.$$

An extension of the Laplace operator are the so-called *weighted Laplace operators*

$$\Delta_w = \partial_z w(z)^{-1}\bar{\partial}_z,$$

in a domain Ω of the complex plane $\mathbb{C}$ which is equipped with a weight function $w : \Omega \to (0,\infty)$. We mention that weighted Laplacians seem to have been first studied systematically by P. Garabedian [3].

In the study of Bergman spaces on the unit disc $\mathbb{D}$ one often considers so-called standard weights, which are weight functions of the form

$$w(z) := w_\alpha(z) = (1-|z|^2)^\alpha,$$

where $\alpha > -1$ is a real constant. For an account of recent developments in Bergman space theory we mention the monograph [6] by Hedenmalm, Korenblum and Zhu. The case $\alpha = 0$ is commonly referred to as the unweighted case, whereas the case $\alpha = 1$ has attracted special attention recently, with contributions by Hedenmalm, Shimorin and others (see for instance [18], [19], [20], [32] in [12]).

For $\alpha > -1$, we will denote the weighted Laplace operator corresponding to the weight w_α by Δ_α:

$$\Delta_\alpha = \partial_z(1-|z|^2)^{-\alpha}\bar{\partial}_z \quad \text{for} \quad z \in \mathbb{D}.$$

A function u that satisfies the equation $\Delta_\alpha u = 0$ on $\mathbb{D}$ is called *α-harmonic*. In particular, the case $\alpha = 0$ yields the harmonic functions. Properties of α-harmonic functions have recently been investigated in a number of papers. For instance, their Lipschitz continuity was investigated in [10].

The associated Dirichlet boundary value problem is

$$\lim_{r\to 1-} u(re^{i\theta}) = f(e^{i\theta}) \quad \text{for} \quad z \in \partial\mathbb{D} \qquad \text{and} \qquad \Delta_\alpha u = 0, \tag{2}$$

where f is a distribution on $\mathbb{D}$. It is shown in [12] that the solution to the boundary problem (2) is given by

$$u(z) = \mathcal{P}_\alpha[f](z) = \frac{1}{2\pi}\int_0^{2\pi} P_\alpha(ze^{-i\theta})f(e^{i\theta})d\theta \quad \text{for} \quad z \in \mathbb{D}, \tag{3}$$

where P_α is the α-harmonic Poisson kernel in $\mathbb{D}$:

$$P_\alpha(z) = \frac{(1-|z|^2)^{\alpha+1}}{(1-z)(1-\bar{z})^{\alpha+1}}. \tag{4}$$

A basis in the space of α-harmonic functions is formed by the functions

$$e_{\alpha,k}(z) = \mathcal{P}_\alpha[e^{ik\theta}](z), \quad \text{for an integer} \quad k.$$

Then $P_\alpha(z) = \sum_{k=-\infty}^{\infty} e_{\alpha,k}(z)$ for $z \in \mathbb{D}$. We have

$$e_{\alpha,k}(z) = z^k, \quad k = 0,1,2,\ldots, \qquad \text{and} \qquad e_{\alpha,-k}(z) = \frac{\bar{z}^k}{B(k,\alpha+1)}\int_0^1 t^{k-1}(1-t|z|^2)^\alpha dt, \quad k = 1,2,\ldots. \tag{5}$$

Since $e_{\alpha,k}(e^{i\theta}z) = e^{ik\theta}e_{\alpha,k}(z)$, rotation of the variable z about the origin preserves α-harmonicity. However, unlike the harmonic case, α-harmonicity is not preserved under translation of the variable.

When D is any open, bounded and simply connected region in the xy-plane, assuming that its boundary ∂D is a rectifiable Jordan curve, a numerical approach to the boundary value problem $\Delta u = 0$ with $u \equiv f$ on ∂D was discussed in [1, 7]. Namely, for a given $\zeta \in D$, the value of $u(\zeta)$ can be approximated by an interpolation formula of the form

$$u(\zeta) \approx \sum_{k=1}^{n} A_k u(z_k), \tag{6}$$

where the n nodes $z_1, \dots, z_n$ lie on the boundary ∂D and the weight coefficients A_k are constants. An n-node formula (6) is a *Gauss harmonic interpolation formula* if it gives the correct result whenever $u(z)$ is of the form $P(z) + Q(\bar{z})$ for some polynomials P and Q of degree at most $n-1$. Barrow and Stroud [1] established the existence of an n-node Gauss harmonic interpolation formula with positive real weights A_k. They further note that, under the assumption that $u(z)$ is continuous on ∂D, the positivity of the weights A_k implies convergence of Gauss harmonic interpolation formulas to $u(\zeta)$ as $n \to \infty$. When D is a circular region, Johnson and Riess [7] developed a procedure for computing nodes and weights for a Gauss formula. Harmonic interpolation has applications e.g. in computer graphics, see for instance [5] where an arbitrary curve is approximated by harmonic interpolation.

In this paper we investigate interpolation formulas of the form (6) for α-harmonic functions $u(z)$ when $\alpha = 1$ (here called simply 1-*harmonic functions*) and the region D is the unit disc $\mathbb{D}$. In this case, formula (6) is said to have the *degree of exactness* d if it gives the correct result whenever $u(z)$ is a linear combination of the base functions e_k given by (5) for $-d \leqslant k \leqslant d$. We will prove that there is no n-node 1-harmonic interpolation formula of degree of exactness $n-1$ (that would be called a "Gauss 1-harmonic formula"), but there is a unique n-node 1-harmonic interpolation formula of degree $n-2$. Although its weights are not positive nor real, we will prove convergence of these formulas to $u(\zeta)$ as $n \to \infty$.

2. 1-harmonic interpolation formulas

Our objective are interpolation formulas of the type (6) when u is a 1-harmonic function on the unit disc $\mathbb{D}$ and ζ inside the unit circle. Thus, we will require the nodes $z_1, z_2, \dots, z_n$ to lie on the unit circle.

The basis (5) for $\alpha = 1$ becomes

$$e_k(z) = z^k \qquad \text{and} \qquad e_{-k}(z) = \left(k + 1 - k|z|^2\right)\bar{z}^k \qquad \text{for } k \geqslant 0. \tag{7}$$

We observe that for $|z| = 1$ we have $e_k(z) = z^{-k}$. Suppose that

$$\sum_{j=1}^{n} w_j z_j^k = e_k(\zeta) \tag{8}$$

holds for $-d_1 \leqslant k \leqslant d_2$, where d_1, d_2 are nonnegative integers. The formula (6) has the degree of exactness d if $\min\{d_1, d_2\} \geqslant d$. Consider the polynomial

$$P(z) = (z - z_1)(z - z_2) \cdots (z - z_n) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \qquad (a_n = 1).$$

Multiplying the equation $\sum_{j=1}^{n} w_j z_j^{r-k} = e_{r-k}(\zeta)$ in (8) by $a_r \zeta^k$ and adding over $r = 0, 1, \dots, n$ yields

$$\sum_{r=0}^{k-1} |\zeta|^{2(k-r)}\Big((k-r+1) - (k-r)|\zeta|^2\Big) \cdot a_r \zeta^r + \sum_{r=k}^{n-1} a_r \zeta^r = -\zeta^n.$$

for $n - d_2 \leqslant k \leqslant d_1$. This can be written as

$$\Big(\underbrace{1, 1, \dots, 1}_{n-k}, |\zeta|^2(2 - |\zeta|^2), |\zeta|^4(3 - 2|\zeta|^2), \dots, |\zeta|^{2k}(k + 1 - k|\zeta|^2)\Big) \cdot (a_{n-1}\zeta^{n-1}, \dots, a_1\zeta, a_0) = -\zeta^n.$$

In other words, we have

$$A\mathbf{p}^T = \mathbf{z}^T,$$

where

$$\mathbf{p} = (a_{n-1}\zeta^{n-1} \;\; \dots \;\; a_1\zeta \;\; a_0) \qquad \text{and} \qquad \mathbf{z} = \zeta^n(C_1 \;\; C_2 \;\; \dots \;\; C_n) \in \mathbb{R}^n, \tag{9}$$

with $C_j = -1$ for $n - d_1 \leqslant j \leqslant d_2$, and

$$A = \begin{bmatrix} 1 & |\zeta|^2(2-|\zeta|^2) & |\zeta|^4(3-2|\zeta|^2) & \cdots & |\zeta|^{2n-2}(n-(n-1)|\zeta|^2) \\ 1 & 1 & |\zeta|^2(2-|\zeta|^2) & \cdots & |\zeta|^{2n-4}((n-1)-(n-2)|\zeta|^2) \\ 1 & 1 & 1 & \cdots & |\zeta|^{2n-6}((n-2)-(n-3)|\zeta|^2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & |\zeta|^2(2-|\zeta|^2) \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Then the polynomial P is given by

$$P(x) = x^n + \mathbf{x}\mathbf{p}^T = x^n + \mathbf{x}A^{-1}\mathbf{z}^T,$$

where $\mathbf{z}$ and $\mathbf{p}$ are given by (9) and

$$\mathbf{x} = \Big((x/\zeta)^{n-1} \;\; \dots \;\; x/\zeta \;\; 1\Big).$$

The inverse of the matrix A is

$$A^{-1} = \frac{1}{(1-|\zeta|^2)^2}\begin{bmatrix} 1 & -2|\zeta|^2 & |\zeta|^4 & & & & & \mathbf{0} \\ -1 & 2|\zeta|^2+1 & -|\zeta|^4-2|\zeta|^2 & |\zeta|^4 & & & & \\ & -1 & 2|\zeta|^2+1 & -|\zeta|^4-2|\zeta|^2 & \ddots & & & \\ & & -1 & 2|\zeta|^2+1 & \ddots & \ddots & & \\ & & & \ddots & \ddots & \ddots & |\zeta|^4 & \\ & & & & \ddots & 2|\zeta|^2+1 & -|\zeta|^4-2|\zeta|^2 & |\zeta|^4 \\ & & & & & -1 & 2|\zeta|^2+1 & -2|\zeta|^2 \\ \mathbf{0} & & & & & & -1 & 1 \end{bmatrix}.$$

In the case $d = n - 1$ we obtain

$$P(x) = x^{n-1}(x-\zeta) - a(\bar{\zeta}x - 1)^2, \tag{10}$$

where $a = (1 + C_n)\frac{\zeta^n}{(1-|\zeta|^2)^2} \in \mathbb{C}$ is a constant.

Theorem 2.1. *The polynomial $P(x)$ given by* (10) *has at most two distinct zeros on the unit circle.*

Proof. Suppose that x_0 is a zero of $P(x)$ with $|x_0| = 1$. Conjugating the equation $x_0^{n-1}(x_0 - \zeta) = a(1 - \bar{\zeta}x_0)^2$ yields $1 - \bar{\zeta}x_0 = \bar{a}(x_0 - \zeta)^2 x_0^{n-2}$. These two equation together give $a\bar{a}^2(x_0 - \zeta)^3 x_0^{n-3} = 1$, and hence $|x_0 - \zeta| = \frac{1}{|a|}$. It follows that $P(x)$ has at most two zeros of unit modulus. □

Corollary 2.2. *There is no n-node* 1*-harmonic interpolation formula or type* (14) *of degree of exactness $n - 1$ for $n > 2$.*

We proceed to the case $d = n-2$. Now the vector $\mathbf{z}$ takes the form $= \zeta^n(C_1, -1, \dots, -1, C_{n-1}, C_n)$, where C_1, C_{n-1}, C_n are arbitrary constants, and we obtain

$$P(x) = x^{n-2}(x+a)(x-\zeta) + (bx+c)(1-\bar{\zeta}x)^2, \tag{11}$$

where $a = \frac{(C_1+1)\zeta}{(1-|\zeta|^2)^2}$, $b = \frac{(C_{n-1}+1)\zeta^{n-1}}{(1-|\zeta|^2)^2}$ and $c = \frac{(C_n - C_{n-1})\zeta^n}{(1-|\zeta|^2)^2}$ are some complex constants.

Theorem 2.3. *The polynomial* (11) *has n distinct zeros on the unit circle if and only if* $a = -\zeta$, $b = 0$ *and* $|c| = 1$.

Proof. *"Only if" part:* Complex numbers of unit modulus are characterized by $\frac{1}{z} = \bar{z}$. It follows that, if all zeros of the polynomial $P(x)$ are of unit modulus, the monic polynomials $\frac{1}{P(0)}x^n P(\frac{1}{x})$ and $\overline{P(\bar{x})}$ have the same zeros, and therefore coincide. Thus, comparing the coefficients of

$$\frac{1}{P(0)}x^n P(\tfrac{1}{x}) = x^{n-3}(x+\tfrac{b}{c})(x-\bar{\zeta})^2 + \tfrac{1}{c}(1+ax)(1-\zeta x) \quad \text{and}$$
$$\overline{P(\bar{x})} = x^{n-2}(x+\bar{a})(x-\bar{\zeta}) + (\bar{b}x+\bar{c})(1-\zeta x)^2$$

we easily obtain $a = -\zeta$, $b = 0$ and $|c| = 1$. Therefore,

$$P(x) = P_n(x) = x^{n-2}(x-\zeta)^2 + c(1-\bar{\zeta}x)^2, \qquad \text{where} \qquad |c| = 1. \tag{12}$$

"If" part: Let P_n be given by (12). Since obviously $P_n(\bar{\zeta}^{-1}) \neq 0$, the equation $P_n(x) = 0$ can be written as

$$x^{n-2} = -cf(x)^2, \quad \text{where} \quad f(x) = \frac{1-\bar{\zeta}x}{x-\zeta}.$$

We observe that the Möbius transformation $f(x)$ bijectively maps the unit circle onto itself, and since $|f(0)| > 1$, it maps the interior of the unit circle to its exterior and vice-versa. It follows that for $|x| > 1$ we have $|x^{n-2}| > 1 > |cf(x)|$, and for $|x| < 1$ we have $|x^{n-2}| < 1 < |cf(x)|$. Therefore, $x^{n-2} = f(x)$ can hold only if $|x| = 1$.

It remains to show that P_n has no multiple roots. Any such root x would also be a root of $P_n'(x) = (n-2)x^{n-3}(x-\zeta)^2 + 2x^{n-2}(x-\zeta) - 2c\bar{\zeta}(1-\bar{\zeta}x)$, and since $x^{n-2} = -cf(x)^2$, substituting will reduce the last equation to

$$(n-2)\Big(\tfrac{|\zeta|}{\zeta}x\Big)^2 - \Big(\tfrac{n}{|\zeta|} + (n-4)|\zeta|\Big)\Big(\tfrac{|\zeta|}{\zeta}x\Big) + (n-2) = 0,$$

which is a real quadratic in $|\zeta|x/\zeta$ with a positive discriminant, and hence has two positive real roots different from 1, contrary to the assumption that $|x| = 1$. □

The polynomial (12) is obtained for $b = 0$ in (11), that is, for $C_{n-1} = -1$. By (9), this means that (8) actually holds for $-(n-2) \leqslant k \leqslant (n-1)$, so

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{n-1} & z_2^{n-1} & \cdots & z_n^{n-1} \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} 1 \\ \zeta \\ \vdots \\ \zeta^{n-1} \end{bmatrix}.$$

From here we find

$$w_i = \frac{P_n(\zeta)}{(\zeta - z_i)P_n'(z_i)} = \frac{c(1-|\zeta|^2)^2}{(\zeta - z_i)P_n'(z_i)}. \tag{13}$$

To sum up:

Theorem 2.4. *For each $n > 2$ and z on the unit circle, there is a unique n-node 1-harmonic interpolation formula (6) of degree of exactness at least $n-2$ whose nodes $z_1, \dots, z_n$ lie on the unit circle and $z_n = z$. This formula is given by*

$$\mathcal{K}_n(u) = \sum_{j=1}^{n} w_j u(z_j), \tag{14}$$

where the nodes z_j are the zeros of the polynomial (12) with $c = -z^{n-2}(\frac{z-\zeta}{1-\bar{\zeta}z})^2$, and the weights w_j are given by (13). Then $\mathcal{K}_n(u) = u(\zeta)$ whenever u is a linear combination of the functions e_k given by (7) for $-(n-2) \leqslant k \leqslant n-1$.

Since for any $\varepsilon > 0$ and $n \geqslant 32/(1-|\zeta|)^2$

$$\begin{aligned}|P_n'(z_i)| &= \left|(n-2)z_i^{n-3}(z_i-\zeta)^2 + 2z_i^{n-2}(z_i-\zeta) - 2c\bar{\zeta}(1-\bar{\zeta}z_i)\right| \\ &\geqslant (n-2)|z_i-\zeta|^2 - 2|z_i-\zeta| - 2|\zeta||z_i-\zeta| \geqslant n(1-|\zeta|)^2 - 16 \geqslant \tfrac{1}{2}(1-|\zeta|)^2 n,\end{aligned}$$

we have $|w_i| \leqslant \frac{2(1+|\zeta|)^2}{(1-|\zeta|)n}$ and hence the sum of weights is bounded:

$$\sum_{i=1}^{n} |w_i| \leqslant \frac{2(1+|\zeta|)^2}{(1-|\zeta|)}$$

for n large enough, which is a well-known criterion for convergence of a sequence of quadratures as $n \to \infty$ (see, e.g., [11, p.203]). Therefore:

Theorem 2.5. *If u is 1-harmonic in $\mathbb{D}$ and continuous on $\partial\mathbb{D}$, then the sequence of interpolation formulas $\mathcal{K}_n(u)$ given by (14) converges to $u(\zeta)$ as $n \to \infty$.*

We end this section with a statement that provides closer information about the location of the nodes $z_1, \dots, z_n$ on the unit circle.

Theorem 2.6. *Every arc of length $\frac{2\pi}{n-2}$ on the unit circle $\partial\mathbb{D}$ contains a zero of the polynomial $P_n(x)$ (12).*

Proof. A complex number $x = \cos t + i \cdot \sin t$ is a zero of P_n if $g(t) := (n-2)t - 2f(t)$ is a multiple of 2π, where

$$f(t) = \arg \frac{1 - \bar{\zeta}(\cos t + i \cdot \sin t)}{\cos t + i \cdot \sin t - \zeta}.$$

Clearly, we can define the argument so as to make $f(t)$ continuous for $t \in [0, 2\pi]$. It is easy to show that then $f(t)$ is a decreasing function with $f(2\pi) = f(0) - 2\pi$. Therefore $g'(t) \geqslant n-2$ for each t, so when t passes an interval of length $\frac{2\pi}{n-2}$, the function $g(t)$ takes at least one value that is a multiple of 2π. □

It follows that the nodes $z_1, \dots, z_n$ are distributed fairly uniformly on the unit circle, even if ζ is close to the boundary. In fact, with more careful computation, one could refine the statement of Theorem 2.6 to give that the distance between any two consecutive zeros of P_n along the circle is between $2\pi/\left(n - \frac{4|\zeta|}{1+|\zeta|}\right)$ and $2\pi/\left(n + \frac{4|\zeta|}{1-|\zeta|}\right)$.

3. Numerical examples

Example 3.1. (a) *We will use the interpolation formula (14) for $n = 5$ and $n = 10$, with the parameter c set to -1, to approximate the value $u(\frac{1}{2})$, where u is the 1-harmonic function*

$$u(z) = 2(z+1)\ln|z+1| - |z|^2 \quad \textit{for} \quad z \neq -1, \qquad \textit{with} \qquad u(-1) := -1.$$

The correct value is $u(\frac{1}{2}) \approx 0.9663953243$. The nodes z_j and the weights w_j of the formulas $\mathcal{K}_5$ and $\mathcal{K}_{10}$ are shown in Table 1 and graphically in Figure 1.

$n = 5:$	$\mathcal{K}_5(u) = 0.9648\ldots$
$z_1 = -0.6614 - i \cdot 0.75$	$w_1 = 0.0541 - i \cdot 0.0153$
$z_2 = -0.6614 + i \cdot 0.75$	$w_2 = 0.0541 + i \cdot 0.0153$
$z_3 = 0.6614 - i \cdot 0.75$	$w_3 = 0.1959 - i \cdot 0.1097$
$z_4 = 0.6614 + i \cdot 0.75$	$w_4 = 0.1959 + i \cdot 0.1097$
$z_5 = 1$	$w_5 = 0.5$

$n = 10:$	$\mathcal{K}_{10}(u) = 0.9666\ldots$
$z_1 = -1$	$w_1 = 0.0192$
$z_2 = -0.75 - i \cdot 0.6614$	$w_2 = 0.0221 - i \cdot 0.0053$
$z_3 = -0.75 + i \cdot 0.6614$	$w_3 = 0.0221 + i \cdot 0.0053$
$z_4 = -0.1404 - i \cdot 0.9901$	$w_4 = 0.0343 - i \cdot 0.0159$
$z_5 = -0.1404 + i \cdot 0.9901$	$w_5 = 0.0343 + i \cdot 0.0159$
$z_6 = 0.5 - i \cdot 0.8660$	$w_6 = 0.0750 - i \cdot 0.0433$
$z_7 = 0.5 + i \cdot 0.8660$	$w_7 = 0.0750 + i \cdot 0.0433$
$z_8 = 0.8904 - i \cdot 0.4552$	$w_8 = 0.1983 - i \cdot 0.0813$
$z_9 = 0.8904 + i \cdot 0.4552$	$w_9 = 0.1983 + i \cdot 0.0813$
$z_{10} = 1$	$w_{10} = 0.3214$

Table 1: The nodes z_j and weights w_j for the formula (14) in Example 3.1 for $n \in \{5, 10\}$.

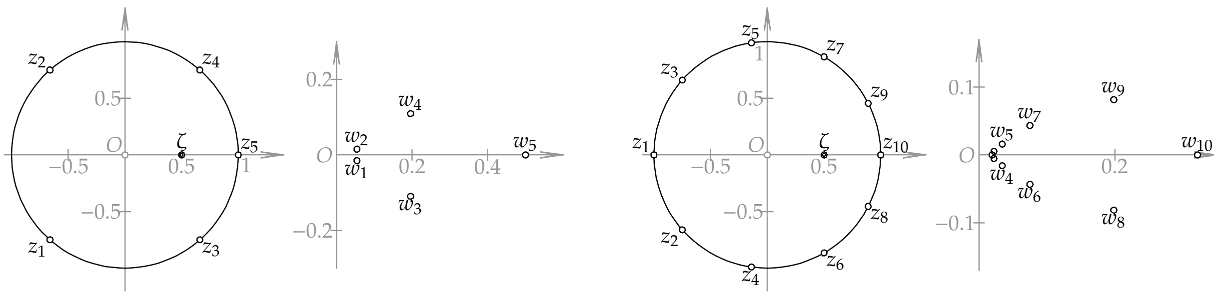

Figure 1: The nodes z_j and weights w_j for the formula (14) in Example 3.1 for $n \in \{5, 10\}$.

(b) *In order to compare the errors, we now use the formula (14) for $n = 5$ and $n = 10$ with $c = -1$ for different values of ζ to approximate $u(\zeta)$. We obtain the following results.*

ζ	$u(\zeta)$	$\mathcal{K}_5(u)$	$\lvert\mathcal{K}_5(u) - u(\zeta)\rvert$	$\mathcal{K}_{10}(u)$	$\lvert\mathcal{K}_{10}(u) - u(\zeta)\rvert$
0.99	1.7586758622	1.7586758512	1.10×10^{-8}	1.7586758633	1.11×10^{-9}
0.9	1.6290447675	1.6290335082	1.13×10^{-5}	1.6290459451	1.18×10^{-6}
0.75	1.3961552578	1.3959711650	1.84×10^{-4}	1.3961756559	2.04×10^{-5}
0.5	0.9663953243	0.9648017744	1.59×10^{-3}	0.9665917001	1.96×10^{-4}
0.25	0.4953588783	0.4895176678	5.84×10^{-3}	0.4961723873	8.14×10^{-4}
0	0	−0.0150733146	1.51×10^{-2}	0.0024250561	2.43×10^{-3}

(c) *The same nodes and weights from part* **(a)** *can be used if z is any point with $|z| = \frac{1}{2}$. Indeed, since α-harmonicity is invariant under rotation of z, we can write $z = e^{i\theta}|z|$, define $v(x) = u(e^{i\theta}x)$ and then evaluate $v(|z|)$ instead.*

For example, suppose that we need $u(\frac{1}{2}e^{i\frac{\pi}{7}})$ for the 1-harmonic function $u(z) = e^z$. We can use the formula (14) for $n = 5$ and $n = 10$ on the function $v(z) = e^{ze^{i\pi/7}}$ to approximate $v(\frac{1}{2})$. The results obtained for $n = 5$ and $n = 10$ are

$$\mathcal{K}_5(v) \approx 1.5281282288 + i \cdot 0.3418971610 \quad \textit{and} \quad \mathcal{K}_{10}(v) \approx 1.5322934416 + i \cdot 0.3377334760,$$

whereas the exact value is $u(\frac{1}{2}e^{i\frac{\pi}{7}}) \approx 1.5322934702 + i \cdot 0.3377336491$.

Example 3.2. **(a)** *We will now apply the formula (14) for $n = 5$ and $n = 10$ on the function $v(z) = e^{ze^{i\pi/7}}$ from Example 3.1(b), but with different choices of the parameter c, to approximate $v(\frac{1}{2})$.*

c	$\mathcal{K}_5(v)$	$\lvert\mathcal{K}_5(v) - v(\frac{1}{2})\rvert$	$\mathcal{K}_{10}(v)$	$\lvert\mathcal{K}_{10}(v) - v(\frac{1}{2})\rvert$
−1	$1.5281282288 + i \cdot 0.3418971610$	5.89×10^{-3}	$1.5322934416 + i \cdot 0.3377334760$	1.75×10^{-7}
1	$1.5364531904 + i \cdot 0.3335681930$	5.89×10^{-3}	$1.5322934987 + i \cdot 0.3377338223$	1.76×10^{-7}
$e^{i\frac{\pi}{3}}$	$1.5379834822 + i \cdot 0.3392543157$	5.89×10^{-3}	$1.5322933345 + i \cdot 0.3377337605$	1.76×10^{-7}
$\frac{3+4i}{5}$	$1.5381242523 + i \cdot 0.3385625645$	5.89×10^{-3}	$1.5322933488 + i \cdot 0.3377337759$	1.76×10^{-7}

Curiously, the error is in each case almost the same in modulus, and moreover, the difference of the arguments of the error and the parameter c is almost constant. However, this phenomenon seems to be due to the nature of the function u.

(b) *Applying the formula* (14) *for* $n = 5$ *and* $n = 10$ *on the function* $u(z) = 2(z+1)\ln|z+1| - |z|^2$ *from Example 3.1(a) with the same choices of the parameter c gives larger errors:*

c	$\mathcal{K}_5(u)$	$\lvert\mathcal{K}_5(u) - u(\frac{1}{2})\rvert$	$\mathcal{K}_{10}(u)$	$\lvert\mathcal{K}_{10}(u) - u(\frac{1}{2})\rvert$
-1	0.9648017744	1.59×10^{-3}	0.9665917001	1.96×10^{-4}
1	0.9686305590	2.24×10^{-3}	0.9662496647	1.46×10^{-4}
$e^{i\frac{\pi}{3}}$	$0.9670826366 + i \cdot 0.0231630420$	2.32×10^{-2}	$0.9663084775 - i \cdot 0.0029317982$	2.93×10^{-3}
$\frac{3+4i}{5}$	$0.9673073392 + i \cdot 0.0231187990$	2.31×10^{-2}	$0.9662961073 - i \cdot 0.0026367261$	2.64×10^{-3}

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