Filomat 38:3 (2024), 803–809
https://doi.org/10.2298/FIL2403803W

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Greville type $\{1, 2, 3\}$-generalized inverses for rectangular matrices

Cang Wu[a], **Jianlong Chen**[a,*]

[a]*School of Mathematics, Southeast University, Nanjing 210096, China*

Abstract. For any complex matrices A and W, $m \times n$ and $n \times m$, respectively, it is proved that there exists a complex matrix X such that $AXA = A$, $XAX = X$, $(AX)^* = AX$ and $XA(WA)^k = (WA)^k$, where k is the index of WA. When A is square and W is the identity matrix, such an X reduces to Greville's spectral $\{1, 2, 3\}$-inverse of A. Various expressions of such generalized inverses are established.

1. Introduction

Throughout the paper, let $\mathbb{C}^{m\times n}$ be the set of all $m \times n$ complex matrices and I_n be the $n \times n$ identity matrix. For $A \in \mathbb{C}^{m\times n}$, the symbols A^* and $\mathrm{rank}(A)$ will denote the conjugate transpose and the rank of A, respectively. When A is square, $\mathrm{Ind}(A)$ denotes the index of A, i.e., the smallest nonnegative integer k such that $\mathrm{rank}(A^k) = \mathrm{rank}(A^{k+1})$.

For $A \in \mathbb{C}^{m\times n}$, recall the four Penrose equations [17]

$$\text{(i) } AXA = A, \quad \text{(ii) } XAX = X, \quad \text{(iii) } (AX)^* = AX, \quad \text{(iv) } (XA)^* = XA. \tag{1}$$

As usual, a common solution of the i-th, $\cdots$, j-th equations in (1) is called an $\{i, \cdots, j\}$-inverse of A and denoted by $A^{(i,\cdots,j)}$, and the set of all $\{i, \cdots, j\}$-inverses of A is denoted by $A\{i, \cdots, j\}$. It is known that the set $A\{1, 2, 3, 4\}$ is nonempty and it consists of a single element $A^\dagger$, called the Moore–Penrose inverse of A.

For $A \in \mathbb{C}^{n\times n}$, recall that the Drazin inverse A^D of A is the unique common solution of the equations

$$XA^{k+1} = A^k, \qquad XAX = X, \qquad AX = XA, \tag{2}$$

where $k = \mathrm{Ind}(A)$ [6]. The Drazin inverse of A always exists, and in the special case of $\mathrm{Ind}(A) \leq 1$, the Drazin inverse of A is called the group inverse of A and denoted by $A^\#$. The spectral idempotent $I_n - AA^D$ will be denoted by A^π.

The equation $XA^{k+1} = A^k$ in (2) is closely related to spectral properties of generalized inverses. For example, if G is a solution of $XA^{k+1} = A^k$, then every λ-vector of A of grade p for $\lambda \neq 0$ is a λ^{-1}-vector of G of grade p (see, e.g., [3, p. 162]). Following Campbell and Meyer [4], any solution of $XA^{k+1} = A^k$ is called a weak Drazin inverse of A.

2020 *Mathematics Subject Classification*. Primary 15A09; Secondary 15A24.
Keywords. Spectral $\{1, 2, 3\}$-inverse; Moore–Penrose inverse; Drazin inverse; Core inverse.
Received: 28 April 2023; Accepted: 25 July 2023
Communicated by Dragana Cvetković-Ilić
This research was supported by the National Natural Science Foundation of China (Nos. 12171083 and 12071070) and the Qing Lan Project of Jiangsu Province.
* Corresponding author: Jianlong Chen
Email addresses: cang_wu@hotmail.com (Cang Wu), jlchen@seu.edu.cn (Jianlong Chen)

Although the Moore–Penrose inverse $A^\dagger$ is in general not a weak Drazin inverse of A, Greville [7] showed that there exists a class of $\{1,2,3\}$-inverses of A that are weak Drazin inverses of A. According to [7, Theorem 1], for a $\{1\}$-inverse $A^{(1)}$ of A, the composite generalized inverse $A^DAA^\dagger + A^{(1)}(A - AA^DA)A^\dagger$ is a common solution of equations

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad XA^{k+1} = A^k; \tag{3}$$

and conversely, any solution of (3) is of the form $A^DAA^\dagger + A^{(1)}(A - AA^DA)A^\dagger$ for some $\{1\}$-inverse $A^{(1)}$ of A (see also [3, p.173, Ex. 52]). These composite generalized inverses will hereafter be referred to as spectral $\{1,2,3\}$-inverses of A.

Spectral $\{1,2,3\}$-inverses can be used like Moore–Penrose inverses when studying the least-squares problem of linear equations [3], and like Drazin inverses when studying systems of differential equations with singular coefficients or Markov chains [4, 5].

Unaware of Greville's work, the present authors studied solutions of (3) under the name of $\{1,2,3,1^k\}$-inverses [22]. A main idea is that if X is a $\{1,2,3\}$-inverse of A and Y is a weak Drazin inverse of A, then $X + (I_n - XA)YAX$ is a spectral $\{1,2,3\}$-inverse of A. Also, it was shown that A has a unique spectral $\{1,2,3\}$-inverse if and only if $\mathrm{Ind}(A) \le 1$; in this case the unique spectral $\{1,2,3\}$-inverse is exactly the core inverse of Baksalary and Trenkler [1], which has attracted much attention in the last decade (see, e.g., [2, 8–12, 15, 16, 18–21]).

In this paper, the notion of spectral $\{1,2,3\}$-inverses is extended to rectangular matrices. Let $A \in \mathbb{C}^{m\times n}$. It is proved that for any $W \in \mathbb{C}^{n\times m}$, there exists a class of $\{1,2,3\}$-inverses X of A such that $XA(WA)^k = (WA)^k$, where k is the index of WA. This class of $\{1,2,3\}$-inverses, called W-spectral $\{1,2,3\}$-inverses of A, reduces to spectral $\{1,2,3\}$-inverses of A when $m = n$ and $W = I_n$, and becomes the Moore–Penrose $A^\dagger$ when $W = A^*$. Some characterizations of W-spectral $\{1,2,3\}$-inverses are presented, and the set of all W-spectral $\{1,2,3\}$-inverses is described. Moreover, a canonical form for W-spectral $\{1,2,3\}$-inverses is given by using the singular value decomposition.

2. Spectral $\{1,2,3\}$-inverses for rectangular matrices

We begin with the following definition.

Definition 2.1. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$. Then $X \in \mathbb{C}^{n\times m}$ is called a W-spectral $\{1,2,3\}$-inverse of A if it satisfies*

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad XA(WA)^k = (WA)^k, \tag{4}$$

where k is the index of WA.

Example 2.2. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$.*

(i) *When $m = n$ and $W = I_n$ (or more generally, W is a nonsingular matrix commuting with A), W-spectral $\{1,2,3\}$-inverses of A reduce to its spectral $\{1,2,3\}$-inverses.*

(ii) *When $W = A^*$, the equation $XA(WA)^k = (WA)^k$ becomes $XAA^*A = A^*A$ since $\mathrm{Ind}(A^*A) \le 1$. Multiplying by $A^\dagger$ from the right and using $A^*AA^\dagger = A^*$, we get $XAA^* = A^*$, which is equivalent to X being a $\{1,4\}$-inverse of A. Thus, the A^*-spectral $\{1,2,3\}$-inverse of A is exactly the $\{1,2,3,4\}$-inverse $A^\dagger$ and so it is unique.*

The next result shows the existence of W-spectral $\{1,2,3\}$-inverses by giving an explicit construction of them.

Theorem 2.3. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$. Then the following statements are equivalent:*

(i) *X is a W-spectral $\{1,2,3\}$-inverse of A.*

(ii) *$X = A^{(1,2,3)} + (I_n - A^{(1,2,3)}A)[(WA)^DW]AA^{(1,2,3)}$ for some $\{1,2,3\}$-inverse $A^{(1,2,3)}$ of A.*

(iii) $X = A^{(1,2,3)} + (I_n - A^{(1,2,3)}A)ZAA^{(1,2,3)}$ *for some* $\{1,2,3\}$*-inverse* $A^{(1,2,3)}$ *of* A *and some* Z *satisfying* $ZA(WA)^k = (WA)^k$, *where* $k = \text{Ind}(WA)$.

Proof. Let $k = \text{Ind}(WA)$.

(i)⇒(ii). Assume that X is a W-spectral $\{1,2,3\}$-inverse of A. Since $XA(WA)^k = (WA)^k$, we have $XA(WA)^D = (WA)^D$, i.e., $(I_n - XA)(WA)^D = 0$. It follows that

$$X = X + (I_n - XA)[(WA)^D W]AX.$$

Thus (ii) follows by taking $A^{(1,2,3)} = X$.

(ii)⇒(iii). It is clear by noting that $(WA)^D W$ is just such a Z.

(iii)⇒(i). For any $\{1,2,3\}$-inverse $A^{(1,2,3)}$ of A and Z satisfying $ZA(WA)^k = (WA)^k$, let

$$X = A^{(1,2,3)} + (I_n - A^{(1,2,3)}A)ZAA^{(1,2,3)}.$$

Since $AX = AA^{(1,2,3)}$, it is easy to see that X is a $\{1,2,3\}$-inverse of A. Moreover,

$$\begin{aligned} XA(WA)^k &= A^{(1,2,3)}A(WA)^k + (I_n - A^{(1,2,3)}A)ZA(WA)^k \\ &= A^{(1,2,3)}A(WA)^k + (I_n - A^{(1,2,3)}A)(WA)^k = (WA)^k. \end{aligned}$$

Therefore, X is a W-spectral $\{1,2,3\}$-inverse of A by the definition. □

For a $\{1,2,3\}$-inverse of A and a solution of the equation $XA(WA)^k = (WA)^k$, we may think of Theorem 2.3 as a way to construct a matrix that is simultaneously a $\{1,2,3\}$-inverse of A and a solution of $XA(WA)^k = (WA)^k$.

Remark 2.4. *In a similar vein, taking*

$$Y = A^{(1,2,4)} + A^{(1,2,4)}A[W(AW)^D](I_m - AA^{(1,2,4)}),$$

one can show that Y *is a* $\{1,2,4\}$*-inverse of* A *and satisfies* $(AW)^l AY = (AW)^l$, *where* l *is the index of* AW. *This type of* $\{1,2,4\}$*-inverses is dual to* W*-spectral* $\{1,2,3\}$*-inverses.*

Also, by Cline's formula $(WA)^D = W[(AW)^D]^2 A$, we know that

$$(WA)^D W = W(AW)^D. \tag{5}$$

The next result gives a characterization of the set of all $\{1,2,3\}$-inverses of A; it is a slight modification of [3, p. 56, Exercise 12].

Lemma 2.5. *For any fixed* $\{1\}$*-inverse* $A^{(1)}$ *and* $\{1,2,3\}$*-inverse* $A^{(1,2,3)}$ *of* $A \in \mathbb{C}^{m\times n}$, *the set of all* $\{1,2,3\}$*-inverses of* A *is given by*

$$A\{1,2,3\} = \{A^{(1,2,3)} + (I_n - A^{(1)}A)ZAA^{(1,2,3)} : \ Z \in \mathbb{C}^{n\times m}\}. \tag{6}$$

Proof. For any $Z \in \mathbb{C}^{n\times m}$, it is direct to verify that

$$A^{(1,2,3)} + (I_n - A^{(1)}A)ZAA^{(1,2,3)} \in A\{1,2,3\}.$$

Also, for any $X \in A\{1,2,3\}$, since $AX = AA^{(1,2,3)}$, it follows that $A(X - A^{(1,2,3)}) = 0$ and $XAA^{(1,2,3)} = X$. Thus,

$$\begin{aligned} &A^{(1,2,3)} + (I_n - A^{(1)}A)(X - A^{(1,2,3)})AA^{(1,2,3)} \\ &= A^{(1,2,3)} + (X - A^{(1,2,3)})AA^{(1,2,3)} = A^{(1,2,3)} + X - A^{(1,2,3)} = X, \end{aligned}$$

which shows that $X \in \{A^{(1,2,3)} + (I_n - A^{(1)}A)WAA^{(1,2,3)} : \ W \in \mathbb{C}^{n\times n}\}$. The proof is completed. □

Observe Greville's construction for spectral $\{1,2,3\}$-inverses:

$$A^D AA^\dagger + A^{(1)}(A - AA^D A)A^\dagger = A^{(1)}AA^\dagger + (I_n - A^{(1)}A)A^D AA^\dagger.$$

Here, $A^{(1)}AA^\dagger$ is a $\{1,2,3\}$-inverse of A.

We next present a characterization of the set of all W-spectral $\{1,2,3\}$-inverses of A.

Theorem 2.6. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$, and let $A^{(1)}$ be a $\{1\}$-inverse of A. Then the set of all W-spectral $\{1,2,3\}$-inverses of A is given by*

$$\{A^{(1)}AA^\dagger + (I_n - A^{(1)}A)\Big[W(AW)^D + S(AW)^\pi\Big]AA^\dagger : \ S \in \mathbb{C}^{n\times m}\}.$$

Proof. Let $k = \mathrm{Ind}(WA)$. For any $S \in \mathbb{C}^{n\times m}$, take

$$X_s = A^{(1)}AA^\dagger + (I_n - A^{(1)}A)\Big[W(AW)^D + S(AW)^\pi\Big]AA^\dagger.$$

Then X_s is a $\{1,2,3\}$-inverse of A by Lemma 2.5, and

$$\begin{aligned}
X_sA(WA)^k &= A^{(1)}A(WA)^k + (I_n - A^{(1)}A)\Big[W(AW)^D + S(AW)^\pi\Big]A(WA)^k \\
&= A^{(1)}A(WA)^k + (I_n - A^{(1)}A)\Big[W(AW)^D A(WA)^k + S(AW)^\pi A(WA)^k\Big] \\
&\overset{(5)}{=} A^{(1)}A(WA)^k + (I_n - A^{(1)}A)\Big[(WA)^D WA(WA)^k + SA(WA)^\pi (WA)^k\Big] \\
&= A^{(1)}A(WA)^k + (I_n - A^{(1)}A)(WA)^k = (WA)^k.
\end{aligned}$$

Therefore, X_s is a W-spectral $\{1,2,3\}$-inverse of A. In particular,

$$X_0 = A^{(1)}AA^\dagger + (I_n - A^{(1)}A)[W(AW)^D]AA^\dagger$$

is a W-spectral $\{1,2,3\}$-inverse of A.

On the other hand, for any W-spectral $\{1,2,3\}$-inverse X of A, since X and X_0 are $\{1,2,3\}$-inverses of A, it follows by Lemma 2.5 that there exists a $T \in \mathbb{C}^{n\times m}$ such that

$$X = X_0 + (I_n - A^{(1)}A)TAX_0.$$

Since $XA(WA)^k = (WA)^k$ and $X_0A(WA)^k = (WA)^k$, it follows that

$$[(I_n - A^{(1)}A)TAX_0]A(WA)^k = 0,$$

and so

$$[(I_n - A^{(1)}A)T]A(WA)^k = 0.$$

Thus we get

$$\begin{aligned}
[(I_n - A^{(1)}A)T]AW(AW)^D &\overset{(5)}{=} [(I_n - A^{(1)}A)T]A(WA)^D W \\
&= [(I_n - A^{(1)}A)T]A[(WA)^k[(WA)^D]^{k+1}]W = 0,
\end{aligned}$$

which implies that $(I_n - A^{(1)}A)T = (I_n - A^{(1)}A)T(AW)^\pi$. Therefore,

$$\begin{aligned}
X &= X_0 + (I_n - A^{(1)}A)TAA^\dagger \\
&= X_0 + (I_n - A^{(1)}A)T(AW)^\pi AA^\dagger \\
&= A^{(1)}AA^\dagger + (I_n - A^{(1)}A)[W(AW)^D + T(AW)^\pi]AA^\dagger.
\end{aligned}$$

The proof is completed. □

In particular, when $m = n$ and $W = I_n$, we obtain the following characterization for the set of all spectral $\{1, 2, 3\}$-inverses. It is a supplement to Greville's construction.

Corollary 2.7. *Let $A \in \mathbb{C}^{n\times n}$ and let $A^{(1)}$ be a $\{1\}$-inverse of A. Then the set of all spectral $\{1, 2, 3\}$-inverses of A is given by $\{A^{(1)}AA^{\dagger} + (I_n - A^{(1)}A)(A^D + SA^{\pi})AA^{\dagger} : S \in \mathbb{C}^{n\times n}\}$.*

Using Theorem 2.6, we next consider a canonical form for W-spectral $\{1, 2, 3\}$-inverses.

For $A \in \mathbb{C}^{m\times n}$, the singular value decomposition states that there exist two unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that

$$A = U\begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}V^*, \tag{7}$$

where $\Sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_r)$ is the diagonal matrix of singular values of A, $r = \mathrm{rank}(A)$. Moreover,

$$A^{\dagger} = V\begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix}U^* \tag{8}$$

and a general $\{1, 2, 3\}$-inverse of A is of the form $V\begin{bmatrix} \Sigma^{-1} & 0 \\ Z & 0 \end{bmatrix}U^*$ for some Z; see [3, p. 208].

The next result shows how to choose a Z to get a W-spectral $\{1, 2, 3\}$-inverse for A.

Proposition 2.8. *Let $A \in \mathbb{C}^{m\times n}$ be as in* (7) *and $W \in \mathbb{C}^{n\times m}$ be partitioned as*

$$W = V\begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}U^*,$$

where $W_1 \in \mathbb{C}^{r\times r}$, and W_2, W_3, W_4 are of appropriate sizes. Then $X \in \mathbb{C}^{n\times m}$ is a W-spectral $\{1, 2, 3\}$-inverse of A if and only if there is $T \in \mathbb{C}^{(n-r)\times r}$ such that

$$X = V\begin{bmatrix} \Sigma^{-1} & 0 \\ W_3(\Sigma W_1)^D + T(\Sigma W_1)^{\pi} & 0 \end{bmatrix}U^*. \tag{9}$$

Proof. First, direct calculation shows that

$$AA^{\dagger} = U\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}U^*, \quad A^{\dagger}A = V\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}V^*, \quad AW = U\begin{bmatrix} \Sigma W_1 & \Sigma W_2 \\ 0 & 0 \end{bmatrix}U^*.$$

By [12, Eq. 14], we have

$$(AW)^D = U\begin{bmatrix} (\Sigma W_1)^D & [(\Sigma W_1)^D]^2\Sigma W_2 \\ 0 & 0 \end{bmatrix}U^*,$$

and so

$$W(AW)^D = V\begin{bmatrix} W_1(\Sigma W_1)^D & W_1[(\Sigma W_1)^D]^2\Sigma W_2 \\ W_3(\Sigma W_1)^D & W_3[(\Sigma W_1)^D]^2\Sigma W_2 \end{bmatrix}U^*, \tag{10}$$

$$(AW)^{\pi} = U\begin{bmatrix} (\Sigma W_1)^{\pi} & -(\Sigma W_1)^D\Sigma W_2 \\ 0 & I_{m-r} \end{bmatrix}U^*. \tag{11}$$

By Theorem 2.6, X is a W-spectral $\{1, 2, 3\}$-inverse of A if and only if there is $S \in \mathbb{C}^{n\times m}$ such that

$$X = A^{\dagger} + (I_n - A^{\dagger}A)[W(AW)^D + S(AW)^{\pi}]AA^{\dagger}.$$

Let S be partitioned as $S = V\begin{bmatrix} S_1 & S_2 \\ T & S_3 \end{bmatrix}U^*$, where $T \in \mathbb{C}^{(n-r)\times r}$. Then by (10) and (11),

$$W(AW)^D + S(AW)^\pi = V\begin{bmatrix} \star_1 & \star_2 \\ W_3(\Sigma W_1)^D + T(\Sigma W_1)^\pi & \star_3 \end{bmatrix}U^*.$$

It follows that

$$(I_n - A^\dagger A)[W(AW)^D + S(AW)^\pi]AA^\dagger = V\begin{bmatrix} 0 & 0 \\ W_3(\Sigma W_1)^D + T(\Sigma W_1)^\pi & 0 \end{bmatrix}U^*,$$

and therefore

$$X = A^\dagger + (I_n - A^\dagger A)[W(AW)^D + S(AW)^\pi]AA^\dagger = V\begin{bmatrix} \Sigma^{-1} & 0 \\ W_3(\Sigma W_1)^D + T(\Sigma W_1)^\pi & 0 \end{bmatrix}U^*,$$

which completes the proof. □

In the rest of the paper, we study two special cases: matrices which possess a unique W-spectral $\{1,2,3\}$-inverse, and matrices for which every $\{1,2,3\}$-inverse is a W-spectral $\{1,2,3\}$-inverse. To these ends, we need the following well known result.

Lemma 2.9. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$. If $ASB = 0$ for all $S \in \mathbb{C}^{n\times p}$, then $A = 0$ or $B = 0$.*

Proof. If $A, B \neq 0$, then there exist invertible matrices $P_1 \in \mathbb{C}^{m\times m}, Q_1 \in \mathbb{C}^{n\times n}$ and $P_2 \in \mathbb{C}^{p\times p}, Q_2 \in \mathbb{C}^{q\times q}$ such that $A = P_1\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}Q_1$ and $B = P_2\begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}Q_2$, where $r = \text{rank}(A) > 0$ and $t = \text{rank}(B) > 0$. Now let S_{11} be an $n \times p$ matrix whose entries are all zeros except the $(1,1)$-entry s_{11}. Then

$$A(Q_1^{-1}S_{11}P_2^{-1})B = P_1\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}S_{11}\begin{bmatrix} I_t & 0 \\ 0 & 0 \end{bmatrix}Q_2 = P_1\begin{bmatrix} s_{11} & 0 \\ 0 & 0 \end{bmatrix}Q_2 \neq 0,$$

contradicting $ASB = 0$ for all $S \in \mathbb{C}^{n\times p}$. Thus $A = 0$ or $B = 0$. □

By Meyer and Painter [14], A has a unique $\{1,3\}$-inverse if and only if it is of full column rank. Analogously, A has a unique $\{1,2,3\}$-inverse if and only if A is of full column rank or $A = 0$. In [22], it was shown that a square matrix A has a unique spectral $\{1,2,3\}$-inverse if and only if $\text{rank}(A) = \text{rank}(A^2)$, in which case the unique spectral $\{1,2,3\}$-inverse is exactly the core inverse $A^\# AA^\dagger$. Now we consider the class of matrices which have a unique W-spectral $\{1,2,3\}$-inverse.

Proposition 2.10. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$. Then A has a unique W-spectral $\{1,2,3\}$-inverse if and only if A is of full column rank or $\text{rank}(A) = \text{rank}(AWA)$. Moreover, if A is of full column rank, then $A^\dagger$ is the unique W-spectral $\{1,2,3\}$-inverse of A; if $\text{rank}(A) = \text{rank}(AWA)$, then $W(AW)^\# AA^\dagger$ is the unique W-spectral $\{1,2,3\}$-inverse of A.*

Proof. Let $A^{(1)}$ be a fixed $\{1\}$-inverse of A. By Theorem 2.6, $X_0 = A^{(1)}AA^\dagger + (I_n - A^{(1)}A)[W(AW)^D]AA^\dagger$ is a W-spectral $\{1,2,3\}$-inverse of A, and so is $X_s = X_0 + (I_n - A^{(1)}A)S(AW)^\pi AA^\dagger$ for any $S \in \mathbb{C}^{n\times m}$.

Now assume that A has a unique W-spectral $\{1,2,3\}$-inverse. Then it follows that $X_0 = X_s$, i.e., $(I_n - A^{(1)}A)S(AW)^\pi AA^\dagger = 0$ for any $S \in \mathbb{C}^{n\times m}$. Thus by Lemma 2.9, we have $I_n - A^{(1)}A = 0$ or $(AW)^\pi AA^\dagger = 0$. When $I_n - A^{(1)}A = 0$, A is of full column rank. When $(AW)^\pi AA^\dagger = 0$, we have $AA^\dagger = [(AW)^D AW]AA^\dagger$. Post-multiplication this equation by A yields $A = (AW)^D AWA$, so $\text{rank}(A) = \text{rank}(AWA)$.

Conversely, if A is of full column rank, then every $\{1\}$-inverse of A is a left inverse and thus a $\{1,2,4\}$-inverse. So a $\{1,3\}$-inverse of A must be the unique $\{1,2,3,4\}$-inverse $A^\dagger$, and $A^\dagger A(WA)^k = (WA)^k$ holds, where k is the index of WA. It follows that $A^\dagger$ is the unique W-spectral $\{1,2,3\}$-inverse of A. If $\text{rank}(A) = \text{rank}(AWA)$, then

$$(AW)^\# \text{ and } (WA)^\# \text{ exist, } A = AW(AW)^\# A \text{ and } (AW)^\pi A = 0.$$

It follows that

$$\begin{aligned} X_0 &= A^{(1)}AA^{\dagger} + (I_n - A^{(1)}A)[W(AW)^{\#}]AA^{\dagger} \\ &= A^{(1)}AA^{\dagger} + [W(AW)^{\#}]AA^{\dagger} - A^{(1)}[AW(AW)^{\#}A]A^{\dagger} \\ &= A^{(1)}AA^{\dagger} + [W(AW)^{\#}]AA^{\dagger} - A^{(1)}AA^{\dagger} = W(AW)^{\#}AA^{\dagger}, \end{aligned}$$

and $X_s = X_0 + (I_n - A^{(1)}A)S(AW)^{\pi}AA^{\dagger} = X_0$. Therefore, by Theorem 2.6, $W(AW)^{\#}AA^{\dagger}$ is the unique W-spectral $\{1,2,3\}$-inverse of A. □

Similarly, we have the next result.

Proposition 2.11. *Let $A \in \mathbb{C}^{m\times n}$, $W \in \mathbb{C}^{n\times m}$. Then every $\{1,2,3\}$-inverse of A is its W-spectral $\{1,2,3\}$-inverse if and only if A is of full column rank or WA is nilpotent.*

Proof. The "if" part is clear. For the "only if" part, let X be a fixed $\{1,2,3\}$-inverse of A. Then by Lemma 2.5, $X + (I_n - XA)SX$ is a $\{1,2,3\}$-inverse of A for every $S \in \mathbb{C}^{n\times n}$. Assume that every $\{1,2,3\}$-inverse of A is its W-spectral $\{1,2,3\}$-inverse. Let $k = \mathrm{Ind}(WA)$. Then we have

$$XA(WA)^k = (WA)^k \text{ and } [X + (I_n - XA)SX]A(WA)^k = (WA)^k,$$

which imply that $(I_n - XA)S(WA)^k = 0$ for every $S \in \mathbb{C}^{n\times n}$. Thus, we get $(I_n - XA) = 0$ or $(WA)^k = 0$ by Lemma 2.9, and so A is of full column rank or WA is nilpotent. The proof is completed. □

References

[1] O. M. Baksalary, G. Trenkler, *Core inverse of matrices*, Linear Multilinear Algebra, 58(6) (2010), 681–697.

[2] O. M. Baksalary, G. Trenkler, *On a generalized core inverse*, Appl. Math. Comput., 236 (2014), 450–457.

[3] A. Ben-Israel, T. N. E. Greville, *Generalized Inverses: Theory and Applications*, (2nd edition), Springer, New York, 2003.

[4] S. L. Campbell, C. D. Meyer, *Weak Drazin inverses*, Linear Algebra Appl., 20(2) (1978), 167–178.

[5] D. S. Cvetković-Ilić, Y. M. Wei, *Algebraic Properties of Generalized Inverses*, Springer, Singapore, 2017.

[6] M. P. Drazin, *Pseudo-inverses in associative rings and semigroups*, Amer. Math. Monthly, 65 (1958), 506–514.

[7] T. N. E. Greville, *Some new generalized inverses with spectral properties*, Proc. Sympos. Theory and Application of Generalized Inverses of Matrices (Lubbock, Texas, 1968), pp. 26–46.

[8] H. F. Ma, *Optimal perturbation bounds for the core inverse*, Appl. Math. Comput., 336 (2018), 176–181.

[9] H. F. Ma, *Displacement structure of the core inverse*, Linear Multilinear Algebra, 70(2) (2022), 203–214.

[10] S. B. Malik, *Some more properties of core partial order*, Appl. Math. Comput., 221 (2013), 192–201.

[11] S. B. Malik, L. Rueda, N. Thome, *Further properties on the core partial order and other matrix partial orders*, Linear Multilinear Algebra, 62(12) (2014), 1629–1648.

[12] S. B. Malik, N. Thome, *On a new generalized inverse for matrices of an arbitrary index*, Appl. Math. Comput., 226 (2014), 575–580.

[13] C. D. Meyer, *Generalized inverses of block triangular matrices*, SIAM J. Appl. Math., 19 (1970), 741–750.

[14] C. D. Meyer, R. J. Painter, *Note on a least squares inverse for a matrix*, J. Assoc. Comput. Mach., 17 (1970), 110–112.

[15] D. Mosić, *Condition numbers related to the core inverse of a complex matrix*, Filomat, 36(11) (2022), 3785–3796.

[16] D. Mosić, C. Y. Deng, H. F. Ma, *On a weighted core inverse in a ring with involution*, Comm. Algebra, 46(6) (2018), 2332–2345.

[17] R. Penrose, *A generalized inverse for matrices*, Proc. Cambridge Philos. Soc., 51 (1955), 406–413.

[18] K. M. Prasad, K. S. Mohana, *Core-EP inverse*, Linear Multilinear Algebra, 62(6) (2014), 792–802.

[19] D. S. Rakić, N. Č. Dinčić, D. S. Djordjevic, *Group, Moore–Penrose, core and dual core inverse in rings with involution*, Linear Algebra Appl., 463 (2014), 115–133.

[20] X. P. Sheng, D. W. Xin, *Methods of Gauss–Jordan elimination to compute $A^{\text{\textcircled{\#}}}$ and dual core inverse $A_{\text{\textcircled{\#}}}$*, Linear Multilinear Algebra, 70(12) (2022), 2354–2366.

[21] H. X. Wang, X. J. Liu, *Characterizations of the core inverse and the core partial ordering*, Linear Multilinear Algebra, 63(9) (2015), 1829–1836.

[22] C. Wu, J. L. Chen, *The $\{1,2,3,1^m\}$-inverses: A generalization of core inverses for matrices*, Appl. Math. Comput., 427 (2022), No. 127149, 8 pp.