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On (B, C)-MP-inverses of rectangular matrices

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Abstract. For any $A \in \mathbb{C}^{n\times m}$, the set of all n by m complex matrices, Mosić and Stanimirović [14] introduced the composite OMP inverse of A by its outer inverse with the prescribed range, null space and Moore-Penrose inverse. This inverse unifies the core inverse, DMP inverse and Moore-Penrose inverse. In this paper, we mainly introduce and investigate a class of generalized inverses in complex matrices. Also, it is proved that this generalized inverse coincides with the OMP inverse. Finally, the defined inverse is related to OMP-inverses, W-core inverses and (b, c)-core inverses in the context of matrices.

1. Introduction and notation

For complex matrix A, the Moore-Penrose inverse $A^\dagger$ [15] and the Drazin inverse A^D [6] are two classical generalized inverses. In the last decade, several new types of mixed generalized inverses were introduced by combining the Moore-Penrose inverse and the Drazin inverse (or the group inverse). For instance, in 2010, Baksalary and Trenkler [1] introduced the core inverse $A^{\circledast}$ of A with index one (i.e., rank(A)=rank(A^2)). In 2014, Malik and Thome [11] defined the DMP-inverse $A^{D,\dagger}$ of A with index $m \geq 1$ (i.e., m is the smallest positive integer such that rank(A^k)=rank(A^{k+1})), extending the core inverse.

In order to unify the core inverse, the DMP inverse and so on, Mosić and Stanimirović [14] introduced the composite OMP inverse of a complex matrix by its outer inverse with the prescribed range, null space and Moore-Penrose inverse.

Motivated by [14], we mainly investigate a special case of OMP inverses, called (B, C)-MP-inverses. The paper is organized as follows. In Section 2, given $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$, the (B, C)-Moore-Penrose inverse (abbr. (B, C)-MP-inverse) of A is given. Also, we characterize the (B, C)-MP-inverse of A by its range and null spaces. But beyond that, it is shown in Theorem 2.8 that X is the (B, C)-MP inverse of A if and only if X is an outer inverse of A with prescribed range $\mathcal{T}$ and null space $\mathcal{S}$. In Section 3, the (b, c)-core inverse in $*$-semigroups [21] is investigated in the context of rectangular matrices. Also, the (B, C)-MP-inverse is related to other generalized inverses.

Throughout this paper, $\mathbb{C}^{n\times m}$ denotes the set of $n \times m$ complex matrices. The symbol I_n stands for the identity matrix of order n.

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For any $A \in \mathbb{C}^{n\times m}$, the column space and the null space of A are respectively defined as $\mathcal{R}(A) = \{Ax : x \in \mathbb{C}^{m\times 1}\}$ and $\mathcal{N}(A) = \{x \in \mathbb{C}^{m\times 1} : Ax = 0\}$. The symbols A^* and $\mathrm{rk}(A)$ stand for the conjugate transpose and the rank of A, respectively.

Three basic facts are given as follows: $\mathcal{N}(A^*) = \mathcal{R}(A)^{\perp}, \mathcal{R}(A^*) = \mathcal{N}(A)^{\perp}$ and $\mathrm{rk}(A) + \dim\mathcal{N}(A) = n$. Let $A, B \in \mathbb{C}^{n\times m}$. Then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ (resp., $\mathcal{N}(B) \subseteq \mathcal{N}(A)$) if and only if there exists some $X \in \mathbb{C}^{m\times m}$ (resp., $Y \in \mathbb{C}^{n\times n}$) such that $A = BX$ (resp., $A = YB$).

Let us now recall several notions of generalized inverses. For any $A \in \mathbb{C}^{n\times m}$, the Moore-Penrose inverse $A^{\dagger}$ [15] of A is the unique matrix $X \in \mathbb{C}^{m\times n}$ satisfying

$$\text{(i) } AXA = A, \text{ (ii) } XAX = X, \text{ (iii) } (AX)^* = AX, \text{ (iv) } (XA)^* = XA.$$

More generally, a matrix $X \in \mathbb{C}^{m\times n}$ satisfying (i) $AXA = A$ is called an inner inverse of A and is denoted by A^{-}. A matrix $X \in \mathbb{C}^{m\times n}$ satisfying (ii) $XAX = X$ is called an outer inverse of A and is denoted by $A^{(2)}$.

Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$. The matrix A is said to be (B, C)-invertible (see [2]) if there exists a matrix $X \in \mathbb{C}^{m\times n}$ such that $XAB = B, CAX = C, \mathcal{R}(X) \subseteq \mathcal{R}(B)$ and $\mathcal{N}(C) \subseteq \mathcal{N}(X)$. Such a matrix X is called a (B, C)-inverse of A. It is unique if it exists and is denoted by $A^{\|(B,C)}$. One knows that the inverse along a matrix is an instance of the (B, C)-inverse. The inverse of A along D is denoted by $A^{\|D}$. The standard notion for the inverse along a matrix can be referred to [2].

Given $A \in \mathbb{C}^{n\times n}$, the Drazin inverse of A [6] is the unique matrix $A^D \in \mathbb{C}^{n\times n}$ satisfying $A^D AA^D = A^D$, $AA^D = A^D A$ and $A^D A^{k+1} = A^k$, where $k = \mathrm{ind}(A)$. The smallest positive integer k such that $\mathrm{rk}(A^k) = \mathrm{rk}(A^{k+1})$ is called the index of A and is denoted by $\mathrm{ind}(A)$. In particular, if $\mathrm{ind}(A) \le 1$, then A is called group invertible. It is well known that A is group invertible if and only if $\mathrm{rk}(A) = \mathrm{rk}(A^2)$.

Following [1], a matrix $A \in \mathbb{C}^{n\times n}$ is called core invertible if there exists some $X \in \mathbb{C}^{n\times n}$ such that $AX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$, where P_A represents the orthogonal projector onto $\mathcal{R}(A)$. Such an X is called a core inverse of A [1]. The core inverse of A is unique if it exists and is denoted by $A^{\circledast}$. One knows from [1] that A is core invertible if and only if A is group invertible. In this case, we have $A^{\circledast} = A^{\#}AA^{\dagger}$.

Let $A \in \mathbb{C}^{n\times n}$ with index m. The DMP-inverse (denoted by $A^{D,\dagger}$) of $A \in \mathbb{C}^{n\times n}$ is defined as the unique matrix $X \in \mathbb{C}^{n\times n}$ satisfying $XAX = X, XA = A^D A$ and $A^m X = A^m A^{\dagger}$. Also, it is shown that $A^{D,\dagger} = A^D AA^{\dagger}$.

Suppose that $\mathcal{T}$ and $\mathcal{S}$ are subspaces of $\mathbb{C}^{m\times 1}$ and $\mathbb{C}^{n\times 1}$, respectively. Given $A \in \mathbb{C}^{n\times m}$, a matrix $X \in \mathbb{C}^{m\times n}$ is called an outer inverse of A with prescribed range $\mathcal{T}$ and null space $\mathcal{S}$ if $X = XAX, \mathcal{R}(X) = \mathcal{T}$ and $\mathcal{N}(X) = \mathcal{S}$ (see e.g., [20]). The outer inverse of A with prescribed range $\mathcal{T}$ and null space $\mathcal{S}$ is unique if it exists, and is denoted by $A^{(2)}_{\mathcal{T},\mathcal{S}}$. Some types of generalized inverses are characterized by $A^{(2)}_{\mathcal{T},\mathcal{S}}$. Here are several well known characterizations for generalized inverses :

(1) $A^{\dagger} = A^{(2)}_{\mathcal{R}(A^*),\mathcal{N}(A^*)}$ for $A \in \mathbb{C}^{n\times m}$ [20].

(2) $A^D = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k)}$ for $A \in \mathbb{C}^{n\times n}$ and $k = \mathrm{ind}(A)$ [20].

(3) $A^{\|D} = A^{(2)}_{\mathcal{R}(D),\mathcal{N}(D)}$ for $A \in \mathbb{C}^{n\times m}$ and $D \in \mathbb{C}^{m\times n}$ [2].

(4) $A^{\|(B,C)} = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ for $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ [2].

(5) $A^{D,\dagger} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k A^{\dagger})}$ for $A \in \mathbb{C}^{n\times n}$ and $k = \mathrm{ind}(A)$ [24].

Let $A \in \mathbb{C}^{n\times m}$ be of rank r, let T be of dimension $s \le r$ and let S be of dimension $m - s$. Suppose $A^{(2)}_{\mathcal{T},\mathcal{S}}$ exists. A matrix $X \in \mathbb{C}^{m\times n}$ is called an OMP inverse of A if it satisfies the system of equations $XAX = X$, $AX = AA^{(2)}_{\mathcal{T},\mathcal{S}}AA^{\dagger}$ and $XA = A^{(2)}_{\mathcal{T},\mathcal{S}}A$. This inverse is unique if it exists. Also, it was shown in [14] that $X = A^{(2)}_{\mathcal{T},\mathcal{S}}AA^{\dagger}$ is solution to the system above.

Several known generalized inverses are listed as special cases of OMP inverses.

(1) For $m = n$ and $A^{(2)}_{\mathcal{T},\mathcal{S}} = A^{\#}$, then the OMP inverse of A coincides with its core inverse.

(2) For $m = n$ and $A^{(2)}_{\mathcal{T},\mathcal{S}} = A^D$, then the OMP inverse of A coincides with its DMP-inverse.

2. The (B, C)-MP-inverse of a matrix

As defined in [14], the OMP inverse of a rectangular matrix A was given by combining its outer inverse $A^{(2)}_{\mathcal{T},\mathcal{S}}$ and Moore-Penrose inverse $A^{\dagger}$. The main goal in this section is to introduce and investigate a type of generalized inverses, called the (B, C)-MP-inverse of A (See Definition 2.1 below).

Definition 2.1. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ exists. The matrix A is called* (B, C)-MP-*invertible if there exists some matrix $X \in \mathbb{C}^{m\times n}$ satisfying the system of equations*

$$XAX = X,\ XA = A^{\|(B,C)}A \text{ and } CAX = CAA^{\dagger}. \qquad (1)$$

Such an X is called a (B, C)-MP-*inverse of A.*

Following [14], a matrix $A \in \mathbb{C}^{n\times m}$ is called (B, C)-MP-invertible (in the sense of Mosić and Stanimirović) if there exists some $X \in \mathbb{C}^{m\times n}$ such that $XAX = X$, $AX = AA^{\|(B,C)}AA^{\dagger}$ and $XA = A^{\|(B,C)}A$. Such an X is called the (B, C)-MP-inverse of A. We remark here the readers that the defined (B, C)-MP-inverse is equivalent to Mosić and Stanimirović's (B, C)-MP-inverse [14]. Suppose $X \in \mathbb{C}^{m\times n}$ satisfy $XAX = X$, $AX = AA^{\|(B,C)}AA^{\dagger}$ and $XA = A^{\|(B,C)}A$. Then it satisfies $XAX = X$, $XA = A^{\|(B,C)}A$ and $CAX = CAA^{\dagger}$. Conversely, given $XAX = X$, $XA = A^{\|(B,C)}A$ and $CAX = CAA^{\dagger}$, then by Theorem 2.2 below, $X = A^{\|(B,C)}AA^{\dagger}$, and consequently $AX = AA^{\|(B,C)}AA^{\dagger}$.

Recently, Hernández, Lattanzi and Thome [8, 9] introduced two more general 1MP-inverses and 2MP-inverses of A, where 1MP-inverses (resp., 2MP-inverses) of A are given by its inner inverses (resp., outer inverses) and Moore-Penrose inverse. More details on these generalized inverses can be found in [3–5, 7, 14, 16, 18, 22, 23].

Needless to say, the (B, C)-MP-inverse belongs to 2MP-inverses. However, 2MP-inverses do not have many properties owned by the (B, C)-MP-inverse, such as the most fundamental uniqueness. It is known that the OMP inverse is unique whenever it exists, and so is the (B, C)-MP-inverse. We denote the (B, C)-MP-inverse of A by $A^{\|(B,C),\dagger}$.

The following theorem gives the expression for the (B, C)-MP inverse of A.

Theorem 2.2. *The system* (1) *has a unique solution*: $X = A^{\|(B,C)}AA^{\dagger}$.

Proof. Suppose $X = A^{\|(B,C)}AA^{\dagger}$. Then one can directly check that X satisfies the system (1). □

Several known generalized inverses are listed as special cases of (B, C)-MP-inverses.

(1) For $m = n$ and $B = C = A$, then $A^{\|(B,C)} = A^{\#}$ and (A, A)-MP inverse of A coincides with its core inverse.

(1′) For $m = n$, $B = A$ and $C = A^*$, then by [17, Theorem 4.4], we have $A^{\|(B,C)} = A^{\textcircled{\#}}$ and (A, A^*)-MP inverse of A coincides with its core inverse.

(2) Let $\mathrm{ind}(A) = k$, $m = n$ and $B = C = A^k$. Then $A^{\|(B,C)} = A^D$, so that (A^k, A^k)-MP inverse of A coincides with its DMP-inverse.

(3) If $B = C$, then $A^{\|(B,C)} = A^{\|B}$ and (B, B)-MP inverse of A coincides with its MMP-inverse along B.

(4) Suppose $B = C = A^*$. Then $A^{\|(B,C)} = A^{\dagger}$ and (A^*, A^*)-MP inverse of A coincides with its Moore-Penrose inverse.

In [2], the writers derived the criterion for the (B, C)-inverse by rank conditions in complex matrices as follows.

Lemma 2.3. [2, Theorem 4.4] *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$. Then the following statements are equivalent*:

(i) *A is (B, C)-invertible.*

(ii) $\mathrm{rk}(C) = \mathrm{rk}(B) = \mathrm{rk}(CAB)$.

In this case, $A^{\|(B,C)} = B(CAB)^{\dagger}C$.

Lemma 2.4. [2, Corollary 4.5] *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ exists. Then* $\mathrm{rk}(AB) = \mathrm{rk}(CA) = \mathrm{rk}(C) = \mathrm{rk}(B)$.

Based on the above results, we obtain the following theorem, which plays an important role in the sequel.

Theorem 2.5. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ exists. Then*
(i) $\mathcal{R}(A^{\|(B,C),\dagger}A) = \mathcal{R}(A^{\|(B,C),\dagger}) = \mathcal{R}(B)$ *and* $\mathcal{R}(AA^{\|(B,C),\dagger}) = \mathcal{R}(AB)$.
(ii) $\mathcal{N}(AA^{\|(B,C),\dagger}) = \mathcal{N}(A^{\|(B,C),\dagger}) = \mathcal{N}(CAA^{\dagger})$ *and* $N(A^{\|(B,C),\dagger}A) = N(CA)$.
(iii) $\mathrm{rk}(AB) = \mathrm{rk}(B) = \mathrm{rk}(AA^{\|(B,C),\dagger}) = \mathrm{rk}(A^{\|(B,C),\dagger}) = \mathrm{rk}(A^{\|(B,C),\dagger}A) = \mathrm{rk}(CAA^{\dagger}) = \mathrm{rk}(CA)$.

Proof. (i) Since $A^{\|(B,C),\dagger}AA^{\|(B,C),\dagger} = A^{\|(B,C),\dagger}$, one has $\mathcal{R}(A^{\|(B,C),\dagger}A) = \mathcal{R}(A^{\|(B,C),\dagger})$. From [2, Theorem 6.6], it follows that $\mathcal{R}(A^{\|(B,C)}A) = \mathcal{R}(B)$ and $\mathcal{R}(AA^{\|(B,C)}) = \mathcal{R}(AB)$, whence $\mathcal{R}(A^{\|(B,C),\dagger}A) = \mathcal{R}(A^{\|(B,C)}A) = \mathcal{R}(B)$ and $\mathcal{R}(AB) = \mathcal{R}(AA^{\|(B,C),\dagger}AB) \subseteq \mathcal{R}(AA^{\|(B,C),\dagger}) = \mathcal{R}(AA^{\|(B,C)}AA^{\dagger}) \subseteq \mathcal{R}(AA^{\|(B,C)}) = \mathcal{R}(AB)$. So, $\mathcal{R}(AA^{\|(B,C),\dagger}) = \mathcal{R}(AB)$.

(ii) We have $\mathcal{N}(AA^{\|(B,C),\dagger}) = \mathcal{N}(A^{\|(B,C),\dagger})$ since $A^{\|(B,C),\dagger}AA^{\|(B,C),\dagger} = A^{\|(B,C),\dagger}$. Again by [2, Theorem 6.6], we have $\mathcal{N}(A^{\|(B,C)}A) = \mathcal{N}(CA)$, so that $\mathcal{N}(A^{\|(B,C),\dagger}A) = \mathcal{N}(A^{\|(B,C)}A) = \mathcal{N}(CA)$. As $\mathcal{N}(C) \subseteq \mathcal{N}(A^{\|(B,C)})$, then there exists some $T \in \mathbb{C}^{m\times m}$ such that $A^{\|(B,C)} = TC$. So, $\mathcal{N}(CAA^{\dagger}) = \mathcal{N}(CAA^{\|(B,C),\dagger}) \subseteq \mathcal{N}(TCAA^{\|(B,C),\dagger}) = \mathcal{N}(A^{\|(B,C),\dagger}) \subseteq (CAA^{\|(B,C),\dagger}) = \mathcal{N}(CAA^{\dagger})$. Therefore, $\mathcal{N}(AA^{\|(B,C),\dagger}) = \mathcal{N}(A^{\|(B,C),\dagger}) = \mathcal{N}(CAA^{\dagger})$.

(iii) It follows from (i) and (ii). □

A matrix $A \in \mathbb{C}^{n\times n}$ is called Hermitian if $A^* = A$. A Hermitian projector matrix is called an orthogonal projector. It is known that $AA^{\|(B,C),\dagger}$ and $A^{\|(B,C),\dagger}A$ are both projectors. However, they may not be orthogonal projectors. We next show under what conditions $AA^{\|(B,C),\dagger}$ and $A^{\|(B,C),\dagger}A$ are orthogonal projectors.

Theorem 2.6. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ exists. Then the following statements are equivalent:*
(i) $AA^{\|(B,C),\dagger}$ *is an orthogonal projector.*
(ii) $\mathcal{R}(AB) = \mathcal{R}(AA^{\dagger}C^*)$.
(iii) $\mathcal{R}(AA^{\dagger}C^*) \subseteq \mathcal{R}(AB)$.
(iv) $\mathcal{R}(AB) \subseteq \mathcal{R}(AA^{\dagger}C^*)$.

Proof. To begin with, (ii) ⇒ (iii) and (ii) ⇒ (iv) are obvious.

(i) ⇒ (ii) Given (i), then $AA^{\|(B,C),\dagger} = (AA^{\|(B,C),\dagger})^*$, so that $\mathcal{R}(AA^{\|(B,C),\dagger}) = \mathcal{R}((AA^{\|(B,C),\dagger})^*) = \mathcal{N}(AA^{\|(B,C),\dagger})^{\perp}$.
By Theorem 2.5, we have

$$\begin{aligned}\mathcal{R}(AB) &= \mathcal{R}(AA^{\|(B,C),\dagger}) = \mathcal{N}(AA^{\|(B,C),\dagger})^{\perp} = \mathcal{N}(CAA^{\dagger})^{\perp}\\ &= \mathcal{R}((CAA^{\dagger})^*) = \mathcal{R}(AA^{\dagger}C^*).\end{aligned}$$

(iii) ⇒ (i) Since $AA^{\|(B,C),\dagger} = (AA^{\|(B,C),\dagger})^2$, to prove (i), it suffices to show $(AA^{\|(B,C),\dagger})^* = AA^{\|(B,C),\dagger}$. As $\mathcal{R}(AA^{\dagger}C^*) \subseteq \mathcal{R}(AB)$, then by Theorem 2.5, we have

$$\begin{aligned}\mathcal{R}((AA^{\|(B,C),\dagger})^*) &= \mathcal{N}(AA^{\|(B,C),\dagger})^{\perp} = \mathcal{N}(CAA^{\dagger})^{\perp} = \mathcal{R}((CAA^{\dagger})^*)\\ &= \mathcal{R}(AA^{\dagger}C^*) \subseteq \mathcal{R}(AB) = \mathcal{R}(AA^{\|(B,C),\dagger}).\end{aligned}$$

Hence, there exists some $D \in \mathbb{C}^{n\times n}$ such that $(AA^{\|(B,C),\dagger})^* = AA^{\|(B,C),\dagger}D = AA^{\|(B,C),\dagger}AA^{\|(B,C),\dagger}D = AA^{\|(B,C),\dagger}(AA^{\|(B,C),\dagger})^* = AA^{\|(B,C),\dagger}$, as required.

(iv) ⇒ (ii) It follows from Theorem 2.5 (iii) that $\mathrm{rk}(AB) = \mathrm{rk}(CAA^{\dagger}) = \mathrm{rk}(AA^{\dagger}C^*)$, whence $\mathcal{R}(AB) = \mathcal{R}(AA^{\dagger}C^*)$ since $\mathcal{R}(AB) \subseteq \mathcal{R}(AA^{\dagger}C^*)$. □

In Theorem 2.7 below, we derive the necessary and sufficient conditions such that $A^{\|(B,C),\dagger}A$ is an orthogonal projector, whose proof is similar to that of Theorem 2.6. We herein leave it to the readers.

Theorem 2.7. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ exists. Then the following statements are equivalent:*
(i) $A^{\|(B,C),\dagger}A$ *is an orthogonal projector.*
(ii) $\mathcal{R}((CA)^*) = \mathcal{R}(B)$.
(iii) $\mathcal{R}((CA)^*) \subseteq \mathcal{R}(B)$.
(iv) $\mathcal{R}(B) \subseteq \mathcal{R}((CA)^*)$.

As stated in Section 1, several types of generalized inverses are described by $A^{(2)}_{\mathcal{T},\mathcal{S}}$. We next establish the criterion of the (B,C)-MP-inverse of A using its $A^{(2)}_{\mathcal{T},\mathcal{S}}$.

Theorem 2.8. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ exists. Then $X = A^{\|(B,C),\dagger}$ if and only if $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(CAA^\dagger)}$.*

Proof. Suppose $X = A^{\|(B,C),\dagger}$. Then, by Theorem 2.5, we have $XAX = X$, $\mathcal{R}(X) = \mathcal{R}(B)$ and $\mathcal{N}(X) = \mathcal{N}(CAA^\dagger)$, so that $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(CAA^\dagger)}$.

Conversely, if $X = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(CAA^\dagger)}$, then $XAX = X, \mathcal{R}(X) = \mathcal{R}(B)$ and $\mathcal{N}(X) = \mathcal{N}(CAA^\dagger)$, and hence $\mathcal{R}(AX - I_n) \subseteq \mathcal{N}(X) = \mathcal{N}(CAA^\dagger)$. This implies $CAX = CAA^\dagger$. The inclusion $\mathcal{N}(C) \subseteq \mathcal{N}(A^{\|(B,C)})$ gives $A^{\|(B,C)} = SC$ for some $S \in \mathbb{C}^{m\times m}$. Also, from $\mathcal{R}(X) = \mathcal{R}(B)$, it follows that $X = A^{\|(B,C)}AX = SCAX = SCAA^\dagger = A^{\|(B,C)}AA^\dagger = A^{\|(B,C),\dagger}$. □

We denote by $P_{M,N}$ the projector onto M along N, where M, N are two complementary subspaces of $\mathbb{C}^{n\times 1}$, namely $\mathbb{C}^{n\times 1} = M \oplus N$.

It follows from Theorem 2.5 that $\mathcal{R}(AA^{\|(B,C),\dagger}) = \mathcal{R}(AB)$, $N(AA^{\|(B,C),\dagger}) = \mathcal{N}(CAA^\dagger)$ and $\mathcal{R}(A^{\|(B,C),\dagger}) \subseteq \mathcal{R}(B)$. So, $\mathcal{R}(AB) \oplus \mathcal{N}(CAA^\dagger) = \mathbb{C}^{n\times 1}$. Let $X = A^{\|(B,C)}AA^\dagger$. Then $AX = P_{\mathcal{R}(AB),\mathcal{N}(CAA^\dagger)}$ is a projector onto $\mathcal{R}(AB)$ along $\mathcal{N}(CAA^\dagger)$.

We next give show that $X = A^{\|(B,C)}AA^\dagger$ is the unique solution of the following system consisting of $P_{\mathcal{R}(AB),\mathcal{N}(CAA^\dagger)}$.

Theorem 2.9. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ exists. Then*

$$AX = P_{\mathcal{R}(AB),\mathcal{N}(CAA^\dagger)}, \mathcal{R}(X) \subseteq \mathcal{R}(B). \tag{2}$$

is consistent and has the unique solution $X = A^{\|(B,C),\dagger}$.

Proof. We assume that X_1, X_2 satisfy (2). Then $AX_1 = AX_2 = P_{\mathcal{R}(AB),\mathcal{N}(CAA^\dagger)}$, $\mathcal{R}(X_1) \subseteq \mathcal{R}(B)$ and $\mathcal{R}(X_2) \subseteq \mathcal{R}(B)$. We have at once $A(X_1 - X_2) = 0$, $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{N}(A)$ and $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{R}(B)$. Consequently, it follows that $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{N}(A) \cap \mathcal{R}(B)$.

Given any $X \in \mathcal{N}(A)\cap\mathcal{R}(B)$, then there exists some $T \in \mathbb{C}^{n\times n}$ such that $X = BT = A^{\|(B,C)}ABT = A^{\|(B,C)}AX = 0$ and $\mathcal{N}(A) \cap \mathcal{R}(B) = \{0\}$. Hence $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{N}(A) \cap \mathcal{R}(B) = \{0\}$ and $X_1 = X_2$. □

Remark 2.10. In Theorem 2.9, $\mathcal{R}(X) \subseteq \mathcal{R}(B)$ is equivalent to the condition $X = A^{\|(B,C)}AX$. Indeed, if $\mathcal{R}(X) \subseteq \mathcal{R}(B)$, then $X = BT = A^{\|(B,C)}ABT = A^{\|(B,C)}AX$ for some $T \in \mathbb{C}^{n\times n}$. For the converse statement, if $X = A^{\|(B,C)}AX$ then $\mathcal{R}(X) \subseteq \mathcal{R}(A^{\|(B,C)})$, so that $\mathcal{R}(X) \subseteq \mathcal{R}(B)$ since $\mathcal{R}(A^{\|(B,C)}) \subseteq \mathcal{R}(B)$.

Let $\mathbb{C}^P_n$ be the set of $n \times n$ projector matrices. Given $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ exists, then $AA^{\|(B,C),\dagger} \in \mathbb{C}^P_n$, $A^{\|(B,C),\dagger}A \in \mathbb{C}^P_m$.

The following result presents characterizations for the (B,C)-MP-inverse of A using projectors $AA^{\|(B,C),\dagger}$ and $A^{\|(B,C),\dagger}A$.

Theorem 2.11. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ exists. Then the following conditions are equivalent:*
(i) $X = A^{\|(B,C),\dagger}$.
(ii) $CAX = CAA^\dagger, \mathcal{R}(X) = \mathcal{R}(B)$.
(iii) $CAX = CAA^\dagger, X = A^{\|(B,C)}AX$.
(iv) $XAB = B, \mathcal{N}(X) = \mathcal{N}(CAA^\dagger)$.
(v) $XAA^{\|(B,C)} = A^{\|(B,C)}, \mathrm{rk}(X) = \mathrm{rk}(B), CAX = CAA^\dagger$.
(vi) $AX = AA^{\|(B,C)}AA^\dagger, \mathcal{R}(X) = \mathcal{R}(B)$.
(vii) $AX = AA^{\|(B,C)}AA^\dagger, X = A^{\|(B,C)}AX$.
(viii) $AX \in \mathbb{C}^P_n, \mathcal{R}(X) = \mathcal{R}(B), \mathcal{N}(X) = \mathcal{N}(CAA^\dagger)$.
(ix) $AX \in \mathbb{C}^P_n, X = A^{\|(B,C)}AX, \mathcal{N}(X) = \mathcal{N}(CAA^\dagger)$.

(x) $AXA = AA^{\|(B,C)}A, \mathcal{R}(X) = \mathcal{R}(B), \mathcal{N}(X) = \mathcal{N}(CAA^{\dagger})$.
(xi) $XA = A^{\|(B,C)}A, \mathcal{N}(X) = \mathcal{N}(CAA^{\dagger})$.
(xii) $XA \in \mathbb{C}_m^P, \mathcal{R}(X) = \mathcal{R}(B), \mathcal{N}(X) = \mathcal{N}(CAA^{\dagger})$.

Proof. (i) implies these items (ii)-(xii) by Theorems 2.5 and 2.8; (ii) ⇒ (iii), (vi) ⇒ (vii), (viii) ⇒ (ix), (x) ⇒ (xi) follow from Remark 2.10.

(iii) ⇒ (i) It follows from $\mathcal{N}(C) \subseteq \mathcal{N}(A^{\|(B,C)})$ that $X = A^{\|(B,C)}AX = SCAX = SCAA^{\dagger} = A^{\|(B,C)}AA^{\dagger} = A^{\|(B,C),\dagger}$ for some $S \in \mathbb{C}^{m\times m}$.

(iv) ⇒ (v) Since $\mathcal{R}(A^{\|(B,C)}) \subseteq \mathcal{R}(B)$, we have $A^{\|(B,C)} = BS$ for suitable $S \in \mathbb{C}^{n\times n}$, This combines with $XAB = B$ to imply $XAA^{\|(B,C)} = A^{\|(B,C)}$. According to $\mathcal{N}(CAA^{\dagger}) = \mathcal{N}(X)$ and Theorem 2.5, we have $\text{rk}(X) = \text{rk}(CAA^{\dagger}) = \text{rk}(B)$. Also, $XAB = B$ implies $\mathcal{R}(B) \subseteq \mathcal{R}(X)$. So, $\mathcal{R}(X) = \mathcal{R}(B)$. Then X can be written as the form of BT for suitable $T \in \mathbb{C}^{n\times n}$. Post-multiplying $XAB = B$ by T gives $XAX = X$. So, $\mathcal{R}(I_n - AX) \subseteq \mathcal{N}(X) = \mathcal{N}(CAA^{\dagger})$. Therefore, $CAX = CAA^{\dagger}$.

(v) ⇒ (ii) Post-Multiplying $XAA^{\|(B,C)} = A^{\|(B,C)}$ by AB implies $XAB = B$. Then we have at once $\mathcal{R}(B) \subseteq \mathcal{R}(X)$, which combines with $\text{rk}(X) = \text{rk}(B)$ to ensure $\mathcal{R}(X) = \mathcal{R}(B)$.

(vii) ⇒ (i) Given $AX = AA^{\|(B,C)}AA^{\dagger}$, then it follows that $X = A^{\|(B,C)}AX = A^{\|(B,C)}AA^{\|(B,C)}AA^{\dagger} = A^{\|(B,C)}AA^{\dagger} = A^{\|(B,C),\dagger}$.

(ix) ⇒ (iii) By $AX \in \mathbb{C}_n^P$, we have $X = A^{\|(B,C)}AX = A^{\|(B,C)}AXAX = XAX$. Hence, $\mathcal{R}(I_n - AX) \subseteq \mathcal{N}(X) = \mathcal{N}(CAA^{\dagger})$ and $CAX = CAA^{\dagger}$.

(xi) ⇒ (iv) is obvious.

(xii) ⇒ (ii) As $XA \in \mathbb{C}_m^P$, then $\mathcal{R}(A - AXA) \subseteq \mathcal{N}(X) = \mathcal{N}(CAA^{\dagger})$, so that $CA = CAXA$. Post-multiplying $CA = CAXA$ by $A^{\dagger}$ gives $CAA^{\dagger} = CAXAA^{\dagger}$. From $\mathcal{R}(I_n - AA^{\dagger}) \subseteq \mathcal{N}(CAA^{\dagger}) = \mathcal{N}(X)$, one has $X = XAA^{\dagger}$ and $CAA^{\dagger} = CA(XAA^{\dagger}) = CAX$. □

Remark 2.12. In Theorem 2.11 above, the condition $\mathcal{R}(X) = \mathcal{R}(B)$ can be weaken to the inclusion $\mathcal{R}(X) \subseteq \mathcal{R}(B)$.

3. Connections with other generalized inverses

Let $A \in \mathbb{C}^{n\times m}$ and $B, C, B', C' \in \mathbb{C}^{m\times n}$. Benítez et al. in [2, Remark 4.3] proved that if $\mathcal{R}(B) = \mathcal{R}(B')$, $\mathcal{N}(C) = \mathcal{N}(C')$, then the existence of $A^{\|(B,C)}$ coincides with that of $A^{\|(B',C')}$ and $A^{\|(B,C)} = A^{\|(B',C')}$.

The following result shows that the converse statement also holds.

Lemma 3.1. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C, B', C' \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ and $A^{\|(B',C')}$ exist. Then the following conditions are equivalent*:
(i) $A^{\|(B,C)} = A^{\|(B',C')}$.
(ii) $\mathcal{R}(B) = \mathcal{R}(B')$, $\mathcal{N}(C) = \mathcal{N}(C')$.
(iii) $\mathcal{R}(B) \subseteq \mathcal{R}(B')$, $\mathcal{N}(C) \subseteq \mathcal{N}(C')$.

Proof. (i) ⇒ (ii) Post-multiplying $A^{\|(B,C)} = A^{\|(B',C')}$ by AB gives $B = A^{\|(B',C')}AB$, and $\mathcal{R}(B) \subseteq \mathcal{R}(A^{\|(B',C')}) \subseteq \mathcal{R}(B')$. Pre-multiplying $A^{\|(B,C)} = A^{\|(B',C')}$ by CA yields $C = CAA^{\|(B',C')}$, so that $\mathcal{N}(C') \subseteq \mathcal{N}(A^{\|(B',C')}) \subseteq \mathcal{N}(C)$. Dually, one can get $\mathcal{R}(B') \subseteq \mathcal{R}(B)$ and $\mathcal{N}(C) \subseteq \mathcal{N}(C')$. Consequently, $\mathcal{R}(B) = \mathcal{R}(B')$, $\mathcal{N}(C) = \mathcal{N}(C')$.

(ii) ⇒ (iii) is obvious.

(iii) ⇒ (i) Note that $\mathcal{R}(B) \subseteq \mathcal{R}(B')$ implies $\text{rk}(B) \le \text{rk}(B')$, and $\mathcal{N}(C) \subseteq \mathcal{N}(C')$ gives $\text{rk}(C') \le \text{rk}(C)$. By Lemma 2.3, one knows that $\text{rk}(B) = \text{rk}(C)$ and $\text{rk}(B') = \text{rk}(C')$. So, $\text{rk}(B) = \text{rk}(B') = \text{rk}(C') = \text{rk}(C)$ and hence $\mathcal{R}(B) = \mathcal{R}(B')$, $\mathcal{N}(C) = \mathcal{N}(C')$. Hence $A^{\|(B,C)} = A^{\|(B',C')}$ from [2, Remark 4.3]. □

It is known from [4] that $A^{\dagger} = A^{\|(A^*,A^*)}$ for $A \in \mathbb{C}^{n\times m}$. Taking $B' = C' = A^*$ in Lemma 3.1, we have the following result.

Lemma 3.2. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ exists. Then the following statements are equivalent*:
(i) $A^{\|(B,C)} = A^{\dagger}$.
(ii) $\mathcal{R}(B) = \mathcal{R}(A^*)$, $\mathcal{N}(C) = \mathcal{N}(A^*)$.
(iii) $\mathcal{R}(B) \subseteq \mathcal{R}(A^*)$, $\mathcal{N}(C) \subseteq \mathcal{N}(A^*)$.

It is worth pointing out that if $A^{\|(B,C)} = A^\dagger$ then $A^{\|(B,C),\dagger} = A^{\|(B,C)}AA^\dagger = A^\dagger AA^\dagger = A^\dagger$. However, the converse statement may not be true, namely $A^{\|(B,C),\dagger} = A^\dagger$ does not imply $A^{\|(B,C)} = A^\dagger$ in general. A counterexample is given below.

Example 3.3. Set $A = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}, B = \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{C}^{2\times 2}$. As $\mathrm{rk}(CAB) = \mathrm{rk}(B) = \mathrm{rk}(C)$, then, by Lemma 2.3, $A^{\|(B,C)}$ exists. A simple computation gives $A^{\|(B,C)} = C = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}$, $A^\dagger = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ 0 & 0 \end{bmatrix}$ and hence $A^{\|(B,C),\dagger} = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ 0 & 0 \end{bmatrix}$, so that $A^\dagger = A^{\|(B,C),\dagger}$. However, $A^\dagger \neq A^{\|(B,C)}$.

The following theorem presents the necessary and sufficient conditions such that $A^\dagger = A^{\|(B,C),\dagger}$.

Theorem 3.4. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ exists. Then the following statements are equivalent*:
(i) $A^{\|(B,C),\dagger} = A^\dagger$.
(ii) $\mathcal{R}(A^*) = \mathcal{R}(B)$.
(iii) $\mathcal{R}(A^*) \subseteq \mathcal{R}(B)$.

Proof. (i) ⇒ (ii) Multiplying $A^{\|(B,C),\dagger} = A^\dagger$ by AA^* on the right side yields $A^{\|(B,C)}AA^* = A^\dagger AA^* = A^*$ and $\mathcal{R}(A^*) \subseteq \mathcal{R}(A^{\|(B,C)}) \subseteq \mathcal{R}(B)$. By Lemma 2.4, $\mathrm{rk}(B) \leq \mathrm{rk}(A) = \mathrm{rk}(A^*)$. Consequently, $\mathcal{R}(A^*) = \mathcal{R}(B)$.

(ii) ⇒ (iii) is clear.

(iii) ⇒ (i) As $\mathcal{R}(A^*) \subseteq \mathcal{R}(B)$, then there exists some $T \in \mathbb{C}^{n\times n}$ such that $A^* = BT = A^{\|(B,C)}ABT = A^{\|(B,C)}AA^*$. Multiplying $A^* = A^{\|(B,C)}AA^*$ by $(A^\dagger)^*A^\dagger$ on the right side gives $A^\dagger = A^{\|(B,C)}AA^\dagger = A^{\|(B,C),\dagger}$. □

Theorem 3.5. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$ such that $A^{\|(B,C)}$ exists. Then $A^{\|(B,C),\dagger} = A^{\|(B,C)}$ if and only if $C = CAA^\dagger$.*

Proof. Suppose $A^{\|(B,C),\dagger} = A^{\|(B,C)}$. Pre-multiplying $A^{\|(B,C),\dagger} = A^{\|(B,C)}$ by CA yields $CAA^\dagger = CAA^{\|(B,C),\dagger} = CAA^{\|(B,C)} = C$.

Conversely, since $\mathcal{N}(C) \subseteq \mathcal{N}(A^{\|(B,C)})$, there exists some $S \in \mathbb{C}^{m\times m}$ such that $A^{\|(B,C)} = SC = SCAA^\dagger = A^{\|(B,C)}AA^\dagger = A^{\|(B,C),\dagger}$. □

Suppose S is a $*$-semigroup and $a, b, c \in S$. We recall from [21] that a is (b, c)-core invertible if there exists some $x \in S$ such that $caxc = c$, $xS = bS$ and $Sx = Sc^*$. The (b, c)-core inverse x of a is uniquely determined (if it exists) and is denoted by $a^{\#}_{(b,c)}$. It is shown in [21] that a is (b, c)-core invertible if and only if a is (b, c)-invertible and c is $\{1, 3\}$-invertible. Moreover, $a^{\#}_{(b,c)} = a^{\|(b,c)}c^{(1,3)}$.

We next give the notion of (b, c)-core inverses in complex matrices.

Definition 3.6. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$. The matrix A is called (B, C)-core invertible if there exists some $X \in \mathbb{C}^{m\times m}$ such that $CAXC = C, \mathcal{R}(X) = \mathcal{R}(B)$ and $\mathcal{N}(X) = \mathcal{N}(C^*)$. Such an X is called a (B, C)-core inverse of A.*

Given any $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$, it can be proved that the (B, C)-core inverse of A is unique if it exists. As usual, we denote by $A^{\#}_{(B,C)}$ the (B, C)-core inverse of A. Moreover, one has the following equivalence: A is (B, C)-core invertible if and only if A is (B, C)-invertible. In this case, $A^{\#}_{(B,C)} = A^{\|(B,C)}C^\dagger$.

Theorem 3.7. *Let $A \in \mathbb{C}^{n\times m}$ and $B, C \in \mathbb{C}^{m\times n}$. Then the following conditions are equivalent*:
(i) *A is (B, C)-MP-invertible.*
(ii) *A is (B, C)-core invertible.*
In this case, $A^{\|(B,C),\dagger} = A^{\#}_{(B,C)}CAA^\dagger$.

Proof. The equivalence between (i) and (ii) is obvious. It next suffices to give the formula. As $A^{\#}_{(B,C)}$ exists, then $A^{\#}_{(B,C)} = A^{\|(B,C)}C^\dagger$, $A^{\#}_{(B,C)}C = A^{\|(B,C)}C^\dagger C = A^{\|(B,C)}$ and post-multiplying $A^{\#}_{(B,C)} = A^{\|(B,C)}C^\dagger$ by $CAA^\dagger$ yields $A^{\|(B,C),\dagger} = A^{\|(B,C)}AA^\dagger = A^{\#}_{(B,C)}CAA^\dagger$. □

As pointed out in [4], $A^{\|D}$ is the (D, D)-inverse of A for any $A, D \in \mathbb{C}^{n\times n}$, and hence $A^{\|(D,D),\dagger} = A^{\|(D,D)}AA^{\dagger} = A^{\|D}AA^{\dagger} = A_D^{\|,\dagger}$.

The following theorem presents the criterion such that $A^{\|(B,C),\dagger} = A_D^{\|,\dagger}$.

Theorem 3.8. *Let $A, B, C, D \in \mathbb{C}^{n\times n}$ such that $A^{\|(B,C)}$ and $A^{\|D}$ exist. Then the following conditions are equivalent*:
(i) $A^{\|(B,C),\dagger} = A_D^{\|,\dagger}$.
(ii) $\mathcal{R}(D) = \mathcal{R}(B), \mathcal{N}(DA) = \mathcal{N}(CA)$.
(iii) $\mathcal{R}(D) \subseteq \mathcal{R}(B), \mathcal{N}(DA) \subseteq \mathcal{N}(CA)$.

Proof. (i) $\Rightarrow$ (ii) As $A^{\|(B,C),\dagger} = A_D^{\|,\dagger}$, i.e., $A^{\|(B,C)}AA^{\dagger} = A^{\|D}AA^{\dagger}$, then $A^{\|(B,C)}A = A^{\|D}A$. Post-multiplying $A^{\|(B,C)}A = A^{\|D}A$ by B and D give $B = A^{\|D}AB$ and $A^{\|(B,C)}AD = D$, respectively. Then $\mathcal{R}(B) \subseteq \mathcal{R}(A^{\|D}) \subseteq \mathcal{R}(D)$ and $\mathcal{R}(D) \subseteq \mathcal{R}(A^{\|(B,C)}) \subseteq \mathcal{R}(B)$. So, $\mathcal{R}(B) = \mathcal{R}(D)$. Pre-multiplying $A^{\|(B,C)}A = A^{\|D}A$ by CA and DA yield $CA = CAA^{\|D}A$ and $DAA^{\|(B,C)}A = DA$, respectively. It follows that $\mathcal{N}(A^{\|D}A) \subseteq \mathcal{N}(CA)$ and $\mathcal{N}(A^{\|(B,C)}A) \subseteq \mathcal{N}(DA)$. By [2, Theorem 6.6], we have $\mathcal{N}(CA) = \mathcal{N}(A^{\|(B,C)}A)$ and $\mathcal{N}(DA) = \mathcal{N}(A^{\|D}A)$. Consequently, $\mathcal{N}(CA) = \mathcal{N}(DA)$.
(ii) $\Rightarrow$ (iii) is trivial.
(iii) $\Rightarrow$ (i) Since $\mathcal{R}(D) \subseteq \mathcal{R}(B)$, there exists some $T \in \mathbb{C}^{n\times n}$ such that $D = BT = A^{\|(B,C)}ABT = A^{\|(B,C)}AD$, which combines with $\mathcal{R}(A^{\|D}) \subseteq \mathcal{R}(D)$ to lead $A^{\|D} = A^{\|(B,C)}AA^{\|D}$. Similarly, as $\mathcal{N}(DA) \subseteq \mathcal{N}(CA)$, then $CA = SDA = SDAA^{\|D}A = CAA^{\|D}A$ for suitable $S \in \mathbb{C}^{n\times n}$. From $\mathcal{N}(C) \subseteq \mathcal{N}(A^{\|(B,C)})$, it follows that $A^{\|(B,C)}A = A^{\|(B,C)}AA^{\|D}A$. Consequently, $A^{\|(B,C)}A = A^{\|D}A$ and $A^{\|(B,C),\dagger} = A_D^{\|,\dagger}$. □

In Theorem 3.8, taking $D = A^m$ ($m = \text{ind}(A)$), then $A^{\|A^m} = A^D$, so that $A_{A^m}^{\|,\dagger} = A^{\|A^m}AA^{\dagger} = A^DAA^{\dagger} = A^{D,\dagger}$. Note that $\mathcal{N}(A^m) = \mathcal{N}(A^{m+1})$ since $A^m = A^DA^{m+1}$. So, we have the following result.

Corollary 3.9. *Let $A, B, C \in \mathbb{C}^{n\times n}$ and $\text{ind}(A) = m$. Suppose $A^{\|(B,C)}$ exists. Then the following conditions are equivalent*:
(i) $A^{\|(B,C),\dagger} = A^{D,\dagger}$.
(ii) $\mathcal{R}(A^m) = \mathcal{R}(B), \mathcal{N}(A^m) = \mathcal{N}(CA)$.
(iii) $\mathcal{R}(A^m) \subseteq \mathcal{R}(B), \mathcal{N}(A^m) \subseteq \mathcal{N}(CA)$.

Setting $m = 1$ in Corollary 3.9, then A is group invertible and hence $A^{D,\dagger} = A^{\#}AA^{\dagger} = A^{\textcircled{\#}}$. We claim herein that $\mathcal{N}(CA) = \mathcal{N}(A)$ in the item (ii) and $\mathcal{N}(A^m) \subseteq \mathcal{N}(CA)$ in the item (iii) can be dropped. Indeed, the condition $\mathcal{N}(A) \subseteq \mathcal{N}(CA)$ is evident. By Lemma 2.4, one knows that $\text{rk}(CA) = \text{rk}(A)$, and consequently the condition $\mathcal{N}(CA) = \mathcal{N}(A)$ in the item (ii) can be dropped.

Corollary 3.10. *Let $A, B, C \in \mathbb{C}^{n\times n}$ and $\text{ind}(A) = 1$. Suppose $A^{\|(B,C)}$ exists. Then the following statements are equivalent*:
(i) $A^{\|(B,C),\dagger} = A^{\textcircled{\#}}$.
(ii) $\mathcal{R}(A) = \mathcal{R}(B)$.
(iii) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$.

Recently, the author Zhu et al. [22] introduced W-core inverses in complex matrices. For $A, W \in \mathbb{C}^{n\times n}$, A is called W-core invertible if there exists an $X \in \mathbb{C}^{n\times n}$ satisfying $AWX = P_A$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$. Such an X is called a W-core inverse of A. It is unique if it exists and is denoted by $A_W^{\textcircled{\#}}$. It is proved that A is W-core invertible if and only if W is invertible along A (i.e., W is (A, A)-invertible). In this case, $A_W^{\textcircled{\#}} = W^{\|A}A^{\dagger}$.

We close this section with the following result, relating the (B, C)-MP-inverse and the W-core inverse.

Theorem 3.11. *Let $A, W \in \mathbb{C}^{n\times n}$. Then the following conditions are equivalent*:
(i) *W is (A, A)-MP-invertible.*
(ii) *A is W-core invertible.*
In this case, $W^{\|(A,A),\dagger} = A_W^{\textcircled{\#}}AWW^{\dagger}$.

Proof. It is known that W is (A, A)-MP-invertible if and only if W is invertible along A if and only if A is W-core invertible. It next only need to give the formula. Since $A_W^{\textcircled{\#}} = W^{\|A}A^{\dagger} = W^{\|(A,A)}A^{\dagger}$ and $W^{\|(A,A)} = W^{\|(A,A)}A^{\dagger}A$, post-multiplying $A_W^{\textcircled{\#}} = W^{\|(A,A)}A^{\dagger}$ by $AWW^{\dagger}$ gives the equality $A_W^{\textcircled{\#}}AWW^{\dagger} = W^{\|(A,A)}A^{\dagger}AWW^{\dagger} = W^{\|(A,A)}WW^{\dagger} = W^{\|(A,A),\dagger}$. □

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