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# Surface family interpolating a common spherical indicatrix curve

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**Abstract.** The trajectory of a moving particle in space forms a curve. By moving a line along a curve, a surface called ruled surface is obtained. The striction point on a ruled surface is the foot of the common normal between two consecutive generators or ruling. The set of striction points defines the striction curve. In the present paper, we obtain surfaces passing through the spherical indicatrix curves formed on the unit sphere by the end points of the geodesic Frenet frame formed on this curve. We present conditions for these curves to be asymptotic curves or geodesic on the surface. We illustrate the method with several examples.

## 1. Introduction

The study of curves and surfaces has wide application areas such as architectural design, computer aided design, astronomy, astrophysics and genetics [1–14]. We encounter curves and surfaces in every differential geometry book. Traditional studies focus on special surface curves such as geodesic, line of curvature, asymptotic curve etc. There are vast studies dealing with these special curves and their properties. However, there is an increasing interest of finding surfaces interpolating a given curve as a special curve. Recently, Güler [16] presented the geometric relationship between the focal surfaces and the original surface. Bayram [7] constructed surfaces with constant mean curvature along a timelike curve. Surfaces with a common adjoint curve are obtained in [9]. Güler et. al. [10] presented conditions for offset surfaces with a common asymptotic curve. Bayram et. al. [11] studied magnetic flux surfaces.

Another interesting curve is the spherical indicatrix curve. Let $\alpha(s)$ be a unit speed curve ($\|\alpha'(s)\| = 1,\ \forall s$) and $T(s) = \alpha'(s)$ be its tangent vector field. The tangent indicatrix of $\alpha(s)$ is the curve $\gamma(s) = T(s)$. Geometrically, $\gamma(s)$ is obtained by moving every $T(s)$ to the origin of $\mathbb{R}^3$. By definition, it lies on the unit sphere and its motion shows the turning of $\alpha(s)$. Given a timelike space curve, its directional spherical indicatrices are introduced in [17]. A new formula for binormal spherical indicatrices of magnetic curves presented by Körpınar and Baş [18]. Ateş et. al. studied tubular surfaces whose centers are semi-spherical indicatrices of a spatial curve [19]. Şahiner defined some new associated curves by Frenet vectors of tangent indicatrix of a curve in 3 dimensional Euclidean space [14].

On the other hand, curvature theory examines the simple geometric properties of lines and planes. It is the easiest way to determine solid space motions. It also deals with the velocity and acceleration distribution of the moving rigid body. Results from the curvature theory are applied to the synthesis and

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analysis of spherical, planar and spatial mechanisms. A point and a line fixed to a rigid body in space draw a ruled surface. With the help of a common perpendicular line between two adjacent main lines, a striction curve forms, which is defined as the shortest distance. On this curve, the geodesic Frenet frame is defined with the help of the director vector. The curve drawn by the end points of the director vector on the sphere is the spherical indicatrix curve. We can think of [20] as the paper that best explains the curvature theory. They characterize the shape of the ruled surface in two different ways. Firstly, the authors define a sequence of ruled surfaces associated with the trajectory ruled surface and secondly use dual vector algebra to transform the differential geometry of ruled surfaces into that of spherical curves. There has recently been an ascending interest regarding curvature theory [21–24]. Since the curvature properties in curvature theory characterize a point trajectory, the curvature theory is quite useful for the path planning of robot trajectory. When we look at the studies in the literature, we can see the curvature theory in robot trajectory motion. Some of these papers are [25–30].

In this study, the curves, that is, the spherical indicatrix curves drawn by the end points of the geodesic Frenet frame vectors on the sphere during the formation of the ruled surface using the curvature theory is discussed. We obtain the conditions for surface family passing through these curves for a common asymptotic curve or geodesic. Also, we present examples to illustrate the method.

## 2. Preliminaries

Let $p$ and $p'$ be two points in a rigid body and $P(\psi)$ and $P'(\psi)$ be their trajectories, respectively. Then, $\overline{R}(\psi) = P'(\psi) - P(\psi)$ is called the spherical indicatrix curve or the director vector which is on the surface of a sphere of radius $|p' - p|$. Now

$$L(\psi, v) = \alpha(\psi) + v\overline{R}(\psi)$$

defines a ruled surface in the parametric form, where $\alpha(\psi)$ is the base curve of it [20].

Since the shape of the ruled surface $L(\psi, v)$ is independent of the parameter $\psi$ chosen to identify, we take a standard parametrization, i.e. the arc-length parameter as

$$s(\psi) = \int_0^{\psi} \left\| d\overline{R}/dt \right\| dt, \tag{1}$$

where $R = \left\| d\overline{R}/d\psi \right\|$ is called the speed of the spherical indicatrix curve $\overline{R}(\psi)$. If $R \neq 0$, then Eqn. (1) can be revised to yield $\psi(s)$ allowing the definition of $\overline{R}(\psi(s)) = \overline{R}(s)$.

A frame $\{e, t, g\}$ called the geodesic Frenet frame is formed on the striction curve of the ruled surface $L(\psi, v)$. $e(s) = \overline{R} / \left\|\overline{R}\right\|$, $t(s) = d\overline{R}/ds$ and $g(s) = e \times t$ are unit vector fields and they are called the unit vector field along the directrix, the center normal vector field and the asymptotic normal vector field, respectively. Derivative formulas of the geodesic Frenet frame are

$$\begin{aligned} de(s)/ds &= \tfrac{1}{R} t(s), \\ dt(s)/ds &= -\tfrac{1}{R} e(s) + \tfrac{\gamma}{R} g(s), \\ dg(s)/ds &= -\tfrac{\gamma}{R} t(s), \end{aligned}$$

where $\gamma(s) = \left\langle d^2\overline{R}(s)/ds^2 \times \overline{R}(s), d\overline{R}(s)/ds \right\rangle$ is the geodesic curvature of the spherical indicatrix curve $\overline{R}(\psi)$ [20].

## 3. Surface family interpolating a common spherical indicatrix curve

### 3.1. Surface family interpolating the spherical indicatrix curve drawn by the director curve $e(s)$ of the ruled surface

In this section, we obtain surfaces interpolating the spherical indicatrix curve $\overline{R}_e(s) = e(s)$. They are given by

$$\begin{gathered}\varphi_1(s,v) = \overline{R}_e(s) + x_1(s,v)\,e(s) + x_2(s,v)\,t(s) + x_3(s,v)\,g(s), \\ A_1 \le s \le A_2,\ B_1 \le v \le B_2,\end{gathered} \tag{2}$$

where $x_i(s,v)$, $i = 1,2,3$, are the so-called marching scale functions. We assume that the curve $\overline{R}_e(s)$ is a parameter curve on the surface (2). Thus, we have

$$x_1(s,v_0) = x_2(s,v_0) = x_3(s,v_0) = 0,$$

for some $v_0 \in [B_1,B_2]$. The partial derivatives of (2) are calculated as

$$\begin{aligned}\frac{\partial\varphi_1}{\partial s}(s,v) &= \left(\frac{\partial x_1}{\partial s}(s,v) - \frac{1}{R}x_2(s,v)\right)e(s) \\ &\quad + \left(\frac{1}{R} + x_1(s,v)\frac{1}{R} + \frac{\partial x_2}{\partial s}(s,v) - \frac{\gamma}{R}x_3(s,v)\right)t(s) \\ &\quad + \left(\frac{\gamma}{R}x_2(s,v) + \frac{\partial x_3}{\partial s}(s,v)\right)g(s)\end{aligned}$$

$$\frac{\partial\varphi_1}{\partial v}(s,v) = \frac{\partial x_1}{\partial v}(s,v)\,e(s) + \frac{\partial x_2}{\partial v}(s,v)\,t(s) + \frac{\partial x_3}{\partial v}(s,v)\,g(s)$$

The normal vector field $\widehat{n_1}(s,v)$ of the surface (2) is

$$\begin{aligned}\widehat{n_1}(s,v) &= \left[\frac{\partial x_3}{\partial v}(s,v)\left(\frac{\gamma}{R}x_3(s,v) - \frac{x_1(s,v)}{R} - \frac{\partial x_2}{\partial s}(s,v) - \frac{1}{R}\right)\right. \\ &\quad \left. - \frac{\partial x_2}{\partial v}(s,v)\left(\frac{\gamma}{R}x_2(s,v) + \frac{\partial x_3}{\partial s}(s,v)\right)\right]e(s) \\ &\quad + \left[\frac{\partial x_1}{\partial v}(s,v)\left(\frac{\gamma}{R}x_2(s,v) + \frac{\partial x_3}{\partial s}(s,v)\right)\right. \\ &\quad \left. + \frac{\partial x_3}{\partial v}(s,v)\left(\frac{1}{R}x_2(s,v) - \frac{\partial x_1}{\partial s}(s,v)\right)\right]t(s) \\ &\quad + \left[\frac{\partial x_2}{\partial v}(s,v)\left(\frac{\partial x_1}{\partial v}(s,v) - \frac{1}{R}x_2(s,v)\right)\right. \\ &\quad \left. + \frac{\partial x_1}{\partial v}(s,v)\left(\frac{1}{R} + \frac{x_1(s,v)}{R} + \frac{\partial x_2}{\partial s}(s,v) - \frac{\gamma}{R}x_3(s,v)\right)\right]g(s).\end{aligned}$$

The normal vector field along the curve $\overline{R}_e(s)$ is

$$\widehat{n_1}(s,v_0) = \frac{1}{R}\frac{\partial x_1}{\partial v}(s,v_0)\,g(s) - \frac{1}{R}\frac{\partial x_3}{\partial v}(s,v_0)\,e(s) \tag{3}$$

**Theorem 3.1.** *Condition for $\overline{R}_e(s)$ to be an asymptotic curve on the surface* (2) *is*

$$\gamma\frac{\partial x_1}{\partial v}(s,v_0) + \frac{\partial x_3}{\partial v}(s,v_0) = 0,\ \forall s. \tag{4}$$

*Proof.* Since the surface normal vector field along the curve $\overline{R}_e(s)$ is orthogonal to the tangent vector field $t(s)$, we have

$$\langle \widehat{n_1}(s, v_0), t(s) \rangle = 0, \ \forall s.$$

Differentiating both sides with respect to $s$ we obtain

$$\left\langle \frac{\partial \widehat{n_1}}{\partial s}(s, v_0), t(s) \right\rangle + \langle \widehat{n_1}(s, v_0), t'(s) \rangle = 0, \ \forall s. \tag{5}$$

Using Eqn. (5), the normal curvature of the surface (2) along the curve $\overline{R}_e(s)$ is given by

$$\kappa_n = -\left\langle \frac{\partial \widehat{n_1}}{\partial s}(s, v_0), t(s) \right\rangle.$$

$\overline{R}_e(s)$ is an asymptotic curve on the surface (2) if the normal curvature vanishes. By Eqn. (3), we have

$$\left\langle \frac{\partial}{\partial s}\left( \frac{1}{R}\frac{\partial x_1}{\partial v}(s, v_0)\, g(s) - \frac{1}{R}\frac{\partial x_3}{\partial v}(s, v_0)\, e(s) \right), t(s) \right\rangle = 0$$

if and only if

$$\frac{\gamma}{R}\frac{\partial x_1}{\partial v}(s, v_0) + \frac{1}{R}\frac{\partial x_3}{\partial v}(s, v_0) = 0, \ \forall s.$$

Since $R$ is nonzero, we have the desired condition. □

**Theorem 3.2.** *Condition for $\overline{R}_e(s)$ to be a geodesic on the surface* (2) *is*

$$x_3(s, v) = v - v_0, \ x_1(s, v) = \gamma(v - v_0), \ \forall s. \tag{6}$$

*Proof.* $\overline{R}_e(s)$ is a geodesic on the surface (2) if and only if $\overline{R}_e''(s)$ is orthogonal to the surface. To satisfy this condition, one should have $\overline{R}_e''(s) \parallel \widehat{n_1}(s, v_0)$. We have

$$\overline{R}_e''(s) = t'(s) = -\frac{1}{R}e(s) + \frac{\gamma}{R}g(s).$$

Choosing of $x_1(s, v)$ and $x_3(s, v)$ as in Eqn. (6) makes $\overline{R}_e''(s)$ parallel to $\widehat{n_1}(s, v_0)$ completing the proof. □

### 3.2. *Surface family interpolating the spherical indicatrix curve drawn by the center normal vector field $t(s)$ of the geodesic Frenet frame*

Now, we construct surface family interpolating the spherical indicatrix curve $\overline{R}_t(s) = t(s)$. They are given by

$$\varphi_2(s, v) = \overline{R}_t(s) + y_1(s, v)\, e(s) + y_2(s, v)\, t(s) + y_3(s, v)\, g(s), \quad A_1 \le s \le A_2, \ B_1 \le v \le B_2. \tag{7}$$

We assume that the curve $\overline{R}_t(s)$ is a parameter curve on the surface (7). Thus, we have

$$y_1(s, v_0) = y_2(s, v_0) = y_3(s, v_0) = 0,$$

for some $v_0 \in [B_1, B_2]$. The partial derivatives of (7) are calculated as

$$\begin{aligned}\frac{\partial \varphi_2}{\partial s}(s,v) &= \left(\frac{\partial y_1}{\partial s}(s,v) - \frac{1}{R}y_2(s,v) - \frac{1}{R}\right)e(s) \\ &+ \left(\frac{1}{R}y_1(s,v) + \frac{\partial y_2}{\partial s}(s,v) + \frac{\gamma}{R}y_2(s,v) - \frac{\gamma}{R}y_3(s,v)\right)t(s) \\ &+ \left(\frac{\gamma}{R}y_2(s,v) + \frac{\partial y_3}{\partial s}(s,v)\right)g(s)\end{aligned}$$

$$\frac{\partial \varphi_2}{\partial v}(s,v) = \frac{\partial y_1}{\partial v}(s,v)\,e(s) + \frac{\partial y_2}{\partial v}(s,v)\,t(s) + \frac{\partial y_3}{\partial v}(s,v)\,g(s).$$

The normal vector field $\widehat{n_2}(s,v)$ of the surface (7) is

$$\begin{aligned}\widehat{n_2}(s,v) &= \left[\frac{\partial y_3}{\partial v}(s,v)\left(\frac{1}{R}y_1(s,v) + \frac{\partial y_2}{\partial s}(s,v) + \frac{\gamma}{R}y_2(s,v) - \frac{\gamma}{R}y_3(s,v)\right)\right. \\ &\left. -\frac{\partial y_2}{\partial v}(s,v)\left(\frac{\gamma}{R} + \frac{\partial y_3}{\partial s}(s,v)\right)\right]e(s) \\ &+ \left[\frac{\partial y_1}{\partial v}(s,v)\left(\frac{\gamma}{R} + \frac{\partial y_3}{\partial s}(s,v) + \frac{\partial y_1}{\partial s}(s,v) - \frac{1}{R}y_2(s,v) - \frac{1}{R}\right)\right. \\ &\left. +\frac{\partial y_3}{\partial v}(s,v)\left(\frac{1}{R} - \frac{\partial y_1}{\partial s}(s,v) + \frac{1}{R}y_2(s,v)\right)\right]t(s) \\ &+ \left[\frac{\partial y_2}{\partial v}(s,v)\left(\frac{\partial y_1}{\partial s}(s,v) - \frac{1}{R}y_2(s,v) - \frac{1}{R}\right)\right. \\ &\left. -\frac{\partial y_1}{\partial v}(s,v)\left(\frac{1}{R}y_1(s,v) + \frac{\partial y_2}{\partial s}(s,v) + \frac{\gamma}{R}y_2(s,v) - \frac{\gamma}{R}y_3(s,v)\right)\right]g(s).\end{aligned}$$

The normal vector field along the curve $\overline{R}_t(s)$ is

$$\widehat{n_2}(s,v_0) = \frac{1}{R}\left[-\gamma\frac{\partial y_2}{\partial v}(s,v_0)\,e(s) + \left(\gamma\frac{\partial y_1}{\partial v}(s,v_0) - \frac{\partial y_3}{\partial v}(s,v_0)\right)t(s) - \frac{\partial y_2}{\partial v}(s,v_0)\,g(s)\right].$$

**Theorem 3.3.** *Condition for $\overline{R}_t(s)$ to be an asymptotic curve on the surface* (7) *is*

$$\gamma\left(\frac{1+\gamma^2}{R}\right)\frac{\partial y_1}{\partial v}(s,v_0) + \gamma'\frac{\partial y_2}{\partial v}(s,v_0) + \frac{1+\gamma^2}{R}\frac{\partial y_3}{\partial v}(s,v_0) = 0, \ \forall s. \tag{8}$$

**Theorem 3.4.** *Condition for $\overline{R}_t(s)$ to be a geodesic on the surface* (7) *is*

$$\gamma\frac{\partial y_1}{\partial v}(s,v_0) + \frac{\partial y_3}{\partial v}(s,v_0) + \frac{1+\gamma^2}{R} = 0, \ \forall s. \tag{9}$$

### *3.3. Surface family interpolating the spherical indicatrix curve drawn by the asymptotic normal vector field $g(s)$ of the geodesic Frenet frame*

Surface family interpolating the spherical indicatrix curve $\overline{R}_g(s) = g(s)$ is given by

$$\begin{aligned}\varphi_3(s,v) = \overline{R}_g(s) + z_1(s,v)\,e(s) + z_2(s,v)\,t(s) + z_3(s,v)\,g(s), \\ A_1 \le s \le A_2, \ B_1 \le v \le B_2.\end{aligned} \tag{10}$$

We assume that the curve $\overline{R}_g(s)$ is a parameter curve on the surface (10). Thus, we have

$$z_1(s,v_0) = z_2(s,v_0) = z_3(s,v_0) = 0,$$

for some $v_0 \in [B_1, B_2]$. The partial derivatives of (10) are calculated as

$$\begin{aligned}\frac{\partial \varphi_3}{\partial s}(s,v) &= \left(\frac{\partial z_1}{\partial s}(s,v) - \frac{1}{R}z_2(s,v)\right)e(s)\\ &+\frac{1}{R}\left(z_1(s,v) + \frac{\partial z_2}{\partial s}(s,v) - \gamma - \gamma z_3(s,v)\right)t(s)\\ &+\left(\frac{\gamma}{R}z_2(s,v) + \frac{\partial z_3}{\partial s}(s,v)\right)g(s),\end{aligned}$$

$$\frac{\partial \varphi_3}{\partial v}(s,v) = \frac{\partial z_1}{\partial v}(s,v)\,e(s) + \frac{\partial z_2}{\partial v}(s,v)\,t(s) + \frac{\partial z_3}{\partial v}(s,v)\,g(s).$$

The normal vector field $\widehat{n_3}(s,v)$ of the surface (10) is

$$\begin{aligned}\widehat{n_3}(s,v) &= \left[\frac{\partial z_3}{\partial v}(s,v)\left(\frac{1}{R}z_1(s,v) - \frac{\gamma}{R} + \frac{\partial z_2}{\partial s}(s,v) - \frac{\gamma}{R}z_3(s,v)\right)\right.\\ &\left.-\frac{\partial z_2}{\partial v}(s,v)\left(\frac{\gamma}{R}z_2(s,v) + \frac{\partial z_3}{\partial s}(s,v)\right)\right]e(s)\\ &+\left[\frac{\partial z_1}{\partial v}(s,v)\left(\frac{\gamma}{R}z_2(s,v) + \frac{\partial z_3}{\partial s}(s,v)\right)\right.\\ &\left.-\frac{\partial z_3}{\partial v}(s,v)\left(\frac{\partial z_1}{\partial s}(s,v) - \frac{1}{R}z_2(s,v)\right)\right]t(s)\\ &+\left[\frac{\partial z_2}{\partial v}(s,v)\left(\frac{\partial z_1}{\partial s}(s,v) - \frac{1}{R}z_2(s,v)\right)\right.\\ &\left.-\frac{\partial z_1}{\partial v}(s,v)\left(\frac{1}{R}z_1(s,v) - \frac{1}{R} + \frac{\partial z_2}{\partial s}(s,v) - \frac{\gamma}{R}z_3(s,v)\right)\right]g(s).\end{aligned}$$

The normal vector field along the curve $\overline{R}_g(s)$ is

$$\widehat{n_3}(s,v_0) = \frac{\gamma}{R}\left(-\frac{\partial z_3}{\partial v}(s,v_0)\,e(s) + \frac{\partial z_1}{\partial v}(s,v_0)\,g(s)\right).$$

**Theorem 3.5.** *Condition for $\overline{R}_g(s)$ to be an asymptotic curve on the surface* (10) *is*

$$\frac{\partial z_3}{\partial v}(s,v_0) + \gamma\frac{\partial z_1}{\partial v}(s,v_0) = 0,\ \forall s. \tag{11}$$

**Theorem 3.6.** *Condition for $\overline{R}_g(s)$ to be a geodesic curve on the surface* (10) *is*

$$\frac{\partial z_1}{\partial v}(s,v_0) = -\frac{\gamma}{R},\ \frac{\partial z_3}{\partial v}(s,v_0) = -\frac{1}{R}\ \text{and } \gamma = \text{constant}. \tag{12}$$

## 4. Numerical examples

*4.1. Surface family interpolating the spherical indicatrix curve drawn by $e(s)$*

Let $\overline{R}(s) = \left(\frac{1}{2}\sin 2s, \frac{1}{2}\cos 2s, \frac{\sqrt{2}}{2}\right)$ be the director curve. The geodesic Frenet frame $\{e, t, g\}$ is given as follow

$$e(s) = \left(\tfrac{\sqrt{3}}{3}\sin 2s, \tfrac{\sqrt{3}}{3}\cos 2s, \tfrac{\sqrt{6}}{3}\right),$$
$$t(s) = (\cos 2s, -\sin 2s, 0),$$
$$g(s) = \left(\tfrac{\sqrt{6}}{3}\sin 2s, \tfrac{\sqrt{6}}{3}\cos 2s, -\tfrac{\sqrt{3}}{3}\right),$$

where $R = \|\overline{R}\| = \frac{\sqrt{3}}{2}$ and the geodesic curvature $\gamma = \sqrt{2}$.

If we take $x_1(s,v) = sv$, $x_2(s,v) = vs^2$, $x_3(s,v) = \sqrt{2}sv$ and $v_0 = 0$, then Eqn. (4) is satisfied, and we obtain a member of the surface family with a common asymptotic spherical indicatrix curve $\overline{R}_e(s) = e(s)$ as

$$\begin{aligned} L_1(s,v) &= (\frac{\sqrt{3}}{3}\sin 2s + s^2 v\cos 2s + \sqrt{3}sv\sin 2s, \\ &\quad \frac{\sqrt{3}}{3}\cos 2s - s^2 v\sin 2s + \sqrt{3}sv\cos 2s, \frac{\sqrt{6}}{3}), \end{aligned}$$

where $-2 < s < 2$, $-2 < v < 2$ (Fig. 1).

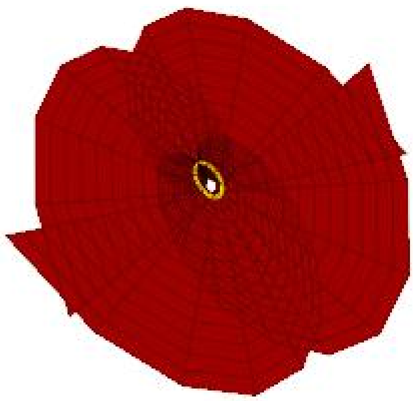

Figure 1: A member of the surface family with a common asymptotic spherical indicatrix curve $\overline{R}_e(s) = e(s)$

If we choose $x_1(s,v) = -\sqrt{2}v$, $x_2(s,v) = vs^2$, $x_3(s,v) = v$ and $v_0 = 0$, then Eqn. (6) is satisfied, and we obtain a member of the surface family with a common geodesic spherical indicatrix curve $\overline{R}_e(s) = e(s)$ as

$$L_2(s,v) = (\frac{\sqrt{3}}{3}\sin 2s + s^2 v\cos 2s, \frac{\sqrt{3}}{3}\cos 2s - s^2 v\sin 2s, \frac{\sqrt{6}}{3} - \sqrt{3}v),$$

where $-2 < s < 2$, $-2 < v < 2$ (Fig. 2).

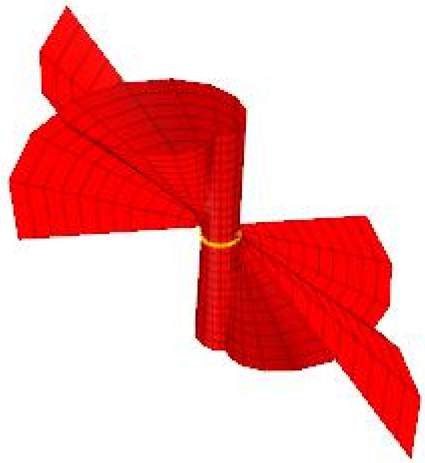

Figure 2: A member of the surface family with a common geodesic spherical indicatrix curve $\overline{R}_e(s) = e(s)$

### *4.2. Surface family interpolating the spherical indicatrix curve drawn by $t(s)$*

Taking $y_1(s,v) = vs^2$, $y_2(s,v) = vs$, $y_3(s,v) = \sqrt{2}vs^2$ and $v_0 = 0$ Eqn. (8) is satisfied, and we obtain a member of the surface family with a common asymptotic spherical indicatrix curve $\overline{R}_t(s) = t(s)$, as

$$\begin{aligned} L_3(s,v) = \; & (\cos 2s + sv\cos 2s + \sqrt{3}s^2 v \sin 2s, \\ & -\sin 2s - sv \sin 2s + \sqrt{3}s^2 v\cos 2s, 0), \end{aligned}$$

where $-2 < s < 2$, $-2 < v < 2$ (Fig. 3).

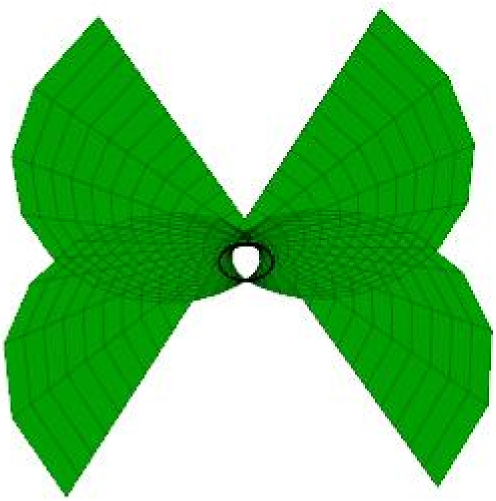

Figure 3: A member of the surface family with a common asymptotic spherical indicatrix curve $\overline{R}_t(s) = t(s)$

If we take $y_1(s,v) = \sqrt{6}v$, $y_2(s,v) = 0$, $y_3(s,v) = 4\sqrt{3}v$ and $v_0 = 0$, then Eqn. (9) is satisfied, and we obtain a member of the surface family with a common geodesic spherical indicatrix curve $\overline{R}_t(s) = t(s)$ as

$$L_4(s,v) = \left(\cos 2s + 5\sqrt{2}v\sin 2s,\ 5\sqrt{2}v\cos 2s - \sin 2s, -2v\right),$$

where $-5 < s < 5$, $-5 < v < 5$ (Fig.4).

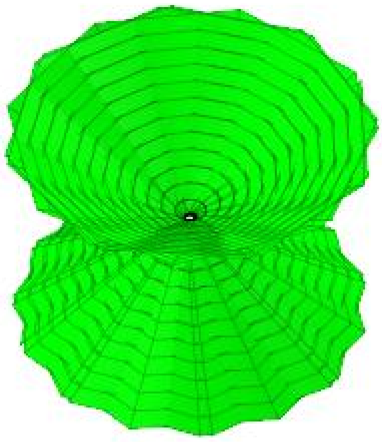

Figure 4: A member of the surface family with a common geodesic spherical indicatrix curve $\overline{R}_t(s) = t(s)$

*4.3. Surface family interpolating the spherical indicatrix curve drawn by $g(s)$*

If we choose $z_1(s,v) = sv$, $z_2(s,v) = vs^2$, $z_3(s,v) = \sqrt{2}sv$ and $v_0 = 0$, then Eqn. (11) is satisfied, and we obtain a member of the surface family with a common asymptotic spherical indicatrix curve $\overline{R}_g(s) = g(s)$ as

$$\begin{aligned} L_5(s,v) &= \Big(\frac{\sqrt{6}}{3}\sin 2s + s^2 v\cos 2s + \sqrt{3}sv\sin 2s, \\ &\quad \frac{\sqrt{6}}{3}\cos 2s - s^2 v\sin 2s + \sqrt{3}sv\cos 2s, -\frac{\sqrt{3}}{3}\Big), \end{aligned}$$

where $-2 < s < 2$, $-2 < v < 2$ (Fig. 5).

If we take $z_1(s,v) = \frac{2\sqrt{6}}{3}v$, $z_2(s,v) = vs^2$, $z_3(s,v) = -\frac{2\sqrt{3}}{3}v$ and $v_0 = 0$, then Eqn. (12) is satisfied, and we obtain a member of the surface family with a common geodesic spherical indicatrix curve $\overline{R}_g(s) = g(s)$ as

$$L_6(s,v) = \Big(\frac{\sqrt{6}}{3}\sin 2s + s^2 v\cos 2s, \frac{\sqrt{6}}{3}\cos 2s - s^2 v\sin 2s, 6v - \frac{\sqrt{3}}{3}\Big),$$

where $-2 < s < 2$, $-2 < v < 2$ (Fig. 6).

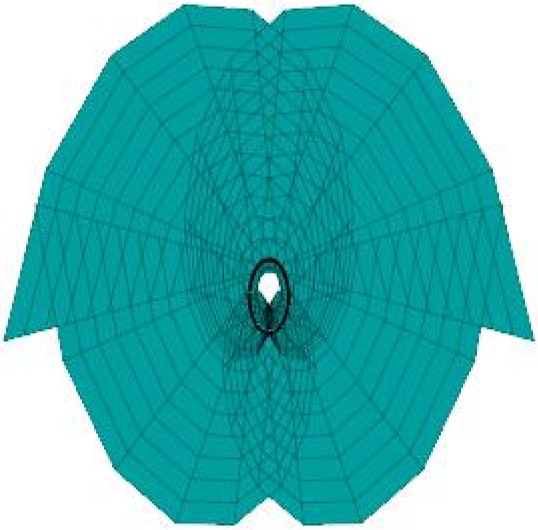

Figure 5: A member of the surface family with a common asymptotic spherical indicatrix curve $\overline{R}_g(s) = g(s)$

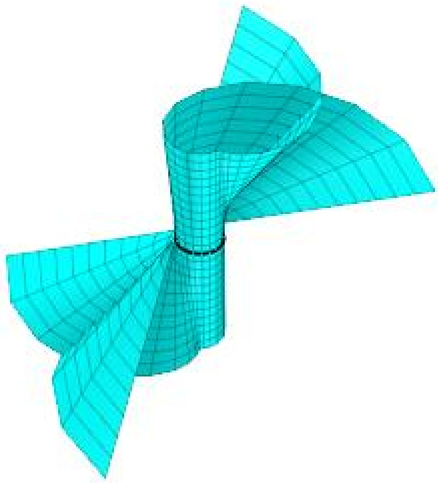

Figure 6: A member of the surface family with a common geodesic spherical indicatrix curve $\overline{R}_g(s) = g(s)$

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